On the stability of swelling porous elastic soils with a viscoelastic damping

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Abstract. The present work studies a swelling porous-elastic system with viscoelastic damping. We establish a general and optimal decay estimate which generalizes some recent results in the literature. Our result is established without imposing the usual equal-wave-speed condition associated with similar problems in literature.
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1. Introduction

The basic field equations describing the linear theory of swelling porous elastic soils, see Ieşan [?] and Quintanilla [?] are given by

$$\begin{aligned} \rho_\varphi \varphi_{tt} &= F_{1x} - K_1 + L_1, \\ \rho_\psi \psi_{tt} &= F_{2x} + K_2 + L_2, \end{aligned} \tag{1.1}$$

where the constituents $\varphi = \varphi(x,t)$ and $\psi = \psi(x,t)$ are respectively, the displacement of the fluid and the elastic solid material. The physical parameters ρ_φ and ρ_ψ are the densities of each constituent. The functions (F_1, K_1, L_1) and (F_2, K_2, L_2) are the partial tensions, internal body forces, and external forces acting on the displacement and the elastic solid respectively. In addition, the constitutive equations of partial tensions are given by

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 & b_2 \\ b_2 & b_3 \end{pmatrix}}_{\Lambda:=} \begin{pmatrix} \varphi_x \\ \psi_x \end{pmatrix} \tag{1.2}$$

where b_1, b_3 are positive constants and $b_2 \neq 0$ is a real number. The matrix Λ is positive definite (that is $b_1 b_3 > b_2^2$. Quintanilla [?] studeid (??) with

$$K_1 = K_2 = \xi(\varphi_t - \psi_t), \quad L_1 = b_3 \varphi_{xxt}, \quad L_2 = 0,$$

where ξ is a positive coefficient and established an exponential stability result. Wang and Guo [?] investigated (??) with

$$K_1 = K_2 = 0, \quad L_1 = -\rho_\varphi \gamma(x) \varphi_t, \quad L_2 = 0,$$

where $\gamma(x)$ is an internal viscous damping function with positive mean. The authors in [?] used the spectral method to prove an exponential stability result. For more related results, we refere the reader to [?]-[?] and the references cited there in.

The present work aims at studying (??) with null internal body forces, where the external force is acting only on the elastic solid as a viscoelastic force, that is:

$$K_1 = K_2 = 0, \quad L_1 = 0, \quad L_2 = -\int_0^t g(t-s)\psi_{xx}(s)ds, \tag{1.3}$$

where g is a given kernel to be specified later (also known as the relaxation function). Substituting (??) into (??), we arrive at

$$\begin{cases} \rho_\varphi \varphi_{tt} - b_1 \varphi_{xx} - b_2 \psi_{xx} = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_\psi \psi_{tt} - b_3 \psi_{xx} - b_2 \varphi_{xx} + \displaystyle\int_0^t g(t-s)\psi_{xx}(x,s)ds = 0, & \text{in } (0,1)\times(0,\infty). \end{cases} \tag{1.4}$$

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We supplement (**??**) with the following boundary conditions

$$\psi(0,t)=\psi(1,t)=\varphi(0,t)=\varphi(1,t)=0, \qquad t\geq 0 \tag{1.5}$$

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$$\varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x),\ x\in[0,1]. \tag{1.6}$$

The novelty of this paper is to improve the work established by Apalara [**?**], where he considered (**??**) and proved a general decay result when g satisfied $g'(t)\leq -\xi(t)g(t)$. This present work considers very general condition with minimal assumption on g and prove an optimal decay estimate from which the result in [**?**] is a particular case. For more related results or background of porous elastic swelling soil theory, we refer the reader to [**?**]-[**?**] and the references cited there.

The rest of this work is organized as follows: In Section **??**, we present preliminary materials which will be helpful in obtaining our results. In Section **??**, we establish some useful lemmas. In Section **??**, we study the decay rate of the energy functional associated to problem (**??**)-(**??**).

2. Assumptions and functional setting

From now on, the variables C denote a positive constants that may change within lines or from line to line. We denote by $\|.\|_2$ the usual norm in $L^2(0,1)$. We consider the following assumptions on g

(A_1) The kernel $g:[0,+\infty)\to(0,+\infty)$ is a C^1 decreasing function satisfying

$$g_0:=\int_0^\infty g(s)ds<b$$

where $b=b_3-\dfrac{b_2^2}{b_1}>0$.

(A_2) There exists a C^1 function $G:[0,+\infty)\to[0,+\infty)$ which is linear or it is strictly convex C^2 function on $(0,\eta]$, $\eta\leq g(t_0)$, for any $t_0>0$ with $\mathrm{G}(0)=\mathrm{G}'(0)=0$ and a positive non-increasing differentiable function $\xi:[0,+\infty)\to(0,+\infty)$, such that

$$g'(t)\leq -\xi(t)G\left(g(t)\right), \quad t\geq 0. \tag{2.1}$$

Remark 2.1. 1. *From assumption (A_1) and (A_2) we see that G is a strictly increasing convex C^2-function on $(0,r]$, with $G(0)=G'(0)=0$, thus, we can find an extension of G say*

$$\bar{G}:[0,+\infty)\to(0,+\infty)$$

which is also strictly increasing and strictly convex C^2-function. For example, for any $t>\eta$, we define $\bar{G}$ by

$$\bar{G}(s)=\frac{G''(\eta)}{2}s^2+(G'(\eta)-G''(\eta)\eta)s+G(\eta)-G'(\eta)\eta+\frac{G''(\eta)}{2}\eta^2. \tag{2.2}$$

2. *For any $t_0>0$, it follows from the fact that g is continuous, positive and $g(0)>0$, that*

$$\int_0^t g(s)ds\geq\int_0^{t_0}g(s)ds=c_0>0,\ \forall t\geq t_0. \tag{2.3}$$

3. Using the fact that g and ξ are positive, continuous, and nonincreasing, and G is continuous and positive, there exists $t_0>0$ such that, for all $t\in[0,t_0]$,

$$0<g(t_0)\leq g(t)\leq g(0), \quad 0<\xi(t_0)\leq\xi(t)\leq\xi(0).$$

It follows that

$$a_1\leq\xi(t)G(g(t))\leq a_2,$$

for some $a_1,a_2>0$. Therefore, $\forall\ t\in[0,t_0]$, we get

$$g'(t)\leq-\xi(t)G(g(t))\leq-\frac{a_1}{g(0)}g(0)\leq-\frac{a_1}{g(0)}g(t) \tag{2.4}$$

and

$$\xi(t)g(t)\leq-\frac{g(0)}{a_1}g'(t). \tag{2.5}$$

The well-posedness reads:

Theorem 2.2. *Let* $(\varphi_0, \varphi_1, \psi_0, \psi_1,) \in H_0^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1)$ *be given and assume conditions* (A_1) *and* (A_2) *hold. Then, problem* (**??**) – (**??**) *has a global weak unique solution* (z,u) *such that*

$$(\varphi, \psi) \in C\left([0,+\infty), H_0^1(0,1) \times H_0^1(0,1)\right) \cap C^1\left([0,+\infty), L^2(0,1) \times L^2(0,1)\right). \tag{2.6}$$

Furthermore, if

$$(\varphi_0, \varphi_1, \psi_0, \psi_1,) \in H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1) \times H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1),$$

then the global unique solution of problem (**??**) – (**??**) *has more regularity in the class*

$$\varphi \in C\left([0,+\infty), H^2(0,1) \cap H_0^1(0,1)\right) \cap C^1\left([0,+\infty), H_0^1(0,1)\right) \cap C^2\left([0,+\infty), L^2(0,1)\right),$$
$$\psi \in C\left([0,+\infty), H^2(0,1) \cap H_0^1(0,1)\right) \cap C^1\left([0,+\infty), H_0^1(0,1)\right) \cap C^2\left([0,+\infty), L^2(0,1)\right).$$

Proof. The result can be establish using the Galerkin approximation method similarly like in [**?**] □

We recall some lemmas which will be useful in establishing the main result.

Lemma 2.3. *Let* $w \in L^2_{loc}([0,+\infty), L^2(0,1))$*, there holds*

$$\int_0^1 \left(\int_0^t g(t-s)(w(t)-w(s))ds\right)^2 dx \leq c_p(a-g_0)(g \diamond w_x)(t), \tag{2.7}$$

$$\int_0^1 \left(\int_0^x w(y,t)dy\right)^2 dx \leq \|w\|_2^2, \tag{2.8}$$

where

$$(g \diamond w)(t) = \int_0^t g(t-s)\|w(t)-w(s)\|_2^2 ds.$$

Proof. The result follows easily by Cauchy-Schwarz and Poincaré's inequalities. □

As in [**?**], for any $0 < \alpha < 1$, let

$$h(t) = \alpha g(t) - g'(t) \quad \text{and} \quad A_\alpha = \int_0^{+\infty} \frac{g^2(s)}{\alpha g(s) - g'(s)} ds.$$

We have the following lemma.

Lemma 2.4. *Let* (φ, ψ) *be the solution of problem* (**??**) – (**??**)*. Then, for any* $0 < \alpha < 1$*, we have*

$$\int_0^1 \left(\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)ds\right)^2 dx \leq A_\alpha \left(h \diamond \psi_x\right)(t), \tag{2.9}$$

where

$$\left(h \diamond \psi_x\right)(t) = \int_0^t h(t-s)\|\psi_x(t)-\psi_x(s)\|_2^2 ds.$$

Proof. Using Cauchy-Schwarz and Poincaré's inequalities, we have

$$\begin{aligned}
&\int_0^1 \left(\int_0^t g(t-s)\left(\psi(t)-\psi(s)\right)dds\right)^2 dx \\
&= \int_0^1 \left(\int_0^t \frac{g(t-s)}{\sqrt{h(t-s)}}\sqrt{h(t-s)}\left(\psi(t)-\psi(s)\right)ds\right)^2 dx \\
&\leq \left(\int_0^{+\infty} \frac{g^2(s)}{h(s)}ds\right)\int_0^1\int_0^t h(t-s)\left(\psi(t)-\psi(s)\right)^2 dsdx \\
&\leq A_\alpha \left(h \diamond \psi_x\right)(t).
\end{aligned} \tag{2.10}$$

□

Lemma 2.5. *Suppose* F *is convex function on a close interval* $[a,b]$ *and let* $f : \Omega \to [a,b]$ *and a function* j *such that* $j(x) \geq 0$ *and* $\int_\Omega j(x)dx = a > 0$*. Then, we have the following Jensen inequality*

$$\frac{1}{a}\int_\Omega F(f(y))j(y)dy \geq F\left(\frac{1}{a}\int_\Omega f(y)j(y)dy\right). \tag{2.11}$$

3. Essential Lemmas

We state and prove some essential lemmas in this section.

Lemma 3.1. [?] *Let (φ,ψ) be the solution given by Theorem ??. Then, the energy functional E, defined by*

$$E(t)=\frac{1}{2}\int_0^1\left[\rho_\varphi\varphi_t^2+b_1\varphi_x^2+\rho_\psi\psi_t^2+\left(b_3-\int_0^t g(s)ds\right)\psi_x^2+2b_2\varphi_x\psi_x\right]dx+\frac{1}{2}\left(g\diamond\psi_x\right)(t), \tag{3.1}$$

satisfies

$$E'(t)=\frac{1}{2}\left(g'\diamond\psi_x\right)(t)-\frac{1}{2}g(t)\int_0^1\psi_x^2dx\leq\frac{1}{2}\left(g'\diamond\psi_x\right)(t)\leq 0,\ \forall\ t\geq 0. \tag{3.2}$$

Lemma 3.2. *Let (φ,ψ) be the solution given by Theorem ??. Then, the functional*

$$F_1(t):=\rho_\psi\int_0^1\psi_t\psi dx-\frac{b_2}{b_1}\rho_\varphi\int_0^1\varphi_t\psi dx$$

satisfies, for any positive σ_1

$$F_1'(t)\leq-\frac{b_0}{2}\int_0^1\psi_x^2dx+\sigma_1\int_0^1\varphi_t^2dx+\left(\rho_\psi+\frac{b_2^2\rho_\varphi^2}{4\sigma_1b_1^2}\right)\int_0^1\psi_t^2dx+\frac{A_\alpha}{2b_0}(h\diamond\psi_x)(t), \tag{3.3}$$

where $b_0>0$ is a constant.

Proof. Direct differentiation and using integration by parts yields

$$F_1'(t)=-\left(b_3-\frac{b_2^2}{b_1}\right)\int_0^1\psi_x^2dx+\rho_\psi\int_0^1\psi_t^2dx\underbrace{-\frac{b_2\rho_\varphi}{b_1}\int_0^1\varphi_t\psi_tdx}_{J_1}+\underbrace{\int_0^1\psi_x\int_0^t g(t-s)\psi_x(s)dsdx}_{J_2}. \tag{3.4}$$

Using Cauchy-Schwarz and Young's inequalities, we get

$$J_1\leq\sigma_1\int_0^1\varphi_t^2dx+\frac{b_2^2\rho_\varphi^2}{4\sigma_1b_1^2}\int_0^1\psi_t^2dx. \tag{3.5}$$

Recalling Lemma ??–??, then Cauchy-Schwarz and Young's inequalities give

$$\begin{aligned}J_2&=\int_0^t g(s)ds\int_0^1\psi_x^2dx-\int_0^1\psi_x\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))dsdx\\&\leq\left(\int_0^t g(s)ds+\varepsilon_0\right)\int_0^1\psi_x^2dx+\frac{1}{4\varepsilon_0}\int_0^1\left(\int_0^t g(t-s)\left(\psi_x(t)-\psi_x(s)\right)dds\right)^2dx\\&\leq\left(\int_0^t g(s)ds+\varepsilon_0\right)\int_0^1\psi_x^2dx+\frac{A_\alpha}{4\varepsilon_0}\left(h\diamond\psi_x\right)(t).\end{aligned} \tag{3.6}$$

By substituting the estimates in (??) and (??) into (??), we arrive at

$$\begin{aligned}F_1'(t)\leq&-\left(b_3-\frac{b_2^2}{b_1}-\int_0^t g(s)ds-\varepsilon_0\right)\int_0^1\psi_x^2dx+\sigma_1\int_0^1\varphi_t^2dx\\&+\left(\rho_\psi+\frac{b_2^2\rho_\varphi^2}{4\sigma_1b_1^2}\right)\int_0^1\psi_t^2dx+\frac{A_\alpha}{4\varepsilon_0}\left(h\diamond\psi_x\right)(t).\end{aligned} \tag{3.7}$$

From condition (A_1), we have

$$\int_0^t g(s)ds<g_0<b=b_3-\frac{b_2^2}{b_1}. \tag{3.8}$$

Thus, it follows that $b_3-\dfrac{b_2^2}{b_1}-\displaystyle\int_0^t g(s)ds=b_0>0$. Therefore, by choosing $\varepsilon_0=\dfrac{b_0}{2}$, we obtain (??). □

Lemma 3.3. *Let (φ,ψ) be the solution given by Theorem ??. Then, for any $t_0>0$, the functional*

$$F_2(t):=-\rho_\psi\int_0^1\psi_t\int_0^t g(t-s)(\psi(t)-\psi(s))dsdx$$

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satisfies, for any positive σ_2, σ_3, *the estimate*

$$\begin{aligned} F_2'(t) \leq & -\frac{\rho_\psi c_0}{2}\int_0^1 \psi_t^2 dx + \sigma_2 \int_0^1 \psi_x^2 dx + \sigma_3 \int_0^1 \varphi_x^2 dx \\ & + \left(\frac{\rho_\psi}{c_0} + A_\alpha\left(1 + \frac{b^2}{\sigma_2} + \frac{b_2^2}{4\sigma_3} + \frac{\rho_\psi \alpha^2}{c_0}\right)\right)(h \diamond \psi_x)(t). \end{aligned} \tag{3.9}$$

where $c_0 = \int_0^{t_0} g(s)ds$.

Proof. Differentiation of F_2, using (**??**), and then integrating by parts, we get

$$\begin{aligned} F_2'(t) = & -\rho_\psi \int_0^t g(s)ds \int_0^1 \psi_t^2 dx + \underbrace{b_3 \int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx}_{J_3} \\ & \underbrace{- \int_0^1 \int_0^t g(t-s)\psi_x(s)ds \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx}_{J_4} \\ & \underbrace{- \rho_\psi \int_0^1 \psi_t \int_0^t g'(t-s)(\psi(t) - \psi(s))dsdx}_{J_5} + \underbrace{b_2 \int_0^1 \varphi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx}_{J_6}. \end{aligned} \tag{3.10}$$

Using Lemma **??**, it follows by Young's, and Cauchy-Schwarz, and Poincaré's inequalities that for any $\sigma_2, \sigma_3, \varepsilon_1 > 0$,

$$\begin{aligned} J_3 &\leq \frac{\sigma_2}{2}\int_0^1 \psi_x^2 dx + \frac{b_3^2}{2\sigma_2}\int_0^1 \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s))ds\right)^2 dx \\ &\leq \frac{\sigma_2}{2}\int_0^1 \psi_x^2 dx + \frac{b_3^2 A_\alpha}{2\sigma_2}(h \diamond \psi_x)(t), \end{aligned} \tag{3.11}$$

$$\begin{aligned} J_4 &= -\int_0^t g(s)ds \int_0^1 \psi_x \int_0^t g(t-s)(\psi_x(t) - \psi_x(s))dsdx + \int_0^1 \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s))ds\right)^2 dx \\ &\leq \frac{\sigma_2}{2}\int_0^1 \psi_x^2 dx + \frac{1}{2\sigma_2}\left(\int_0^t g(s)ds\right)^2 \int_0^1 \left(\int_0^t g(t-s)(\psi_x(t) - \psi_x(s))ds\right)^2 dx + A_\alpha(h \diamond \psi_x)(t) \\ &\leq \frac{\sigma_2}{2}\int_0^1 \psi_x^2 dx + \left(1 + \frac{b^2}{2\sigma_2}\right) A_\alpha(h \diamond \psi_x)(t), \end{aligned} \tag{3.12}$$

$$\begin{aligned} -J_5 &= -\rho_\psi \alpha \int_0^1 \psi_t \int_0^t g(t-s)(\psi(t) - \psi(s))dsdx + \rho_\psi \int_0^1 \psi_t \int_0^t h(t-s)(\psi(t) - \psi(s))dsdx \\ &\leq \rho_\psi \varepsilon_1 \int_0^1 \psi_t^2 dx + \frac{\rho_\psi \alpha^2}{2\varepsilon_1}\int_0^1 \left(\int_0^t g(t-s)(\psi(t) - \psi(s))ds\right)^2 dx \\ &\quad + \frac{\rho_\psi}{2\varepsilon_1}\int_0^1 \left(\int_0^t h(t-s)(\psi(t) - \psi(s))ds\right)^2 dx \\ &\leq \rho_\psi \varepsilon_1 \int_0^1 \psi_t^2 dx + \frac{\rho_\psi \alpha^2}{2\varepsilon_1} A_\alpha(h \diamond \psi_x)(t) + \frac{\rho_\psi}{2\varepsilon_1}(h \diamond \psi_x)(t) \\ &= \rho_\psi \varepsilon_1 \int_0^1 \psi_t^2 dx + \left(\frac{\rho_\psi \alpha^2}{2\varepsilon_1} A_\alpha + \frac{\rho_\psi}{2\varepsilon_1}\right)(h \diamond \psi_x)(t), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} J_6 &\leq \sigma_3 \int_0^1 \varphi_x^2 dx + \frac{b_2^2}{4\sigma_3}\int_0^1 \left(\int_0^t g(t-s)(\psi(t) - \psi(s))ds\right)^2 dx \\ &\leq \sigma_3 \int_0^1 \varphi_x^2 dx + \frac{b_2^2}{4\sigma_3} A_\alpha(h \diamond \psi_x)(t). \end{aligned} \tag{3.14}$$

Substituting (**??**)–(**??**) into (**??**), we end up with

$$\begin{aligned} F_2'(t) \leq -\rho_\psi \left(\int_0^t g(s)ds - \varepsilon_1 \right) \int_0^1 \psi_t^2 dx + \sigma_2 \int_0^1 \psi_x^2 dx + \sigma_3 \int_0^1 \varphi_x^2 dx \\ + \left(\frac{\rho_\psi}{2\epsilon_1} + A_\alpha \left(1 + \frac{b_3^2}{\sigma_2} + \frac{b_2^2}{4\sigma_3} + \frac{\rho_\psi \alpha^2}{2\epsilon_1} \right) \right) (h \diamond \psi_x)(t). \end{aligned} \tag{3.15}$$

Recalling (**??**), it follows that for any $t \geq t_0 > 0$

$$c_0 = \int_0^{t_0} g(s)ds \leq \int_0^t g(s)ds. \tag{3.16}$$

Thus, we select $\varepsilon_1 = \dfrac{c_0}{2}$ to get (**??**). □

Lemma 3.4. *Let (φ, ψ) be the solution given by Theorem **??**. Then, the functional*

$$F_3(t) := -b_2 \int_0^1 (\psi \varphi_t - \psi_t \varphi)\, dx$$

satisfies

$$F_3'(t) \leq -\frac{b_2^2}{2\rho_\psi} \int_0^1 \varphi_x^2 dx + \left(\frac{b_2^2}{\rho_\varphi} + \frac{b^2}{\rho_\psi} + \rho_\psi \left(\frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi} \right)^2 \right) \int_0^1 \psi_x^2 dx + \frac{2A_\alpha}{\rho_\psi} (h \diamond \psi_x)(t). \tag{3.17}$$

Proof. Differentiating F_3, using (**??**) and then integration by parts give

$$\begin{aligned} F_3'(t) = &- \frac{b_2^2}{\rho_\psi} \int_0^1 \varphi_x^2 dx + \frac{b_2^2}{\rho_\varphi} \int_0^1 \psi_x^2 dx + \underbrace{b_2 \left(\frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi} \right) \int_0^1 \varphi_x \psi_x dx}_{J_7} \\ &+ \underbrace{\frac{b_2}{\rho_\psi} \int_0^1 \varphi_x \int_0^t g(t-s)\psi_x(s) ds dx}_{J_8}. \end{aligned} \tag{3.18}$$

Keeping in mind Lemma **??**–**??**, Young's and Cauchy-Schwarz inequalities yield for any $\varepsilon_2 > 0$,

$$\begin{aligned} J_7 \leq & b_2^2 \frac{\varepsilon_2}{2} \int_0^1 \varphi_x^2 dx + \frac{1}{2\varepsilon_2} \left(\frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi} \right)^2 \int_0^1 \psi_x^2 dx, \\ J_8 = & \frac{b_2}{\rho_\psi} \int_0^1 \varphi_x \int_0^t g(t-s) (\psi_x(s) - \psi_x(t)) ds dx + \frac{b_2}{\rho_\psi} \left(\int_0^t g(s)ds \right) \int_0^1 \varphi_x \psi_x dx \\ \leq & b_2^2 \frac{\varepsilon_2}{2} \int_0^1 \varphi_x^2 dx + \frac{1}{\varepsilon_2 \rho_\psi^2} \int_0^1 \left(\int_0^t g(t-s) (\psi_x(t) - \psi_x(s)) ds \right)^2 dx + \frac{1}{\varepsilon_2 \rho_\psi^2} \left(\int_0^t g(s)ds \right)^2 \int_0^1 \psi_x^2 dx \\ \leq & b_2^2 \frac{\varepsilon_2}{2} \int_0^1 \varphi_x^2 dx + \frac{A_\alpha}{\varepsilon_2 \rho_\psi^2} (h \diamond \psi_x)(t) + \frac{b^2}{\varepsilon_2 \rho_\psi^2} \int_0^1 \psi_x^2 dx. \end{aligned}$$

Substituting J_7 and J_8 into (**??**), we end up with

$$F_3'(t) \leq -b_2^2 \left(\frac{1}{\rho_\psi} - \varepsilon_2 \right) \int_0^1 \varphi_x^2 dx + \left[\frac{b_2^2}{\rho_\varphi} + \frac{b^2}{\varepsilon_2 \rho_\psi^2} + \frac{1}{2\varepsilon_2} \left(\frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi} \right)^2 \right] \int_0^1 \psi_x^2 dx + \frac{A_\alpha}{\varepsilon_2 \rho_\psi^2} (h \diamond \psi_x)(t)$$

Finally, by selecting $\varepsilon_2 = \dfrac{1}{2\rho_\psi}$, we obtain (**??**). □

Lemma 3.5. [**?**] *Let (φ, ψ) be the solution given by Theorem **??**. Then, the functional*

$$F_4(t) := -\rho_\varphi \int_0^1 \varphi_t \varphi dx$$

satisfies

$$F_4'(t) \leq -\rho_\varphi \int_0^1 \varphi_t^2 dx + 2b_1 \int_0^1 \varphi_x^2 dx + \frac{b_2^2}{4b_1} \int_0^1 \psi_x^2 dx. \tag{3.19}$$

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Lemma 3.6. Assume conditions (A_1) and (A_2) hold. Then, the functional

$$F_5(t) = \int_0^1 \int_0^t f(t-s)\psi_x^2(s)dsdx,$$

where

$$f(t) = \int_t^{+\infty} g(s)ds$$

satisfies, along the solution of Problem (**??**), the estimate

$$F_5'(t) \leq 3g_0 \int_0^1 \psi_x^2(t)dx - \frac{1}{2}(g \diamond \psi_x)(t). \tag{3.20}$$

Proof. Differentiating F_6 and observing that $f'(t) = -g(t)$, we infer

$$\begin{aligned} F_5'(t) &= \int_0^1 \int_0^t f'(t-s)\psi_x^2(s)dsdx + f(0)\int_0^1 \psi_x^2 dx \\ &= -\int_0^1 \int_0^t g(t-s)\left(\psi_x(s) - \psi_x(t)\right)^2 dsdx \\ &\quad - 2\int_0^1 \psi_x \int_0^t g(t-s)\left(\psi_x(s) - \psi_x(t)\right)dx + f(0)\int_0^1 \psi_x^2(t)dx \end{aligned}$$

But,

$$-2\int_0^1 \psi_x \int_0^t g(t-s)\left(\psi_x(s) - \psi_x(t)\right)dx \leq 2g_0 \int_0^1 \psi_x^2(t)dx + \frac{1}{2}(g \diamond \psi_x)(t).$$

Therefore,

$$F_5'(t) \quad \leq \quad 2g_0 \int_0^1 \psi_x^2(t)dx - \frac{1}{2}(g \diamond \psi_x)(t) + f(0)\int_0^1 \psi_x^2(t)dx.$$

Since f is decreasing, so $f(t) \leq f(0) = g_0$. Thus, we get

$$F_5'(t) \leq 3g_0 \int_0^1 \psi_x^2(t)dx - \frac{1}{2}(g \diamond \psi_x)(t).$$

□

Lemma 3.7. *Let (φ, ψ) be the solution given by Theorem* **??**. *Then, for suitable positive parameters $N, N_j, j = 1, 2, 3, 4$ to be choosen later, the functional*

$$F(t) = NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + N_4F_4(t)$$

satisfies

$$c_1E(t) \leq F(t) \leq c_2E(t), \quad \forall\ t \geq 0 \tag{3.21}$$

and

$$F'(t) \leq -\beta \int_0^1 \left(\varphi_t^2 + \varphi_x^2 + \psi_t^2 + \psi_x^2\right)dx + \frac{1}{2}\left(g \diamond \psi_x\right)(t), \quad \forall\ t \geq t_0, \tag{3.22}$$

where c_1, c_2 and β are some positive constants.

Proof. With simple routine estimation using Young's, Cauchy-Schwarz, and Poincaré's inequalities, we obtain (**??**) easily. Using (**??**), (**??**), (**??**), (**??**), (**??**) and recalling that $h = \alpha g - g'$, we obtain

$$\begin{aligned} F'(t) \leq &-\left[\rho_\varphi N_4 - \sigma_1 N_1\right]\int_0^1 \varphi_t^2 dx - \left[\frac{\rho_\psi c_0 N_2}{2} - N_1\left(\rho_\psi + \frac{b_2^2\rho_\varphi^2}{4\sigma_1 b_1^2}\right)\right]\int_0^1 \psi_t^2 dx \\ &- \left[\frac{b_2^2 N_3}{2\rho_\psi} - \sigma_3 N_2 2 - 2b_1 N_4\right]\int_0^1 \varphi_x^2 dx + \frac{\alpha N}{2}\left(g \diamond \psi_x\right)(t) \\ &- \left[\frac{b_0 N_1}{2} - \sigma_2 N_2 - N_3\left(\frac{b_2^2}{\rho_\varphi} + \frac{b^2}{\rho_\psi} + \rho_\psi\left(\frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi}\right)^2\right) - \frac{b_2^2 N_4}{4b_1}\right]\int_0^1 \psi_x^2 dx \\ &- \left[\frac{N}{2} - A_\alpha\left(\frac{N_1}{2b_0} + N_2\left(1 + \frac{b^2}{\sigma_2} + \frac{b_2^2}{4\sigma_3} + \frac{\rho_\psi\alpha^2}{c_0}\right) + \frac{2N_3}{\rho_\psi}\right) - \frac{\rho_\psi N_2}{c_0}\right]\left(h \diamond \psi_x\right)(t). \end{aligned}$$

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Taking

$$N_4 = 1, \quad \sigma_1 = \frac{\rho_\varphi}{2N_1}, \qquad \sigma_2 = \frac{b_3}{4N_2}, \qquad \sigma_3 = \frac{b_1}{N_2}, \qquad N_3 = \frac{8b_1\rho_\psi}{b_2^2},$$

we obtain

$$\begin{aligned} F'(t) \leq & -\frac{\rho_\varphi}{2}\int_0^1 \varphi_t^2 dx - \left[\frac{\rho_\psi c_0 N_2}{2} - N_1\left(\rho_\psi + \frac{b_2^2\rho_\varphi N_1}{2b_1^2}\right)\right]\int_0^1 \psi_t^2 dx - b_1\int_0^1 \varphi_x^2 dx \\ & - \left[\frac{b_0 N_1}{2} - \frac{8b_1\rho_\psi}{b_2^2}\left(\frac{b_2^2}{\rho_\varphi} + \frac{b^2}{\rho_\psi} + \rho_\psi\left(\frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi}\right)^2\right) - \frac{b_3}{2}\right]\int_0^1 \psi_x^2 dx + \frac{\alpha N}{2}\left(g \diamond \psi_x\right)(t) \\ & - \left[\frac{N}{2} - A_\alpha\left(\frac{N_1}{2b_0} + N_2\left(1 + \frac{b^2 N_2}{4b_3} + \frac{b_2^2 N_2}{4b_1} + \frac{\rho_\psi \alpha^2}{c_0}\right) + \frac{16b_1}{b_2^2}\right) - \frac{\rho_\psi N_2}{c_0}\right]\left(h \diamond \psi_x\right)(t). \end{aligned}$$

Now we choose the remaining constants: First, we select N_1 big enough such that

$$\frac{b_0 N_1}{2} - \frac{8b_1\rho_\psi}{b_2^2}\left(\frac{b_2^2}{\rho_\varphi} + \frac{b^2}{\rho_\psi} + \rho_\psi\left(\frac{b_1}{\rho_\varphi} - \frac{b_3}{\rho_\psi}\right)^2\right) - \frac{b_3}{2} > 0,$$

then we choose N_2 so large that

$$\frac{\rho_\psi c_0 N_2}{2} - N_1\left(\rho_\psi + \frac{b_2^2\rho_\varphi N_1}{2b_1^2}\right) > 0$$

From $h = \alpha g - g'$, it follows that $\dfrac{\alpha g^2(s)}{h(s)} = \dfrac{\alpha g^2(s)}{\alpha g(s) - g'(s)} < g(s)$; thus application of dominated convergence theorem gives

$$\alpha A_\alpha = \int_0^{+\infty} \frac{\alpha g^2(s)}{\alpha g(s) - g'(s)} ds \to 0 \quad as \quad \alpha \to 0. \tag{3.23}$$

Therefore, we can find $0 < \alpha_0 < 1$ so that if $\alpha < \alpha_0 < 1$, then

$$\alpha A_\alpha < \frac{1}{4\left(\frac{N_1}{2b_0} + N_2\left(1 + \frac{b^2 N_2}{4b_3} + \frac{b_2^2 N_2}{4b_1} + \frac{\rho_\psi \alpha^2}{c_0}\right) + \frac{16b_1}{b_2^2}\right)}.$$

Finally, we choose N so large and take $\alpha = \dfrac{1}{2N}$ such that (**??**) remains true and

$$\frac{N}{2} - A_\alpha\left(\frac{N_1}{2b_0} + N_2\left(1 + \frac{b^2 N_2}{4b_3} + \frac{b_2^2 N_2}{4b_1} + \frac{\rho_\psi \alpha^2}{c_0}\right) + \frac{16b_1}{b_2^2}\right) - \frac{\rho_\psi N_2}{c_0} > 0$$

□

4. Main decay result

Now, we state the main stability result of this paper.

Theorem 4.1. Under the assumptions (A_1) and (A_2), there exist positive constants m_1 and m_2 such that the solution energy functional (**??**) satisfies

$$E(t) \leq m_2 G_1^{-1}\left(m_1\int_{t_0}^t \xi(s)ds\right), \quad \text{where} \quad G_1(t) = \int_t^r \frac{1}{sG'(s)}ds \tag{4.1}$$

and G_1 is a strictly convex function that is non-increasing on $(0, r]$, where $r = g(t_0) > 0$ with $\lim\limits_{t\to 0} G_1(t) = +\infty$.

Proof. Using (**??**) and (**??**), if follows that for any $t \geq t_0$,

$$\int_0^{t_0} g(s)\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds \leq -\frac{g(0)}{a_1}\int_0^{t_0} g'(s)\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds \leq -cE'(t). \tag{4.2}$$

Thus, estimates (**??**) and (**??**) give

$$F'(t) \leq -\beta E(t) + \frac{1}{2}\int_0^{t_0} g(s)\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds + \frac{1}{2}\int_{t_0}^t g(s)\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds$$

$$\leq -\beta E(t) - cE'(t) + \frac{1}{2}\int_{t_0}^{t} g(s)\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds.$$

This yields

$$R_1'(t) \leq -\beta E(t) + \frac{1}{2}\int_{t_0}^{t} g(s)\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds, \ \forall t \geq t_0, \tag{4.3}$$

where $R_1 = F + cE$ and $R_1 \sim E$ by virtue of (**??**). The proof completed by considering two cases:
Case I: G is linear. Multiplying (**??**) by $\xi(t)$, recalling (**??**) and (A_2), we have

$$\begin{aligned}
\xi(t)R_1'(t) &\leq -\beta\xi(t)E(t) + \frac{1}{2}\xi(t)\int_{t_0}^{t} g(s)\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds \\
&\leq -\beta\xi(t)E(t) + \frac{1}{2}\int_{t_0}^{t} \xi(s)g(s)\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds \\
&\leq -\beta\xi(t)E(t) - \frac{1}{2}\int_{t_0}^{t} g'(s)\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds \\
&\leq -\beta\xi(t)E(t) - cE'(t), \ \forall\ t \geq t_0.
\end{aligned} \tag{4.4}$$

From (A_2), ξ is non-increasing, so we get

$$(\xi R_1 + cE)'(t) \leq -\beta\xi(t)E(t), \quad \forall\ t \geq t_0 \tag{4.5}$$

and

$$\xi R_1 + cE \sim E \quad \text{since} \quad R_1 \sim E. \tag{4.6}$$

It follows that

$$R_2'(t) \leq -\beta\xi(t)E(t) \leq -m\xi(t)R_2(t), \quad \forall\ t \geq t_0, \tag{4.7}$$

where $R_2 = \xi R_1 + cE$ and m is some positive constant. Integrating (**??**) over (t_0, t) and recalling (**??**), we obtain

$$E(t) \leq m_2 e^{-m_1\int_{t_0}^{t}\xi(s)ds} = m_2 G_1^{-1}\left(m_1\int_{t_0}^{t}\xi(s)ds\right).$$

Case II: G **is nonlinear.** We define the functional $L(t) = F(t) + F_5(t)$. Then, it follows from Lemma **??** and estimate (**??**), that for some $\lambda > 0$,

$$L'(t) \leq -\lambda E(t), \quad \forall\ t \geq t_0, \tag{4.8}$$

which gives

$$\lambda\int_{t_0}^{t} E(s)ds \leq L(t_0) - L(t) \leq L(t_0).$$

It follows that

$$\int_0^{+\infty} E(s)ds < \infty. \tag{4.9}$$

Using (**??**), we can choose $0 < \mu < 1$ such that

$$q(t) := \mu\int_{t_0}^{t}\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds,$$

satisfies

$$q(t) < 1, \forall\ t \geq t_0. \tag{4.10}$$

From here onward, we assume $q(t) > 0$ for all $t \geq t_0$; otherwise estimate (**??**) yields an exponential decay result. Now, we define the function $u(t)$ by

$$u(t) = -\int_{t_0}^{t} g'(s)\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds.$$

It follows from (**??**) that $u(t) \leq -cE'(t)$, $\forall\ t \geq t_0$. Thanks to remark **??**, condition (A_2), (**??**), and Jensen's inequality, we obtain

$$u(t) = \frac{1}{\mu q(t)}\int_{t_0}^{t} q(t)(-g'(s))\mu\int_0^1 \left(\psi_x(t) - \psi_x(t-s)\right)^2 dxds$$

$$\begin{aligned}
&\geq \frac{1}{\mu q(t)}\int_{t_0}^{t} q(t)\xi(s)G(g(s))\mu\int_0^1 \left(\psi_x(t)-\psi_x(t-s)\right)^2 dxds\\
&\geq \frac{\xi(t)}{\mu q(t)}\int_{t_0}^{t} G(q(t)g(s))\mu\int_0^1 \left(\psi_x(t)-\psi_x(t-s)\right)^2 dxds\\
&\geq \frac{\xi(t)}{\mu}G\left(\mu\int_{t_0}^{t} g(s)\int_0^1 \left(\psi_x(t)-\psi_x(t-s)\right)^2 dxds\right)\\
&= \frac{\xi(t)}{\mu}\bar{G}\left(\mu\int_{t_0}^{t} g(s)\int_0^1 \left(\psi_x(t)-\psi_x(t-s)\right)^2 dxds\right),
\end{aligned} \tag{4.11}$$

where $\bar{G}$ is the convex extension of G on $(0,+\infty)$. It follows from estimate (**??**) that

$$\int_{t_0}^{t} g(s)\int_0^1 \left(\psi_x(t)-\psi_x(t-s)\right)^2 dxds \leq \frac{1}{\mu}\bar{G}^{-1}\left(\frac{\mu u(t)}{\xi(t)}\right),\ \forall\ t\geq t_0.$$

Thus estimate (**??**) takes the form

$$R_1'(t)\leq -\beta E(t)+c\bar{G}^{-1}\left(\frac{\mu u(t)}{\xi(t)}\right),\quad \forall\ t\geq t_0. \tag{4.12}$$

Let $\eta_0<\eta\leq g(t_0)$, to be chosen later, and define $R_2(t)$ by

$$R_2(t):=\bar{G}'\left(\eta_0\frac{E(t)}{E(0)}\right)R_1(t)+E(t)\sim E(t)\ \text{ since } R_1\sim E.$$

By using (**??**) and recalling that

$$E'(t)\leq 0,\ \bar{G}'(t)>0,\ \bar{G}''(t)>0,$$

we obtain

$$\begin{aligned}
R_2'(t) &= \eta_0\frac{E'(t)}{E(0)}\bar{G}''\left(\eta_0\frac{E(t)}{E(0)}\right)R_1(t)+\bar{G}'\left(\eta_0\frac{E(t)}{E(0)}\right)R_1'(t)+E'(t)\\
&\leq -\beta E(t)\bar{G}'\left(\eta_0\frac{E(t)}{E(0)}\right)+\underbrace{c\bar{G}'\left(\eta_0\frac{E(t)}{E(0)}\right)\bar{G}^{-1}\left(\mu\frac{u(t)}{\xi(t)}\right)}_{J}+E'(t),\ \forall\ t\geq t_0.
\end{aligned} \tag{4.13}$$

To estimate the term J, we consider the convex conjugate $\bar{G}^*$ of $\bar{G}$ in the sense of Young, see [**?**], defined by

$$\bar{G}^*(s)=\tau(\bar{G}')^{-1}(s)-\bar{G}\left[(\bar{G}')(s)\right] \tag{4.14}$$

and satisfies the generalized Young inequality

$$AB\leq \bar{G}^*(A)+\bar{G}(B). \tag{4.15}$$

We set $A=\bar{G}'\left(r_0\frac{E(t)}{E(0)}\right)$ and $B=\bar{G}^{-1}\left(B\frac{u(t)}{\xi(t)}\right)$, then using (**??**) and (**??**)-(**??**), we obtain

$$\begin{aligned}
R_2'(t) &\leq -\beta E(t)\bar{G}'\left(\eta_0\frac{E(t)}{E(0)}\right)+c\bar{G}^*\left(\bar{G}'\left(\eta_0\frac{E(t)}{E(0)}\right)\right)+c\mu\frac{u(t)}{\xi(t)}+E'(t)\\
&\leq -\beta E(t)\bar{G}'\left(\eta_0\frac{E(t)}{E(0)}\right)+c\eta_0\frac{E(t)}{E(0)}\bar{G}'\left(\eta_0\frac{E(t)}{E(0)}\right)+c\mu\frac{u(t)}{\xi(t)}+E'(t).
\end{aligned} \tag{4.16}$$

Next, multiplying (**??**) by $\xi(t)$, recalling $\eta_0\frac{E(t)}{E(0)}<\eta$, then

$$\bar{G}'\left(\eta_0\frac{E(t)}{E(0)}\right)=G'\left(\eta_0\frac{E(t)}{E(0)}\right),$$

and we arrive at

$$\begin{aligned}
\xi(t)R_2'(t) &\leq -\beta\xi(t)E(t)G'\left(\eta_0\frac{E(t)}{E(0)}\right)+c\eta_0\frac{E(t)}{E(0)}\xi(t)G'\left(\eta_0\frac{E(t)}{E(0)}\right)+c\mu u(t)+\xi(t)E'(t)\\
&\leq -\beta\xi(t)E(t)G'\left(\eta_0\frac{E(t)}{E(0)}\right)+c\eta_0\frac{E(t)}{E(0)}\xi(t)G'\left(\eta_0\frac{E(t)}{E(0)}\right)-cE'(t).
\end{aligned} \tag{4.17}$$

Let $R_3=\xi R_2+cE$ and using the fact that $R_1\sim E$, we get

$$\kappa_0 R_3(t)\leq E(t)\leq \kappa_1 R_3(t), \tag{4.18}$$

for some $\kappa_0, \kappa_1 > 0$. Thus, from estimate (**??**), we obtain

$$R_3'(t) \leq -(\beta E(0) - c\eta_0)\xi(t)\frac{E(t)}{E(0)}G'\left(\eta_0\frac{E(t)}{E(0)}\right), \ \forall t \geq t_0.$$

Now, we choose $\eta_0 < \eta$ very small so that $\beta E(0) - c\eta_0 > 0$ and we get for some positive m,

$$R_3'(t) \leq -m\xi(t)\frac{E(t)}{E(0)}G'\left(\eta_0\frac{E(t)}{E(0)}\right) = -m\xi(t)G_2\left(\frac{E(t)}{E(0)}\right), \quad \forall t \geq t_0, \tag{4.19}$$

where $G_2(t) = tG'(\eta_0 t)$. Its easy to see that

$$G_2'(t) = G'(\eta_0 t) + \eta_0 G''(\eta_0 t).$$

Therefore, using the fact G is strictly convex on $(0, \eta]$, we have $G_2(t) > 0$, $G_2'(t) > 0$ on $(0, \eta]$. Let

$$R_4(t) = \kappa_0\frac{R_3(t)}{E(0)}.$$

Then, from (**??**) and (**??**), we obtain

$$R_4(t) \sim E(t) \tag{4.20}$$

and

$$R_4'(t) = \kappa_0\frac{R_3'(t)}{E(0)} \leq -m_1\xi(t)G_2(R_4(t)), \ \forall t \geq t_0. \tag{4.21}$$

The integration of (**??**) over (t_0, t) yields

$$m_1\int_{t_0}^{t}\xi(s)ds \leq -\int_{t_0}^{t}\frac{R_4'(s)}{G_2(R_4(s))}ds = \frac{1}{\eta_0}\int_{\eta_0 R_4(t)}^{\eta_0 R_4(t_0)}\frac{1}{sG'(s)}ds. \tag{4.22}$$

This implies

$$R_4(t) \leq \frac{1}{\eta_0}G_1^{-1}\left(m_1\int_{t_0}^{t}\xi(s)ds\right), \quad \text{where} \ \ G_1(t) = \int_t^{\eta}\frac{1}{sG'(s)}ds. \tag{4.23}$$

From (A_2), we see that G_1 is strictly decreasing on $(0, \eta]$ and

$$\lim_{t\longrightarrow 0}G_1(t) = +\infty.$$

Thus, (**??**) follows from (**??**) and (**??**). □

corollary 4.2. Under assumptions (A_1) and (A_2), suppose the function G in assumption (A_2) is defined by

$$G(s) = s^q, \quad p \geq 1. \tag{4.24}$$

Then there exist positive constants m and $\bar{m}$ such that the solution energy (**??**) satisfies

$$E(t) \leq \begin{cases} m\exp\left(-\bar{m}\displaystyle\int_0^t\xi(s)ds\right), & \text{for } p = 1, \\ \dfrac{m}{\left(1+\displaystyle\int_{t_0}^t\xi(s)ds\right)^{\frac{1}{p-1}}}, & \text{for } p > 1. \end{cases} \tag{4.25}$$

5. Concluding Remarks

The present work improves the result in [**?**], where the author established a general decay result. The decay results in Theorem **??** is optimal in the sense that it agrees with the decay rate of the memory term g, see Remark 2.3 in [**?**]. This decay result is paramount to the engineers and architects as they might employ it to attenuate the harmful effects of swelling soils. The result in this paper also holds for some other boundary conditions such as

$$\psi_x(0,t) = \psi_x(1,t) = \varphi_x(0,t) = \varphi_x(1,t) = 0, \qquad \psi(0,t) = \psi_x(1,t) = \varphi(0,t) = \varphi_x(1,t) = 0, \quad \text{and}$$
$$\psi_x(0,t) = \psi(1,t) = \varphi_x(0,t) = \varphi(1,t) = 0.$$

However, there might be some challenges for the following boundary conditions

$$\psi_x(0,t) = \psi_x(1,t) = \varphi(0,t) = \varphi(1,t) = 0 \quad \text{and} \quad \psi(0,t) = \psi(1,t) = \varphi_x(0,t) = \varphi_x(1,t) = 0,$$

except we impose

$$\int_0^1 \psi_0 dx = 0 \text{ and } \int_0^1 \varphi_0 dx = 0.$$

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References

[1] Ieşan, D. *On the theory of mixtures of thermoelastic solids*. J. Thermal stresses, **14**(4), 389–408 (1991).
[2] Quintanilla, R., *Exponential stability for one-dimensional problem of swelling porous elastic soils with fluid saturation*. J. Comput. Appl. Math, **145**(2), 525–533 (2002).
[3] Wang, J. M. and Guo, B. Z. *On the stability of swelling porous elastic soils with fluid saturation by one internal damping*. IMA J. Appl. Math., **71**(4), 565–582 (2006).
[4] Quintanilla, R. *Existence and exponential decay in the linear theory of viscoelastic mixtures*. European J. Mechanics-A/Solids, **24**(2), 311–324 (2005).
[5] Quintanilla, R. *Exponential stability of solutions of swelling porous elastic soils*. Meccanica, **39**(2), 139-145 (2004).
[6] Quintanilla, R. *On the linear problem of swelling porous elastic soils with incompressible fluid*. Int. J. Eng. Sci., **40**(13), 1485–1494 (2002).
[7] Bofill, F. and Quintanilla, R. *Anti-plane shear deformations of swelling porous elastic soils*. Int. J. Eng. Sci., **41**(8), 801–816 (2003).
[8] Murad, M. A. and Cushman, J. H. *Thermomechanical theories for swelling porous media with microstructure*. Int. J. Eng. Sci., **38**(5), 517–564 (2000).
[9] Apalara, T. A. *General stability result of swelling porous elastic soils with a viscoelastic damping*, Z. Angew. Math. Phys. (2020) 71:200.
[10] Eringen, A. C. *A continuum theory of swelling porous elastic soils*. Int. J. Eng. Sci., **32**(8), 1337–1349 (1994).
[11] Bedford, A. and Drumheller, D. S. *Theories of immiscible and structured mixtures*. Int. J. Eng. Sci., **21**(8), 863–960 (1983).
[12] Karalis, T. K., *On the elastic deformation of non-saturated swelling soils*. Acta mechanica, **84**(1–4), 19–45 (1990).
[13] Handy, R. L. *A stress path model for collapsible loess*. In Genesis and properties of collapsible soils. Springer, Dordrecht 33–47 (1995).
[14] Leonard, R.J., *Expansive Soils*. Shallow Foundation. Regent Centre. University of Kansas, Kansas, U.S.A., (1989).
[15] Jones, L. D. and Jefferson, I. *Expansive Soils*. ICE Publishing, 413–441 (2012).
[16] Bowels, J.E., *Foundation Design and Analysis*. McGraw Hill Inc., New York, U.S.A., (1988).
[17] Hung, V.Q., *Hidden Disaster*. University News, University of Saska Techwan. Saskatoon, Canada, (2003).
[18] Kalantari, B. *Engineering significant of swelling soils*. Research J. Appl. Sci. Eng. Tech., **4**(17), 2874–2878 (2012).
[19] Ammar-Khodja, F., Benabdallah, A., Rivera, J. M., and Racke, R., *Energy decay for Timoshenko systems of memory type*, J. Diff. Equa., **194**(1), 82–115 (2003).
[20] Messaoudi, S. A. and Mustafa, M. I., *A stability result in a memory-type Timoshenko system*, Dyn. Sys. Appl., **18**(3), 457 (2009)
[21] Apalara, T. A. *General stability of memory-type thermoelastic Timoshenko beam acting on shear force*. Continuum Mechanics and Thermodynamics, **30**(2), 291–300 (2018).
[22] Apalara, T. A. *General decay of solutions in one-dimensional porous-elastic system with memory*. J. Math. Anal. Appl., **469**(2), 457–471 (2019).
[23] Pamplona, P. X., Muñoz Rivera, J. E., Quintanilla, R., *On the decay of solutions for porous-elastic systems with history*, J. Math. Anal. Appl., **379**(2), 682–705 (2011).
[24] Feng, B. and Apalara, T. A., *Optimal decay for a porous elasticity system with memory*. J. Math. Anal. Appl., **470**(2), 1108-1128 (2019).
[25] Messaoudi, S.A. and Fareh, A., *General decay for a porous thermoelastic system with memory: Case of equal speeds*, Nonlinear Anal.: TMA, **74**(18), 6895–6906 (2011).
[26] Messaoudi, S.A. and Fareh, A., *General decay for a porous-thermoelastic system with memory: The case of nonequal speeds*, Acta Math. Sci. **33** (1), 23–40 (2013).
[27] Guesmia, A. and Messaoudi, S.A., *On the stabilization of Timoshenko systems with memory and different speeds of wave propagation*, Appl. Math. Computation **219** (17), 9424–9437 (2013).
[28] Hassan, J. H., Messaoudi, S. A. and Zahri,M. *Existence and new general decay result for a viscoleastic-type Timoshenko system,* Z. Anal. Anwend., vol. 39, 185–222, (2020). DOI: 10.4171/ZAA/1657.
[29] Mustafa M. I., *General decay result for nonlinear viscoelastic equations,* J. Math. Anal. Appl., vol. 457,no. 1, pp. 134-152, 2018. DOI: 10.1016/j.jmaa.2017.08.019
[30] Arnorld V. I., *Mathematical Methods of Classical Mechanics,* New York: Springer, 1989.