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ψ-Hilfer Fractional Approximations of Csiszar's f-Divergence

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Abstract: Here are given tight probabilistic inequalities that provide nearly best estimates for the Csiszar's f-divergence. These use the right and left ψ-Hilfer fractional derivatives of the directing function f. Csiszar's f-divergence or the so called Csiszar's discrimination is used as a measure of dependence between two random variables which is a very essential aspect of stochastics, we apply our results there. The Csiszar's discrimination is the most important and general measure for the comparison between two probability measures. We give also other applications.

Keywords: Csiszar's discrimination, Csiszar's distance, fractional calculus, ψ-Hilfer fractional derivative.

1. Background - I

Throughout this work we use the following.

Let f be a convex function from $(0, +\infty)$ into $\mathbb{R}$ that is strictly convex at 1 with $f(1) = 0$. Let $(X, \mathcal{A}, \lambda)$ be a measure space, where λ is a finite or a σ-finite measure on $(X, \mathcal{A})$. And let μ_1, μ_2 be two probability measures on $(X, \mathcal{A})$ such that $\mu_1 \ll \lambda$, $\mu_2 \ll \lambda$ (absolutely continuous); for example, $\lambda = \mu_1 + \mu_2$. Denote by $p = \frac{d\mu_1}{d\lambda}$, $q = \frac{d\mu_2}{d\lambda}$ (densities) Radon-Nikodym derivatives of μ_1, μ_2 with respect to λ. Here we assume that

$$0 < a \leq \frac{p}{q} \leq b, \text{ a.e. on } X$$

and $a \leq 1 \leq b$.

The quantity

$$\Gamma_f(\mu_1, \mu_2) = \int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda(x), \quad (1)$$

was introduced by I. Csiszar in 1967 (see [5]), and is called f-divergence of the probability measures μ_1 and μ_2. By Lemma 1.1 of [5], the integral (1) is well defined and $\Gamma_f(\mu_1, \mu_2) \geq 0$ with equality only when $\mu_1 = \mu_2$. In [5] the author without proof mentions that $\Gamma_f(\mu_1, \mu_2)$ does not depend on the choice of λ.

For a proof of the last see [2], Lemma 1.1.

The concept of f-divergence was introduced first in [4] as a generalization of Kullback's "information for discrimination" or I-divergence (generalized entropy) [8,9] and of Rényi's "information gain" (I-divergence of order α) [11]. In fact the I-divergence of order 1 equals

$$\Gamma_{u \log_2 u}(\mu_1, \mu_2).$$

The choice $f(u) = (u-1)^2$ again produces a known measure of difference of distributions called χ^2-divergence; of course the total variation distance $|\mu_1 - \mu_2| = \int_X |p(x) - q(x)| d\lambda(x)$ equals $\Gamma_{|u-1|}(\mu_1, \mu_2)$.

Here by assuming $f(1) = 0$ we can consider $\Gamma_f(\mu_1, \mu_2)$ as a measure of the difference between the probability measures μ_1, μ_2. The f-divergence is in general asymmetric in μ_1 and μ_2. But because f is convex and strictly convex at 1 (see Lemma 2, [2]) so is

$$f^*(u) = u f\left(\frac{1}{u}\right)$$

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and as in [5] we get

$$\Gamma_f(\mu_2, \mu_1) = \Gamma_{f^*}(\mu_1, \mu_2).$$

In information theory and statistics many other concrete divergences are used that are special cases of the above general Csiszar f-divergence, such as Hellinger distance D_H, α-divergence D_α, Bhattacharyya distance D_B, harmonic distance D_{H_α}, Jeffrey's distance D_J, and triangular discrimination D_Δ; for all these, see, for example [3], [6]. The problem of finding and estimating the proper distance (or difference or discrimination) of two probability distributions is one of the major questions in probability theory.

The above f-divergence measures in their various forms have also been applied to anthropology, genetics, finance, economics, political science, biology, approximation of probability distributions, signal processing, and pattern recognition. A great inspiration for this article has been the very important monograph on the topic by S. Dragomir [6].

2. Background - II

Let $-\infty < a < b < \infty$, the left and right Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$ ($\mathcal{R}(\alpha) > 0$) are defined by

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \tag{2}$$

$x > a$; where Γ stands for the gamma function,

and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \tag{3}$$

$x < b$.

The Riemann-Liouville left and right fractional derivatives of order $\alpha \in \mathbb{C}$ ($\mathcal{R}(\alpha) \geq 0$) are defined by

$$\left(\Delta_{a+}^{\alpha} y\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a+}^{n-\alpha} y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} y(t) dt \tag{4}$$

($n = \lceil \mathcal{R}(\alpha) \rceil$, $\lceil \cdot \rceil$ means ceiling of the number; $x > a$)

$$\left(\Delta_{b-}^{\alpha} y\right)(x) = (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{b-}^{n-\alpha} y\right)(x) =$$

$$\frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_x^b (t-x)^{n-\alpha-1} y(t) dt \tag{5}$$

($n = \lceil \mathcal{R}(\alpha) \rceil$; $x < b$), respectively, where $\mathcal{R}(\alpha)$ is the real part of α.

In particular, when $\alpha = n \in \mathbb{Z}_+$, then

$$\left(\Delta_{a+}^{0} y\right)(x) = \left(\Delta_{b-}^{0} y\right)(x) = y(x);$$

$$\left(\Delta_{a+}^{n} y\right)(x) = y^{(n)}(x), \text{ and } \left(\Delta_{b-}^{n} y\right)(x) = (-1)^n y^{(n)}(x), \ n \in \mathbb{N},$$

see [12].

Let $\alpha > 0$, $I = [a, b] \subset \mathbb{R}$, f an integrable function defined on I and $\psi \in C^1(I)$ an increasing function such that $\psi'(x) \neq 0$, for all $x \in I$. Left fractional integrals and left Riemann-Liouville fractional derivatives of a function f with respect to another function ψ are defined as ([7], [12])

$$I_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t) (\psi(x) - \psi(t))^{\alpha-1} f(t) dt, \tag{6}$$

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and

$$\Delta_{a+}^{\alpha,\psi} f(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{a+}^{n-\alpha,\psi} f(x) =$$

$$\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1} f(t)dt, \tag{7}$$

respectively, where $n = \lceil \alpha \rceil$.

Similarly, we define the right ones:

$$I_{b-}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1} f(t)dt, \tag{8}$$

and

$$\Delta_{b-}^{\alpha,\psi} f(x) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{b-}^{n-\alpha,\psi} f(x) =$$

$$\frac{1}{\Gamma(n-\alpha)}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n \int_x^b \psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1} f(t)dt. \tag{9}$$

The following semigroup property holds; if $\alpha, \beta > 0$, $f \in C(I)$, then

$$I_{a+}^{\alpha,\psi} I_{a+}^{\beta,\psi} f = I_{a+}^{\alpha+\beta,\psi} f \quad \text{and} \quad I_{b-}^{\alpha,\psi} I_{b-}^{\beta,\psi} f = I_{b-}^{\alpha+\beta,\psi} f.$$

Next let again $\alpha > 0$, $n = \lceil \alpha \rceil$, $I = [a,b]$, $f, \psi \in C^n(I) : \psi$ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The left ψ-Caputo fractional derivative of f of order α is given by ([1])

$$^C D_{a+}^{\alpha,\psi} f(x) = I_{a+}^{n-\alpha,\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x), \tag{10}$$

and the right ψ-Caputo fractional derivative ([1])

$$^C D_{b-}^{\alpha,\psi} f(x) = I_{b-}^{n-\alpha,\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x). \tag{11}$$

We set

$$f_\psi^{[n]}(x) := f_\psi^{(n)} f(x) := \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n f(x). \tag{12}$$

Clearly, when $\alpha = m \in \mathbb{N}$ we have

$$^C D_{a+}^{\alpha,\psi} f(x) = f_\psi^{[m]}(x) \ \text{ and } \ ^C D_{b-}^{\alpha,\psi} f(x) = (-1)^m f_\psi^{[m]}(x),$$

and if $\alpha \notin \mathbb{N}$, then

$$^C D_{a+}^{\alpha,\psi} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1} f_\psi^{[n]}(t)dt, \tag{13}$$

and

$$^C D_{b-}^{\alpha,\psi} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1} f_\psi^{[n]}(t)dt. \tag{14}$$

If $\psi(x) = x$, then we get the usual left and right Caputo fractional derivatives

$$^C D_{a+}^{m} f(x) = f^{(m)}(x), \ ^C D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x),$$

for $m \in \mathbb{N}$, and ($\alpha \notin \mathbb{N}$)

$$D_{*a}^{\alpha} f(x) = {}^C D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt, \tag{15}$$

$$D^{\alpha}_{b-}(x) = {}^{C}D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1} f^{(n)}(t)dt. \quad (16)$$

Also we set

$${}^{C}D^{0,\psi}_{a+}f(x) = {}^{C}D^{0,\psi}_{b-}f(x) = f(x).$$

Next we will deal with the ψ-Hilfer fractional derivative.

Definition 1. *([14]) Let $n-1<\alpha<n$, $n\in\mathbb{N}$, $I=[a,b]\subset\mathbb{R}$ and $f,\psi\in C^n([a,b])$, ψ is increasing and $\psi'(x)\neq 0$, for all $x\in I$. The ψ-Hilfer fractional derivative (left-sided and right-sided) ${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a+(b-)}f$ of order α and type $0\leq\beta\leq 1$, respectively, are defined by*

$${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a+}f(x) = I^{\beta(n-\alpha);\psi}_{a+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I^{(1-\beta)(n-\alpha);\psi}_{a+}f(x), \quad (17)$$

and

$${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{b-}f(x) = I^{\beta(n-\alpha);\psi}_{b-}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I^{(1-\beta)(n-\alpha);\psi}_{b-}f(x), \quad x\in[a,b]. \quad (18)$$

The original Hilfer fractional derivatives ([13]) come from $\psi(x)=x$, and are denoted by ${}^{H}\mathbb{D}^{\alpha,\beta}_{a+}f(x)$ and ${}^{H}\mathbb{D}^{\alpha,\beta}_{b-}f(x)$.

When $\beta=0$, we get Riemann-Liouville fractional derivatives, while when $\beta=1$ we have Caputo type fractional derivatives.

We define $\gamma=\alpha+\beta(n-\alpha)$. We notice that $n-1<\alpha\leq\alpha+\beta(n-\alpha)\leq\alpha+n-\alpha=n$, hence $\lceil\gamma\rceil=n$. We can easily write that ([14])

$${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a+}f(x) = I^{\gamma-\alpha;\psi}_{a+}\Delta^{\gamma;\psi}_{a+}f(x), \quad (19)$$

and

$${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{b-}f(x) = I^{\gamma-\alpha;\psi}_{b-}\Delta^{\gamma;\psi}_{b-}f(x), \quad x\in[a,b]. \quad (20)$$

We have that ([14])

$$\Delta^{\gamma;\psi}_{a+}f(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I^{(1-\beta)(n-\alpha);\psi}_{a+}f(x), \quad (21)$$

and

$$\Delta^{\gamma;\psi}_{b-}f(x) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I^{(1-\beta)(n-\alpha);\psi}_{b-}f(x). \quad (22)$$

In particular, when $0<\alpha<1$ and $0\leq\beta\leq 1$; $\gamma=\alpha+\beta(1-\alpha)$, we have that

$${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a+}f(x) = \frac{1}{\Gamma(\gamma-\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\gamma-\alpha-1}\Delta^{\gamma;\psi}_{a+}f(t)dt, \quad (23)$$

and

$${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{b-}f(x) = \frac{1}{\Gamma(\gamma-\alpha)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{\gamma-\alpha-1}\Delta^{\gamma;\psi}_{b-}f(t)dt, \quad (24)$$

$x\in[a,b]$.

Remark 1. *([14]) Let $\mu=n(1-\beta)+\beta\alpha$, then $\lceil\mu\rceil=n$.*

Assume that $g(x)=I^{(1-\beta)(n-\alpha);\psi}_{a+}f(x)\in C^n([a,b])$, we have that

$${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a+}f(x) = I^{n-\mu;\psi}_{a+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n g(x). \quad (25)$$

Thus

$${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a+}f = {}^{C}D^{\mu;\psi}_{a+}g(x) = {}^{C}D^{\mu;\psi}_{a+}\left[I^{(1-\beta)(n-\alpha);\psi}_{a+}f(x)\right]. \quad (26)$$

Assume that $w(x) = I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x) \in C^n([a,b])$. *Hence*

$$^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\beta(n-\alpha);\psi} \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n w(x) = I_{b-}^{n-\mu;\psi} \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n w(x). \tag{27}$$

Thus

$$^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f = {}^{C}D_{b-}^{\mu;\psi} w(x) = {}^{C}D_{b-}^{\mu;\psi} \left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x)\right). \tag{28}$$

We mention the simplified ψ-Hilfer fractional Taylor formulae:

Theorem 1. *(see also [14]) Let* $\psi, f \in C^n([a,b])$, *with* ψ *being increasing such that* $\psi'(x) \neq 0$ *over* $[a,b]$, *where* $n-1 < \alpha < n$, $0 \leq \beta \leq 1$, *and* $\gamma = \alpha + \beta(n-\alpha)$, $x \in [a,b]$. *Then*

$$f(x) - \sum_{k=1}^{n-1} \frac{(\psi(x)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} \left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f\right)(a) =$$
$$\frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\, {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t)dt, \tag{29}$$

and

$$f(x) - \sum_{k=1}^{n-1} \frac{(-1)^k(\psi(b)-\psi(x))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} \left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f\right)(b) =$$
$$\frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{\alpha-1}\, {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(t)dt. \tag{30}$$

Here notice that $\left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f\right)(a) = \left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f\right)(b) = 0$.

3. Main Results - I

Here f and the whole setting is as in section 1 the Background - I. Additionally, we assume that $\psi, f \in C^n([a,b])$, with ψ being increasing such that $\psi'(x) \neq 0$ over $[a,b]$, where $n-1 < \alpha < n$, $0 \leq \beta \leq 1$, and $\gamma = \alpha + \beta(n-\alpha)$. Furthermore we stress that $0 < a \leq \frac{p(x)}{q(x)} \leq b$, a.e. on X.

Next we present estimations for $\Gamma_f(\mu_1, \mu_2)$.

We give first left side results:

Theorem 2. *Additionally assume that* $f_{\psi}^{[n-k]} \left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f\right)(a) = 0$, $k = 1, ..., n-1$. *Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \frac{\left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)} \int_X q(x) \left(\psi\left(\frac{p(x)}{q(x)}\right) - \psi(a)\right)^{\alpha} d\lambda. \tag{31}$$

Proof. We have that (by (29))

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1}\, {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t)dt, \tag{32}$$

$\forall\, x \in [a,b]$.

Hence it holds

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} \left| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t) \right| dt \leq$$
$$\frac{\left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)} (\psi(x)-\psi(a))^{\alpha}, \;\; \forall\, x \in [a,b]. \tag{33}$$

Consequently we obtain

$$\Gamma_f(\mu_1,\mu_2) = \int_X q(x) f\left(\frac{p(x)}{q(x)}\right) d\lambda \leq \int_X q(x) \left| f\left(\frac{p(x)}{q(x)}\right) \right| d\lambda \overset{(33)}{\leq}$$

$$\frac{\left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)} \int_X q(x) \left(\psi\left(\frac{p(x)}{q(x)}\right) - \psi(a) \right)^{\alpha} d\lambda, \tag{34}$$

proving (31). □

We continue with the L_1-analog:

Theorem 3. *All as in Theorem 2, with $\alpha > 1$. Then*

$$\Gamma_f(\mu_1,\mu_2) \leq \frac{\left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{L_1([a,b],\psi)}}{\Gamma(\alpha)} \int_X q(x) \left(\psi\left(\frac{p(x)}{q(x)}\right) - \psi(a) \right)^{\alpha-1} d\lambda. \tag{35}$$

Proof. Since $\alpha > 1$, by (33), we have

$$|f(x)| \leq \frac{(\psi(x)-\psi(a))^{\alpha-1}}{\Gamma(\alpha)} \int_a^x \left| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t) \right| d\psi(t) \leq$$

$$\frac{\left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{L_1([a,b],\psi)}}{\Gamma(\alpha)} (\psi(x)-\psi(a))^{\alpha-1}, \ \forall\, x \in [a,b]. \tag{36}$$

Hence it holds

$$\Gamma_f(\mu_1,\mu_2) \leq \int_X q(x) \left| f\left(\frac{p(x)}{q(x)}\right) \right| d\lambda \overset{(36)}{\leq}$$

$$\frac{\left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{L_1([a,b],\psi)}}{\Gamma(\alpha)} \int_X q(x) \left(\psi\left(\frac{p(x)}{q(x)}\right) - \psi(a) \right)^{\alpha-1} d\lambda, \tag{37}$$

proving (35). □

Next comes to L_q-analog:

Theorem 4. *All as in Theorem 2. Let also $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. Then*

$$\Gamma_f(\mu_1,\mu_2) \leq \frac{\left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{L_q([a,b],\psi)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \int_X q(x) \left(\psi\left(\frac{p(x)}{q(x)}\right) - \psi(a) \right)^{\alpha-\frac{1}{q}} d\lambda. \tag{38}$$

Proof. By Hölder's inequality we have

$$|f(x)| \overset{(33)}{\leq} \frac{1}{\Gamma(\alpha)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{\alpha-1} \left| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t) \right| dt =$$

$$\frac{1}{\Gamma(\alpha)} \int_a^x (\psi(x)-\psi(t))^{\alpha-1} \left| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t) \right| d\psi(t) \leq$$

$$\frac{1}{\Gamma(\alpha)} \frac{(\psi(x)-\psi(a))^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{L_q([a,b],\psi)} =$$

$$\frac{\left\| {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{L_q([a,b],\psi)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} (\psi(x)-\psi(a))^{\alpha-\frac{1}{q}}, \ \forall\, x \in [a,b], \tag{39}$$

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and $\alpha > \frac{1}{q}$.

Consequently we derive

$$\Gamma_f(\mu_1, \mu_2) \leq \int_X q(x) \left| f\left(\frac{p(x)}{q(x)}\right) \right| d\lambda \overset{(39)}{\leq}$$

$$\frac{\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{L_q([a,b],\psi)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \int_X q(x) \left(\psi\left(\frac{p(x)}{q(x)}\right) - \psi(a) \right)^{\alpha - \frac{1}{q}} d\lambda, \tag{40}$$

where $\alpha > \frac{1}{q}$, proving (38). □

We continue with right side results:

Theorem 5. *Additionally assume that* $f_\psi^{[n-k]} \left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f \right)(b) = 0$, $k = 1, ..., n-1$. *Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \frac{\left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)} \int_X q(x) \left(\psi(b) - \psi\left(\frac{p(x)}{q(x)}\right) \right)^{\alpha} d\lambda. \tag{41}$$

Proof. We have that (by (30))

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} \, {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(t) dt, \tag{42}$$

$\forall\, x \in [a,b]$.

Hence it holds

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b \psi'(t)(\psi(t) - \psi(x))^{\alpha-1} \left| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(t) \right| dt \leq$$

$$\frac{\left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)} (\psi(b) - \psi(x))^{\alpha}, \quad \forall\, x \in [a,b]. \tag{43}$$

Therefore we derive

$$\Gamma_f(\mu_1, \mu_2) \leq \int_X q(x) \left| f\left(\frac{p(x)}{q(x)}\right) \right| d\lambda \overset{(43)}{\leq}$$

$$\frac{\left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)} \int_X q(x) \left(\psi(b) - \psi\left(\frac{p(x)}{q(x)}\right) \right)^{\alpha} d\lambda, \tag{44}$$

proving (41). □

It follows the L_1-analog:

Theorem 6. *All as in Theorem 5,* $\alpha > 1$. *Then*

$$\Gamma_f(\mu_1, \mu_2) \leq \frac{\left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{L_1([a,b],\psi)}}{\Gamma(\alpha)} \int_X q(x) \left(\psi(b) - \psi\left(\frac{p(x)}{q(x)}\right) \right)^{\alpha-1} d\lambda. \tag{45}$$

Proof. Since $\alpha > 1$, by (43), we have

$$|f(x)| \leq \frac{(\psi(b) - \psi(x))^{\alpha-1}}{\Gamma(\alpha)} \int_x^b \left| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(t) \right| d\psi(t) \leq$$

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$$\frac{\left\| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{L_1([a,b],\psi)}}{\Gamma(\alpha)} (\psi(b) - \psi(x))^{\alpha-1}, \ \forall\, x \in [a,b]. \tag{46}$$

Hence it holds

$$\Gamma_f(\mu_1,\mu_2) \leq \int_X q(x) \left| f\left(\frac{p(x)}{q(x)}\right) \right| d\lambda \overset{(46)}{\leq}$$

$$\frac{\left\| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{L_1([a,b],\psi)}}{\Gamma(\alpha)} \int_X q(x) \left(\psi(b) - \psi\left(\frac{p(x)}{q(x)}\right) \right)^{\alpha-1} d\lambda, \tag{47}$$

proving (45). □

Next comes the L_q-analog from the right side.

Theorem 7. *All as in Theorem 5. Let also $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. Then*

$$\Gamma_f(\mu_1,\mu_2) \leq \frac{\left\| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{L_q([a,b],\psi)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \int_X q(x) \left(\psi(b) - \psi\left(\frac{p(x)}{q(x)}\right) \right)^{\alpha-\frac{1}{q}} d\lambda. \tag{48}$$

Proof. We apply Hölder's inequality to (43) to get

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (\psi(t) - \psi(x))^{\alpha-1} \left| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(t) \right| d\psi(t) \leq$$

$$\frac{1}{\Gamma(\alpha)} \frac{(\psi(b) - \psi(x))^{\frac{p(\alpha-1)+1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{L_q([a,b],\psi)} =$$

$$\frac{\left\| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{L_q([a,b],\psi)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} (\psi(b) - \psi(x))^{\alpha-\frac{1}{q}}, \ \forall\, x \in [a,b],\ \alpha > \frac{1}{q}. \tag{49}$$

Consequently we derive

$$\Gamma_f(\mu_1,\mu_2) \leq \int_X q(x) \left| f\left(\frac{p(x)}{q(x)}\right) \right| d\lambda \overset{(49)}{\leq}$$

$$\frac{\left\| {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{L_q([a,b],\psi)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \int_X q(x) \left(\psi(b) - \psi\left(\frac{p(x)}{q(x)}\right) \right)^{\alpha-\frac{1}{q}} d\lambda, \tag{50}$$

where $\alpha > \frac{1}{q}$, proving (48). □

We continue with

Remark 2. *I) Assume that ${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \underset{(\leq 0)}{\geq} 0$ over $[a,b]$, then*

$$f(x) \underset{(\leq)}{\geq} \sum_{k=1}^{n-1} \frac{(\psi(x) - \psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} \left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f \right)(a), \tag{51}$$

$\forall\, x \in [a,b]$.

Hence

$$qf\left(\frac{p}{q}\right) \underset{(\leq)}{\geq} \sum_{k=1}^{n-1} q \frac{\left(\psi\left(\frac{p}{q}\right) - \psi(a)\right)^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} \left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f \right)(a). \tag{52}$$

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Consequently we derive

$$\Gamma_f(\mu_1,\mu_2) \underset{(\leq)}{\overset{(52)}{\geq}} \sum_{k=1}^{n-1} \frac{f_\psi^{[n-k]}\left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f\right)(a)}{\Gamma(\gamma-k+1)} \int_X q\left(\psi\left(\frac{p}{q}\right)-\psi(a)\right)^{\gamma-k} d\lambda. \tag{53}$$

II) Assume that ${}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \underset{(\leq 0)}{\geq} 0$ *over* $[a,b]$*, then*

$$f(x) \underset{(\leq)}{\overset{(30)}{\geq}} \sum_{k=1}^{n-1} \frac{(-1)^k(\psi(b)-\psi(x))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_\psi^{[n-k]}\left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f\right)(b), \tag{54}$$

$\forall\, x \in [a,b]$.

Hence

$$qf\left(\frac{p}{q}\right) \underset{(\leq)}{\geq} \sum_{k=1}^{n-1} \frac{(-1)^k q\left(\psi(b)-\psi\left(\frac{p}{q}\right)\right)^{\gamma-k}}{\Gamma(\gamma-k+1)} f_\psi^{[n-k]}\left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f\right)(b). \tag{55}$$

Consequently we derive

$$\Gamma_f(\mu_1,\mu_2) \underset{(\leq)}{\overset{(55)}{\geq}} \sum_{k=1}^{n-1} \frac{(-1)^k f_\psi^{[n-k]}\left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f\right)(b)}{\Gamma(\gamma-k+1)} \int_X q\left(\psi(b)-\psi\left(\frac{p}{q}\right)\right)^{\gamma-k} d\lambda. \tag{56}$$

We state

Corollary 1. *(to Theorem 2) Case of* $\psi(x) = x$*. So, additionally assume that* $\left(I_{a+}^{(1-\beta)(n-\alpha)} f\right)^{(n-k)}(a) = 0$*,* $k = 1,...,n-1$*. Then*

$$\Gamma_f(\mu_1,\mu_2) \leq \frac{\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)} \int_X q(x)^{1-\alpha}(p(x)-aq(x))^\alpha d\lambda. \tag{57}$$

It follows

Corollary 2. *(to Theorem 2) Case of* $\psi(x) = e^x$*. So, additionally assume that* $f_{e^x}^{[n-k]}\left(I_{a+}^{(1-\beta)(n-\alpha);e^x} f\right)(a) = 0$*,* $k = 1,...,n-1$*. Then*

$$\Gamma_f(\mu_1,\mu_2) \leq \frac{\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta;e^x} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)} \int_X q(x)\left(e^{\frac{p(x)}{q(x)}} - e^a\right)^\alpha d\lambda. \tag{58}$$

4. Background - III

Next we use the following. Let f by a convex function from $(0,+\infty)$ into $\mathbb{R}$ which is strictly convex at 1 with $f(1) = 0$. Let $(\mathbb{R}^2, \mathcal{B}^2, \lambda)$ be the measure space, where λ is the product Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}^2)$ with $\mathcal{B}$ being the Borel σ-field. And let $X, Y : \Omega \to \mathbb{R}$ be random variables on the probability space (Ω, P). Consider the probability distributions μ_{XY} and $\mu_X \times \mu_Y$ on $\mathbb{R}^2$, where μ_{XY}, $\mu_X \times \mu_Y$ stand for the joint distribution of X and Y and their marginal distributions, respectively.

Here we assume as existing the following probability density functions, the joint pdf of μ_{XY} to be $t(x,y)$, $x,y \in \mathbb{R}$, the pdf of μ_X to be $p(x)$ and pdf of μ_Y to be $q(y)$. Clearly $\mu_X \times \mu_Y$ has pdf $p(x)q(y)$. Here we further assume that $0 < a \leq \frac{t}{pq} \leq b$, a.e. on $\mathbb{R}^2$ and $a \leq 1 \leq b$.

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The quantity

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) = \int_{\mathbb{R}^2} p(x)q(y) f\left(\frac{t(x,y)}{p(x)q(y)}\right) d\lambda(x,y), \tag{59}$$

is the Csiszar's distance or f-divergence between μ_{XY} and $\mu_X \times \mu_Y$.

Here X, Y are less dependent the closer the distributions μ_{XY} and $\mu_X \times \mu_Y$ are, thus $\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y)$ can be considered as a measure of dependence of X and Y. For $f(u) = u \log_2 u$ we obtain the mutual information of X, and Y,

$$I(X,Y) = I(\mu_{XY} || \mu_X \times \mu_Y) = \Gamma_{u \log_2 u}(\mu_{XY}, \mu_X \times \mu_Y),$$

see [5]. For $f(u) = (u-1)^2$ we get the mean square contingency:

$$\varphi^2(X,Y) = \Gamma_{(u-1)^2}(\mu_{XY}, \mu_X \times \mu_Y),$$

see [10]. In the last we need $\mu_{XY} \ll \mu_X \times \mu_Y$, where $\ll$ denotes absolute continuity, but to cover the case of $\mu_{XY} \not\ll \mu_X \times \mu_Y$ we set $\varphi^2(X,Y) = +\infty$, then the last formula is always valid.

Clearly here $\mu_{XY}, \mu_X \times \mu_Y \ll \lambda$, also $\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) > 0$ with equality only when $\mu_{XY} = \mu_X \times \mu_Y$, i.e. when X, Y are independent r.v.'s.

5. Main Results - II

Here f and the whole setting is as in section 4, the Background - III. Additionally, we assume that $\psi, f \in C^n([a,b])$, with ψ being increasing such that $\psi'(x) \neq 0$ over $[a,b]$, where $n-1 < \alpha < n$, $0 \leq \beta \leq 1$, and $\gamma = \alpha + \beta(n-\alpha)$. Furthermore we stress that $0 < a \leq \frac{t}{pq} \leq b$, a.e. on $\mathbb{R}^2$.

Next we present estimations for $\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y)$. We apply our results of section 3, the Main Results-I.

We give first left side results:

Theorem 8. *Additionally assume that* $f_\psi^{[n-k]}\left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f\right)(a) = 0$, $k = 1, ..., n-1$. *Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)}$$

$$\int_{\mathbb{R}^2} (p(x)q(y)) \left(\psi\left(\frac{t(x,y)}{p(x)q(y)}\right) - \psi(a)\right)^{\alpha} d\lambda(x,y). \tag{60}$$

Proof. By Theorem 2. □

We give the L_1-analog:

Theorem 9. *All as in Theorem 8, with* $\alpha > 1$. *Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{L_1([a,b],\psi)}}{\Gamma(\alpha)}$$

$$\int_{\mathbb{R}^2} (p(x)q(y)) \left(\psi\left(\frac{t(x,y)}{p(x)q(y)}\right) - \psi(a)\right)^{\alpha-1} d\lambda(x,y). \tag{61}$$

Proof. By Theorem 3. □

The L_q-analog follows:

Theorem 10. *All as in Theorem 8. Let also $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \right\|_{L_q([a,b],\psi)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\int_{\mathbb{R}^2} (p(x)q(y)) \left(\psi\left(\frac{t(x,y)}{p(x)q(y)} \right) - \psi(a) \right)^{\alpha - \frac{1}{q}} d\lambda(x,y). \quad (62)$$

Proof. By Theorem 4. □

We continue with right side results:

Theorem 11. *Additionally assume that $f_\psi^{[n-k]} \left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f \right)(b) = 0$, $k = 1, ..., n-1$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+1)}$$

$$\int_{\mathbb{R}^2} (p(x)q(y)) \left(\psi(b) - \psi\left(\frac{t(x,y)}{p(x)q(y)} \right) \right)^{\alpha} d\lambda(x,y). \quad (63)$$

Proof. By Theorem 5. □

It follows the L_1-analog:

Theorem 12. *All as in Theorem 11, $\alpha > 1$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{L_1([a,b],\psi)}}{\Gamma(\alpha)}$$

$$\int_{\mathbb{R}^2} (p(x)q(y)) \left(\psi(b) - \psi\left(\frac{t(x,y)}{p(x)q(y)} \right) \right)^{\alpha-1} d\lambda(x,y). \quad (64)$$

Proof. By Theorem 6. □

The L_q-analog comes next from the right side.

Theorem 13. *All as in Theorem 11. Let also $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, with $\alpha > \frac{1}{q}$. Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{L_q([a,b],\psi)}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}}$$

$$\int_{\mathbb{R}^2} (p(x)q(y)) \left(\psi(b) - \psi\left(\frac{t(x,y)}{p(x)q(y)} \right) \right)^{\alpha - \frac{1}{q}} d\lambda(x,y). \quad (65)$$

Proof. By Theorem 7. □

We make

Remark 3. *(see Remark 2) I) Assume that ${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f \underset{(\leq 0)}{\geq} 0$ over $[a, b]$, then*

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$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \underset{(\leq)}{\geq} \sum_{k=1}^{n-1} \frac{f_\psi^{[n-k]}\left(I_{a+}^{(1-\beta)(n-\alpha);\psi} f\right)(a)}{\Gamma(\gamma - k + 1)} \tag{66}$$

$$\int_{\mathbb{R}^2} (p(x)q(y)) \left(\psi\left(\frac{t(x,y)}{p(x)q(y)}\right) - \psi(a)\right)^{\gamma-k} d\lambda(x,y).$$

II) Assume that ${}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \underset{(\leq 0)}{\geq} 0$ *over* $[a,b]$, *then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \underset{(\leq)}{\geq} \sum_{k=1}^{n-1} \frac{(-1)^k f_\psi^{[n-k]}\left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f\right)(b)}{\Gamma(\gamma - k + 1)} \tag{67}$$

$$\int_{\mathbb{R}^2} (p(x)q(y)) \left(\psi(b) - \psi\left(\frac{t(x,y)}{p(x)q(y)}\right)\right)^{\gamma-k} d\lambda(x,y).$$

We state

Corollary 3. *(to Theorem 8) Case of* $\psi(x) = x$. *So, additionally assume that* $\left(I_{a+}^{(1-\beta)(n-\alpha)} f\right)^{(n-k)}(a) = 0$, $k = 1, ..., n-1$. *Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha + 1)} \tag{68}$$

$$\int_{\mathbb{R}^2} (p(x)q(y))^{1-\alpha} (t(x,y) - ap(x)q(y))^{\alpha} d\lambda(x,y).$$

It follows

Corollary 4. *(to Theorem 8) Case of* $\psi(x) = e^x$. *So, additionally assume that* $f_{e^x}^{[n-k]}\left(I_{a+}^{(1-\beta)(n-\alpha);e^x} f\right)(a) = 0$, $k = 1, ..., n-1$. *Then*

$$\Gamma_f(\mu_{XY}, \mu_X \times \mu_Y) \leq \frac{\left\| {}^H\mathbb{D}_{a+}^{\alpha,\beta;e^x} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha + 1)} \tag{69}$$

$$\int_{\mathbb{R}^2} (p(x)q(y)) \left(e^{\frac{t(x,y)}{p(x)q(y)}} - e^a\right)^{\alpha} d\lambda(x,y).$$

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