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New study of existence and dimension of the set of solutions for nonlocal impulsive differential inclusions with sectorial operator

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Abstract: In this article, we are interested in a new generic class of nonlocal fractional impulsive differential inclusions with linear sectorial operator and Lipschitz multivalued function in the setting of finite dimensional Banach spaces. By modifying the definition of PC-mild solutions initiated by Shu, we succeeded to determine new conditions that sufficiently guarantee the existence of the solutions. The results are obtained by combining techniques of fractional calculus and fixed point theorem for contraction maps. We also characterize the topological structure of the set of solutions. Finally, we provide a demonstration to address the applicability of the theoretical results.

Keywords: Impulsive fractional differential inclusions; Nonlocal conditions; Fixed point theorems; Mild solutions

1. Introduction

Fractional calculus is the look at integrations and derivatives in case of non-integer order, which is a generalized shape of classical integrals and derivatives.The significance of fractional calculus is due to its numerous applications in many significant scientific fields such as physics, control theory, fluid dynamics, image processing, computer networking, sign processing, biology and different fields. During the previous couple of decades, fractional calculus has been substantially studied and one can have a look at a quantity of researchers who have demonstrated a deep interest in it, which have brought about the enlargement and improvement of its idea [11,14,17,21,27,28,31,41].

On the alternative hand, there are numerous examples in biotechnology, physics, population dynamics and processes economics which are characterized by the reality that the model parameters are subjected to short-time period perturbations. For instance, in treating some diseases, impulses can correspond to management of a drug treatment; in environmental sciences, impulses can correspond to seasonal modifications or harvesting; in economics, impulses can additionally correspond with sudden prices adjustments. Adequate equipments to remedy such techniques and phenomena are impulsive differential equations and inclusions. The first ones had been extensively researched in finite and infinite-dimensional Banach spaces; see, for instance, the studies [2,8,22,30,33]. Recently, the existence of mild solutions for the impulsive differential equations and inclusions in Banach spaces have been examined through many researchers, we refer the reader to [4–7,13,19,23–25,36,39].

In this paper, we are interested in the existence of mild solutions to the following nonlocal impulsive fractional differential inclusions of the type

$$\begin{cases} {}^c D_t^\alpha x(t) \in Ax(t) + F(t, x(t)), & t \in J = [0, b], \ t \neq t_i, \\ \Delta x(t_i) = I_i(x(t_i^-)), & i = 1, ..., m, \\ x(0) + g(x) = x_0, & x_0 \in \mathbb{R}^n. \end{cases} \quad (1)$$

where $^cD^\alpha$, $0 < \alpha < 1$, is the Caputo derivative, A is sectorial operator on $\mathbb{R}^n$, $F : J \times \mathbb{R}^n \to P(\mathbb{R}^n)$ is a multifunction, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_i : \mathbb{R}^n \to \mathbb{R}^n$, $1 \le i \le m$, are impulsive functions, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $x(t_i^-) = \lim_{s \to t_i^-} x(s)$, $x(t_i^+) = \lim_{s \to t_i^+} x(s)$ and $g : PC(J, \mathbb{R}^n) \to \mathbb{R}^n$.

It is worthy to mention that the topic of evolution inclusions and equations involving sectorial or almost sectorial term has been extensively examined (see [1,29,34,35,37,38,40]).

Shu et al. [34] gave a new and different concept of mild solutions for the following equation:

$$\begin{cases} ^cD^qx(t) = Ax(t) + f(t,x(t)), & t \in J = [0,b], t \neq t_i, \\ \Delta x(t_i) = I_i(x(t_i^-), & i = 1, \cdots, m, \\ x(0) = x_0, \end{cases} \tag{2}$$

where f is a completely continuous single function and A is a sectorial operator with the operators families $\{\mathcal{S}_\alpha(t) : t \ge 0\}$ and $\{\mathcal{T}_\alpha : t \ge 0\}$ are compact.

Agarwal et al. [1] used the definition of mild solution given by Shu in [34] to prove an existence result for the inclusion:

$$\begin{cases} ^cD^qx(t) \in Ax(t) + f(t,x(t)), & t \in J = [0,b], t \neq t_k, \\ x(0) + h(x) = x_0 \in \mathbb{R}^n, \end{cases} \tag{3}$$

where the linear part is fractional sectorial. They also studied the dimension of the set of mild solutions.

Shu et al. [35] proved that the solutions obtained in [34] are not correct and presented the right form of the solutions for (2) when $0 < \alpha < 1$ and $1 < \alpha < 2$.

Motivated from the above mentioned results, we present to study the problem (1); we first introduce the modified form of solutions for (1) based on the definition given by Shu [35]. Second, we use our formula to extend the results obtained in [1] to nonlocal fractional inclusions associated with impulses effect. Moreover, we study the topological structure of the set of solutions.

2. Preliminaries

In this part, we present some primary concepts, definitions and initial facts which are useful for the development of this article.

Let $C(J, \mathbb{R}^n) = \{\mu : J \to \mathbb{R}^n : \mu \text{ is continuous } \}$, with the norm $\|x\| = \sup\{\|x(t)\| : t \in J\}$, $L^1(J, \mathbb{R}^n) = \{\mathcal{G} : J \to \mathbb{R}^n : \mathcal{G} \text{ is Bochner integrable}\}$, $\mathcal{L}(\mathbb{R}^n)$ is the Banach space of all linear operators bounded on $\mathbb{R}^n$. Let (E, d) be a metric space, we denote:

$$P_b(E) = \{X : X \subset E, \ X \neq \varnothing, X \text{ is bounded } \},$$
$$P_{cl}(E) = \{X : X \subset E, \ X \neq \varnothing, X \text{ is closed } \},$$
$$P_k(E) = \{X : X \subset E, \ X \neq \varnothing, X \text{ is compact } \},$$
$$P_c(E) = \{X : X \subset E, \ X \neq \varnothing, X \text{ is convex } \},$$
$$P_{ck}(E) = \{X : X \subset E, \ X \neq \varnothing, X \text{ is convex and compact } \}.$$

Let $H_d : P(E) \times P(E) \to \mathbb{R} \cup \{\infty\}$ define by

$$H_d(X,Y) = \max\left\{ \sup_{x \in X} d(x,Y), \sup_{y \in Y} d(y,X) \right\},$$

where $d(x,Y) = \inf_{y \in Y} d(x,y)$ and $d(y,X) = \inf_{x \in X} d(x,y)$. H_d is the Pompeiu Hausdorff functional. It is well known that $(P_{cl,b}(E), H_d)$ is a metric space and $(P_{cl}(E), H_d)$ is a gener-

alized metric space. For more details see [18].

Let $J_0 = [0, t_1]$, $J_i =]t_i, t_{i+1}]$, $i = 1, \cdots, m$, and

$$PC(J, \mathbb{R}^n) = \{x : J \to \mathbb{R}^n, \ x_{|_{J_i}} \in C(J_i, \mathbb{R}^n), with \ x(t_i^+) \ and \ x(t_i^-) \ exist \ \}.$$

The space $PC(J, \mathbb{R}^n)$ is a Banach space with the norm

$$\|x\|_{PC} = \sup\{\|x(t)\| : t \in J\}.$$

Now, let us recall some important definitions and lemmas on multivalued functions and fractional calculus.

Definition 1. *([15], [20]). If X , Y are topological spaces. $F : X \to P(Y)$ is called:*

1.   *Upper semi-continuous ($u.s.c$) if $F^{-1}(W) \subset X$ is an open for every open subset W of Y.*
2.   *Lower semi-continuous ($l.s.c$) when $F^{+1}(W) = \{x \in X : F(x) \cap W \neq \varnothing\}$ is an open $\forall \ W \subset Y$ such that W is open.*
3.   *Continuous if it is both ($u.s.c$) and ($l.s.c$).*
4.   *Completely continuous if $\forall$ bounded set $W \subset X$, $F(W)$ is relatively compact.*
5.   *x is a fixed point for F, if $x \in F(x)$, $x \in X$.*

Definition 2. *A sequence $\{f_n : n \in \mathbb{N}\} \subset L^1(J, E)$ is called semi-compact if:*

1.   *It is integrably bounded i.e. there is $q \in L^1(J, \mathbb{R}^+)$ such that $\|f_n(t)\| \leq q(t)$ a.e. $t \in J$.*
2.   *The set $\{f_n(t) : n \in \mathbb{N}\}$ is relatively compact in E a.e. $t \in J$.*

Lemma 1. *([20]). In $L^1(J, E)$, any semi-compact sequence is weakly compact.*

Definition 3. *The multivalued map $F : E \to P_{cl}(E)$ is called*

(i) $\eta-$Lipschitz if there is $\eta > 0$ with

$$H_d(F(x), F(y)) \leq \eta \ d(x, y), \quad x, y \in E.$$

(ii) $\eta-$contraction if it is $\eta-$Lipschitz with $\eta < 1$.

Lemma 2. *Let $F : E \to P_k(E)$ be $\eta-$contraction. Then F is continuous with respect to Hausdorff metric.*

Definition 4. *Let $F : J \to P_{cl}(\mathbb{R}^n)$. If the function $t \to d(x, F(t)) = \inf\{\|x - y\| : y \in F(t)\}$ is measurable for all $x \in \mathbb{R}^n$, then F is measurable.*

Lemma 3. *([12]). If $F : J \to P_{ck}(\mathbb{R}^n)$ is a measurable function with $\chi\{t : dimF(t) < 1\} = 0$, where χ is the Lebesgue measure, then there are arbitrarily many linearly independent measurable selections $f_1(\cdot), f_2(\cdot), \cdots, f_m(\cdot)$ of F.*

Lemma 4. *([12]). Let X be a Banach space, $B \subset X$ such that B is a nonempty, convex and closed, and $Q : B \to P_{ck}(B)$ is $\eta-$contraction. If $f : B \to B$ is a continuous selection of Q, then $Fix(f) \neq \varnothing$.*

Lemma 5. *(Saint Raymond [32]). Let $Q : Y \to P_{ck}(X)$ be a lower semi-continuous map with $0 \in Q(x)$ and $dim \ Q(x) \geq n \ \forall x \in Y$. Then, there is a selection g of Q such that g is continuous and $g(x) \neq 0 \ \forall x \in Y$, where X is a Banach space and Y is a compact metric space with dim $Y < n$.*

Lemma 6. *(Michael's selection Theorem, [26]). Let $Q : Y \to P_{c,cl}(Y)$ be lower semi-continuous map. Then, there is a continuous selection $f : Y \to X$ of Q, where X is a Banach space and Y is a metric space.*

Definition 5. *([21]). Let $f \in L^1(J, E)$. Fractional integration of the order $\alpha > 0$ with lower limit zero for f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ \ t > 0.$$

provided that the right-hand side is point-wise defined on $[0, \infty[$.

Definition 6. *([21]). Let $f \in C^n([0, \infty), \mathbb{R})$. The Caputo derivative of the order $\alpha > 0$ for f is defined as*

$$
\begin{aligned}
{}^c D^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^n(s) ds \\
&= I^{n-\alpha} f^n(t), \ \ t > 0, \ n = [\alpha] + 1,
\end{aligned}
$$

where $[\alpha]$ is the integer part of the real number α.

For further details on fractional calculus, we refer to [21,27,28].

Definition 7. *Let $A : D(A) \subset \mathbb{R}^n \to \mathbb{R}^n$ be a linear closed operator. A is said to be sectorial of type (M, θ, ω) if $\exists \ \ \omega \in \mathbb{R}, \ \ \theta \in]0, \frac{\pi}{2}[$ and $M > 0$, with the resolvent of A exists outside the sector*

$$\omega + S_\theta = \{\omega + \lambda : \lambda \in \mathbb{C}, Arg(-\lambda) < \theta\}$$

and

$$\|(\lambda I - A)^{-1}\| \le \frac{M}{|\lambda - \omega|}, \ \ \lambda \notin \omega + S_\theta.$$

Lemma 7. *([35]). Assume that A is sectorial of type (M, θ, ω). If $0 < \alpha < 1$, then*
(i) $\ \ {}^c D^\alpha(\mathcal{T}_\alpha(t)x_0) = A(\mathcal{T}_\alpha(t)x_0)$.
(ii) $\ \ {}^c D^\alpha \left(\int_0^t \mathcal{S}_\alpha(t-s)f(s)ds \right) = A \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds + f(t)$,

where $\mathcal{T}_\alpha(t) = \frac{1}{2\pi i} \int_Y e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda$, $\mathcal{S}_\alpha(t) = \frac{1}{2\pi i} \int_Y e^{\lambda t} R(\lambda^\alpha, A) d\lambda$, and Y is a suitable path with $\lambda^\alpha \notin \omega + S_\theta$ for $\lambda \in Y$ and $R(\lambda^\alpha, A) = (\lambda^\alpha I - A)^{-1}$.

Lemma 8. *([34]). Let $\alpha \in (0, 1)$ and $A \in A^\alpha(\theta_0, \omega_0)$. Then for all $t > 0$ we have*
$\|\mathcal{T}_\alpha(t)\|_{\mathcal{L}(\mathbb{R}^n)} \le Me^{\omega t}, \ \ \|\mathcal{S}_\alpha(t)\|_{\mathcal{L}(\mathbb{R}^n)} \le Ce^{\omega t}(1 + t^{\alpha-1}), \ \ \omega > \omega_0$.
Letting

$$M_{\mathcal{T}_\alpha} = \sup_{0 \le t \le b} \|\mathcal{T}_\alpha(t)\|_{\mathcal{L}(\mathbb{R}^n)}, \ \ M_{\mathcal{S}_\alpha} = \sup_{0 \le t \le b} Ce^{\omega t}(1 + t^{\alpha-1}).$$

Then,

$$\|\mathcal{T}_\alpha(t)\|_{\mathcal{L}(\mathbb{R}^n)} \le M_{\mathcal{T}_\alpha}, \ \ \|\mathcal{S}_\alpha(t)\|_{\mathcal{L}(\mathbb{R}^n)} \le t^{\alpha-1} M_{\mathcal{S}_\alpha}.$$

For more information about the sectorial operator we refer the reader to [34]

Theorem 1. *(Covitz and Nadler [10]). Let (E, d) be a complete metric space. If $Q : E \to P_{cl}(E)$ is η-contraction, then $Fix(Q) \ne \emptyset$.*

3. Main Results
Lemma 9. *Let*

$$(P_1) \begin{cases} {}^c D_t^\alpha x(t) = Ax(t) + f(t), & t \in [0, b], \ t \ne t_i, \ 0 < \alpha < 1, \\ \Delta x(t_i) = I_i(x(t_i^-)), & i = 1, \cdots, m, \\ x(0) = x_0 - g(x), \end{cases}$$

where $f : [0,b] \to \mathbb{R}^n$. If f satisfies the uniform Holder condition with exponent $\delta \in (0,1]$ and A is sectorial operator of type (M,θ,ω), then (P_1) has unique solution $x(t) \in PC(J,\mathbb{R}^n)$ defined as:

$$x(t) = \begin{cases} \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(x_0 - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_i, 1 \le i \le m, \end{cases} \tag{4}$$

Proof. In order to show that $x(t)$ is a solution of (P_1), We consider the following two cases:
Case I. If $t \in [0,t_1]$, we have

$$x(t) = \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds.$$

Case II. If $t \in J_i =]t_i, t_{i+1}]$, we have

$$x(t) = \mathcal{T}_\alpha(t)(x_0 - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds. \tag{5}$$

By taking Caputo derivative to both side of (5) and making use of Lemma 7, we get

$$^cD^\alpha x(t) =^c D^\alpha \left[\mathcal{T}_\alpha(t)(x_0 - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds \right]$$

$$= A\mathcal{T}_\alpha(t)(x_0 - g(x)) + A\sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) + A \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds + f(t)$$

$$= Ax(t) + f(t).$$

When $t = 0$, we have

$$x(0) = \mathcal{T}_\alpha(0)(x_0 - g(x)) + \int_0^0 \mathcal{S}_\alpha(0-s)f(s) = x_0 - g(x).$$

Furthermore,

$$\Delta x(t_i) = x(t_i^+) - x(t_i^-)$$

$$= \sum_{k=1}^{i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) - \sum_{k=1}^{i-1} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-))$$

$$= \mathcal{T}_\alpha(t_i - t_i)I_i(x(t_i^-))$$

$$= \mathcal{T}_\alpha(0)I_i(x(t_i^-))$$

$$= I_i(x(t_i^-)).$$

This shows that (4) is solution for (P_1). □

Now, we will give the definition for mild solution for (1).

Definition 8. *Let A be a sectorial of type (M,θ,ω) and $\alpha \in (0,1)$. A function $x \in PC(J,\mathbb{R}^n)$ is a mild solution for (1) if*

$$x(t) = \begin{cases} \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(x_0 - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_i, 1 \le i \le m, \end{cases}$$

where $f \in S^1_{F(t,x(t))} = \{ f \in L^1(J,\mathbb{R}^n), f(t) \in F(t,x(t)) \text{ for a. e. } t \in J \}.$

We will show our result for (1) by using the following conditions:

H_1 : A is a sectorial operator of type (M, θ, ω).

H_2 : Let $F : J \times \mathbb{R}^n \to P_{ck}(\mathbb{R}^n)$, such that for every $x \in \mathbb{R}^n$, F is measurable to t.

H_3 : There is a function $\vartheta \in C(J, \mathbb{R}^+)$ with

  (a) $H_d(F(t, x), F(t, y)) \leq \vartheta(t)\|x - y\|, \; for\, a.\, e.\, t \in J, \; x, y \in \mathbb{R}^n.$
  (b) $\|F(t, x)\| = \sup\{\|u\| : u \in F(t, x)\} \leq \vartheta(t), \quad a.\, e.\, t \in J \; and \; x \in \mathbb{R}^n.$

H_4 : $g : PC(J, \mathbb{R}^n) \to \mathbb{R}^n$ is continuous and $\exists \; N > 0$ with

$$\|g(x) - g(y)\| \leq N\|x - y\|, \forall \; x, y \in PC(J, \mathbb{R}^n).$$

H_5 : For every $i = 1, \cdots, m$, $I_i : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $\exists \; \zeta_i > 0$ with

$$\|I_i(x) - I_i(y)\| \leq \zeta_i \|x - y\|, \; \forall \, x, y \in \mathbb{R}^n.$$

Theorem 2. *Suppose that H_1 - H_5 hold. Then the problem (1) has a solution on J provided that*

$$M_{\mathcal{T}_\alpha}(N + \zeta) + M_{\mathcal{S}_\alpha} \frac{b^\alpha}{\alpha} \|\vartheta\| < 1, \tag{6}$$

where $\zeta = \sum_{i=1}^{m} \zeta_i$.

Proof. For every $x \in PC(J, \mathbb{R}^n)$, let $S^1_{F(\cdot, x(\cdot))} = \{f \in L^1(J, \mathbb{R}^n) : f(t) \in F(t, x(t)) \, for\, a.\, e.\, t \in J\}$. By using H_2, H_3 and Theorem III.6 in [9], we get $F(t, x(t))$ is measurable. Moreover, $F(t, x(t))$ has measurable selection $f(t)$ and from the condition H_3 part (b) we have $\|f(t)\| \leq \|F(t, x(t))\| \leq \vartheta(t)$. Then, $S^1_{F(\cdot, x(\cdot))} \neq \varnothing \; \forall \, x \in PC(J, \mathbb{R}^n)$. Now, let $Q : PC(J, \mathbb{R}^n) \to 2^{PC(J, \mathbb{R}^n)}$ defined as $y \in Q(x)$ iff:

$$y(t) = \begin{cases} \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t \mathcal{S}_\alpha(t - s)f(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(x_0 - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t - s)f(s)ds, & t \in J_i, 1 \leq i \leq m. \end{cases} \tag{7}$$

It is simple to prove that any fixed point for Q is mild solution for (1). So, we will prove that Q satisfies the assumptions of Theorem 1. As the first step, we prove that $Q(x)$ is closed for every $x \in PC(J, \mathbb{R}^n)$. Let $\{w_n\}_{n \geq 1}$ be a sequence in $Q(x)$ such that $w_n \to w$ in $PC(J, \mathbb{R}^n)$. Our goal to show that $w \in Q(x)$. As $\{w_n\}_{n \geq 1} \in Q(x)$, there is $\{f_n\}_{n \geq 1}$ in $S^1_{F(\cdot, x(\cdot))}$ with

$$w_n(t) = \begin{cases} \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t \mathcal{S}_\alpha(t - s)f_n(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(x_0 - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t - s)f_n(s)ds, & t \in J_i, 1 \leq i \leq m. \end{cases} \tag{8}$$

By (H_3) we have for every $n \geq 1$ and *a.e.* $t \in J$, $\|f_n(t)\| \leq \vartheta(t)$. So, $\{f_n : n \geq 1\}$ is integrable bounded. Also, $\{f_n : n \geq 1\}$ is relatively compact in $\mathbb{R}^n$ for *a.e.* $t \in J$ since $\{f_n(t) : n \geq 1\} \subset F(t, x(t))$. Then, the set $\{f_n : n \geq 1\}$ semicompact. By Lemma 1, it is weakly compact in $L^1(J, \mathbb{R}^n)$. We may assume that $(f_n)_{n \geq 1}$ converges weakly to $f \in L^1(J, \mathbb{R}^n)$. By Mazur's Lemma there exists a sequence $\{\mathcal{V}_n\}_{n=1}^{\infty} \subseteq \overline{conv}\{f_n : n \geq 1\}$ such that $\mathcal{V}_n$ converges strongly to f. Because F has convex and compact values, the set $S^1_{F(\cdot, x(\cdot))}$ is convex and compact. Therefore, $\{\mathcal{V}_n\}_{n=1}^{\infty} \subseteq S^1_{F(\cdot, x(\cdot))}$ and $f \in S^1_{F(\cdot, x(\cdot))}$. Also, by using Lemma 8, we obtain $\forall \, n \geq 1$

$$\|\mathcal{S}_\alpha(t - s)f_n(s)\| \leq |t - s|^{\alpha - 1} M_{\mathcal{S}_\alpha} \vartheta(s) \in L^1(J, \mathbb{R}^+), \quad \forall \, t, s \in J, \, s \in (0, t].$$

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Taking the limit as $n \to \infty$ on each sides (8), we get from the Lebesgue dominated convergence Theorem

$$w(t) = \begin{cases} \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(x_0 - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds, & t \in J_i, 1 \leq i \leq m. \end{cases}$$

Thus, $w \in Q(x)$.
Let $x_1, x_2 \in PC(J, \mathbb{R}^n)$, we show that $\exists \, \eta \in (0,1)$ such that

$$H_d(Q(x_1), Q(x_2)) \leq \eta \, \|x_1 - x_2\|_{PC}, \quad \forall \, x_1, x_2 \in PC(J, \mathbb{R}^n).$$

Let $z_1 \in Q(x_1)$, from (7) there is $f_1 \in S^1_{F(\cdot, x_1(\cdot))}$ with

$$z_1(t) = \begin{cases} \mathcal{T}_\alpha(t)(x_0 - g(x_1)) + \int_0^t \mathcal{S}_\alpha(t-s)f_1(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(x_0 - g(x_1)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t-s)f_1(s)ds, & t \in J_i, \ i = 1, \cdots, m. \end{cases}$$

From H_3 we get

$$H_d(F(t, x_1), F(t, x_2)) \leq \vartheta(t)\|x_1(t) - x_2(t)\|.$$

Then, $\exists \, \mu \in F(t, x_2(t))$ with

$$\|f_1(t) - \mu\| \leq \vartheta(t)\|x_1(t) - x_2(t)\|, \quad t \in J.$$

Now, consider the map $\Lambda : J \to 2^{\mathbb{R}^n}$ defined as

$$\Lambda(t) = \{\mu \in \mathbb{R}^n : \|f_1(t) - \mu\| \leq \vartheta(t)\|x_1(t) - x_2(t)\|\}.$$

Since f_1, ϑ, x_1, x_2 are measurable, then by using proposition 3.4 in [9], $\Lambda(t) \cap F(t, x_2(t))$ is measurable. Moreover, its values are nonempty and compact. So, from Theorem 1.3.1 in [16], there exists $f_2 \in F(t, x_2(t))$ with for every $t \in J$,

$$\|f_1(t) - f_2(t)\| \leq \vartheta(t)\|x_1(t) - x_2(t)\|.$$

For every $t \in J_i$, $i = 1, \cdots, m$, define

$$z_2(t) = \begin{cases} \mathcal{T}_\alpha(t)(x_0 - g(x_2)) + \int_0^t \mathcal{S}_\alpha(t-s)f_2(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(x_0 - g(x_2)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t-s)f_2(s)ds, & t \in J_i. \end{cases}$$

Therefore, if $t \in J_0$ we have

$$\|z_1(t) - z_2(t)\| \leq \|\mathcal{T}_\alpha(t)\| \, \|g(x_1) - g(x_2)\| + \int_0^t \|\mathcal{S}_\alpha(t-s)\| \, \|f_1(t) - f_2(t)\|ds$$

$$\leq \left[M_{\mathcal{T}_\alpha} N + M_{\mathcal{S}_\alpha} \frac{b^\alpha}{\alpha} \|\vartheta\| \right] \|x_1 - x_2\|_{PC}.$$

Similarly, when $t \in J_i$, $i = 1, \cdots, m$, we have

$$\|z_1(t) - z_2(t)\| \leq \left[M_{\mathcal{T}_\alpha}(N + \zeta) + M_{\mathcal{S}_\alpha} \frac{b^\alpha}{\alpha} \|\vartheta\| \right] \|x_1 - x_2\|_{PC}.$$

Consequently,

$$H_d(Q(x_1), Q(x_2)) \leq \eta \, \|x_1 - x_2\|_{PC} \quad \forall \, x_1, x_2 \in PC(J, \mathbb{R}^n),$$

where, $\eta = \left[M_{\mathcal{T}_\alpha}(N + \zeta) + M_{\mathcal{S}_\alpha} \dfrac{b^\alpha}{\alpha} \| \vartheta \| \right]$. From (6) we have $\eta < 1$. Hence, Q is $\eta-$contraction. Applying Theorem 1, Q has a fixed point which is a solution for (1). $\quad \square$

In the next Theorem, we will show that Q has convex and compact values.

Theorem 3. *Let Q be the multivalued map which defined in (7). Then, under the assumptions H_1 - H_5 of Theorem 2, the set $Q(x)$ is convex and compact $\forall \, x \in PC(J, \mathbb{R}^n)$.*

Proof. From Theorem 2, we know that $Q(x) \neq \varnothing$ for each $x \in PC(J, \mathbb{R}^n)$.
At the first, we prove that $Q(x)$ is convex $\forall \, x \in PC(J, \mathbb{R}^n)$. Let $y_1, y_2 \in Q(x)$, and $\lambda \in (0, 1)$.
If $t \in J_0$, then from (7) we obtain

$$\lambda y_1(t) + (1 - \lambda) y_2(t) = \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t \mathcal{S}_\alpha(t - s)[\lambda f_1(s) - (1 - \lambda) f_2(s)] ds,$$

where $f_1, f_2 \in S_F^1$. As F has convex value, S_F^1 is convex. Then, $[\lambda f_1(s) - (1 - \lambda) f_2(s)] \in S_F^1$.
Thus, $\lambda y_1(t) + (1 - \lambda) y_2(t) \in Q(x)$. By the same way we may show that $\lambda y_1(t) + (1 - \lambda) y_2(t) \in Q(x)$ for $t \in J_i$, $i = 1, \cdots, m,$. Hence, $Q(x)$ is convex for each $x \in PC(J, \mathbb{R}^n)$.
Now, we demonstrate Q maps any bounded set into bounded set in $PC(J, \mathbb{R}^n)$.
Let $\mathcal{G}_r = \{x \in PC(J, \mathbb{R}^n) : \|x\|_{PC(J, \mathbb{R}^n)} \leq r\}$, where r is a positive number. Obviously, $\mathcal{G}_r$ is bounded set in $PC(J, \mathbb{R}^n)$. Let $x \in \mathcal{G}_r$, $y \in Q(x)$, then by using H_3, H_4 and Lemma 8 for $t \in J_0$, we obtain

$$\|y(t)\| \leq \|\mathcal{T}_\alpha(t)\| \, \|x_0 - g(x)\| + \| \int_0^t \mathcal{S}_\alpha(t - s) f(s) ds \|$$

$$\leq \|\mathcal{T}_\alpha(t)\| (\|x_0\| + \sup_{x \in \mathcal{G}_r} \|g(x)\|) + \int_0^b \|\mathcal{S}_\alpha(t - s)\| \vartheta(s) ds$$

$$\leq M_{\mathcal{T}_\alpha}(\|x_0\| + \sup_{x \in \mathcal{G}_r} \|g(x)\|) + \frac{b^\alpha}{\alpha} M_{\mathcal{S}_\alpha} \|\vartheta\|.$$

Similarly, if $t \in J_i$, $i = 1, \cdots, m$, from H_3, H_4, H_5 and Lemma 8 we get

$$\|y(t)\| \leq M_{\mathcal{T}_\alpha}(\|x_0\| + \sup_{x \in \mathcal{G}_r} \|g(x)\|) + M_{\mathcal{T}_\alpha} \sum_{k=1}^{k=i} \sup_{x \in \mathcal{G}_r} \|I_k(x(t_k^-))\| + \frac{b^\alpha}{\alpha} M_{\mathcal{S}_\alpha} \|\vartheta\|.$$

This means Q maps bounded sets into bounded sets.
Next, we show that $Q(\mathcal{G}_r)$ is equicontinuous. Let $\mathcal{B} = Q(\mathcal{G}_r)$. We need to show that $\mathcal{B}_{\overline{J_i}}$ is equicontinuou for every $i = 0, 1, \cdots, m$, where

$$\mathcal{B}_{\overline{J_i}} = \{y^* \in C(\overline{J_i}, \mathbb{R}^n) : \, y^*(t) = y(t), \, t \in J_i, \, y^*(t_i) = y(t_i^+), \, y \in \mathcal{B}\}.$$

Let $y \in \mathcal{B}$. Then there exist $x \in \mathcal{G}_r$ with $y \in Q(x)$. Form the definition of Q, there is $f \in S_{F(., x(.))}^1$ with

$$y(t) = \begin{cases} \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t \mathcal{S}_\alpha(t - s) f(s) ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(x_0 - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k) I_k(x(t_k^-)) \\ \quad + \int_0^t \mathcal{S}_\alpha(t - s) f(s) ds, & t \in J_i, 1 \leq i \leq m. \end{cases}$$

We consider the following cases:

Case 1. When $i = 0$, let $t, t + \tau \in J_0 = [0, t_1]$, then

$$\|y^*(t+\tau) - y^*(t)\| = \|y(t+\tau) - y(t)\| \leq \|\mathcal{T}_\alpha(t+\tau)(x_0 - g(x)) - \mathcal{T}_\alpha(t)(x_0 - g(x))\| +$$

$$\| \int_0^{t+\tau} \mathcal{S}_\alpha(t+\tau-s)f(s)ds - \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds \|$$

$$\leq Q_1 + Q_2 + Q_3,$$

where
$Q_1 = \|\mathcal{T}_\alpha(t+\tau)(x_0 - g(x)) - \mathcal{T}_\alpha(t)(x_0 - g(x))\|$,
$Q_2 = \| \int_0^t [\mathcal{S}_\alpha(t+\tau-s) - \mathcal{S}_\alpha(t-s)]f(s)ds \|$,
$Q_3 = \| \int_t^{t+\tau} \mathcal{S}_\alpha(t+\tau-s)f(s)ds \|$.
We will show that $Q_i \to 0$ as $\tau \to 0$ for $i = 1, 2, 3$.

$$\lim_{\tau \to 0} Q_1 \leq (\|x_0\| + \sup_{x \in \mathcal{G}_r} \|g(x)\|) \lim_{\tau \to 0} \|\mathcal{T}_\alpha(t+\tau) - \mathcal{T}_\alpha(t)\| = 0,$$

independently of $x \in \mathcal{G}_r$.
For Q_2, from the definition of $\mathcal{S}_\alpha(t)$ and Lebesgue dominated convergence Theorem we get

$$Q_2 \leq \int_0^t \| [\mathcal{S}_\alpha(t+\tau-s) - \mathcal{S}_\alpha(t-s)]f(s)\|ds \to 0, \text{ as } \tau \to 0,$$

independently of $x \in \mathcal{G}_r$.
For Q_3,

$$\lim_{\tau \to 0} Q_3 = \lim_{\tau \to 0} \| \int_t^{t+\tau} \mathcal{S}_\alpha(t+\tau-s)f(s)ds \| \leq M_{\mathcal{S}_\alpha}\|\vartheta\| \lim_{\tau \to 0} \int_t^{t+\tau} (t-s)^{t+\tau-s}ds = 0,$$

independently of $x \in \mathcal{G}_r$.
Therefore,

$$\lim_{\tau \to 0} \|y(t+\tau) - y(t)\| = 0. \tag{9}$$

Case 2. When $t \in J_i, i \in \{1, 2, \cdots, m\}$. Let $t, t + \tau \in J_i$, from (7) we have

$$\|y^*(t+\tau) - y^*(t)\| = \|y(t+\tau) - y(t)\| \leq \|\mathcal{T}_\alpha(t+\tau)(x_0 - g(x)) - \mathcal{T}_\alpha(t)(x_0 - g(x))\|$$

$$+ \sum_{k=1}^{k=i} \|\mathcal{T}_\alpha(t+\tau-t_k)I_k(x(t_k^-)) - \mathcal{T}_\alpha(t-t_k)I_k(x(t_k^-))\|$$

$$+ \| \int_0^{t+\tau} \mathcal{S}_\alpha(t+\tau-s)f(s)ds - \int_0^t \mathcal{S}_\alpha(t-s)f(s)ds \|.$$

Arguing as in the first case, we obtain

$$\lim_{\tau \to 0} \|y(t+\tau) - y(t)\| = 0. \tag{10}$$

Case 3. When $t = t_i$, $i = 1, 2, \cdots, m$. Let $\tau > 0$ and $\delta > 0$ such that $t_i + \tau \in J_i$ and $t_i < \delta < t_i + \tau \leq t_{i+1}$, then we have

$$\|y^*(t_i + \tau) - y^*(t_i)\| = \lim_{\delta \to t_i^+} \|y(t_i + \tau) - y(\delta)\|.$$

From (7), we obtain

$$\|y(t_i + \tau) - y(\delta)\| \le \|\mathcal{T}_\alpha(t_i + \tau)(x_0 - g(x)) - \mathcal{T}_\alpha(\delta)(x_0 - g(x))\|$$
$$+ \sum_{k=1}^{k=i} \|\mathcal{T}_\alpha(t_i + \tau - t_k)I_k(x(t_k^-)) - \mathcal{T}_\alpha(\delta - t_k)I_k(x(t_k^-))\|$$
$$+ \|\int_0^{t_i+\tau} \mathcal{S}_\alpha(t_i + \tau - s)f(s)ds - \int_0^\delta \mathcal{S}_\alpha(\delta - s)f(s)ds\|.$$

Arguing as in the first case, we get

$$\lim_{\substack{\tau \to 0 \\ \delta \to t_i^+}} \|y(t_i + \tau) - y(\delta)\| = 0. \tag{11}$$

From (9), (10) and (11) $\mathcal{B}_{\overline{J_i}}$ is equicontinuous in $PC(J, \mathbb{R}^n)$ for each $i = 0, 1, \cdots, m$. From Azelà-Ascoli Theorem, Q is completely continuous. Moreover, from Theorem 2 we have $Q(x)$ is closed. So, $Q(x)$ is compact for all $x \in PC(J, \mathbb{R}^n)$. $\square$

Let $\mathcal{E}_{x_0}([0, \gamma])$ be the set of all solution of (1) on $[0, \gamma]$ where $0 < \gamma \le b$. From Theorem 2, we know that $Fix(Q) = \mathcal{E}_{x_0}([0, \gamma]) \ne \emptyset$. In the following Theorem we shall prove that $\mathcal{E}_{x_0}([0, \gamma])$ has an infinite dimension for any x_0.

Theorem 4. *Let $F : [0, \gamma] \times \mathbb{R}^n \to P_{ck}(\mathbb{R}^n)$. Suppose that $H_1 - H_5$ hold and $\chi\{t : dimF(t, x) < 1$ for some $x \in \mathbb{R}^n\} = 0$, where χ is the Lebesgue measure. Then, for all γ, $0 < \gamma < \min\left[\left(\frac{(1-M_{\mathcal{T}_\alpha}N)\alpha}{M_{\mathcal{S}_\alpha}\|\vartheta\|}\right)^{\frac{1}{\alpha}}, b\right]$, the set $\mathcal{E}_{x_0}([0, \gamma])$ has an infinite dimension for any x_0.*

Proof. Let $Q : PC([0, \gamma], \mathbb{R}^n) \to P(PC([0, \gamma], \mathbb{R}^n))$ define by: for every $x \in PC([0, \gamma], \mathbb{R}^n)$ $y \in Q(x)$ iff

$$y(t) = \begin{cases} \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t \mathcal{S}_\alpha(t - s)f(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(x_0 - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t - s)f(s)ds, & t \in J_i, 1 \le i \le p. \end{cases}$$

where $f \in S_{F(\cdot, x(\cdot))}^1$. From Theorem 3.1 we have $Fix(Q) = \mathcal{E}_{x_0}([0, \gamma])$. So, we will prove that $Fix(Q)$ has an infinite dimension. From Theorem 3, $Q(x)$ is convex and compact for all $x \in PC([0, \gamma], \mathbb{R}^n)$. Moreover, from Theorem 2 Q is contraction if $\left(M_{\mathcal{T}_\alpha}(N + \zeta) + M_{\mathcal{S}_\alpha}\frac{b^\alpha}{\alpha}\|\vartheta\|\right) < 1$ or $\gamma < \left(\frac{(1-M_{\mathcal{T}_\alpha}N)\alpha}{M_{\mathcal{S}_\alpha}\|\vartheta\|}\right)^{\frac{1}{\alpha}}$. We will prove that $\dim Q(x) \ge q$ for any $x \in PC([0, \gamma], \mathbb{R}^n)$ and $q \in \mathbb{N}$. Put $R(t) = F(t, x(t))$. From Lemma 3, there are linearly independent measurable selections $f_1(\cdot), f_2(\cdot), \cdots, f_q(\cdot)$ of R. Let $y_j(t) \in Q(x), j = 1, \cdots, q$, such that

$$y_j(t) = \begin{cases} \mathcal{T}_\alpha(t)(x_0 - g(x)) + \int_0^t \mathcal{S}_\alpha(t - s)f_j(s)ds, & t \in J_0, \\ \mathcal{T}_\alpha(t)(x_0 - g(x)) + \sum_{k=1}^{k=i} \mathcal{T}_\alpha(t - t_k)I_k(x(t_k^-)) \\ + \int_0^t \mathcal{S}_\alpha(t - s)f_j(s)ds, & t \in J_i, 1 \le i \le p. \end{cases}$$

We suppose that $\sum_{j=1}^q a_j y_j(t) = 0$, *a.e.* $t \in [0, \gamma]$. By taking Caputo derivatives *a.e.* in $[0, \gamma]$, we obtain $\sum_{j=1}^q a_j f_j(t) = 0$, *a.e.* $t \in [0, \gamma]$. Since $f_1(\cdot), f_2(\cdot), \cdots, f_q(\cdot)$ are linearly independent, $a_j = 0 \ \forall j$. So, $y_j(\cdot)$ are linearly independent. Hence, $Q(x)$ contains $q-$dimensional simplex. Therefore, $\dim Q(x) \ge q$. From Theorem 2, we have $Fix(Q) \ne \emptyset$ and by Definition 2.1, we get $Fix(Q) \subset Q(Fix(Q))$. Then,

$$\Omega(Fix(Q)) \le \Omega(Q(Fix(Q))),$$

where Ω is the Hausdorff measure of noncompactness. Thus, Q is condensing (see [3]). This implies that, $Fix(Q)$ is compact. Let I be the identity operator, consider the map $I - Q : Fix(Q) \to P_{ck}(\mathbb{R}^n)$. Suppose that dim $Fix(Q) < n$, then by using Lemma 5, there is a continuous selection σ of $I - Q$ with $\sigma(x) \neq 0 \ \forall \ x \in Fix(Q)$. So, there is a continuous selection v of $F : Fix(Q) \to P_{ck}(\mathbb{R}^n)$ with no fixed points. Now, define $\mathcal{F} : \mathbb{R}^n \to P_{ck}(\mathbb{R}^n)$ as

$$\mathcal{F}(x) = \begin{cases} Q(x), & x \in \mathbb{R}^n \setminus Fix(Q) \\ v(x), & x \in Fix(Q). \end{cases}$$

By using Lemma 6, $\mathcal{F}$ admits a continuous selection $f : \mathbb{R}^n \to \mathbb{R}^n$. Consequently, f is a continuous selection of Q such that $Fix(f) = \varnothing$ and $v = f$ on $Fix(Q)$, this contradicts Lemma 4. Therefore, the set $\mathcal{E}_{x_0}([0, \gamma])$ of all solutions of (1) has infinite dimensional. $\square$

4. Example

Consider the following fractional impulsive differential inclusion as an application of our results.

$$\begin{cases} {}^c D_t^{\frac{1}{2}} x(t) \in x(t) + F(t, x(t)), & t \in J = [0, 1], \ x \in \mathbb{R}, \\ x(t_i^+) = x(t_i^-) + \frac{1}{2^i}, & i = 1, \cdots, m, \\ x(0) = 0, \end{cases} \tag{12}$$

where $A = 1$.
Let $F : [0, 1] \times \mathbb{R} \to P(\mathbb{R})$ such that

$$F(t, x(t)) = \left(\frac{t}{\sqrt{(t+1)}}, \frac{\cot^{-1} x}{2\sqrt{t^2 + 1}} + \frac{\sin t}{\sqrt{t+1}} \right).$$

Obviously,

$$\|F(t, x)\| \leq \frac{\pi}{2\sqrt{t^2 + 1}} + \frac{\sin t}{\sqrt{t+1}} = \vartheta(t).$$

For each $i = 1, \cdots .m$, we define $I_i : \mathbb{R} \to \mathbb{R}$ as

$$I_i(x(t_i)) = \frac{1}{2^i}.$$

Clearly, I_i satisfies H_5 for every $i = 1, \cdots, m$.
Therefore, by Theorem 2 the system (12) has mild solution on $[0, 1]$.

Conclusion

Nonlocal fractional impulsive differential inclusions involving sectorial term are considered. The existence conditions are established under the situation when the multivalued function is Lipschitz. Also, we investigated the topological properties of the set of solutions. Our obtained results enhance and generalize many results proved in recent papers.

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