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Article

A Non-Standard Finite Difference Scheme for Magneto-Hydro Dynamics Boundary Layer Flows of an Incompressible Fluid Past a Flat Plate

Riccardo Fazio * and Alessandra Jannelli

Department of Mathematics, Computer Science, Physical Sciences and Earth Sciences, University of Messina, Viale F. Stagno D'Alcontres 31, 98166 Messina, Italy; ajannelli@unime.it
* Correspondence: rfazio@unime.it

Abstract: This paper deals with a non-standard finite difference scheme defined on a quasi-uniform mesh for approximate solutions of the Magneto-Hydro Dynamics (MHD) boundary layer flow of an incompressible fluid past a flat plate for a wide range of the magnetic parameter. We show how to improve the obtained numerical results via a mesh refinement and a Richardson extrapolation. The obtained numerical results are favourably compared with those available in the literature.

Keywords: MHD model problem; boundary problem on semi-infinite interval; non-standard finite difference scheme; quasi-uniform mesh; error estimation

MSC: 65L10; 65L12; 65L70

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1. Introduction

The simplest example of the application of the boundary layer theory is related to the celebrated Blasius [1] problem. This problem describes the flow around a very thin flat plate.

The first goal of this paper is to solve numerically, with great accuracy, the MHD boundary layer equation governing the flow of an incompressible fluid past a flat plate by a non-standard finite difference scheme defined on a quasi-uniform mesh. Numerical methods for problems like the one considered in this paper can be classified according to the numerical treatment of the boundary condition imposed at infinity. The oldest and simplest treatment is to replace infinity with a suitable finite value, the so-called truncated boundary. However, being the simplest approach this has revealed within the decades some drawbacks that suggest not to apply it, especially if we have to face a given problem without any clue on its solution behaviour. Several other treatments have been proposed in the literature to overcome the shortcomings of the truncated boundary approach. In this research area they are worthy of consideration: the formulation of so-called asymptotic boundary conditions by de Hoog and Weiss [2], Lentini and Keller [3] and Markowich [4,5]; the reformulation of the given problem in a bounded domain as studied first by de Hoog and Weiss and developed more recently by Kitzhofer et al. [6]; the free boundary formulation proposed by Fazio [7] where the unknown free boundary can be identified with a truncated boundary; the treatment on the original domain via pseudo-spectral collocation methods, see the book by Boyd [8] or the review by Shen and Wang [9] for more details on this topic; and, finally, a non-standard finite difference scheme on a quasi-uniform grid defined on the original domain by Fazio and Jannelli [10]. This non-standard finite difference scheme has been successively modified by Fazio and Jannelli [11]. Moreover, this method has been applied to the numerical solution of the perpetual American put option problem of financial markets, see Fazio [12].

This study concludes by comparing the current numerical results with those given by the integral approximation method (ITM) and the non-integral technique (NIT) used by Singh and Chandarki [13].

2. Model Problem

We consider a steady two-dimensional flow of a viscous fluid on a flat plate in the presence of a given transverse magnetic field with small electric conductivity and large transverse magnetic field. Introducing appropriate similarity variables, the governing equations can be reduced to the following boundary value problem (BVP) [13]

$$\frac{d^3u}{dx^3} + u\frac{d^2u}{dx^2} + \beta\left(1 - \frac{du}{dx}\right) = 0 \tag{1}$$

$$u(0) = \frac{du}{dx}(0) = 0\,, \qquad \frac{du}{dx}(\infty) = 1\,,$$

where β is the magnetic parameter. Let us notice that for $\beta = 0$ the BVP (1) reduces to the celebrated Blasius problem [1]. For a general class of problems, including (1), results on the existence, uniqueness, and boundedness of solutions were obtained by Brighi [14].

3. The Finite Difference Scheme

Without loss of generality, we consider the class of BVPs

$$\begin{aligned} &\frac{d\mathbf{u}}{dx} = \mathbf{f}(x, \mathbf{u})\,, \quad x \in [0, \infty)\,, \\ &\mathbf{g}(\mathbf{u}(0), \mathbf{u}(\infty)) = \mathbf{0}\,, \end{aligned} \tag{2}$$

where $\mathbf{u}(x)$ is a $d-$dimensional vector with ${}^{\ell}u(x)$ for $\ell = 1, \ldots, d$ as components, $\mathbf{f} : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$, and $\mathbf{g} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. Here, and in the following, we use Lambert's notation for the vector components [15, pp. 1–5].

In order to solve the problem (2) on the original domain, we discuss first quasi-uniform grids maps from a reference finite domain and introduce on the original domain a non-standard finite difference scheme that allows us to impose the given boundary conditions exactly. Let us consider the smooth strict monotone quasi-uniform maps $x = x(\xi)$, the so-called grid generating functions, see Boyd [8, pp. 325–326] or Canuto et al. [16, p. 96],

$$x = -c \cdot \ln(1 - \xi)\,, \tag{3}$$

and

$$x = c\frac{\xi}{1 - \xi}\,, \tag{4}$$

where $\xi \in [0, 1]$, $x \in [0, \infty]$, and $c > 0$ is a control parameter. So that, a family of uniform grids $\xi_n = n/N$ defined on interval $[0, 1]$ generates one parameter family of quasi-uniform grids $x_n = x(\xi_n)$ on the interval $[0, \infty]$. The two maps (3) and (4) are referred to as logarithmic and algebraic maps, respectively. As far as the authors' knowledge is concerned, van de Vooren and Dijkstra [17] were the first to use these kinds of maps. We notice that more than half of the intervals are in the domain with length approximately equal to c and $x_{N-1} = c \ln N$ for (3), while $x_{N-1} \approx cN$ for (4). For both maps, the equivalent mesh in x is nonuniform with the most rapid variation occurring with $c \ll x$. The logarithmic map (3) gives slightly better resolution near $x = 0$ than the algebraic map (4), while the algebraic map gives much better resolution than the logarithmic map as $x \to \infty$. In fact, it is easily verified that

$$-c \cdot \ln(1 - \xi) < c\frac{\xi}{1 - \xi}\,,$$

for all ξ, but $\xi = 0$.

The problem under consideration can be discretized by introducing a uniform grid ξ_n of $N+1$ nodes in $[0,1]$ with $\xi_0 = 0$ and $\xi_{n+1} = \xi_n + h$ with $h = 1/N$, so that x_n is a quasi-uniform grid in $[0,\infty]$. The last interval in (3) and (4), namely $[x_{N-1}, x_N]$, is infinite but the point $x_{N-1/2}$ is finite, because the non integer nodes are defined by

$$x_{n+\alpha} = x\left(\xi = \frac{n+\alpha}{N}\right),$$

with $n \in \{0,1,\ldots,N-1\}$ and $0 < \alpha < 1$. These maps allow us to describe the infinite domain by a finite number of intervals. The last node of such grid is placed on infinity so right boundary conditions are taken into account correctly.

We approximate the values of the scalar variable $u(x)$ and its derivative at mid-points of the grid $x_{n+1/2}$, for $n = 0, \cdots, N-1$, using non-standard difference discretizations

$$\begin{aligned} u_{n+1/2} &\approx \frac{x_{n+3/4} - x_{n+1/2}}{x_{n+3/4} - x_{n+1/4}} u_n + \frac{x_{n+1/2} - x_{n+1/4}}{x_{n+3/4} - x_{n+1/4}} u_{n+1}, \\ \left.\frac{du}{dx}\right|_{n+1/2} &\approx \frac{u_{n+1} - u_n}{2(x_{n+3/4} - x_{n+1/4})}. \end{aligned} \tag{5}$$

We emphasize that the key advantage of our non-standard finite difference formulation is to overcome the difficulty of the numerical treatment of the boundary conditions at the infinity. In fact, the formulae (5) use the value $u_N = u(\infty)$, but not $x_N = \infty$ and then, the boundary conditions at infinity are taken into account in a natural way.

For the class of BVPs (2), a non-standard finite difference scheme on a quasi-uniform grid can be defined by using the approximations given by (5) above, and it can be written as follows

$$\begin{gathered} \mathbf{U}_{n+1} - \mathbf{U}_n - a_{n+1/2}\mathbf{f}(x_{n+1/2}, b_{n+1/2}\mathbf{U}_{n+1} + c_{n+1/2}\mathbf{U}_n) = \mathbf{0}, \\ \text{for} \quad n = 0,1,\ldots,N-1 \\ \mathbf{g}(\mathbf{U}_0, \mathbf{U}_N) = \mathbf{0}, \end{gathered} \tag{6}$$

where

$$\begin{aligned} a_{n+1/2} &= 2(x_{n+3/4} - x_{n+1/4}), \\ b_{n+1/2} &= \frac{x_{n+1/2} - x_{n+1/4}}{x_{n+3/4} - x_{n+1/4}}, \\ c_{n+1/2} &= \frac{x_{n+3/4} - x_{n+1/2}}{x_{n+3/4} - x_{n+1/4}}, \end{aligned} \tag{7}$$

for $n = 0,1,\ldots,N-1$. The finite difference formulation (6) has order of accuracy $O(N^{-2})$. It is evident that (6) is a nonlinear system of $d\ (N+1)$ equations in the $d\ (N+1)$ unknowns $\mathbf{U} = (\mathbf{U}_0, \mathbf{U}_1, \ldots, \mathbf{U}_N)^T$. For the solution of (6) we can apply the classical Newton's method along with the simple termination criterion

$$\frac{1}{d(N+1)} \sum_{\ell=1}^{d} \sum_{n=0}^{N} |\Delta^\ell U_n| \le \text{TOL}, \tag{8}$$

where $\Delta^\ell U_n$, for $n = 0,1,\ldots,N$ and $\ell = 1,2,\ldots,d$, is the difference between two successive iterate components and TOL is a fixed tolerance.

4. Numerical Results and Comparison

In this Section, we present the numerical results obtained by solving the mathematical model (1) using the non-standard finite difference scheme (6) on the quasi-uniform

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grid defined by the logarithmic map (3) with control parameter $c = 2$. Now, let us rewrite the model (1) as a first-order system as follows

$$\begin{aligned} \frac{d^1u}{dx} &= {}^2u, \\ \frac{d^2u}{dx} &= {}^3u, \qquad x \in (0,\infty) \\ \frac{d^3u}{dx} &= -{}^1u\,{}^3u - \beta(1-{}^2u), \end{aligned} \tag{9}$$

with

$${}^1u(0) = {}^2u(0) = 0\,, \qquad {}^2u(\infty) = 1\,,$$

or, in an equivalent form,

$$\begin{aligned} \mathbf{u} &= ({}^1u, {}^2u,, {}^3u)^T, \\ \mathbf{f}(x,\mathbf{u}) &= \left({}^2u, {}^3u, -{}^1u\,{}^3u - \beta(1-{}^2u)\right)^T, \\ \mathbf{g}(\mathbf{u}(0),\mathbf{u}(\infty)) &= ({}^1u(0), {}^2u(0), {}^2u(\infty) - 1)^T\,, \end{aligned}$$

where $\mathbf{u}(x)$ is a three-dimensional vector with components ${}^\ell u(x)$ for $\ell = 1,2,3$, and $\mathbf{f} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ and $\mathbf{g} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, with $d = 3$.

For all tests we consider the control parameter $c = 2$, a fixed tolerance TOL $= 10^{-8}$ and, as a first guess for Newton's iteration, the following initial data

$${}^1u(x) = 0.5\,x\,, \qquad {}^2u(x) = 1\,, \qquad {}^3u(x) = \exp(-x).$$

In Table 1, we report the numerical results obtained for the missing initial data $\frac{d^2u}{dx^2}$ for increasing points number N and $\beta = 1.2$. Here, "iter "stands for the number of the Newton's iterations.

Table 1. Numerical results related to $\frac{d^2u}{dx^2}(0)$ for different values of N.

N	iter	$\frac{d^2u}{dx^2}(0)$
10	6	1.1790527
20	6	1.1776846
40	6	1.1773411
80	6	1.1772552
160	6	1.1772336
320	7	1.1772282
640	7	1.1772269
1280	7	1.1772261

In Figure 1, we show the numerical solution obtained for $\beta = 1.2$ and $N = 80$. The recovered value of the second order derivative of the solution at the origin is $\frac{d^2u}{dx^2}(0) = 1.177255155089504$, obtained in 6 iterations.

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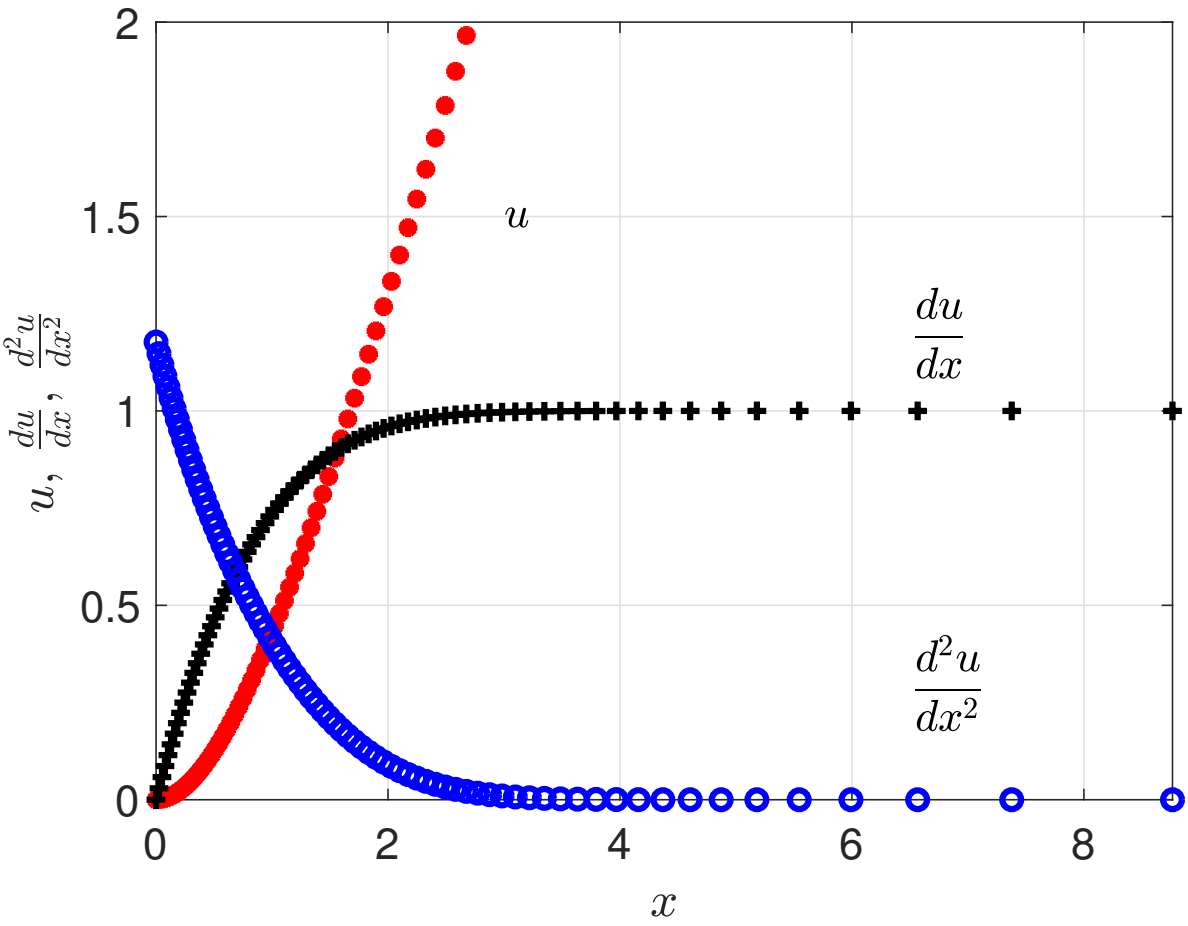

Figure 1. Numerical solution for the problem (1) for $\beta = 1.2$ and $N = 80$.

The Table 2 lists the obtained numerical results for different values of parameter β, with $\beta = 0, 0.2, \cdots, 2$ and $N = 100$. For the sake of brevity, we have chosen to report only the values of the wall shear stress, that is the second derivative value at the origin. Within the same table, we can compare our results with those reported by Singh and Chandarki [13]. The problem with $\beta = 0$ corresponding to the Blasius problem, and in this case, the computed missing initial condition $\frac{d^2u}{dx^2}(0) = 0.4695839$ can be compared with the value 0.469599988361 computed by Fazio [7] by a free boundary formulation of the Blasius problem.

Table 2. Numerical results, related to $\frac{d^2u}{dx^2}(0)$, and comparison.

β	DTM [13]	NIT [13]	FD (this study)
0.0	0.46910	0.46920	0.4695839
0.2	0.66343	0.64819	0.6389912
0.4	0.80009	0.78749	0.7749739
0.6	0.91659	0.90562	0.8917536
0.8	1.01988	1.01002	0.9956343
1.0	1.11362	1.10460	1.0900815
1.2	1.20006	1.19170	1.1772448
1.4	1.28068	1.27285	1.2585668
1.6	1.35652	1.34913	1.3350710
1.8	1.42834	1.42132	1.4075142
2.0	1.49671	1.49002	1.4764751

In order to improve the accuracy of the computed solution, we apply Richardson's extrapolation, using several refinements of the computational domain. On the computational domain of the problem, we build a quasi-uniform grid with a mesh-points number equal to N_0 and proceed with subsequent grid refinements by constructing meshes with grid-point numbers N_g for $g = 1, 2, \cdots$, where $N_{g+1} = rN_g$ with refinement factor $r = 2$. On each grid, the numerical solution U_g, $g = 0, 1, \cdots, G$ is computed using the non-standard finite difference method. In order to reduce the calculations, we adopt a continuation strategy, in fact, we use the final solution U_g obtained on the grid g as initial guess for calculating the solution U_{g+1} on the grid $g + 1$. where the new grid values are approximated by linear interpolations. We define the level of the Richardson's extrapolation by the index k and, the two numerical solutions related to the grids g and $g + 1$ at the extrapolated level k by $U_{g,k}$ and $U_{g+1,k}$. We use the following formula to calculate a more accurate approximation

$$U_{g+1,k+1} = U_{g+1,k} + \frac{U_{g+1,k} - U_{g,k}}{2^{p_k} - 1} \qquad k = 0, 1, \cdots, G - 1 \,. \tag{10}$$

In Table 3, we report the extrapolated values with $N = 100, 200, 400$ grid points for $\beta = 1$. The last extrapolated value is ${}^3U_{2,2} = 1.090064908$ and can be considered as our benchmark value for $\frac{d^2u}{dx^2}(0)$. We can conclude that the reported extrapolated value is correct up to 9 decimal places.

Table 3. Extrapolated values at origin $x = 0$ for $\frac{d^2u}{dx^2}(0)$ with $\beta = 1$.

N_g	${}^3U_{g,0}$	${}^3U_{g,1}$	${}^3U_{g,2}$
100	1.090081494	-	-
200	1.090069055	1.090064908	-
400	1.090065945	1.090064908	1.090064908

5. Concluding Remarks

In this paper, the problem (1), that describes the MHD boundary layer flow of an incompressible fluid past a flat plate, is solved by the non-standard finite difference method on a quasi-uniform grid for the different magnetic parameters β. The values of the second-order derivative of the solution at the origin, for different values of parameter β, are reported in the Table 2. In order to verify the accuracy of the proposed method, the results are compared with those by Singh and Chandarki [13]. Moreover, in the case of the Blasius problem, we compare the missing initial condition with the one computed by Fazio [7] by a free boundary formulation of the Blasius problem. The computed values are found to be really accurate.

Data Availability Statement: The data that support the findings of this study are available from the corresponding author upon reasonable request.

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