Article

Pareto Efficiency of Mixed Quantum Strategy Equilibria

Marek Szopa [1]

[1] Department of Operations Research, University of Economics in Katowice, Bogucicka 3, 40-287 Katowice, Poland; marek.szopa@uekat.pl

Abstract: The aim of the paper is to investigate Nash equilibria and correlated equilibria of classical and quantum games in the context of their Pareto optimality. We study four games: the prisoner's dilemma, battle of the sexes and two versions of the game of chicken. The correlated equilibria usually improve Nash equilibria of games but require a trusted correlation device. We analyze the quantum extension of these games in the Eisert-Wilkens-Lewenstein formalism with the full SU(2) space of players' strategy parameters. It has been shown that the Nash equilibria of these games in quantum mixed Pauli strategies are closer to Pareto optimal results than their classical counterparts. The relationship of mixed Pauli strategies equilibria and correlated equilibria is also analyzed.

Keywords: game theory, quantum games, Nash equilibrium, Pareto-efficiency, correlated equilibria.

1. Introduction

Game theory analyzes and models the behavior of agents in the context of strategic thinking and interactive decision making. It is essential in making choices and considering opportunities not only in business but in everyday life. Examples of situations requiring strategic thinking can be found in economics [1], political science [2], biology [3,4] or military applications [5]. The participating sites have their own sets of possible actions, called strategies, and have preferences over these actions defined by the payoff matrix. Game theory deals with modeling these activities and searching for optimal strategies. Among all notions and concepts of game theory, the Nash equilibrium plays a key role. It describes the optimal decisions with regard to the moves of other players. In a Nash equilibrium no player has anything to gain by changing only his own strategy [6].

Game theory results favorable to the whole group of players are called Pareto-efficient. From an economic point of view, they are the most desirable results. However, in many cases, what is beneficial individually is not always also Pareto-efficient. It is often the opposite - striving to meet ones own interests does not lead to the best solution for all players. This type of dilemma occurs in many real situations regarding e.g. traffic organization [7], excessive exploitation of natural resources [8] or public procurement regulation [9].

The purpose of this work is to analyze game mechanisms, that allow players to regulate their choices in such a way that, attempting to optimize their individual interests, they do not create a disadvantage for the group. In the language of game theory, we will strive to reformulate games in such a way that the participants act individually in a favorable manner, i.e. achieve the Nash equilibrium state, and at the same time obtain results as close as possible to the Pareto-efficient results for the group.

Quantum game theory allows to study interactive decision making by players with access to quantum technology. This technology can be used in both of two ways: as a quantum communication protocol and as a way to randomize players' strategies more efficiently than in classical games [10]. Better randomization of game results by quantum strategies is the key to achieving Pareto-efficient solutions. In this paper we use the Eisert-Wilkens-Lewenstein (EWL) quantization protocol [11], which is the most studied protocol

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in games of quantum communication. In the EWL approach with the SU(2) strategy set, obtaining Pareto-efficient solutions is feasible but the problem is that this 3-parameter strategy space yield only trivial Nash equilibria. On the other hand many authors tried to investigate EWL scheme with a 2-parameter strategy space. This however leads to an undesirable dependence of the equilibria on the selected parameterization [12]. To resolve this dilemma we propose using mixed strategies based on 3-parameter SU(2) pure strategies, which allow for non-trivial NE and, at the same time, are not dependent on the choice of 2-dimensional parameter subspace.

In the present work, we study four games in which the problem of suboptimal Nash's equilibrium arises: the prisoner's dilemma, battle of the sexes and two versions of the game of chicken. Thanks to the use of mixed quantum strategies, we obtain both: non-trivial Nash equilibria and that they are closer to Pareto-efficient solutions than classical equilibria. The ultimate goal is to design a quantum device, the input of which is operated by players, parties to the conflict, economic institutions, and the output, through the collapse of the wave function, determines the result of the game, the solution of the dispute or conflict between the parties. The speed with which quantum technologies are currently developing allows us to assume that the efficient quantum strategies may soon be applicable to real practical problems [13].

In the second section the basic concepts of games and their payouts in pure, mixed strategies and general probability distributions are defined. We also define the concepts of the Nash equilibrium, Pareto-efficiency and correlated equilibrium. The third section, presents four classical games, discuss their Nash equilibria and analyzes their Pareto-optimality. We also discuss their correlated equilibria, which thanks to the use of additional mechanisms of correlation of players' behavior, allow for better Pareto optimization of the results of these games. The fourth section is devoted to defining the concept of quantum game in the EWL scheme with the full SU(2) parameter space. Part five of the paper presents our proposals for new Nash equilibria in quantum mixed strategies and their comparison with correlated equilibria. In the last part we discuss the applicability of both correlation mechanisms and the perspective of physical implementation of quantum games.

2. Game theory preliminaries

Let us consider a two player, two strategy game $G = (N, \{S_X\}_{X\epsilon N}, \{P_X\}_{X\epsilon N})$, where $N = \{A, B\}$ is the set of players (Alice and Bob), $S_A = \{A_0, A_1\}$, $S_B = \{B_0, B_1\}$ are sets of their possible *pure strategies* (or actions) and

$$P_X: S_A \times S_B \to \{v_{ij}^X \,\epsilon\, \mathbb{R} \mid i,j = 0,1\},$$

are respective payoff functions for Player X, $X = A, B$, usually represented by a game bimatrix $\begin{pmatrix} (v_{00}^A, v_{00}^B) & (v_{01}^A, v_{01}^B) \\ (v_{10}^A, v_{10}^B) & (v_{11}^A, v_{11}^B) \end{pmatrix}$. Let us denote by

$$\Delta(S_A \times S_B) = \left\{\textstyle\sum_{i,j=0,1} \sigma_{ij} A_i B_j \,\middle|\, \sigma_{ij} \geq 0, \sum_{i,j=0,1} \sigma_{ij} = 1\right\} \tag{1}$$

the set of all probability distributions over $S_A \times S_B$. The payoff of a Player X, corresponding to a given distribution $\sigma = \{\sigma_{ij}\}_{i,j=0,1}$ is

$$\Delta P_X(\sigma) = \textstyle\sum_{i,j=0,1} \sigma_{ij}\, v_{ij}^X. \tag{2}$$

Let us now restrict the set of all probability distributions to distributions, that can be factorized, i.e. presented in a form

$$\begin{pmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{pmatrix} = \begin{pmatrix} \sigma_A \sigma_B & \sigma_A(1-\sigma_B) \\ (1-\sigma_A)\sigma_B & (1-\sigma_A)(1-\sigma_B) \end{pmatrix}. \tag{3}$$

They define *mixed strategy spaces*

$$\Delta S_X \equiv \Delta(S_X) = \{\sigma_X X_0 + (1-\sigma_X) X_1 \mid 0 \leq \sigma_X \leq 1\} \equiv [0,1],\ X = A, B$$

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which are defined by a single number $\sigma_X \epsilon [0,1]$. Note that the product of mixed strategy spaces is a subset of the set of all probability distributions $\Delta S_A \times \Delta S_B \subset \Delta(S_A \times S_B)$.

Given a profile $\sigma = (\sigma_A, \sigma_B) \epsilon \Delta S_A \times \Delta S_B$ of mixed strategies of both players, Player X obtains an expected payoff which is an element of $\Delta(\mathrm{Im}P_X)$ - the set of probability distributions over the outcomes of G. It leads to the notion of the *mixed classical game* $G^{mix} = (N, \Delta S_A, \Delta S_B, \Delta P_A, \Delta P_B)$, where payoffs $\Delta P_X: [0,1] \times [0,1] \to \Delta(\mathrm{Im}P_X)$ are defined by (2) and (3).

Let us define a vector valued payoff function $\Delta P: \Delta(S_A \times S_B) \to \mathbb{R}^2$ by $\Delta P(\sigma) = \left(\Delta P_A(\sigma), \Delta P_B(\sigma)\right)$. The range of the payoff function of the mixed game is $R_{G^{mix}} = \Delta P(\Delta_A \times \Delta_B)$. The range of all probability distributions (1) over $S_A \times S_B$ is $R_{PD} = \Delta P\left(\Delta(S_A \times S_B)\right)$. Note that $R_{G^{mix}}$ is usually a proper subset of the range of all probability distributions $R_{G^{mix}} \subset R_{PD}$.

The pair of strategies $(\sigma_A^*, \sigma_B^*) \epsilon \Delta S_A \times \Delta S_B$ is a *Nash equilibrium* (NE), if for each strategy $\sigma_X \epsilon \Delta S_X$, $X = A, B$,

$$\Delta P_A(\sigma_A^*, \sigma_B^*) \geq \Delta P_A(\sigma_A, \sigma_B^*) \text{ and } \Delta P_B(\sigma_A^*, \sigma_B^*) \geq \Delta P_B(\sigma_A^*, \sigma_B),$$

i.e. no player has a profitable unilateral deviation from his strategy, while the other stays with his [14]. Thus, NE is such a pair of players' strategies for which they all achieve their optimal (for a given strategy of other player) individual efficiency.

From the viewpoint of mutual efficiency, the concept of Pareto optimality plays an important role. Let S be an arbitrary set of strategies. A pair of strategies $(\sigma_A, \sigma_B) \epsilon S$ is *not Pareto optimal* in S if there exists another pair, $(\sigma_A{}', \sigma_B{}') \epsilon S$ that is better for one of the players $\Delta P_X(\sigma_A, \sigma_B) < \Delta P_X(\sigma_A{}', \sigma_B{}')$, and not worse for the other Player $\Delta P_{-X}(\sigma_A, \sigma_B) \leq \Delta P_{-X}(\sigma_A{}', \sigma_A{}')$, where $-X$ is the remaining player for player $X = A, B$, otherwise the pair (σ_A, σ_B) is called *Pareto optimal* (or *Pareto-efficient*) in S. A set of all Pareto optimal strategies for a given set of strategies S is denoted $\mathcal{PO}(S)$. For instance a pair of strategies $(\sigma_A, \sigma_B) \epsilon \Delta S_A \times \Delta S_B$ is Pareto optimal in $\Delta S_A \times \Delta S_B$ if there exist no other set of mixed strategies, that would be better for at least one of players and not worse for the other. Note that the Pareto optimal strategy in a set S is not necessarily optimal in a larger set $S' \supset S$.

An interesting concept of optimizing equilibria beyond the classical game theory was put forward by R. Aumann . By correlated equilibrium we understand a situation in which players make their optimal decisions, guided by an external signal, transmitted to them by a trusted correlating device according to a given probability distribution. Each player maximizes his expected payoff by following this recommendation. Formally, probability distribution $\{\sigma_{ij}\}_{i,j=0,1}$ over the set of action vectors $(A_i, B_j)_{i,j=0,1}$ of the game G is called a *correlated equilibrium* [15], if for every strategy $A_i \epsilon S_A$ and $B_i \epsilon S_B$

$$\sum_{j=0,1} \sigma_{ij} \, v_{ij}^A \geq \sum_{j=0,1} \sigma_{ij} \, v_{-ij}^A \quad \text{and} \quad \sum_{j=0,1} \sigma_{ji} \, v_{ji}^B \geq \sum_{j=0,1} \sigma_{ji} \, v_{j(-i)}^B \tag{4}$$

respectively, where $X_{-i} \epsilon S_X$ is the remaining strategy $-i \neq i$.

One of the advantages of correlated equilibria is that they are computationally easier than Nash equilibria. Computing the correlated equilibrium requires only solving the linear problem, while solving the Nash equilibrium requires solving the equations that make each player's payoffs independent of the others.

3. The efficiency of classical games

The most contrasting example of the lack of Pareto optimality for Nash equilibria is the prisoner's dilemma (PD) game [16]. The game is universal in nature and describes many decision-making dilemma commonly found in different situations of social life. It is defined by $PD = (N, \{S_X\}_{X\epsilon N}, \{P_X\}_{X\epsilon N})$ and the payoffs are defined by the bimatrix in Table 1, where $t > r > p > s$ and $r > \frac{s+t}{2}$ [17]. A typical scenario assumes that two players, Alice and Bob, independently of each other, choose one of two strategies - "cooperation" A_0 and B_0 or "defection" A_1 and B_1.

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Table 1. The payoff matrix for the prisoner's dilemma.

Alice \ Bob	B_0	B_1
A_0	(r,r)	(s,t)
A_1	(t,s)	(p,p)

It is easy to see that regardless of the opponent's choice, the dominant strategy of each player is to "defect" and the pair of mutual defection strategies $(A_1,\ B_1)$ is the Nash equilibrium of the game. On the other hand the Pareto-efficient solutions are all the remaining pairs of pure strategies. Moreover, when allowing the players to randomize their strategies, the Nash equilibrium remains the same and the set of all Pareto optimal strategies is $\mathcal{PO}(\Delta S_A \times \Delta S_B) = A_0 \times \Delta S_B \cup \Delta S_A \times B_0$. In case of typical game payoffs: $t = 5, r = 3, p = 1, s = 0$, the Nash equilibrium $(A_1,\ B_1)$ with a payoff of (1,1) is far from the Pareto optimal (A_0, B_0) with a payoff of (3,3).

The second game under consideration is battle of the sexes (BoS), defined by the payoff bimatrix in Table 2. Alice and Bob plan to spend the evening together, for which they can get paid 2. However, Alice would prefer to go to the theater X_0, whereas Bob would prefer the football game X_1, $X = A, B$. Going to a preferred place gives players an additional bonus of +1.

Table 2. The payoff matrix of battle of the sexes.

Alice \ Bob	B_0	B_1
A_0	$(3,2)$	$(1,1)$
A_1	$(0,0)$	$(2,3)$

This game has two Nash equilibria (A_0, B_0) and (A_1, B_1) in pure strategies. Both of them form a set of Pareto optimal solutions $\mathcal{PO}(\Delta S_A \times \Delta S_B) = (A_0, B_0) \cup (A_1, B_1)$ but the problem, which gives the name to the game, is that they can not be both satisfied with a just solution. One player consistently does better than the other. BoS has also one NE in mixed strategies, in which players go to their preferred event more often than the other. It is given by a pair of strategies $\sigma_A = \frac{3}{4}A_0 + \frac{1}{4}A_1$, $\sigma_B = \frac{1}{4}B_0 + \frac{3}{4}B_1$, for Alice and Bob, respectively. The mixed strategy NE, where they both get the same payoff $\left(\frac{3}{2}, \frac{3}{2}\right)$ is however not Pareto-efficient in $\Delta S_A \times \Delta S_B$ because e.g. each of the pure strategy NE is better for both players.

The last of the classical games we consider is the game of chicken CG (chicken game), with the payoff bimatrix defined in Table 3. This game describes, e.g. the behavior of two drivers approaching, one from the south and one from the west, at the same time to the intersection. They both have two options: to cross the intersection X_1 or to stop X_0, $X = A, B$ before it. If both of them choose the option to drive, they will collide and both lose 10. If only one of them passes and the other stops, the passing one wins (1,0). If both of them stop, the result is neutral (0,0).

Table 3. The payoff matrix of the game of chicken.

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Driver A \ Driver B	B_0	B_1
A_0	$(0, 0)$	$(0, 1)$
A_1	$(1, 0)$	$(-10, -10)$

CG has two Nash equilibria in pure strategies (A_0, B_1) and (A_1, B_0), which are Pareto-efficient. However, none of these equilibria, just like in BoS, satisfy both players. The game also has the third equilibrium in mixed strategies: each car passes a crossroads with a probability of $1/11$. This equilibrium is fair - both players receive equal payouts, but the trouble is that both payouts are equal to 0, and therefore not optimal in $\Delta S_A \times \Delta S_B$ - each player can increase his payout by increasing the frequency of crossing, while the other stops at the junction.

Let's consider again the chicken game but with different, positive payoffs:

Table 4. The payoff matrix of the game of chicken II

Player A \ Player B	B_0	B_1
A_0	$(4, 4)$	$(1, 5)$
A_1	$(5, 1)$	$(0, 0)$

As in the previous game, the winner is the player who chooses the X_1 option while the other one plays X_0, $X = A, B$. The best solution is for both players to choose (A_0, B_0) but it is not an equilibrium. As before, this game has three Nash equilibria: two in pure strategies (A_1, B_0) and (A_0, B_1) and one in a mixed strategy, in which both players choose X_0 and X_1 with equal probabilities $\sigma_X = \frac{1}{2}X_0 + \frac{1}{2}X_1$, $X = A, B$. The payoffs for these Nash equilibria are: (5,1), (1,5) and $(2\frac{1}{2}, 2\frac{1}{2})$ respectively. As before, Pareto-efficient equilibria are not fair (in the sense that one player wins and the other loses), and the fair equilibrium is not Pareto-efficient (because both players can score better in $\Delta S_A \times \Delta S_B$ by choosing (A_0, B_0). It follows from (4) that the correlated equilibrium for this game should obey four inequalities: $\sigma_{00} \leq \sigma_{01}$, $\sigma_{00} \leq \sigma_{10}$, $\sigma_{11} \leq \sigma_{01}$ and $\sigma_{11} \leq \sigma_{10}$. It is easy to show that the correlated equilibrium corresponding to the highest equal payoffs for both players is

$$\sigma_C = \begin{pmatrix} \sigma_{00} & \sigma_{01} \\ \sigma_{10} & \sigma_{11} \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 \\ 1/3 & 0 \end{pmatrix},$$

and the corresponding payoffs are $(3\frac{1}{3}, 3\frac{1}{3})$, better then in the symmetric Nash equilibrium. Aumann [18] proposed the following mechanism of correlated equilibrium realization. Let's consider the third side (or some natural event), which with a probability of $1/3$ draws one of three cards marked: (0,0), (0,1) and (1,0). After the card is drawn, the third party informs the players about the strategy assigned to them on the card (but not about the strategy assigned to the opponent). Suppose one player is assigned "1", knowing that the other player saw "0" (because there is only one card that assigns him "0"), he should play "1" because he will receive the highest possible payout 5. Let's assume that the player was assigned "0". Then he knows, that the other player has received "0" or "1" commands, with probabilities 1/2. The expected payoff for playing "1" (contrary to the recommendation) is therefore $5 * \frac{1}{2} + 0 * \frac{1}{2} = \frac{5}{2}$, and the expected payout for playing as recommended "0" is the same $4*\frac{1}{2}+1*\frac{1}{2}=\frac{5}{2}$. Because none of the players has motivation to play differently than was recommended by the third party, the result of the draw is the correlated equilibrium. The probability distribution $\sigma_C \in \Delta(S_A \times S_B)$ can not be factorized as in equation (3) and therefore is not a mixed game strategy $\sigma_C \notin \Delta S_A \times \Delta S_B$. It is also not Pareto-efficient $\sigma_C \notin \mathcal{PO}\big(\Delta(S_A \times S_B)\big)$ in the set of all probability distributions.

Back to the first example of a chicken game, the traffic light installed at the intersection may act as a correlation mechanism given by the matrix $\sigma_{CG} = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$, i.e. each of the drivers with a probability of ½ meets the green light. It is a correlated equilibrium because none of the drivers is interested in running a red light, knowing that the other one is green at that time. If they both comply with the traffic rules, they will receive a payment of ½, i.e. higher than the mixed strategy Nash equilibrium. It has the highest, equal for both players payoff because it is Pareto-efficient in the set of all probability distributions $\sigma_{CG} \in \mathcal{PO}\big(\Delta(S_A \times S_B)\big)$ but not accessible by any mixed strategy as $\sigma_{CG} \notin \Delta S_A \times \Delta S_B$.

One can also find an optimal correlated equilibrium for battle of the sexes game. The correlated equilibrium definition (4) yields inequalities: $3\sigma_{00} \geq \sigma_{01}$, $\sigma_{00} \geq 3\sigma_{10}$, $3\sigma_{11} \geq \sigma_{01}$ and $\sigma_{11} \geq 3\sigma_{10}$. The equal payoff optimal solution is then $\left(2\frac{1}{2}, 2\frac{1}{2}\right)$, achievable for the distribution $\sigma_{BoS} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$. It means that they come out together to the theater or the football game depending on the coin toss. This payout is higher than the Nash equilibrium in mixed strategies and is Pareto-optimal $\sigma_{BoS} \in \mathcal{PO}\big(\Delta(S_A \times S_B)\big)$ in the set of all probability distributions and again, not accessible by any mixed strategy $\sigma_{BoS} \notin \Delta S_A \times \Delta S_B$.

In case of the prisoner's dilemma the conditions (4) show that there are no correlated equilibria other that its NE. It is because both cooperation strategies A_0 and B_0 are strictly dominated and therefore can never be played in a correlated equilibrium.

The disadvantage of correlated equilibria is the need to use an external signal that must be generated by an independent device that can be manipulated. It is not difficult to imagine that if one of the drivers is, for example, an important politician, then when he passes through an intersection where a policeman (acting as a correlating device) is standing, he has priority and the other driver has to wait. Therefore, it is worth looking for correlation mechanisms that would be safe and not susceptible to manipulation. As in the field of cryptography [19], such a solution may be transferring games to the quantum domain

4. Quantum games

In recent years, we have witnessed the rapid development of research on quantum information processing [20,21] and successful experiments related to the engineering of entangled qubits [22,23]. In the laboratories of Google Quantum AI [24], IBM [25], D-wave and several other companies [26], there is a race to achieve the so-called quantum supremacy. Google AI Quantum managed to construct a quantum processor based on 53 qubits, which in 200 seconds solved a problem that a classical computer would solve in 10 thousand years [24]. In the field of possible applications of quantum engineering, quantum games are also attracting much attention [27,28]. Apart from their own intrinsic interest, quantum games explore the fascinating world of quantum information [29–31].

The idea of quantum computers use to extend classical games to the quantum domain was put forward at the end of the 20th century. In his groundbreaking work on the theory of quantum games [32], Meyer proposed a simple coin toss game and showed that a player using quantum superposition will always win against a classical player. A general protocol for quantum games was proposed by Eisert, Wilkens and Lewenstein (EWL) [11]. This model has been widely discussed [33] and, e.g. extended to multiplayer games [34].

In this approach, players' strategies are operators in a certain vector space known as a *Bloch sphere* [35]. This space is a set of *qubits* - normalized vectors with complex

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coefficients spanned on a two-element basis $\{|0\rangle, |1\rangle\}$ which, up to the phase, can be represented in the form

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \tag{5}$$

where $\theta \in [0, \pi]$ and $\phi \in [-\pi, \pi]$.

Qubits $|\psi\rangle$ represent superposition of the basis states $|0\rangle$ and $|1\rangle$ are pure quantum states. A qubit in a state (5) does not have any value "between" $|0\rangle$ and $|1\rangle$. It means that before the measurement is carried out, it is not defined and only the measurement yields a value of $|0\rangle$ or $|1\rangle$ with probabilities $\cos^2\frac{\theta}{2}$ and $\sin^2\frac{\theta}{2}$ respectively. This process is called the collapse of the wave function. For example, all qubits representing states with $\theta = \pi/2$, i.e. at the equator of the Bloch sphere represent a quantum state which, after measurement, collapses to the state $|0\rangle$ or $|1\rangle$ with probabilities equal to $\frac{1}{2}$.

Now let us consider a space of pairs of qubits, one for each player. In this product space the standard *observational basis* is $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, where the first (second) qubit belongs to the first (second) player. Then let's use the entangling operator $\hat{J} = \cos\left(\frac{\gamma}{2}\right)\hat{I}\otimes\hat{I} + i\sin\left(\frac{\gamma}{2}\right)\sigma_x\otimes\sigma_x$, where $\hat{I}$ is the unit operator, $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the Pauli matrix and $\gamma \in \left[0, \frac{\pi}{2}\right]$, represents the entanglement level, to prepare the initial quantum state $|\psi_0\rangle = \hat{J}|00\rangle$. For $\gamma = 0$, this state is separable $|\psi_0\rangle = |00\rangle$, whereas for $\gamma = \frac{\pi}{2}$, the initial state $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)$ is the maximally entangled (Bell) state [36]. From now on we assume that $\gamma = \frac{\pi}{2}$, i.e. the initial state is fully entangled. We also assume that the initial state $|\psi_0\rangle$ is known to both players.

In quantum game theory, players' strategies are unitary transformations $\hat{U}_A$ i $\hat{U}_B$ operating on the initial state $|\psi_0\rangle$. Transformations $\hat{U}_X \in SU(2)$, $X = A, B$ are defined by unitary matrices

$$\hat{U}_X(\theta_X, \alpha_X, \beta_X) = \begin{pmatrix} e^{i\alpha_X}\cos\frac{\theta_X}{2} & ie^{i\beta_X}\sin\frac{\theta_X}{2} \\ ie^{-i\beta_X}\sin\frac{\theta_X}{2} & e^{-i\alpha_X}\cos\frac{\theta_X}{2} \end{pmatrix}, \tag{6}$$

where, $\theta_X \in [0, \pi]$ and $\alpha_X, \beta_X \in [0, 2\pi]$, $X = A, B$. The players apply strategies by using a $\hat{U}_X$ transformation on their part of an entangled qubit. The quantum state obtained in this way is then disentangled by the $\hat{J}^\dagger$ (Hermitian conjugate of $\hat{J}$) operator. The final state of this operation is

$$|\psi_f\rangle = \hat{J}^\dagger(\hat{U}_A \otimes \hat{U}_B)\hat{J}\,|00\rangle, \tag{7}$$

and can be expressed in an observational basis by $|\psi_f\rangle = \sum_{i,j=0,1} p_{ij}\,|ij\rangle$, where $|p_{ij}|^2$, $i, j = 0,1$ are probabilities that the final state measurement will give one of four vectors in the observational basis.

The sequence of operations that makes up the quantum game is schematically represented in Fig. 1.

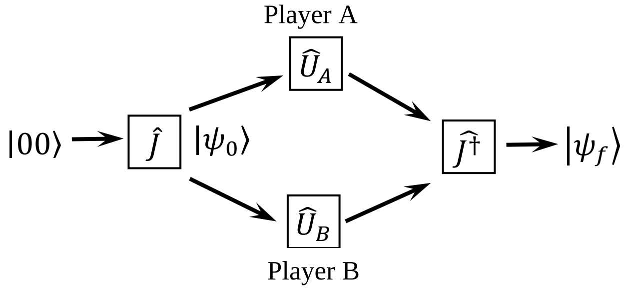

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Figure 1. The quantum game in EWL protocol.

The quantum game in the Eisert-Wilkens-Lewenstein protocol is defined as a triple $\Gamma_{EWL} = (N, \{U_X\}_{X\epsilon N}, \{\Pi_X\}_{X\epsilon N})$, where $N = \{A, B\}$ is the set of players, U_X are sets of unitary transformations (6) $\hat{U}_X \in U_X$ – pure strategies of the players and $\Pi_X: SU(2) \times SU(2) \to \mathbb{R}$ is the payoff function defined by

$$\Pi_X(\hat{U}_A, \hat{U}_B) = \sum_{i,j=0,1} |p_{ij}|^2 \, v_{ij}^X, \quad X = A, B \tag{8}$$

where $\{v_{ij}^X\}$ is the payoff bimatrix of the corresponding classical game and the observational basis probabilities are:

$$\begin{aligned}
|p_{00}|^2 &= \cos\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\cos(\alpha_A+\alpha_B) + \sin\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\sin(\beta_A+\beta_B),\\
|p_{01}|^2 &= \cos\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\cos(\alpha_A-\beta_B) + \sin\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\sin(\alpha_B-\beta_A),\\
|p_{10}|^2 &= \cos\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\sin(\alpha_A-\beta_B) + \sin\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\cos(\alpha_B-\beta_A),\\
|p_{11}|^2 &= \cos\frac{\theta_A}{2}\cos\frac{\theta_B}{2}\sin(\alpha_A+\alpha_B) - \sin\frac{\theta_A}{2}\sin\frac{\theta_B}{2}\cos(\beta_A+\beta_B).
\end{aligned} \tag{9}$$

In the original formulation of the EWL model, transformations (6) are limited to the two-dimensional parameter space where $\beta = -\frac{3}{2}\pi$ is constant. However, Benjamin & Hayden [37] observed that the set of 2 dimensional quantum strategies is not closed under composition and Frąckiewicz [38] showed, that only the full $SU(2)$ strategy parameter space provides a strong isomorphism of the classical and quantum game.

In the special case where the players' strategies are defined only by the angle θ, with fixed $\alpha = \beta = 0$, they can be expressed by $\hat{U}(\theta, 0,0) = \cos\frac{\theta}{2}\hat{I} + i\sin\frac{\theta}{2}\sigma_x$. In this case, $\hat{U}(0,0,0) = \hat{I}$ is the unit matrix corresponding to the classical A_0 (B_0) strategy and $\hat{U}(\pi, 0,0) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ is the matrix that is flipping (up to a constant) $|0\rangle$ and $|1\rangle$ qubits and corresponds to the classical A_1 (B_1) strategy. General one-parameter strategy $\hat{U}(\theta, 0,0)$ is equivalent to the classical mixed strategy for which the probabilities of both pure strategies A_0 (B_0) and A_1 (B_1) are $\cos^2\frac{\theta}{2}$ and $\sin^2\frac{\theta}{2}$ respectively. In this way the classical game becomes a special case of the quantum game.

Quantum games can be physically implemented by a quantum computer operating according to the above algorithm. Such an algorithm was carried out experimentally [39,40] in EPR-type experiments based on measurements of the Stern Gerlach effect. The players initially share an entangled pure quantum state $|\psi_0\rangle$. Each of them apply his strategy by performing arbitrary local unitary operations on his own qubit, but no direct communication between players is allowed. The result of the game is revealed, by measuring the final state (7) which, as a result of the collapse of the wave function, will give one of the four possible states with the appropriate probability. Due to the fact that players use quantum strategies, entanglement offers opportunities for players to interact with each other, which has no analogue in classical games.

The probability distribution leading to the payoff of the quantum game (8) is, in general non-factorizable and therefore can play a role of the external device correlating player actions proposed by Aumann. There is no need to use cryptographic protocols to replace the trusted mediator [41]. In this case, quantum mechanics offers the possibility of randomizing players' strategies better than classical methods.

5. Nash equilibria of quantum games

Let's go back to optimization of game equilibria. In the classical prisoner's dilemma (Table 1), the only Nash equilibrium is the mutual defection (A_1, B_1). In the EWL quantization scheme with 2D parameter space, there is a new Nash equilibrium, the "magic" strategy denoted by $\hat{Q} \equiv \hat{U}\left(0, \frac{\pi}{2}, 0\right) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ corresponding to the Pareto-efficient payoff $(3, 3)$. However, if we consider the above strategy in the full SU (2) space, then the "Nash equilibrium" obtained in this way ceases to be the equilibrium. Indeed, for any strategy $\widehat{U_A}(\theta, \alpha, \beta), \in \mathrm{SU}(2)$ there is a strategy $\widehat{U_B} = \hat{U}(\theta + \pi, \beta - \pi/2, \alpha)$ which "cancels" the action $\hat{U}$ of the Player A and changes the game result to $(0, 5)$ in favor of the Player B. The result is the same if the answer of the Player B is $\widehat{U'}_B = \hat{U}\left(\theta + \pi, \beta\text{-} + \frac{\pi}{2}, \alpha + \pi\right)$. It is easy to show, that in $SU(3)$ case of EWL the Nash equilibrium can exist only if the original game bimatrix has the result with maximal payoffs for both players.

However, a Nash equilibrium can be built by mixing pure quantum strategies, what leads to mixed quantum strategies [42,43]. Let us consider a set of quantum strategies:

$$\begin{aligned} \widehat{P_0} &= \hat{U}(0,0,\beta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \widehat{P_x} &= \hat{U}(\pi, \alpha, \pi) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \\ \widehat{P_y} &= \hat{U}(\pi, \alpha, \pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \widehat{P_z} &= \hat{U}(0, \pi/2, \beta) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \end{aligned} \tag{10}$$

The names of these strategies refer to their similarity to the Pauli matrices $\widehat{P_x} = -i\widehat{\sigma_x}$, $\widehat{P_x} = -i\widehat{\sigma_y}$ and $\widehat{P_x} = i\widehat{\sigma_z}$, and therefore can be named Pauli strategies. They form a basis of infinitesimal generators of SU(2).

Let us consider a quantum game Γ_{EWL}, where the set of unitary strategies is $U_X = \{\widehat{P_0}, \widehat{P_x}, \widehat{P_y}, \widehat{P_z}\}$. The final state of the game $|\psi_f\rangle = \hat{J}^{\dagger}(\hat{P}_\alpha \otimes \hat{P}_\beta)\hat{J}\,|00\rangle$, where $\alpha, \beta \in \{0, x, y, z\}$, can be expanded in terms of a single vector of an observational basis. Therefore payoffs corresponding to this game (Table 5.) are single bimatrix pairs of the original classical game. Note that for any strategy of Player A, there is such a strategy of Player B, that the result of the quantum game is any pair of payoffs of the original game.

Table 5. The payoff matrix of Pauli strategies in the EWL scheme.

		Player B			
		$\widehat{P_0}$	$\widehat{P_x}$	$\widehat{P_y}$	$\widehat{P_z}$
Player A	$\widehat{P_0}$	(v_{00}^A, v_{00}^B)	(v_{01}^A, v_{01}^B)	(v_{10}^A, v_{10}^B)	(v_{11}^A, v_{11}^B)
	$\widehat{P_x}$	(v_{10}^A, v_{10}^B)	(v_{11}^A, v_{11}^B)	(v_{00}^A, v_{00}^B)	(v_{01}^A, v_{01}^B)
	$\widehat{P_y}$	(v_{01}^A, v_{01}^B)	(v_{00}^A, v_{00}^B)	(v_{11}^A, v_{11}^B)	(v_{10}^A, v_{10}^B)
	$\widehat{P_z}$	(v_{11}^A, v_{11}^B)	(v_{10}^A, v_{10}^B)	(v_{01}^A, v_{01}^B)	(v_{00}^A, v_{00}^B)

Having this matrix, one can now construct mixed Pauli strategies defined by quadruples of coefficients $\sigma^X = (\sigma_\alpha^X)_{\alpha=0,x,y,z}$,

$$\Delta U_X \equiv \Delta(U_X) = \left\{\textstyle\sum_{\alpha=0,x,y,z} \sigma_\alpha^X \widehat{P_\alpha} \,\middle|\, 0 \le \sigma_\alpha^X;\ \sum_{\alpha=0,x,y,z} \sigma_\alpha^X = 1\right\},\ X = A, B,$$

Subsequently one can define a mixed quantum game in the EWL protocol $\Gamma_{EWL}^{mix} = (N, \{\Delta U_X\}_{X\epsilon N}, \{\Delta\Pi_X\}_{X\epsilon N})$, where the payoffs are defined by

$$\Delta\Pi_X(\sigma^A, \sigma^B) = \textstyle\sum_{\alpha,\beta=0,x,y,z} \sigma_\alpha^A \sigma_\beta^B\, \Pi_X\left(\widehat{P_\alpha}, \widehat{P_\beta}\right).$$

Now it is possible to construct nontrivial Nash equilibria in mixed Pauli strategies. For the prisoner's dilemma game from Table 1, the pair of strategies $\sigma^A = (\frac{1}{2}, 0, 0, \frac{1}{2})$ and $\sigma^B = (0, \frac{1}{2}, \frac{1}{2}, 0)$ (or equivalently $\sigma'^A = (0, \frac{1}{2}, \frac{1}{2}, 0)$ and $\sigma'^B = (\frac{1}{2}, 0, 0, \frac{1}{2})$ the game has a

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Nash equilibrium with payoffs $(\Delta\Pi_A, \Delta\Pi_B) = \left(\frac{5}{2}, \frac{5}{2}\right)$. Note that this quantum equilibrium gives both players a much higher payoff than the Nash equilibrium and the best correlated equilibrium, both yielding a payoff of $(1,1)$.

Similarly, we can find a Nash equilibrium for battle of the sexes game from Table 2. Likewise the quantum PD, this game has no equilibrium in pure quantum strategies. One can check that, the highest payouts of the game occur in two subgames defined by pairs of quantum strategies $\{\widehat{P_0}, \widehat{P_z}\}$ and $\{\widehat{P_x}, \widehat{P_y}\}$. Therefore, one can be built two pairs of equilibria in mixed Pauli strategies $\sigma^A = \sigma^B = (\frac{1}{2}, 0, 0, \frac{1}{2})$ and $\sigma'^A = \sigma'^B = (0, \frac{1}{2}, \frac{1}{2}, 0)$, that is, unlike the PD example, the Nash equilibrium is the case when Alice and Bob simultaneously play the same pair of strategies. The payoff for both pairs is then equal to $(\Delta\Pi_A, \Delta\Pi_B) = \left(\frac{5}{2}, \frac{5}{2}\right)$, so exactly as much as can be obtained for classical correlated equilibrium for this game.

For the chicken game the pair of Nash equilibrium mixed Pauli strategies is the same as in the prisoner's dilemma $\sigma^A = (\frac{1}{2}, 0, 0, \frac{1}{2})$ and $\sigma^B = (0, \frac{1}{2}, \frac{1}{2}, 0)$ (or equivalently $\sigma'^A = (0, \frac{1}{2}, \frac{1}{2}, 0)$ and $\sigma'^B = (\frac{1}{2}, 0, 0, \frac{1}{2})$. In this equilibrium, drivers receive equal payoffs $(\Delta\Pi_A, \Delta\Pi_B) = \left(\frac{1}{2}, \frac{1}{2}\right)$, the same, which provide the usual traffic lights and at the same time the best available in correlated equilibria. In the chicken game II the above pair of mixed Pauli strategies yields the payoffs $(\Delta\Pi_A, \Delta\Pi_B) = (3, 3)$, Among equilibria giving both players equal payoffs, the above equilibrium gives the highest result and is better than the mixed strategy Nash equilibrium of the classical game $(2\frac{1}{2}, 2\frac{1}{2})$. It is however worse than maximal correlated equilibrium $(3\frac{1}{3}, 3\frac{1}{3})$. The comparison of the obtained results is presented in Table 6.

Table 6. Comparison of the best symmetric game results.

Game name	Table nos.	Best symmetrical Pareto-efficient payoffs in $\Delta(S_A \times S_B)$	Best symmetrical payoffs for the		
			Nash equilibrium	correlated equilibrium	NE in mixed Pauli strategies
Prisoner's dilemma	1	3	1	1	$2\,^1/_2$
Battle of the sexes	2	$2\,^1/_2$	$1\,^1/_2$	$2\,^1/_2$	$2\,^1/_2$
The game of chicken	3	$^1/_2$	0	$^1/_2$	$^1/_2$
The game of chicken II	4	4	$2\,^1/_2$	$3\,^1/_3$	3

Interestingly in the family of all mixed Pauli strategic equilibria, there is e.g. $\sigma^A = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $\sigma^B = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ which yields the payoff $(2\frac{1}{2}, 4\frac{1}{2})$, or symmetrically $\sigma'^A = (\frac{1}{2}, \frac{1}{2}, 0, 0)$ and $\sigma'^B = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ with payoff $(4\frac{1}{2}, 2\frac{1}{2})$, i.e. of the sum of payoffs higher then in the correlated equilibrium.

6. Conclusions

In this paper, we were looking for game solutions that would be closer to the Pareto-efficient results than classical game solutions. We took into account: the prisoner's dilemma game, battle of the sexes and two versions of the chicken game. For most of these games (apart from PD), we have shown that there are correlated equilibria that improve Nash equilibria of these games. However, obtaining results in this way requires the introduction of an external device that correlates the actions of players. Such a device, sending signals to players, would be vulnerable to manipulation and difficult to use. Therefore, we proposed to use the quantum domain extension of games. In the paper we adopted the most common formalism of Eisert-Wilkens-Lewenstein quantum games, extended to the full space of SU(2) strategies. We have shown that the games under consideration have, in the mixed strategies, Nash equilibria much closer to Pareto-efficient solutions than the equilibria of classical games. These equilibria are comparable to correlated equilibria.

In the case of the prisoner's dilemma, the Nash equilibrium of the quantum game corresponds to mixing with equal probability of cooperation and defection. Although this result is not Pareto-efficient, like mutual cooperation, the players' payoffs obtained in this way are better than the best correlated equilibrium, which is equal to the Nash equilibrium of the classical game. In the case of battle of the sexes, the best correlated equilibrium coincides with the quantum NE, it is fully fair for both partners and Pareto-efficient. For the chicken game, the best Pareto optimal, correlated equilibrium also coincides with the Nash equilibrium of quantum game. This solution is unattainable in classical mixed strategies. In the second version of the chicken game, the best equal solution obtained in mixed Pauli strategies is better than classical NE but worse than the one achievable in correlated equilibria. However, there is also an asymmetric solution with payoffs, the sum of which is greater than the sum for the correlated equilibrium. Neither of these solutions is Pareto efficient.

The question, whether quantum versions of games can contribute to solving practical economic situations, naturally arises. Examples from other fields indicate that quantum strategies can solve the problems of market games [44], duopoly problems [45,46], auctions and competitions [47] or gambling [48] better than classical strategies. As is clear from this study, a solution of games with the help of quantum strategy can give better results than conventional solutions.

A general question can be asked: are there any connections between classical games and quantum phenomena? As a mathematical theory, classical games turn out to be a special case of quantum games. Do real classical games played by people every day have anything to do with physical quantum processes? The answer to this question may be surprising. A quantum phenomenon “suspected” of combining both realities is the collapse of the wave function. According to a recent hypothesis, the quantum fluctuations cause macroscopic phenomena that we consider random, such as, for example, tossing a coin or a die [49]. Moreover, every practical use of probability has its source in quantum phenomena. If this point of view were taken, any use of mixed strategy in a classical game would in fact be a quantum phenomenon.

In quantum games, an important element of the game mechanism is a quantum coherence. This phenomenon, by nature, has no analogue to the classical game. Problems with the decoherence of the wave function make it difficult to maintain two entangled qubits even at the level of strictly controlled experiments, taking place under extreme conditions of isolation from the environment. Building a quantum computer based on a register of many entangled qubits, subjected to unitary quantum gate operations and capable of solving practical problems or simulating quantum games with quantum algorithms is a real challenge. However, in recent years we have seen more and more successful attempts to build such a computer and use it to implement quantum games.

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