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Local and global stability analysis for Gilpin-Ayala competition model involved in harmful species via LMI approach and variational methods

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Abstract

In this paper, stability of reaction-diffusion Gilpin-Ayala competition model with Dirichlet boundary value, involved in harmful species, was investigated. Employing Mountain Pass Lemma and linear approximation principle results in the local stability criterion of the null solution of the ecosystem which owns at least three stationary solutions. On the other hand, globally asymptotical stability criterion for the null solution of the ecosystem was derived by variational methods and LMI approach. It is worth mentioning that the stability criteria of null solution presented some useful hints on how to eliminate pests and bacteria. Finally, two numerical examples show the effectiveness of the proposed method.

Keywords: Gilpin-Ayala competition model; LMI approach; Mountain Pass lemma; variational methods; Markovian jumping

1. Introduction

In 1920, Lotka and Volterra proposed the famous population competition model ([1,2]):

$$\begin{cases} \dot{x}_1(t) = x_1(t)[b_1 - a_{11}x_1(t) - a_{12}x_2(t)], \\ \dot{x}_2(t) = x_2(t)[b_2 - a_{21}x_1(t) - a_{22}x_2(t)], \end{cases} \tag{1.1}$$

where $x_i(t)$ represents the population density of the ith population at time $t(i = 1, 2)$, $b_i > 0$ represents the birth rate of the population of the ith population, $a_{ij} > 0$ represents the competition parameter of two populations, which is recognized and cited by many scholars. Diffusion is usually considered reasonably, for example in the reference [3] and related references:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\Delta u_1 + u_1(t)[b_1 - a_{11}u_1(t) - a_{12}u_2(t)], & x \in \Omega, t > 0, \\ \dfrac{\partial u_2}{\partial t} = d_2\Delta u_2 + u_2(t)[b_2 - a_{21}u_1(t) - a_{22}u_2(t)], & x \in \Omega, t > 0, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, & \\ u_1(x, 0) = u_0(x),\ u_2(x, 0) = v_0(x), \quad x \in \Omega. & \end{cases} \tag{1.2}$$

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and

$$\begin{cases} \frac{\partial u}{\partial t} =\Delta[(d_1 + a_{11}u + a_{12}v)u] + \mu_1 u(1 - u - a_1 v), \quad x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} =\Delta[(d_2 + a_{21}u + a_{22}v)v] + \mu_2 v(1 - v - a_2 u), \quad x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} =\frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \\ u(x, 0) =u_0(x),\ v(x, 0) = v_0(x), \quad x \in \Omega. \end{cases} \tag{1.3}$$

In 2017, Yuanyuan Liu and Youshan Tao studied the linear competition model of cross diffusion of two populations under Neumann boundary conditions ([4]):

$$\begin{cases} \frac{\partial u}{\partial t} =\Delta[(d_1 + a_{12}v)u] + \mu_1 u(1 - u - a_1 v), \quad x \in \Omega, t > 0, \\ 0 =\Delta v + \mu_2 v(1 - v - a_2 u), \quad x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} =\frac{\partial v}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \\ u(x, 0) =u_0(x), \quad x \in \Omega. \end{cases} \tag{1.4}$$

In 1973, Gilpin and Ayala found that the linear competition model was not consistent with the experimental results ([5]). Through accurate data analysis, they proposed a nonlinear competition model of two populations:

$$\begin{cases} \dot{x}_1(t) =x_1(t)[b_1 - a_{11}x_1^{\theta_1}(t) - a_{12}x_2(t)], \\ \dot{x}_2(t) =x_2(t)[b_2 - a_{21}x_1(t) - a_{22}x_2^{\theta_2}(t)], \end{cases} \tag{1.5}$$

where θ_1, θ_2 represents the nonlinear density constraint parameter. As pointed out in [6-9], when the parameter θ_i is much less than 1, the nonlinear density constrained model can well simulate the population ecology of Drosophila melanogaster, and the diffusion type Gilpin Ayala competition model under Neumann boundary value condition has also been studied by scholars:

$$\begin{cases} \frac{\partial u_1}{\partial t} =d_1\Delta u_1 + u_1(b_1 - a_{11}u_1^{\theta_1} - a_{12}u_2), \\ \frac{\partial u_2}{\partial t} =d_2\Delta u_2 + u_2(b_2 - a_{22}u^{\theta_2} - a_{21}u_1), \\ \frac{\partial u_1}{\partial \nu} =\frac{\partial u_2}{\partial \nu} = 0, \quad x \in \partial\Omega, t > 0, \end{cases} \tag{1.7}$$

where μ_1, μ_2, a, b, c and d all are positive numbers.

It is noted that the diffusion ecosystem with Neumann boundary value has been widely studied ([3,4,8] and related references), but the diffusive ecosystem under Dirichlet boundary value is rarely studied. In fact, the Drichlet boundary value diffusion ecosystem can better reflect the actual population ecology ([20,21]). Recently, the double positive solutions of the following delay feedback Gilpin-Ayala competition model has been studied in [20]:

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$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\Delta u_1 + u_1(b_1 - a_{11}u_1^{\theta_1} - a_{12}u_2) + k_1(r(t))[u_1 - u_1(t-\tau_1(t),x)] + \chi_1, & t \geqslant 0,\ x\in\Omega, \\ \dfrac{\partial u_2}{\partial t} = d_2\Delta u_2 + u_2(b_2 - a_{21}u_1 - a_{22}u_2^{\theta_2}) + k_2(r(t))[u_2 - u_2(t-\tau_2(t),x)] + \chi_2, & t \geqslant 0,\ x\in\Omega, \\ u_1(t,x) = u_2(t,x) = 0, \quad t\geqslant 0,\ x\in\partial\Omega, \end{cases} \tag{1.8}$$

Therefore, in this paper, the author will study the dynamic behavior of nonlinear Gilpin Ayala competition model with Dirichlet zero boundary value. This paper includes two main purposes: Under some assumptions, the author will give the existence of two nonzero steady-state solutions for this model, and the global asymptotical stability of null solution can not hold so that the local stability criterion was derived ([23]). On the other hand, under another reasonable assumptions, the author will consider the globally asymptotical stability, which may present some good suggestions on how to eliminate pests and bacteria. This is the second main objective of this paper.

Denote by λ_1 the first positive eigenvalue of the Laplace operator $-\Delta$ in $H_0^1(\Omega)$, and by $\|u\| = \sqrt{\int_\Omega |\nabla u(x)|^2 dx}$ the norm of Sobolev space $H_0^1(\Omega)$.

2. Preparation

Consider the nonlinear Gilpin-Ayala competition model under Dirichlet boundary value:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\Delta u_1 + u_1(b_1 - a_{11}u_1^{\theta_1} - a_{12}u_2), & t \geqslant 0,\ x\in\Omega, \\ \dfrac{\partial u_2}{\partial t} = d_2\Delta u_2 + u_2(b_2 - a_{21}u_1 - a_{22}u_2^{\theta_2}), & t \geqslant 0,\ x\in\Omega, \\ u_1(t,x) = u_2(t,x) = 0, \quad t\geqslant 0,\ x\in\partial\Omega, \\ u_1(0,x) = \xi_1(x), \quad u_2(0,x) = \xi_2(x), \end{cases} \tag{2.1}$$

where Ω is a domain in $\mathbb{R}^n (n \in \{2,3\})$ with the smooth boundary $\partial\Omega$.

Besides, Mountain Pass Lemma is given as follows ([12]).

Lemma 2.1 (Mountain Pass Lemma without the (PS) condition). Let X is a Banach space, $\Psi \in C^1(X,\mathbb{R})$, satisfying $\Psi(0) = 0$, and there exists $\rho > 0$ such that $\Psi|_{\partial B_\rho(0)} \geqslant \alpha > 0$. Besides, there is $e \in X \setminus \overline{B_\rho(0)}$ such that $\Psi(e) \leqslant 0$. Let Γ be the set of all paths connecting 0 and e. That is,

$$\Gamma = \{\psi \in C([0,1], H_0^1(\Omega)) : \psi(0) = 0,\ \psi(1) = e\}.$$

Set

$$c_* = \inf_{\psi\in\Gamma} \max_{s\in[0,1]} \Psi(\psi(s)).$$

Then $c_* \geqslant \alpha$, and Ψ possesses a critical sequence on c_*.

Remark 1. Lemma 2.1 is the Mountain Pass Lemma without the (PS) condition (see,e.g. [11,12]). If, in addition, Ψ satisfies the (PS) condition, then c_* is a critical value of Ψ.

3. Main results

As pointed out in [6-9], when the parameter θ_i is much less than 1, the nonlinear density constrained model can well simulate the population ecology of Drosophila melanogaster. So I assume $\theta_i \in (0,1)$, and

(H1) For each $i \in \{1,2\}$, there are positive numbers p_i, q_i such that $\frac{p_i}{q_i} - 2 = \theta_i \in (0,1)$, where p_i and q_i are a pair of Coprime odd numbers.

Theorem 3.1. Suppose (H1) holds, $b_i < d_i\lambda_1$ and $0 < \theta_i < 1$, $\forall\, i = 1,2$. Then the system (2.1) possesses at least three stationary solutions $(0,0)$, $(u_{1*}(x), 0)$ and $(0, u_{2*}(x))$, where $u_{i*}(x) \neq 0$, $\forall\, i = 1,2$.

Proof. Firstly, $(0,0)$ is a trivial solution of the system (2.1).

Next, if $(u_1(x), 0)$ is a stationary solution of the system (2.1),

$$\begin{cases} d_1\Delta u_1(x) + b_1u_1(x) - a_{11}u_1(x)^{1+\theta_1} = 0, & a.e.\ x \in \Omega, \\ u_1(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{3.1}$$

Similarly, if $(0, u_2(x))$ is a stationary solution of the system (2.1),

$$\begin{cases} d_2\Delta u_2(x) + b_2u_2(x) - a_{22}u_2(x)^{1+\theta_2} = 0, \\ u_2(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{3.2}$$

Obviously,

$$J(u_1) = \frac{1}{2}d_1\|u_1\|^2 - \frac{1}{2}b_1\int_\Omega u_1^2dx + \frac{a_{11}}{2+\theta_1}\int_\Omega u_1^{2+\theta_1}dx \tag{3.3}$$

is the functional corresponding to the equation (3.1), and $J \in C^1(H_0^1(\Omega), \mathbb{R}^1)$.

Besides, $J(0) = 0$. And Sobolev embedding theorem yields that there is $c > 0$ such that

$$\begin{aligned} J(u_1) =& \frac{1}{2}d_1\|u_1\|^2 - \frac{1}{2}b_1\int_\Omega u_1^2dx + \frac{a_{11}}{2+\theta_1}\int_\Omega u_1^{2+\theta_1}dx \geqslant \frac{1}{2}d_1\|u_1\|^2 - \frac{b_1}{2\lambda_1}\|u_1\|^2 - \frac{a_{11}}{2+\theta_1}\int_\Omega |u_1|^{2+\theta_1}dx \\ \geqslant& \frac{1}{2}d_1(1 - \frac{b_1}{d_1\lambda_1})\|u_1\|^2 - \frac{ca_{11}}{2+\theta_1}\|u_1\|^{2+\theta_1}. \end{aligned} \tag{3.4}$$

Let $\rho > 0$ small enough such that

$$J|_{\partial B_\rho(0)} \geqslant \alpha, \tag{3.5}$$

where $\alpha = \frac{1}{2}d_1(1 - \frac{b_1}{d_1\lambda_1})\rho^2 - \frac{ca_{11}}{2+\theta_1}\rho^{2+\theta_1} > 0$. Denote by $\varphi_1(x) > 0$ the eigenfunction of λ_1 , satisfying $\|\varphi_1\| = 1$ ([11, 17]). Then

$$J(-s\varphi_1) = \frac{1}{2}d_1\|-s\varphi_1\|^2 - \frac{1}{2}b_1\int_\Omega(-s\varphi_1)^2dx + \frac{a_{11}}{2+\theta_1}\int_\Omega(-s\varphi_1)^{2+\theta_1}dx \to -\infty, \quad s \to +\infty, \tag{3.6}$$

Thereby, there is a s_0 such that $s_0 > \rho$ and $J(-s_0\varphi_1) < 0$, where $\|-s_0\varphi_1\| = s_0 > \rho$.

Let Γ be the set of all paths connecting 0 and $-s_0\varphi_1$, i.e.,

$$\Gamma = \{\psi \in C([0,1], H_0^1(\Omega)) : \ \psi(0) = 0,\ \psi(1) = -s_0\varphi_1\}. \tag{3.7}$$

Set

$$c_0 = \inf_{\psi\in\Gamma}\max_{s\in[0,1]} J(\psi(s)). \tag{3.8}$$

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then

$$c_0 \geqslant \frac{1}{2}d_1(1-\frac{b_1}{d_1\lambda_1})\rho^2 - \frac{ca_{11}}{2+\theta_1}\rho^{2+\theta_1} > 0, \tag{3.9}$$

Lemma 2.1 yields that there is a sequence $\{u_{1n}\}_{n=1}^{\infty} \subset H_0^1(\Omega)$ such that

$$J(u_{1n}) \to c_0, \quad \text{and} \quad J'(u_{1n}) \to 0, \quad n \to \infty. \tag{3.10}$$

Below, similarly as those of [18], I will prove the sequence $\{u_{1n}\}_{n=1}^{\infty} \subset H_0^1(\Omega)$ satisfying (3.10) must be bounded.

In fact, (3.10) yields

$$\frac{1}{2}d_1\|u_{1n}\|^2 - \frac{1}{2}b_1\int_\Omega u_{1n}^2 dx + \frac{a_{11}}{2+\theta_1}\int_\Omega u_{1n}^{2+\theta_1}dx = c_0 + o(1) \tag{3.11}$$

and

$$d_1\|u_{1n}\|^2 - b_1\int_\Omega u_{1n}^2 dx + a_{11}\int_\Omega u_{1n}^{2+\theta_1}dx = \langle J'(u_{1n}), u_{1n}\rangle, \tag{3.12}$$

and for $\varepsilon > 0$ small enough suc that there exits a n big enough such that

$$|\langle J'(u_{1n}), u_{1n}\rangle| \leqslant \varepsilon\|u_{1n}\|. \tag{3.13}$$

So I have

$$d_1(\frac{1}{2} - \frac{1}{2+\theta_1})(1-\frac{b_1}{d_1\lambda_1})\|u_{1n}\|^2 \leqslant c_0 + o(1) - \frac{\varepsilon}{2+\theta_1}\|u_{1n}\|,$$

which means the boundedness of $\{u_{1n}\}_{n=1}^{\infty}$.

Now I shall prove that the bounded sequence $\{u_{1n}\}_{n=1}^{\infty}$ must be compact sequentially. This is only a conventional proof. However, in view of the completeness of the proof, I am willing to give the proof:

In fact, (H1) means $\frac{1}{d_1}(b_1u_1(x) - a_{11}u_1(x)^{1+\theta_1})$ satisfies the Caratheodory condition:

$$\Big|\frac{1}{d_1}(b_1u_1(x) - a_{11}u_1(x)^{1+\theta_1})\Big| \leqslant c_1 + c_2|u_1|^2, \quad \forall\,(x,u_1) \in \Omega\times\mathbb{R},$$

where c_1, c_2 are positive numbers big enough. Due to $\Omega \subset \mathbb{R}^3$, the critical Sobolev exponent is 6, and hence the operator J' : $H_0^1(\Omega) \to (H_0^1(\Omega))^*$ is compact, where the functional

$$\tilde{J} = \int_\Omega \Big(\frac{1}{2}b_1u_1^2 - \frac{a_{11}}{2+\theta_1}u_1^{2+\theta_1}\Big)dx.$$

Moreover,

$$\langle \tilde{J}'(u_1), \varphi\rangle = \int_\Omega \Big(b_1u_1(x)\varphi - a_{11}u_1(x)^{1+\theta_1}\varphi\Big)dx, \quad \forall\,\varphi \in H_0^1(\Omega).$$

and then the bounded sequence $\{u_{1n}\}_{n=1}^{\infty}$ possesses a subsequence, say, $\{u_{1n}\}_{n=1}^{\infty}$, satisfying $J'(u_{1n}) \to J'(u_{1*})$ in $(H_0^1(\Omega))^*$, $n \to \infty$, where $u_{1*} \in H_0^1(\Omega)$. For any $\varphi \in H_0^1(\Omega)$,

$$\langle J'(u_{1n}) - J'(u_{1m}), \varphi\rangle = d_1\int_\Omega (\nabla u_{1n} - \nabla u_{1m})\cdot\nabla\varphi dx - \langle \tilde{J}'(u_{1n}) - \tilde{J}'(u_{1m}), \varphi\rangle,$$

which together with $\{u_{1n}\}_{n=1}^{\infty} \subset H_0^1(\Omega)$, (3.10) and the arbitrariness of φ implies

$$\begin{aligned}\|u_{1n} - u_{1m}\|^2 &\leqslant (\|J'(u_{1n})\| + \|J'(u_{1m}\|)\|u_{1n} - u_{1m}\| + \|\tilde{J}'(u_{1n}) - \tilde{J}'(u_{1m})\|\,\|u_{1n} - u_{1m}\| \\ &\leqslant (\|J'(u_{1n})\| + \|J'(u_{1m}\| + \|\tilde{J}'(u_{1n}) - \tilde{J}'(u_{1m})\|)(\|u_{1n}\| + \|u_{1m}\|) \to 0, \quad n\to\infty,\ m\to\infty,\end{aligned}$$

This shows that $\{u_{1n}\}_{n=1}^{\infty}$ is compact sequentially. And then there exists a subsequence of $\{u_{1n}\}_{n=1}^{\infty}$ convergent to a point in $H_0^1(\Omega)$, say, $u_{1*} \in H_0^1(\Omega)$, Due to $J(u_{1*}) = c_0 \geqslant \frac{1}{2}d_1(1 - \frac{b_1}{d_1\lambda_1})\rho^2 - \frac{ca_{11}}{2+\theta_1}\rho^{2+\theta_1} > 0$, I see $u_{1*} \neq 0$, which shows that $(u_{1*}, 0) \neq (0, 0)$. Similarly, I can similarly prove there is at least another stationary solution $(0, u_{2*}) \neq (0, 0)$ for the system (2.1).

Theorem 3.1 tells us that under the assumptions of Theorem 3.1, $(0, 0)$ can not be global stable. And so the local stability is considered.

Theorem 3.2. Under the assumptions of Theorem 3.1, the zero solution $(0, 0)$ of the system (2.1) is locally asymptotically stable .

Proof. Firstly, the condition $b_i < \lambda_1 d_i$ yields,

$$B < \lambda_1 D, \tag{3.14}$$

where

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \ B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}. \tag{3.15}$$

Next, consider the following linear system:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\Delta u_1 + b_1 u_1, & t \geqslant 0, \ x \in \Omega, \\ \dfrac{\partial u_2}{\partial t} = d_2\Delta u_2 + b_2 u_2, & t \geqslant 0, \ x \in \Omega, \\ u_1(t,x) = u_2(t,x) = 0, & t \geqslant 0, \ x \in \partial\Omega, \\ u_1(0,x) = \xi_1(x), \quad u_2(0,x) = \xi_2(x), \end{cases} \tag{3.16}$$

Consider the Lyapunov function:

$$V = \int_\Omega (u_1^2 + u_2^2) dx.$$

The condition (3.14) yields

$$\begin{aligned} \frac{dV}{dt}\Big|_{(3.16)} &= \int_\Omega \Big(2d_1 u_1 \Delta u_1 + 2b_1 u_1^2 + 2d_2 u_2 \Delta u_2 + 2b_2 u_2^2\Big) dx \\ &\leqslant \int_\Omega u^T\Big(-2\lambda_1 D + 2B\Big) u dx \leqslant 0, \end{aligned} \tag{3.17}$$

where $u = (u_1, u_2)^T$. Then (3.17) yields that the zero solution $(0, 0)$ of the linear system (3.16) is asymptotically stable ([19]). And hence, the zero solution $(0, 0)$ of the nonlinear system (2.1) is locally asymptotically stable.

The main results of this section originate in the author's another work ([23]).

4. Global Stability with boundedness assumption on population densities

In this section, a suitable assumption may proposed on population densities u_1, u_2 :

$$0 \leqslant u_1 \leqslant M_1, \quad 0 \leqslant u_2 \leqslant M_2, \tag{4.1}$$

where M_i is a given positive number for a given $i \in \{1, 2\}$. Remark, this assumption is reasonable due to the limit resources ([20]). Next, the system (2.1) can be rewrite as follows,

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\Delta u_1 + b_1 u_1 - a_{11} f_1(u_1) + 0 \cdot f_2(u_2), & t \geqslant 0,\ x \in \Omega, \\ \dfrac{\partial u_2}{\partial t} = d_2\Delta u_2 + b_2 u_2 + 0 \cdot f_1(u_1) - a_{22} f_2(u_2), & t \geqslant 0,\ x \in \Omega, \\ u_1(t,x) = u_2(t, x) = 0, \quad t \geqslant 0,\ x \in \partial\Omega, \\ u_1(0,x) = \xi_1(x), \quad u_2(0, x) = \xi_2(x), \end{cases} \tag{4.2}$$

where

$$f_1(u_1) = u_1^{1+\theta_1} + \frac{a_{12}}{a_{11}} u_1 u_2, \tag{4.3}$$

$$f_2(u_2) = u_2^{1+\theta_2} + \frac{a_{21}}{a_{22}} u_1 u_2. \tag{4.4}$$

And the system (4.2) can be rewritten as follows,

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + Bu - Af(u), & t \geqslant 0,\ x \in \Omega, \\ u(t, x) = 0, & t \geqslant 0,\ x \in \partial\Omega, \\ u(0, x) = \xi(x), & x \in \Omega, \end{cases} \tag{4.5}$$

where the matrices D and B is defined in (3.15), $u = (u_1, u_2)^T$, $f(u) = (f_1(u_1), f_2(u_2))^T$, $\xi = (\xi_1, \xi_2)^T$, and

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}. \tag{4.6}$$

Moreover, since the larva usually does not have the competitive ability, and the larva matures to the adult needs a period of time, which is usually closely related to the climate, temperature, humidity and other random factors, so the delayed feedback stochastic model may be considered ([20]). Denote by $(\Upsilon,\ \mathcal{F},\ \mathbb{P})$ the complete probability space with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ (see, e.g. [22]). Let $S = \{1,\ 2,\ \cdots,\ n_0\}$ and the random form process $\{r(t) : [0,\ +\infty) \to S\}$ be a homogeneous, finite-state Markovian process with right continuous trajectories with generator $\Pi = (\gamma_{ij})_{n_0 \times n_0}$ and transition probability from mode i at time t to mode j at time $t + \delta$, $i,\ j \in S$,

$$\mathbb{P}(r(t + \delta) = j \mid r(t) = i) = \begin{cases} \gamma_{ij}\delta + o(\delta), & j \neq i \\ 1 + \gamma_{ij}\delta + o(\delta), & j = i \end{cases}$$

where $\gamma_{ij} \geqslant 0$ is transition probability rate from i to j $(j \neq i)$ and $\gamma_{ii} = -\sum_{j=1,\, j\neq i}^{n_0} \gamma_{ij}$, $\delta > 0$ and $\lim\limits_{\delta\to 0} o(\delta)/\delta = 0$.

Consider the following delayed feedback system :

$$\begin{cases} \dfrac{\partial u(t, x)}{\partial t} = D\Delta u(t, x) + Bu(t, x) - Af(u(t, x)) + K(r(t))(u(t, x) - u(t - \tau(t), x)), & t \geqslant 0,\ x \in \Omega, \\ u(t, x) = 0, & t \geqslant 0,\ x \in \partial\Omega, \\ u(0, x) = \xi(x), & x \in \Omega, \end{cases} \tag{4.7}$$

where

$$K_r = K(r(t)) = \begin{pmatrix} k_1(r(t)) & 0 \\ 0 & k_2(r(t)) \end{pmatrix},$$

$k_1(r(t))$ and $k_2(r(t))$ are feedback benefit coefficients at mode $r(t) = r \in S$. Denote $k_1(r(t)) = k_{1r}$, $k_2(r(t)) = k_{2r}$ for simple.

Theorem 4.1. If

$$\lambda_1 D > B + AF \tag{4.8}$$

then the null solution is the unique stationary solution of the ecosystem (4.7) under the restrictive condition (4.1). If, in addition, there are a sequences of positive definite diagonal matrices $P_r(r \in S), Q_i > 0(i \in \{1, 2\})$ and $W > 0$ such that

$$\begin{pmatrix} -\Theta_r & \frac{1}{2}FQ_1 - P_rA & 0 & 0 \\ * & -Q_1 & 0 & 0 \\ * & * & -(1-\tau_*)W & \frac{1}{2}FQ_2 \\ * & * & * & -Q_2 \end{pmatrix} < 0, \quad \forall\, r \in S, \tag{4.9}$$

then the null solution of the ecosystem (4.7) is globally exponentially stable, where $\tau(t) \in [0, \tau]$ with $\dot{\tau}(t) \leqslant \tau_* < 1$, $\Theta_r = 2\lambda_1 PD - 2PB - \sum_{j\in S} \gamma_{rj}P_j - W$, and

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} \tag{4.10}$$

with

$$F_1 = (1+\theta_1)M_1^{\theta_1} + \frac{a_{12}}{a_{11}} M_2 \tag{4.11}$$

and

$$F_2 = (1+\theta_2)M_2^{\theta_2} + \frac{a_{21}}{a_{22}} M_1. \tag{4.12}$$

Proof. Firstly, it follows from (4.1) and (4.3)-(4.4) that $f_i(0) = 0$, $i = 1, 2$, and

$$0 \leqslant \frac{f_1(r) - f_1(s)}{r - s} \leqslant (1+\theta_1)M_1^{\theta_1} + \frac{a_{12}}{a_{11}} M_2 \tag{4.13}$$

and

$$0 \leqslant \frac{f_2(r) - f_2(s)}{r - s} \leqslant (1+\theta_2)M_2^{\theta_2} + \frac{a_{21}}{a_{22}} M_1. \tag{4.14}$$

Next, one can see that under the restrictive condition (4.1) on the state variable u, the null solution $(0, 0)$ is the unique stationary solution of the system (4.7)

Indeed, let $u \equiv u(x)$ be a stationary solution, satisfying (4.1), then it is obvious that

$$0 = D\Delta u + Bu - Af(u), \tag{4.15}$$

which together with the definition of f, Poincare inequality and boundary value condition implies

$$\int_\Omega |u|^T(B + AF)|u|dx \geqslant \int_\Omega (|u|^T B|u| + |u|^T A|f(u)|)dx \geqslant \lambda_1 \int_\Omega |u|^T D|u|dx. \tag{4.16}$$

Combining (4.8) and (4.16) results in $u = 0$, and hence, the null solution solution must be the unique stationary solution of the ecosystem (4.7) under the restrictive condition (4.1) on the state variable u.

Let $P_r(r \in S)$ and W be positive definite matrices such that

$$V(t, r, u) = \int_\Omega u^T(t, x)P_r u(t, x)dx + \int_{t-\tau(t)}^t \int_\Omega u^T(s, x)Wu(s, x)dxds$$

since (4.8) $\Rightarrow u^T\Big(2\lambda_1 D - 2B\Big)u > 0 \Rightarrow u^T\Big(2\lambda_1 D - 2B\Big)u = |u|^T\Big(2\lambda_1 D - 2B\Big)|u|$, one can deduce

On the other hand, it follows from (4.13) and (4.14) that

$$\begin{pmatrix} u^T, f(u)^T \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2}FQ_1 \\ * & -Q_1 \end{pmatrix} \begin{pmatrix} u \\ f(u) \end{pmatrix} \geqslant 0,$$

similarly,

$$\begin{pmatrix} u^T(t-\tau(t),x), f^T(u(t-\tau(t),x)) \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2}FQ_2 \\ * & -Q_2 \end{pmatrix} \begin{pmatrix} u(t-\tau(t),x) \\ f(u(t-\tau(t),x)) \end{pmatrix} \geqslant 0,$$

Let $\mathcal{L}$ be the weak infinitesimal operator (see, e.g. [15]) such that

$$\begin{aligned}
\frac{\mathcal{L}V}{dt} \leqslant & -\int_\Omega u^T\Big(2\lambda_1 PD - 2PB - \sum_{j\in S}\gamma_{rj}P_j - W\Big)udx - 2\int_\Omega u^T P_r A f(u)dx \\
& -(1-\tau_*)\int_\Omega u^T(t-\tau(t))Wu(t-\tau(t))dx \\
\leqslant & -\int_\Omega u^T\Big(2\lambda_1 PD - 2PB - \sum_{j\in S}\gamma_{rj}P_j - W\Big)udx - 2\int_\Omega u^T P_r A f(u)dx \\
& -(1-\tau_*)\int_\Omega u^T(t-\tau(t))Wu(t-\tau(t))dx + \begin{pmatrix} u \\ f(u) \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2}FQ_1 \\ * & -Q_1 \end{pmatrix} \begin{pmatrix} u \\ f(u) \end{pmatrix} \\
& + \begin{pmatrix} u(t-\tau(t),x) \\ f(u(t-\tau(t),x)) \end{pmatrix}^T \begin{pmatrix} 0 & \frac{1}{2}FQ_2 \\ * & -Q_2 \end{pmatrix} \begin{pmatrix} u(t-\tau(t),x) \\ f(u(t-\tau(t),x)) \end{pmatrix} \\
= & \begin{pmatrix} u \\ f(u) \\ u(t-\tau(t),x) \\ f(u(t-\tau(t),x)) \end{pmatrix}^T \begin{pmatrix} -\Theta_r & \frac{1}{2}FQ_1 - P_r A & 0 & 0 \\ * & -Q_1 & 0 & 0 \\ * & * & -(1-\tau_*)W & \frac{1}{2}FQ_2 \\ * & * & * & -Q_2 \end{pmatrix} \begin{pmatrix} u \\ f(u) \\ u(t-\tau(t),x) \\ f(u(t-\tau(t),x)). \end{pmatrix}
\end{aligned} \tag{4.17}$$

Moreover,

$$\mathbb{E}V(t+\varepsilon) - \mathbb{E}V(t) = \int_t^{t+\varepsilon} \mathbb{E}\mathcal{L}V(s)ds.$$

Let $\varepsilon \to 0$, then combining (4.17) and (4.9) leads to

$$D^+\mathbb{E}V < 0. \tag{4.18}$$

Obviously there exist positive constants $\sigma_1, \sigma_2 > 0$ such that

$$\sigma_1 \|u\|^2_{L^2(\Omega)} \leqslant V(t,u) \leqslant \sigma_2\|u\|^2_\tau, \tag{4.19}$$

where

$$\|u\|^2_\tau = \sup_{s\in[-\tau,0]}\int_\Omega |u(t+s,x)|^2 dx.$$

Therefore, one can see it from (4.18) and (4.19) that the null solution of the ecosystem (4.7) is globally exponentially stable.

5. Numerical examples

Example 5.1. In the system (2.1), set $\Omega = (-\frac{1}{2},\frac{1}{2}) \times (-\frac{1}{2},\frac{1}{2}) \times (-\frac{1}{2},\frac{1}{2})$, then $\lambda_1 \geqslant 3$ ([11, Remark 14]). Set $\theta_1 = \frac{1}{3}, \theta_2 = \frac{1}{5}$, then the condition (H1) is satisfied. Assume, in addition,

$$D = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.2 \end{pmatrix},$$

then $b_i < d_i\lambda_1$ and $0 < \theta_i < 1$, $\forall\, i = 1,2$. Theorem 3.1 tells that the system (2.1) possesses at least three stationary solutions $(0,0)$, $(u_{1*}(x),0)$ and $(0,u_{2*}(x))$, where $u_{i*}(x) \neq 0$, $\forall\, i = 1,2$. Moreover , Theorem 3.2 yields that the zero solution $(0,0)$ of the system (2.1) is locally asymptotically stable.

Example 5.2. Set $S = \{1,2\}$, and $\gamma_{11} = -0.5, \gamma_{12} = 0.5, \gamma_{21} = 0.3, \gamma_{22} = -0.3$, $\Omega = [0,1]\times[0,1]$, and hence $\lambda_1 = 19.7392$ ([11, Remark 13]). Let $\tau = 1.5, \tau_* = 0.85$ Set

$$D = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.35 \end{pmatrix},\ B = \begin{pmatrix} 1.5 & 0 \\ 0 & 1.2 \end{pmatrix},\ A = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.55 \end{pmatrix},$$

and $a_{12} = 0.56, a_{21} = 0.53$, $M_1 = 1 = M_2$, $\theta_1 = 0.1, \theta_2 = 0.2$. Direct computation yields that (4.8) holds, which together with Theorem 4.1 implies that the null solution the the null solution is the unique stationary solution of the ecosystem (4.7) under the restrictive condition (4.1).

Moreover, applying computer Matlab LMI toolbox to (4.9) results in the feasible data:

$$P_1 = \begin{pmatrix} 1.0017 & 0 \\ 0 & 1.0015 \end{pmatrix},\ P_2 = \begin{pmatrix} 0.9987 & 0 \\ 0 & 1.0003 \end{pmatrix},\ Q_1 = \begin{pmatrix} 1.1177 & 0 \\ 0 & 0.9996 \end{pmatrix},\ Q_2 = \begin{pmatrix} 0.9993 & 0 \\ 0 & 1.139 \end{pmatrix},$$

$$W = \begin{pmatrix} 1.1177 & 0 \\ 0 & 0.9996 \end{pmatrix},\ Q_2 = \begin{pmatrix} 0.9993 & 0 \\ 0 & 1.1033 \end{pmatrix}.$$

Then Theorem 4.1 yields that the null solution of the ecosystem (4.7) is globally exponentially stable.

6. Conclusions

In this paper, the author investigated the local and global stability for Gilpin-Ayala competition model involved in harmful species via LMI approach and variational methods. Using the mountain pass lemma, the existence of multiple stationary solutions of ecosystem is given, which shows that the global stability of the zero solution of the pest model is difficult to achieve. And applying the principle of linear approximation results in the local stability criterion. Moreover, due to the limited resources of nature, the population densities of species are reasonably assumed to be limited. Based on this boundedness assumption, the author employ variational methods to prove that the null solution is the unique stationary solution of the ecosystem, and so the global stability can be considered. Furthermore, utilizing LMI technique gives the globally asymptotical stability criterion of the unique stationary solution. The obtained Theorems and numerical examples illuminate that improving the diffusion of bacteria or pests is conducive to the elimination of pests or bacteria. For example, more ventilation in the area where bacteria are located is conducive to preventing the multiplication of bacteria and ultimately eliminating them.

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