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Analysis and computation of solutions for a class of nonlinear SBVPs arising in epitaxial growth

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Abstract: In this work the existence and nonexistence of stationary radial solutions to the elliptic partial differential equation arising in the molecular beam epitaxy are studied. Since we are interested in radial solutions we arrive at the following fourth-order differential equation

$$\frac{1}{r}\left\{r\left[\frac{1}{r}(r\phi')'\right]'\right\}' = \frac{1}{r}\phi'\phi'' + \lambda G(r),\ 0 < r < 1.$$

It is non-self adjoint and it does not have exact solutions and also it admits multiple solutions. Here $\lambda \in \mathbb{R}$ measures the intensity of the flux and G is stationary flux. Solution depends on the size of the parameter λ. We use monotone iterative technique and integral equations along with upper and lower solutions to prove that solutions exist. We establish the qualitative properties of the solutions and provide bounds for the values of the parameter λ, which help us to separate the existence from nonexistence. These results complement some existing results in the literature. To verify the analytical results we also propose a new computational iterative technique and use it to verify the bounds on λ and dependence of solutions for these computed bounds on λ.

Keywords: Radial solutions; singular boundary value problems; non-self-adjoint operator; Green's function; lower solution; upper solution; iterative numerical approximations.

0. Introduction

Epitaxy means the growth of a single thin film on top of a crystalline substrate. It is crucial for semiconductor thin-film technology, hard and soft coatings, protective coatings, optical coatings and etc. Epitaxial growth technique is used to produce the growth of semiconductor films and multi-layer structures under high vacuum conditions ([5]). The major advantages of epitaxial growth are to reduces the growth time, better structural and superior electrical properties, eliminates the wastages caused during growth, wafering cost, cutting, polishing and etc. Several types of epitaxial growth techniques like the Hybrid vapor phase epitaxy ([19]), Chemical beam epitaxy ([12]), Molecular beam epitaxy (MBE), etc have been used for the growth of compound semiconductors and other materials. In this work, we strictly focus on MBE, and we restrict our attention to the differential

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equation model, which is proposed by Escudero et al. in [8–11]. The mathematical description of epitaxial growth is carried out by means of a function σ defined as

$$\sigma : \Omega \subset \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R},$$

which describes the height of the growing interface in the spatial point $x \in \Omega \subset \mathbb{R}^2$ at time $t \in \mathbb{R}^+$. Authors ([8–11]) show that the function σ obeys the fourth order partial differential equation

$$\partial_t \sigma + \Delta^2 \sigma \; = \; \det(D^2\sigma) + \lambda \eta(x,t), \quad x \in \Omega \subset \mathbb{R}^2, \tag{1}$$

where $\eta(x,t)$ models the incoming mass entering the system through epitaxial deposition, λ measures the intensity of this flux and the determinat of Hessian matrix is

$$\det(D^2\sigma) = \frac{\partial^2 \sigma}{\partial x_1^2} \times \frac{\partial^2 \sigma}{\partial x_2^2} - \left(\frac{\partial^2 \sigma}{\partial x_1 \partial x_2} \right)^2. \tag{2}$$

Stationary counterpart of the partial differential equation (1) subject to homogeneous Dirichlet boundary condition (4) and homogeneous Navier boundary condition (5) is defined as (see [10])

$$\Delta^2 \sigma \; = \; \det(D^2\sigma) + \lambda G(x), \quad x \in \Omega \subset \mathbb{R}^2, \tag{3}$$

$$\sigma = 0, \quad \frac{\partial \sigma}{\partial n} = 0 \;\text{ on }\; \partial\Omega, \tag{4}$$

$$\sigma = 0, \quad \Delta\sigma = 0 \;\text{ on }\; \partial\Omega. \tag{5}$$

where $\eta(x,t) \equiv G(x)$ is a stationary flux, n is unit out drawn normal to $\partial\Omega$.

By using the transformation $r = |x|$ and $\sigma(x) = \phi(|x|)$, due to symmetry the above set of equations are transformed into the following set of equations

$$\frac{1}{r}\left\{ r\left[\frac{1}{r}(r\phi')' \right]' \right\}' = \frac{1}{r}\phi'\phi'' + \lambda G(r), \tag{6}$$

$$\phi'(0) = 0, \; \phi(1) = 0, \; \phi'(1) = 0, \; \lim_{r\to 0} r\phi'''(r) = 0, \tag{7}$$

$$\phi'(0) = 0, \; \phi(1) = 0, \; \phi'(1) + \phi''(1) = 0, \; \lim_{r\to 0} r\phi'''(r) = 0, \tag{8}$$

where $' = \dfrac{d}{dr}$.

Now, we impose the following boundary conditions which complements the work of (see [10])

$$\phi'(0) = 0, \; \phi(1) = 0, \; \phi''(1) = 0, \; \lim_{r\to 0} r\phi'''(r) = 0. \tag{9}$$

For simplicity we take $G(r) = 1$, which physically mean that the new material is being deposited uniformly on the unit disc.

Now, by using $\lim_{r\to 0} r\phi'''(r) = 0$, $w = r\phi'$ and integrating by parts from equation (6), we have

$$r^2 w'' - r w' = \frac{1}{2}w^2 + \frac{1}{2}\lambda r^4. \tag{10}$$

By using the transformation $t = \frac{r^2}{2}$ and $u(t) = w(r)$, it is posible to reduce the equation (10) into the following equation

$$u'' = \frac{u^2}{8t^2} + \frac{\lambda}{2}, \text{ for } t \in \left(0, \frac{1}{2}\right]. \tag{11}$$

Corresponding to (11), we define the following three boundary value problems:

$$\text{Problem 1: } \begin{cases} u'' = \frac{u^2}{8t^2} + \frac{\lambda}{2}, \text{ for } t \in \left(0, \frac{1}{2}\right] \\ \lim_{t\to 0^+} \sqrt{t}u'(t) = 0, \ u(\frac{1}{2}) = 0, \end{cases} \tag{12}$$

$$\text{Problem 2: } \begin{cases} u'' = \frac{u^2}{8t^2} + \frac{\lambda}{2}, \text{ for } t \in \left(0, \frac{1}{2}\right] \\ \lim_{t\to 0^+} \sqrt{t}u'(t) = 0, \ u'(\frac{1}{2}) = 0, \end{cases} \tag{13}$$

$$\text{Problem 3: } \begin{cases} u'' = \frac{u^2}{8t^2} + \frac{\lambda}{2}, \text{ for } t \in \left(0, \frac{1}{2}\right] \\ \lim_{t\to 0^+} \sqrt{t}u'(t) = 0, \ u(\frac{1}{2}) = u'(\frac{1}{2}). \end{cases} \tag{14}$$

The BVPs (12), (13) and (14) can equivalantly be described as the following integral equations (IE):

- IE corresponding to Problem 1:

$$u(t) = -\left[\left(\frac{1}{2} - t\right) \int_0^t \frac{u^2}{4s} ds + t \int_t^{\frac{1}{2}} \frac{u^2}{4s^2} \left(\frac{1}{2} - s\right) ds + \frac{\lambda}{4} t \left(\frac{1}{2} - t\right)\right], \tag{15}$$

- IE corresponding to Problem 2:

$$u(t) = -\left[\int_0^t \frac{u^2}{8s} ds + t \int_t^{\frac{1}{2}} \frac{u^2}{8s^2} ds + \frac{\lambda}{4} t(1 - t)\right], \tag{16}$$

- IE corresponding to Problem 3:

$$u(t) = -\left[\left(t + \frac{1}{2}\right) \int_0^t \frac{u^2}{4s} ds + t \int_t^{\frac{1}{2}} \frac{u^2}{4s^2} \left(s + \frac{1}{2}\right) ds + \frac{\lambda}{4} t \left(\frac{3}{2} - t\right)\right]. \tag{17}$$

We assume that $u \in C^2_{loc}\left(\left(0, \frac{1}{2}\right]; \mathbb{R}\right)$, where $C^2_{loc}\left(\left(0, \frac{1}{2}\right]; \mathbb{R}\right)$ is defined as

$$\left\{u : \left(0, \frac{1}{2}\right] \to \mathbb{R} | u \in C^2([a, b], \mathbb{R}) \text{ for everycompact set } [a, b] \subset \left(0, \frac{1}{2}\right]\right\}.$$

In [10], Escudero et al. proved the existence and nonexistence of solutions of Problem 1 and Problem 3 by using upper and lower solution techniques. Corresponding to Problem 1 and 3, they have also provided the rigorous bounds of the values of the parameter λ, which helps us to separate the existence from nonexistence. In [16] Verma et al. provide numerical illustrations via VIM to verify the results of Escudero et al. [10]. To verify their numerical results, they provided other iterative schemes based on homotopy ([28]) and Adomian decomposition method ([17]).

The equation (13) has not been investigated theoretically in the existing literature to the best of our belief. Also corresponding to BVPs (12), (13) and (14) lot of investigations are still pending. Here, we focus on both theoretical as well as numerical work. We derive the sign of solution and prove its existence in continuous space. We also compute the bounds of the parameter λ. The results of this paper complements existing theoretical

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results. We also provide an iterative scheme based on Green's function to compute the bounds and solutions to demonstrate the existence and non existence which is depending on λ.

To prove the existence of solutions, here we use the monotone iterative technique ([7,18,23–26,29]). Recently, many researchers applied this technique on the initial value problem (IVP) for the nonlinear noninstantaneous impulsive differential equation (NIDE) ([3]), p-Laplacian boundary value problems with the right-handed Riemann-Liouville fractional derivative ([30]), etc to prove the existence of the solution. Here, we also present numerical results to verify the theoretical results. To develop the iterative scheme based on Green's function, we consider the equations (12), (13) and (14). Recently, many authors have used numerical approximate methods like the VIM [16], Adomian decomposition method (ADM), homotopy perturbation method (HPM), etc to find approximate solution for different models involving differential equations ([21,22]), integral equation ([1,4,20]), fractional differential equations ([2,15]) etc. After that, Waleed Al Hayani ([14]) and Singh et al. ([27]) applied ADM with Green's function to compute the approximate solution. They focused on the BVPs which have a unique solution. The major advantage of our proposed technique is to capture multiple solutions together with desired accuracy.

The remainder of the paper has been focused on both theoretical and numerical results. We have proved some basic properties of the BVPs in section 1. The monotone iterative technique is presented in section 2, to prove the existence of a solution. A wide range of λ of equation (6) corresponding to different types of boundary conditions are shown in section 3. In section 4, we apply our proposed technique on the integral equations and show a wide range of numerical results. Finally in section 5, we draw our main conclusions.

1. Preliminary

Corresponding to $\lambda \geq 0$, we prove some basic qualitative properties of the solution $u \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$, which satisfies the following inequality

$$u'' \geq \frac{u^2}{8t^2} + \frac{\lambda}{2}, \text{ for } t \in \left(0, \frac{1}{2}\right]. \tag{18}$$

Here, we omit the proof of lemma 1.1, lemma 1.2, lemma 1.3, corollary 1.1, lemma 1.4 which has been done by Escudero et. al. in [10].

Lemma 1.1. *Let* $u \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ *satisfy* $\lim_{t\to 0^+} \sqrt{t}u'(t) = 0$ *and equation (18), then* $\lim_{t\to 0} u(t) = 0$.

Lemma 1.2. *Let* $u \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ *satisfy* $\lim_{t\to 0} u(t) = 0$, $u\left(\frac{1}{2}\right) = 0$ *and equation (18), then* $u(t) \leq 0$ *for all* $t \in \left(0, \frac{1}{2}\right]$.

Lemma 1.3. *Let* $u \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ *satisfy* $\lim_{t\to 0} u(t) = 0$, $u\left(\frac{1}{2}\right) = u'\left(\frac{1}{2}\right)$ *and equation (18), then* $u(t) \leq 0$ *for all* $t \in \left(0, \frac{1}{2}\right]$.

Corollary 1.1. *Let* $u \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ *satisfy* $\lim_{t\to 0} u(t) = 0$, $u(t) \leq 0$ *and equation (18), then* $\lim_{t\to 0^+} \sqrt{t}u'(t) = 0$ *if and only if* $\lim_{t\to 0^+} \frac{u(t)}{\sqrt{t}} = 0$.

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Lemma 1.4. *Let* $u \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ *satisfy* $\lim_{t\to 0^+} \frac{u(t)}{\sqrt{t}} = 0$. *Then for every* $\mu \in [0, 1)$, *we have*

$$\lim_{t\to 0^+} t^{1-\mu} \int_t^{\frac{1}{2}} \frac{u^2}{s^2} ds = 0. \tag{19}$$

Lemma 1.5. *Let* $u \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ *satisfy* $\lim_{t\to 0} u(t) = 0$, $u'\left(\frac{1}{2}\right) = 0$ *and equation (18), then* $u(t) \leq 0$ *for all* $t \in \left(0, \frac{1}{2}\right]$.

Proof. First, we show that $u\left(\frac{1}{2}\right) \leq 0$. Assume $u\left(\frac{1}{2}\right) > 0$. Since $\lim_{t\to 0} u(t) = 0$, therefore, there exist a $t_0 \in \left(0, \frac{1}{2}\right]$ such that $u(t_0) < u\left(\frac{1}{2}\right)$. Now from (18), $u'(t)$ is increasing function on $\left(0, \frac{1}{2}\right]$. Again by mean value theorem, we have

$$\frac{u\left(\frac{1}{2}\right) - u(t_0)}{\frac{1}{2} - t_0} = u'(\xi), \ \min\left\{\frac{1}{2}, t_0\right\} \leq \xi \leq \max\left\{\frac{1}{2}, t_0\right\}. \tag{20}$$

Since $u'\left(\frac{1}{2}\right) = 0$, therefore we have $\left(u\left(\frac{1}{2}\right) - u(t_0)\right) \leq u'\left(\frac{1}{2}\right)\left(\frac{1}{2} - t_0\right) = 0$. Hence we get $u\left(\frac{1}{2}\right) \leq u(t_0)$, which is a contradiction. So, we have $u\left(\frac{1}{2}\right) \leq 0$. Furthermore, $u(t)$ is a convex function along with $u'\left(\frac{1}{2}\right) = 0$. Also $u'(t)$ is increasing, which implies $u'(t) \leq 0$. Again $u(t)$ is decreasing function on $\left(0, \frac{1}{2}\right]$. Therefore $\lim_{t\to 0} u(t) = 0$ and $u\left(\frac{1}{2}\right) \leq 0$ leads to $u(t) \leq 0$ on $\left(0, \frac{1}{2}\right]$. □

Lemma 1.6. *Let* $u \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ *be the solution of Problem 3, then* $u(t)$ *satisfies the following integral equation*

$$u(t) = -\left[\left(t + \frac{1}{2}\right)\int_0^t \frac{u^2}{4s} ds + t \int_t^{\frac{1}{2}} \frac{u^2}{4s^2}\left(s + \frac{1}{2}\right) ds + \frac{\lambda}{4} t\left(\frac{3}{2} - t\right)\right], \tag{21}$$

and

$$\lim_{t\to 0^+} \frac{|u(t)|}{t} < +\infty. \tag{22}$$

Proof. The Green's function of the Problem 3 can be written as

$$G(s,t) = \begin{cases} -2s\left(t + \frac{1}{2}\right) & 0 \leq s \leq t, \\ -2t\left(s + \frac{1}{2}\right) & t \leq s \leq \frac{1}{2}. \end{cases} \tag{23}$$

Therefore from equation (23) and Problem 3, we can easily deduce the integral equation (21). Now, by using the result of Lemma 1.1, we have

$$\lim_{t\to 0^+} \frac{|u(t)|}{t} = \left| \int_0^{\frac{1}{2}} \frac{u^2}{4s^2}\left(s + \frac{1}{2}\right) ds + \frac{3\lambda}{8} \right|. \tag{24}$$

Now, put

$$f(t) = \frac{u^2}{t}, \ g(t) = \frac{1}{t^\mu} \text{ and } h(t) = \frac{1}{t^{1-\mu}} \ \text{ for } t \in \left(0, \frac{1}{2}\right]. \tag{25}$$

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Therefore we get $fg \in L\left((0,\frac{1}{2}]\right)$ provided $\mu \in (0,1)$. Consequently, we have

$$\int_0^{\frac{1}{2}} \frac{u^2}{s^2} ds < \infty. \tag{26}$$

Hence, from equation (24), we get the equation (22). □

Lemma 1.7. *Let $u(t) \in C^2_{loc}\left((0,\frac{1}{2}],\mathbb{R}\right)$ be the solution of Problem 2, then $u(t)$ can be written as in the following form*

$$u(t) = -\left[\int_0^t \frac{u^2}{8s} ds + t\int_t^{\frac{1}{2}} \frac{u^2}{8s^2} ds + \frac{\lambda}{4}t(1-t)\right], \tag{27}$$

and also satisfies

$$\lim_{t\to 0^+} \frac{|u(t)|}{t} < \infty. \tag{28}$$

Proof. By using the boundary condition and properties of Green's function, we have

$$G(s,t) = \begin{cases} -s & 0 \le s \le t, \\ -t & t \le s \le \frac{1}{2}. \end{cases} \tag{29}$$

From equation (29) and Problem 2, we can easily derive the equation (27). Now, by using the result of Lemma 1.1, we have

$$\lim_{t\to 0^+} \frac{|u(t)|}{t} = \left| \int_0^{\frac{1}{2}} \frac{u^2}{8s^2} ds + \frac{\lambda}{4} \right|. \tag{30}$$

Therefore, from equations (30) and by similar analysis as in Lemma 1.6, we can prove the result (28). □

Lemma 1.8. *Let $u(t) \in C^2_{loc}\left((0,\frac{1}{2}],\mathbb{R}\right)$ be the solution of Problem 1, then $u(t)$ can be written as in the following form*

$$u(t) = -\left[\left(\frac{1}{2}-t\right)\int_0^t \frac{u^2}{4s} ds + t\int_t^{\frac{1}{2}} \frac{u^2}{4s^2}\left(\frac{1}{2}-s\right) ds + \frac{\lambda}{4}t\left(\frac{1}{2}-t\right)\right], \tag{31}$$

and satisfies

$$\lim_{t\to 0^+} \frac{|u(t)|}{t} < \infty. \tag{32}$$

Proof. The Green's function of the Problem 1 is given by

$$G(s,t) = \begin{cases} -2s\left(\frac{1}{2}-t\right) & 0 \le s \le t, \\ -2t\left(\frac{1}{2}-s\right) & t \le s \le \frac{1}{2}. \end{cases} \tag{33}$$

Again, from equation (33) and Problem 3, we derive integral equation (31). Furthermore, by using the result of Lemma 1.1, we have

$$\lim_{t\to 0^+} \frac{|u(t)|}{t} = \left| \int_0^{\frac{1}{2}} \frac{u^2}{4s^2}\left(\frac{1}{2}-s\right) ds + \frac{\lambda}{8} \right|. \tag{34}$$

Again, by similar analysis as in Lemma 1.7, we get the inequality (32). □

2. Existence of solutions

In this section, we apply the monotone iterative technique coupled with lower and upper solutions to prove the existence of at least one solution

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of Problem 1, Problem 2 and Problem 3. For this purpose, we need to prove some lemmas, which help us to prove the main results of this paper.

2.1. Construction of Green's function

To investigate the Problem 1, Problem 2 and Problem 3, we consider the corresponding nonlinear singular boundary value problems, which are given by

$$\text{Problem } 1(a)\colon \begin{cases} u''+ku=h(t), \text{ for } t\in\left(0,\frac{1}{2}\right], \\ \lim_{t\to 0^+}\sqrt{t}u'(t)=0,\ u\left(\frac{1}{2}\right)=-b_1, \end{cases} \tag{35}$$

$$\text{Problem } 2(a)\colon \begin{cases} u''+ku=h(t), \text{ for } t\in\left(0,\frac{1}{2}\right], \\ \lim_{t\to 0^+}\sqrt{t}u'(t)=0,\ u'\left(\frac{1}{2}\right)=-b_2, \end{cases} \tag{36}$$

$$\text{Problem } 3(a)\colon \begin{cases} u''+ku=h(t), \text{ for } t\in\left(0,\frac{1}{2}\right], \\ \lim_{t\to 0^+}\sqrt{t}u'(t)=0,\ u\left(\frac{1}{2}\right)-b_3=u'\left(\frac{1}{2}\right), \end{cases} \tag{37}$$

where $h(t)=\dfrac{u^2}{8t^2}+\dfrac{\lambda}{2}+ku$, $k\in\mathbb{R}$, $b_1,\ b_2,\ b_3\geq 0$ and $\lambda\in\mathbb{R}$. Throughout the paper we assume the following conditions

- $H_0=\{k\in\mathbb{R}: k<0\}$,
- $H_1=\left\{k\in\mathbb{R}: k<0 \ \&\ \sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2}\right)-\sinh\left(\frac{\sqrt{|k|}}{2}\right)>0\right\}$,
- $H_2=\{k\in\mathbb{R}: 0<k<4\pi^2\}$,
- $H_3=\{k\in\mathbb{R}: 0<k<\pi^2\}$,
- $H_4=\left\{k\in\mathbb{R}: 0<k<\frac{\pi^2}{4} \ \&\ \sqrt{k}\cos\left(\frac{\sqrt{k}}{2}\right)-\sin\left(\frac{\sqrt{k}}{2}\right)>0\right\}$.

Lemma 2.1. *Let k satisfy H_0 and $u(t)\in C^2_{loc}\left((0,\frac{1}{2}],\mathbb{R}\right)$ be the solution of Problem $1(a)$, then*

$$u(t)=-\frac{b_1\sinh\left(\sqrt{|k|}t\right)}{\sinh\left(\frac{\sqrt{|k|}}{2}\right)}+\int_0^{\frac{1}{2}}G(s,t)h(s)ds, \tag{38}$$

where Green's function $G(s,t)$ is given by

$$G(s,t)=\begin{cases} -\dfrac{\sinh\left(\frac{\sqrt{|k|}}{2}-\sqrt{|k|}t\right)\sinh\left(\sqrt{|k|}s\right)}{\sqrt{|k|}\sinh\left(\frac{\sqrt{|k|}}{2}\right)} & 0\leq s\leq t, \\ -\dfrac{\sinh\left(\frac{\sqrt{|k|}}{2}-\sqrt{|k|}s\right)\sinh\left(\sqrt{|k|}t\right)}{\sqrt{|k|}\sinh\left(\frac{\sqrt{|k|}}{2}\right)} & t\leq s\leq\frac{1}{2}, \end{cases} \tag{39}$$

and $G(s,t)\leq 0$ for all $0\leq s\leq\frac{1}{2}$ and $0\leq t\leq\frac{1}{2}$.

Proof. By using the boundary condition of Problem $1(a)$ and properties of Green's function, we can easily prove the equation (39). Furthermore we have $G(s,t)\leq 0$ for all $0\leq s\leq\frac{1}{2}$ and $0\leq t\leq\frac{1}{2}$. □

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Lemma 2.2. *Let k satisfy H_0 and $u(t) \in C^2_{loc}\left((0,\frac{1}{2}],\mathbb{R}\right)$ be the solution of Problem $2(a)$, then*

$$u(t) = -\frac{b_2 \sinh\left(\sqrt{|k|}t\right)}{\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2}\right)} + \int_0^{\frac{1}{2}} G(s,t)h(s)ds, \tag{40}$$

where Green's function $G(s,t)$ is given by

$$G(s,t) = \begin{cases} -\dfrac{\cosh\left(\frac{\sqrt{|k|}}{2} - \sqrt{|k|}t\right)\sinh\left(\sqrt{|k|}s\right)}{\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2}\right)} & 0 \le s \le t, \\ -\dfrac{\cosh\left(\frac{\sqrt{|k|}}{2} - \sqrt{|k|}s\right)\sinh\left(\sqrt{|k|}t\right)}{\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2}\right)} & t \le s \le \frac{1}{2}, \end{cases} \tag{41}$$

and $G(s,t) \le 0$ for all $0 \le s \le \frac{1}{2}$ and $0 \le t \le \frac{1}{2}$.

Proof. In a similar manner as in Lemma 2.1, we can easily get the equation (41), and prove $G(s,t) \le 0$ for all $0 \le s \le \frac{1}{2}$ and $0 \le t \le \frac{1}{2}$. □

Lemma 2.3. *Let k satisfy H_1 and $u(t) \in C^2_{loc}\left((0,\frac{1}{2}],\mathbb{R}\right)$ be the solution of Problem $3(a)$, then*

$$u(t) = -\frac{b_3 \sinh\left(\sqrt{|k|}t\right)}{\left[\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2}\right) - \sinh\left(\frac{\sqrt{|k|}}{2}\right)\right]} + \int_0^{\frac{1}{2}} G(s,t)h(s)ds, \tag{42}$$

where Green's function $G(s,t)$ is given by

$$G(s,t) = \begin{cases} -\dfrac{\left[\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2} - \sqrt{|k|}t\right) - \sinh\left(\frac{\sqrt{|k|}}{2} - \sqrt{|k|}t\right)\right]\sinh\left(\sqrt{|k|}s\right)}{\sqrt{|k|}\left[\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2}\right) - \sinh\left(\frac{\sqrt{|k|}}{2}\right)\right]} & 0 \le s \le t, \\ -\dfrac{\left[\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2} - \sqrt{|k|}s\right) - \sinh\left(\frac{\sqrt{|k|}}{2} - \sqrt{|k|}s\right)\right]\sinh\left(\sqrt{|k|}t\right)}{\sqrt{|k|}\left[\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2}\right) - \sinh\left(\frac{\sqrt{|k|}}{2}\right)\right]} & t \le s \le \frac{1}{2}, \end{cases} \tag{43}$$

and $G(s,t) \le 0$ for all $0 \le s \le \frac{1}{2}$ and $0 \le t \le \frac{1}{2}$.

Proof. Again by similar analysis, we can easily derive the the Green's function. Now,

$$\begin{aligned} &\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2} - \sqrt{|k|}t\right) - \sinh\left(\frac{\sqrt{|k|}}{2} - \sqrt{|k|}t\right), \\ &= \sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2}\right)\cosh\left(\sqrt{|k|}t\right) - \sqrt{|k|}\sinh\left(\frac{\sqrt{|k|}}{2}\right)\sinh\left(\sqrt{|k|}t\right) \\ &\quad - \sinh\left(\frac{\sqrt{|k|}}{2}\right)\cosh\left(\sqrt{|k|}t\right) + \cosh\left(\frac{\sqrt{|k|}}{2}\right)\sinh\left(\sqrt{|k|}t\right), \\ &= \left[\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2}\right) - \sinh\left(\frac{\sqrt{|k|}}{2}\right)\right]\cosh\left(\sqrt{|k|}t\right) \\ &\quad - \sqrt{|k|}\sinh\left(\frac{\sqrt{|k|}}{2}\right)\sinh\left(\sqrt{|k|}t\right) + \cosh\left(\frac{\sqrt{|k|}}{2}\right)\sinh\left(\sqrt{|k|}t\right), \end{aligned}$$

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$$\geq \left(\sqrt{|k|}\cosh\left(\frac{\sqrt{|k|}}{2}\right) - \sinh\left(\frac{\sqrt{|k|}}{2}\right)\right)$$
$$\left(\cosh\left(\sqrt{|k|}t\right) - \sinh\left(\sqrt{|k|}t\right)\right),$$
$$\geq 0, \text{ since } \tanh\left(\sqrt{|k|}t\right) \leq 1 \text{ for all } t \in \left(0, \frac{1}{2}\right].$$

Hence from (43), we have $G(s,t) \leq 0$ for all $0 \leq s \leq \frac{1}{2}$ and $0 \leq t \leq \frac{1}{2}$. □

Lemma 2.4. *Let k satisfy H_2 and $u(t) \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ be the solution of Problem 1(a), then*

$$u(t) = -\frac{b_1 \sin\left(\sqrt{k}t\right)}{\sin\left(\frac{\sqrt{k}}{2}\right)} + \int_0^{\frac{1}{2}} G(s,t)h(s)ds, \tag{44}$$

where Green's function $G(s,t)$ is given by

$$G(s,t) = \begin{cases} -\dfrac{\sin\left(\frac{\sqrt{k}}{2} - \sqrt{k}t\right)\sin\left(\sqrt{k}s\right)}{\sqrt{k}\sin\left(\frac{\sqrt{k}}{2}\right)} & 0 \leq s \leq t, \\ -\dfrac{\sin\left(\frac{\sqrt{k}}{2} - \sqrt{k}s\right)\sin\left(\sqrt{k}t\right)}{\sqrt{k}\sin\left(\frac{\sqrt{k}}{2}\right)} & t \leq s \leq \frac{1}{2}, \end{cases} \tag{45}$$

and $G(s,t) \leq 0$ for all $0 \leq s \leq \frac{1}{2}$ and $0 \leq t \leq \frac{1}{2}$.

Proof. Proof is similar as in Lemma 2.1. □

Lemma 2.5. *Let k satisfy H_3 and $u(t) \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ be the solution of Problem 2(a), then*

$$u(t) = -\frac{b_2 \sin\left(\sqrt{k}t\right)}{\sqrt{k}\cos\left(\frac{\sqrt{k}}{2}\right)} + \int_0^{\frac{1}{2}} G(s,t)h(s)ds, \tag{46}$$

where Green's function $G(s,t)$ is given by

$$G(s,t) = \begin{cases} -\dfrac{\cos\left(\frac{\sqrt{k}}{2} - \sqrt{k}t\right)\sin\left(\sqrt{k}s\right)}{\sqrt{k}\cos\left(\frac{\sqrt{k}}{2}\right)} & 0 \leq s \leq t, \\ -\dfrac{\cos\left(\frac{\sqrt{k}}{2} - \sqrt{k}s\right)\sin\left(\sqrt{k}t\right)}{\sqrt{k}\cos\left(\frac{\sqrt{k}}{2}\right)} & t \leq s \leq \frac{1}{2}, \end{cases} \tag{47}$$

and $G(s,t) \leq 0$ for all $0 \leq s \leq \frac{1}{2}$ and $0 \leq t \leq \frac{1}{2}$.

Proof. Proof is similar as in Lemma 2.2. □

Lemma 2.6. *Let k satisfy H_4 and $u(t) \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ be the solution of Problem 3(a), then*

$$u(t) = -\frac{b_3 \sin\left(\sqrt{k}t\right)}{\left[\sqrt{k}\cos\left(\frac{\sqrt{k}}{2}\right) - \sin\left(\frac{\sqrt{k}}{2}\right)\right]} + \int_0^{\frac{1}{2}} G(s,t)h(s)ds, \tag{48}$$

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where Green's function $G(s,t)$ is given by

$$G(s,t) = \begin{cases} -\dfrac{\left[\sqrt{k}\cos\left(\frac{\sqrt{k}}{2}-\sqrt{k}t\right)-\sin\left(\frac{\sqrt{k}}{2}-\sqrt{k}t\right)\right]\sin\left(\sqrt{k}s\right)}{\sqrt{k}\left[\sqrt{k}\cos\left(\frac{\sqrt{k}}{2}\right)-\sin\left(\frac{\sqrt{k}}{2}\right)\right]} & 0 \le s \le t, \\ -\dfrac{\left[\sqrt{k}\cos\left(\frac{\sqrt{k}}{2}-\sqrt{k}s\right)-\sin\left(\frac{\sqrt{k}}{2}-\sqrt{k}s\right)\right]\sinh\left(\sqrt{k}t\right)}{\sqrt{k}\left[\sqrt{k}\cos\left(\frac{\sqrt{k}}{2}\right)-\sin\left(\frac{\sqrt{k}}{2}\right)\right]} & t \le s \le \frac{1}{2}, \end{cases} \tag{49}$$

and $G(s,t) \le 0$ for all $0 \le s \le \frac{1}{2}$ and $0 \le t \le \frac{1}{2}$.

Proof. Proof is similar as in Lemma 2.3. □

Proposition 2.1. *Let k satisfy H_0 or H_2 (respectively, H_0 or H_3 and H_1 or H_4) and $h(t) \in C^2_{loc}\left((0,\frac{1}{2}],\mathbb{R}\right)$ is such that $h(t) \ge 0$, then the solutions of Problem $1(a)$ (respectively, Problem $2(a)$ and Problem 3a) is non positive.*

2.2. Monotone iterative technique

Here, we define lower and upper solutions corresponding to Problem 1, Problem 2 and Problem 3.

Definition 2.1. *([6]) A function $\alpha \in C^2_{loc}\left((0,\frac{1}{2}],\mathbb{R}\right)$ is the upper solution of Problem 1 (respectively Problem 2 and Problem 3) if*

$$\alpha'' \le \frac{\alpha^2}{8t^2} + \frac{\lambda}{2}, \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{50}$$

with $\lim\limits_{t\to 0} \frac{\alpha(t)}{\sqrt{t}} = 0$ and $\alpha\left(\frac{1}{2}\right) \ge 0$ (respectively $\alpha'\left(\frac{1}{2}\right) \ge 0$ and $\alpha\left(\frac{1}{2}\right) \le \alpha'\left(\frac{1}{2}\right)$).

Definition 2.2. *([6]) A function $\beta \in C^2_{loc}\left((0,\frac{1}{2}],\mathbb{R}\right)$ is the lower solution of Problem 1 (respectively Problem 2 and Problem 3) if*

$$\beta'' \ge \frac{\beta^2}{8t^2} + \frac{\lambda}{2}, \text{ for } t \in \left(0, \frac{1}{2}\right]. \tag{51}$$

with $\lim\limits_{t\to 0} \frac{\beta(t)}{\sqrt{t}} = 0$ and $\beta\left(\frac{1}{2}\right) \le 0$ (respectively $\beta'\left(\frac{1}{2}\right) \le 0$ and $\beta\left(\frac{1}{2}\right) \ge \beta'\left(\frac{1}{2}\right)$).

Now, we construct two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ corresponding to Problem $1(a)$ (respectively Problem $2(a)$ and Problem $3(a)$), which are defined by

$$\alpha_0 = \alpha,$$

$$\alpha''_{n+1} + k\alpha_{n+1} = \frac{\alpha_n^2}{8t^2} + \frac{\lambda}{2} + k\alpha_n, \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{52}$$

$$\lim_{t\to 0} \frac{\alpha_{n+1}(t)}{\sqrt{t}} = 0 \text{ and } \alpha_{n+1}\left(\frac{1}{2}\right) = 0, \tag{53}$$

$\left(\text{respectively } \alpha'_{n+1}\left(\frac{1}{2}\right) = 0 \text{ and } \alpha_{n+1}\left(\frac{1}{2}\right) = \alpha'_{n+1}\left(\frac{1}{2}\right)\right)$ and

$$\beta_0 = \beta,$$

$$\beta''_{n+1} + k\beta_{n+1} = \frac{\beta_n^2}{8t^2} + \frac{\lambda}{2} + k\beta_n, \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{54}$$

$$\lim_{t\to 0} \frac{\beta_{n+1}(t)}{\sqrt{t}} = 0 \text{ and } \beta_{n+1}\left(\frac{1}{2}\right) = 0, \tag{55}$$

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$\left(\text{respectively } \beta'_{n+1}\left(\frac{1}{2}\right) = 0 \text{ and } \beta_{n+1}\left(\frac{1}{2}\right) = \beta'_{n+1}\left(\frac{1}{2}\right)\right)$. We assume the following properties:

- P_1: α_0 and β_0 satisfies

$$\lim_{t\to 0} \frac{|\alpha_0(t)|}{t} < \infty, \ \lim_{t\to 0} \alpha_0(t) = 0, \ \alpha_0(t) \le 0, \tag{56}$$

  and

$$\lim_{t\to 0} \frac{|\beta_0(t)|}{t} < \infty, \ \lim_{t\to 0} \beta_0(t) = 0, \tag{57}$$

- P_2: $h(t,u)$ is continuous on D_0 where

$$D_0 = \left\{(t,u) \in \left(0, \frac{1}{2}\right] \times \mathbb{R} : \beta_0 = \beta \le u \le \alpha_0\right\}$$

  .

Now we state our main existence theorems.

Theorem 2.1. *Assume H_0 (respectively, H_0 and H_1) is true and there exist α_0 and $\beta_0 \in C^2_{loc}\left((0,\frac{1}{2}],\mathbb{R}\right)$ are upper and lower solutions of Problem 1 (respectively, Problem 2 and Problem 3) which satisfy the properties P_1 and P_2 such that $\beta_0 \le \alpha_0 = 0$, then the Problem 1 (respectively, Problem 2 and Problem 3) has at least one solution in the region D_0 and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ defined by (52), (53) and (54), (55) converges to a solutions u, v uniformly as well as monotonically respectively, such that*

$$\beta \le u \le v \le \alpha = 0, \ \forall t \in \left(0, \frac{1}{2}\right]. \tag{58}$$

Proof. We divide the proof into three parts. In the first part, we prove that

$$\beta_n \text{ is a lower solution of problem 1, } \beta_n \le \beta_{n+1} \text{ and } \beta_{n+1} \le \alpha_0 \ \forall n \in \mathbb{N}\cup\{0\}. \tag{59}$$

We apply mathematical induction on n. For $n = 0$, from (54) and (55), we have

$$\beta_1'' + k\beta_1 = \frac{\beta_0^2}{8t^2} + \frac{\lambda}{2} + k\beta_0, \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{60}$$

$$\lim_{t\to 0} \frac{\beta_1(t)}{\sqrt{t}} = 0 \text{ and } \beta_1\left(\frac{1}{2}\right) = 0. \tag{61}$$

Now, from equation (51), we have

$$(\beta_0 - \beta_1)'' + k(\beta_0 - \beta_1) = -\frac{\beta_0^2}{8t^2} - \frac{\lambda}{2} + \beta_0'' \ge 0, \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{62}$$

$$\lim_{t\to 0} \frac{\beta_0 - \beta_1(t)}{\sqrt{t}} = 0 \text{ and } (\beta_0 - \beta_1)\left(\frac{1}{2}\right) \le 0. \tag{63}$$

Therefore by proposition 2.1, we have $\beta_0 \le \beta_1$. Again from (50) and (60), we have

$$(\beta_1 - \alpha_0)'' + k(\beta_1 - \alpha_0) = \frac{\beta_0^2}{8t^2} + \frac{\lambda}{2} + k(\beta_0 - \alpha_0) - \alpha_0'', \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{64}$$

$$\ge \left(\frac{\beta_0 + \alpha_0}{8t} + kt\right)\left(\frac{\beta_0 - \alpha_0}{t}\right). \tag{65}$$

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Since $\beta_0 \leq \alpha_0$, therefore we have

$$(\beta_1 - \alpha_0)'' + k(\beta_1 - \alpha_0) \geq 0, \ \forall t \in \left(0, \frac{1}{2}\right], \tag{66}$$

$$\lim_{t\to 0} \frac{(\beta_1 - \alpha_0)(t)}{\sqrt{t}} = 0 \text{ and } (\beta_1 - \alpha_0)\left(\frac{1}{2}\right) \leq 0. \tag{67}$$

Hence by proposition 2.1, we have $\beta_1 \leq \alpha_0$. So our assumptions are true for $n = 0$. Let our assumptions be true up to $n = m$. Therefore, we have

$$\beta_n \text{ is a lower solution of problem 1, } \beta_n \leq \beta_{n+1} \text{ and } \beta_{n+1} \leq \alpha_0 \tag{68}$$

for $n = 1, 2, \cdots, m$. Now we want to show that our assumptions are true for $n + 1$. Therefore from equation (54), we have

$$\beta''_{n+1} - \frac{\beta^2_{n+1}}{8t^2} - \frac{\lambda}{2} = \frac{\beta_n^2 - \beta^2_{n+1}}{8t^2} + k(\beta_n - \beta_{n+1}), \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{69}$$

$$\geq \left(\frac{\beta_n + \beta_{n+1}}{8t} + kt\right)\left(\frac{\beta_n - \beta_{n+1}}{t}\right). \tag{70}$$

Again by using conditions (68), we have

$$\beta''_{n+1} \geq \frac{\beta^2_{n+1}}{8t^2} + \frac{\lambda}{2}, \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{71}$$

$$\lim_{t\to 0} \frac{\beta_{n+1}(t)}{\sqrt{t}} = 0 \text{ and } \beta_{n+1}\left(\frac{1}{2}\right) \leq 0. \tag{72}$$

Hence β_{n+1} is a lower solution of Problem 1. Now, from equation (54) and (71), we have

$$(\beta_{n+1} - \beta_{n+2})'' + k(\beta_{n+1} - \beta_{n+2}) = -\frac{\beta^2_{n+1}}{8t^2} - \frac{\lambda}{2} + \beta''_{n+1} \geq 0, \tag{73}$$

$$\lim_{t\to 0} \frac{(\beta_{n+1} - \beta_{n+2})(t)}{\sqrt{t}} = 0 \text{ and } (\beta_{n+1} - \beta_{n+2})\left(\frac{1}{2}\right) \leq 0. \tag{74}$$

So by proposition 2.1, we have $\beta_{n+1} \leq \beta_{n+2}$. Again from (50) and (54), we have

$$(\beta_{n+2} - \alpha_0)'' + k(\beta_{n+2} - \alpha_0) = \frac{\beta^2_{n+1}}{8t^2} + \frac{\lambda}{2} + k(\beta_{n+1} - \alpha_0) - \alpha_0'', \tag{75}$$

$$\geq \left(\frac{\beta_{n+1} + \alpha_0}{8t} + kt\right)\left(\frac{\beta_{n+1} - \alpha_0}{t}\right) \tag{76}$$

By similar analysis, we have $\beta_{n+2} \leq \alpha_0$. Hence by mathematical induction, we have

$$\beta_n \text{ is a lower solution of Problem 1, } \beta_n \leq \beta_{n+1} \text{ and } \beta_{n+1} \leq \alpha_0 \ \forall n \in \mathbb{N}. \tag{77}$$

In the second part of the proof, we have to show

$$\alpha_n \text{ is a upper solution of Problem 1 and } \alpha_{n+1} \leq \alpha_n \ \forall n \in \mathbb{N}. \tag{78}$$

Now from (52) and (53), we have

$$\alpha_1'' + k\alpha_1 = \frac{\alpha_0^2}{8t^2} + \frac{\lambda}{2} + k\alpha_0, \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{79}$$

$$\lim_{t\to 0} \frac{\alpha_1(t)}{\sqrt{t}} = 0 \text{ and } \alpha_1\left(\frac{1}{2}\right) = 0. \tag{80}$$

Therefore, by using (50) we have

$$(\alpha_1 - \alpha_0)'' + k(\alpha_1 - \alpha_0) = \frac{\alpha_0^2}{8t^2} + \frac{\lambda}{2} - \alpha_0'', \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{81}$$

$$\geq 0. \tag{82}$$

Again,

$$\lim_{t\to 0} \frac{(\alpha_1 - \alpha_0)(t)}{\sqrt{t}} = 0 \text{ and } (\alpha_1 - \alpha_0)\left(\frac{1}{2}\right) \leq 0. \tag{83}$$

Hence by proposition 2.1, we have $\alpha_1 \leq \alpha_0$. So our assumptions are true for $n = 0$. Let our assumptions be true up to $n = m$. So, we have

$$\alpha_n \text{ is a upper solution of Problem 1 and } \alpha_{n+1} \leq \alpha_n \text{ for } n = 1, 2, \cdots, m. \tag{84}$$

Now, for $n + 1$ we have

$$\alpha_{n+1}'' - \frac{\alpha_{n+1}^2}{8t^2} - \frac{\lambda}{2} = \frac{\alpha_n^2 - \alpha_{n+1}^2}{8t^2} + k(\alpha_n - \alpha_{n+1}), \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{85}$$

$$\leq \left(\frac{1}{8}\frac{\alpha_n + \alpha_{n+1}}{t} + kt\right)\left(\frac{\alpha_n - \alpha_{n+1}}{t}\right), \tag{86}$$

$$\leq 0. \tag{87}$$

Therefore,

$$\alpha_{n+1}'' \leq \frac{\alpha_{n+1}^2}{8t^2} + \frac{\lambda}{2}, \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{88}$$

and

$$\lim_{t\to 0} \frac{\alpha_{n+1}(t)}{\sqrt{t}} = 0, \ \alpha_{n+1}\left(\frac{1}{2}\right) \geq 0. \tag{89}$$

Hence, we have α_{n+1} is a upper solution of Problem 1. Therefore, by using (88), (52) and (53), we have

$$(\alpha_{n+2} - \alpha_{n+1})'' + k(\alpha_{n+2} - \alpha_{n+1}) = \frac{\alpha_{n+1}^2}{8t^2} + \frac{\lambda}{2} - \alpha_{n+1}'', \text{ for } t \in \left(0, \frac{1}{2}\right], \tag{90}$$

$$\geq 0. \tag{91}$$

and

$$\lim_{t\to 0} \frac{(\alpha_{n+2} - \alpha_{n+1})(t)}{\sqrt{t}} = 0, \ (\alpha_{n+2} - \alpha_{n+1})\left(\frac{1}{2}\right) \leq 0. \tag{92}$$

Therefore by proposition 2.1, we have $\alpha_{n+2} \leq \alpha_{n+1}$. Hence by mathematical induction we conclude that

$$\alpha_n \text{ is a upper solution of Problem 1 and } \alpha_{n+1} \leq \alpha_n \ \forall n \in \mathbb{N}. \tag{93}$$

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In the last part of the proof, we want to show $\beta_n \le \alpha_n$ for all $n \in \mathbb{N}$. Again from (71) and (88), we have

$$(\beta_{n+1}-\alpha_{n+1})'' + k(\beta_{n+1}-\alpha_{n+1}) = \frac{\beta_n^2}{8t^2} + \frac{\lambda}{2} + k(\beta_n - \alpha_n) - \alpha_n'', \quad (94)$$
$$\ge \left(\frac{\beta_n+\alpha_n}{8t} + kt\right)\left(\frac{\beta_n-\alpha_n}{t}\right) \quad (95)$$

Since $\beta_n \le \alpha_n \le 0$, therefore we have

$$(\beta_{n+1}-\alpha_{n+1})'' + k(\beta_{n+1}-\alpha_{n+1}) \ge 0, \quad (96)$$

and

$$\lim_{t\to 0} \frac{\beta_{n+1}-\alpha_{n+1}(t)}{\sqrt{t}} = 0, \ (\beta_{n+1}-\alpha_{n+1})\left(\frac{1}{2}\right) \le 0. \quad (97)$$

Hence by proposition 2.1, we have $\beta_{n+1} \le \alpha_{n+1}$. Finally we have

$$\beta = \beta_0 \le \beta_1 \le \cdots \le \beta_n \le \cdots \le \alpha_n \le \cdots \le \alpha_1 \le \alpha_0 = 0. \quad (98)$$

Let $t_n \in \left(0, \frac{1}{2}\right)$ for $n \in \mathbb{N}$ such that

$$t_{n+1} < t_n \text{ for } n \in \mathbb{N}, \ \lim_{n\to+\infty} t_n = 0. \quad (99)$$

Therefore, for every $n \in \mathbb{N}$ there exists a solution α_n and β_n to equations (52), (53) and (54), (55) respectively satisfy the inequality (98) on the interval $[t_n, \frac{1}{2}]$. Since $\{\alpha_n\}$ and $\{\beta_n\}$ are monotone and bounded, therefore they converge to function $u(t)$ and $v(t)$ respectively. Therefore, by Dini's theorem we have, there exists $u(t)$ and $v(t)$ such that

$$\lim_{n\to\infty} \alpha_n = u \text{ and } \lim_{n\to\infty} \beta_n = v \text{ uniformly on every compact interval } \left[t_n, \tfrac{1}{2}\right] \quad (100)$$

of $\left(0, \frac{1}{2}\right]$. Hence, from (52), (53), (54), (55) and (38), we have there exists solutions $v(t) \in C^2_{loc}\left((0,\frac{1}{2}], \mathbb{R}\right)$ and $u(t) \in C^2_{loc}\left((0,\frac{1}{2}], \mathbb{R}\right)$ to Problem 1 satisfying

$$\beta \le u \le v \le \alpha_0 = 0, \ \forall t \in \left(0, \frac{1}{2}\right]. \quad (101)$$

Hence the proof is complete. □

Now, we assume the following conditions

- $H_5 = \left\{k \in \mathbb{R} : 0 < k < k', \text{ where } k' = \min\left\{4\pi^2, -\max_{t\in(0,\frac{1}{2}]} \frac{\alpha_0}{2t}\right\}\right\}$,
- $H_6 = \left\{k \in \mathbb{R} : 0 < k < k', \text{ where } k' = \min\left\{\pi^2, -\max_{t\in(0,\frac{1}{2}]} \frac{\alpha_0}{2t}\right\}\right\}$,
- $H_7 = \{k \in \mathbb{R} : 0 < k < k'\}$,

$$\text{where } k' = \min\left\{\frac{\pi^2}{4}, -\max_{t\in(0,\frac{1}{2}]} \frac{\alpha_0}{2t}\right\} \ \& \ \sqrt{k}\cos\left(\frac{\sqrt{k}}{2}\right) - \sin\left(\frac{\sqrt{k}}{2}\right) > 0$$

.

Theorem 2.2. *Let $\alpha_0, \beta_0 \in C^2_{loc}\left((0,\frac{1}{2}], \mathbb{R}\right)$ are the upper and lower solutions of Problem 1 (respectively, Problem 2 and Problem 3) which satisfy the properties P_1 and P_2 such that $\beta_0 \le \alpha_0$. Assume H_5 (respectively, H_6 and H_7) is true and*

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$\lambda \in \mathbb{R}$. Then the Problem 1 (respectively, Problem 2 and Problem 3) has at least one solution in the region D_0 and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ defined by (52), (53) and (54), (55) converges to a solutions u, v uniformly as well as monotonically respectively, such that

$$\beta \leq u \leq v \leq \alpha, \ \forall t \in \left(0, \frac{1}{2}\right]. \tag{102}$$

Proof. The proof is same as in Theorem 2.1. □

3. Estimations of λ

The objective of this section is to derive some qualitative bounds of the parameter λ, from which we can conclude about the nonexistence of solutions. The equation (11) can be written as in the following form:

$$(tu' - u)' = \frac{u^2}{8t} + \frac{\lambda t}{2}, \ \forall t \in \left(0, \frac{1}{2}\right]. \tag{103}$$

Put $v(t) = -\dfrac{u(t)}{t}$ and integrating from 0 to t, the equation (103) becomes

$$v'(t) = -\frac{1}{8t^2} \int_0^t v^2(s) s \, ds - \frac{\lambda}{4}, \ \forall t \in \left(0, \frac{1}{2}\right]. \tag{104}$$

Therefore, we have

$$v(t) \geq 0, \ \forall t \in \left(0, \frac{1}{2}\right]. \tag{105}$$

In view of the transformation, the boundary condition at $r = 1$ becomes

$$\text{BC of Problem 1: } v\left(\frac{1}{2}\right) = 0, \tag{106}$$

$$\text{BC of Problem 2: } v\left(\frac{1}{2}\right) = -\frac{1}{2} v'\left(\frac{1}{2}\right), \tag{107}$$

$$\text{BC of Problem 3: } v\left(\frac{1}{2}\right) = -v'\left(\frac{1}{2}\right). \tag{108}$$

Escudero et. al. in [10] prove the following two lemmas.

Lemma 3.1. *The set of numbers $\lambda \geq 0$, for which there exists a solution $u(t) \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ of equation (11) satisfying $\lim_{t \to 0} \dfrac{u(t)}{\sqrt{t}} = 0$ and $u(t) \leq 0$, is nonempty and bounded from above.*

Lemma 3.2. *If the Problem 1, Problem 2 and Problem 3 are solvable for some $\lambda_0 \geq 0$, then these are solvable for every $0 \leq \lambda \leq \lambda_0$.*

We present the following results which complements the results proved by Escudero et al. [10].

Lemma 3.3. *Let there exist a function $u \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ satisfying equations (104), (105) and (107), then*

$$\lambda \leq \frac{384}{11} \approx 34.91. \tag{109}$$

Proof. Now from equation (104), we have

$$v'(t) \leq 0, \ \forall t \in \left(0, \frac{1}{2}\right]. \tag{110}$$

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Again from equation (104), we get

$$v''(t) = \frac{1}{4t^3}\int_0^t v^2(s)s\,ds - \frac{v^2(t)}{8t},\ \forall t \in \left(0, \frac{1}{2}\right]. \tag{111}$$

Therefore by using (110) and (105), from (111) we have

$$v''(t) \geq 0,\ \forall t \in \left(0, \frac{1}{2}\right]. \tag{112}$$

Therefore $v'(t)$ is increasing in $\left(0, \frac{1}{2}\right]$. Now

$$v'(t) \leq v'\left(\frac{1}{2}\right) = -\frac{1}{2}\int_0^{\frac{1}{2}} v^2(s)s\,ds - \frac{\lambda}{4},\ \forall t \in \left(0, \frac{1}{2}\right]. \tag{113}$$

Therefore, we have

$$v'(t) \leq -c,\ \forall t \in \left(0, \frac{1}{2}\right], \tag{114}$$

where

$$c = \frac{1}{2}\int_0^{\frac{1}{2}} v^2(s)s\,ds + \frac{\lambda}{4}. \tag{115}$$

Now, integrating equation (114) from 0 to t and by using equation (107), we have

$$v(t) \geq c(1-t),\ \forall t \in \left(0, \frac{1}{2}\right]. \tag{116}$$

Therefore, from equations (115) and (116), we get

$$\frac{11}{384}c^2 - c + \frac{\lambda}{4} \leq 0, \tag{117}$$

which implies the equation (109). □

Lemma 3.4. *Let*

$$0 \leq \lambda \leq 2C \text{ and } C \leq \frac{128}{9}, \tag{118}$$

then there exists a solution $\beta \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ *satisfies equation (51), the assumption* P_1, $\beta'\left(\frac{1}{2}\right) = 0$ *and* $\beta(t) \leq 0$.

Proof. We put

$$\beta(t) = -Ct(A - \sqrt{2t}),\ \forall t \in \left(0, \frac{1}{2}\right] \text{ and } C \geq 0. \tag{119}$$

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Obviously $\beta(t)$ satisfy assumption P_1. Now, $\beta'\left(\frac{1}{2}\right) = 0$ implies $A = \dfrac{3}{2}$. Therefore $\beta(t) \leq 0$ is also fulfilled. Now, we have

$$\beta''(t) - \frac{\beta^2(t)}{8t^2} - \frac{\lambda}{2} \tag{120}$$

$$= \frac{3C}{2\sqrt{2t}} - \frac{C^2t^2\left(\frac{3}{2} - \sqrt{2t}\right)^2}{8t^2} - \frac{\lambda}{2}, \tag{121}$$

$$= \frac{C}{\sqrt{2t}}\left(\frac{3}{2} - \frac{\lambda\sqrt{2t}}{2C}\right) - \frac{C^2\left(\frac{3}{2} - \sqrt{2t}\right)^2}{8}, \tag{122}$$

$$\geq \frac{C}{\sqrt{2t}}\left(\frac{3}{2} - \sqrt{2t}\right) - \frac{C^2\left(\frac{3}{2} - \sqrt{2t}\right)^2}{8}, \text{ since } \lambda \leq 2C \tag{123}$$

$$= \frac{C^2}{8\sqrt{2t}}\left(\frac{3}{2} - \sqrt{2t}\right)\left(\left(\sqrt{2t} - \frac{3}{4}\right)^2 + \frac{-9C+128}{16C}\right), \tag{124}$$

$$\geq 0, \ \forall t \in \left(0, \frac{1}{2}\right], \text{ since } C \leq \frac{128}{9}. \tag{125}$$

Hence the inequality (51) is satisfied. □

Lemma 3.5. *Let*

$$0 \leq \lambda \leq 3C \text{ and } C \leq 48, \tag{126}$$

then there exists a solution $\beta \in C^2_{loc}\left((0, \frac{1}{2}], \mathbb{R}\right)$ *satisfies equation (51), the assumption* P_1, $\beta\left(\frac{1}{2}\right) = 0$ *and* $\beta(t) \leq 0$.

Proof. We put

$$\beta(t) = -Ct(A - \sqrt{2t}), \ \forall t \in \left(0, \frac{1}{2}\right] \text{ and } C \geq 0. \tag{127}$$

Again, $\beta(t)$ satisfy assumption P_1. Now, $\beta\left(\frac{1}{2}\right) = 0$ implies $A = 1$. Hence, $\beta(t) \leq 0$ is also fulfilled. Now, we have

$$\beta''(t) - \frac{\beta^2(t)}{8t^2} - \frac{\lambda}{2} \tag{128}$$

$$= \frac{3C}{2\sqrt{2t}} - \frac{C^2t^2\left(1 - \sqrt{2t}\right)^2}{8t^2} - \frac{\lambda}{2}, \tag{129}$$

$$= \frac{3C}{2\sqrt{2t}}\left(1 - \frac{\lambda\sqrt{2t}}{3C}\right) - \frac{C^2\left(1 - \sqrt{2t}\right)^2}{8}, \tag{130}$$

$$\geq \frac{3C}{2\sqrt{2t}}\left(1 - \sqrt{2t}\right) - \frac{C^2\left(1 - \sqrt{2t}\right)^2}{8}, \text{ since } \lambda \leq 3C \tag{131}$$

$$= \frac{C^2}{8\sqrt{2t}}\left(1 - \sqrt{2t}\right)\left(\left(\sqrt{2t} - \frac{1}{2}\right)^2 + \frac{-C+48}{4C}\right), \tag{132}$$

$$\geq 0, \ \forall t \in \left(0, \frac{1}{2}\right], \text{ since } C \leq 48. \tag{133}$$

This completes the proof. □

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Lemma 3.6. *Let*
$$0 \le \lambda \le \frac{3C}{2} \text{ and } C \le 6, \tag{134}$$
then there exists a solution $\beta \in C^2_{loc}\left((0,\frac{1}{2}],\mathbb{R}\right)$ *satisfies equation (51), the assumption* P_1, $\beta\left(\frac{1}{2}\right) = \beta'\left(\frac{1}{2}\right)$ *and* $\beta(t) \le 0$.

Proof. We put
$$\beta(t) = -Ct(A-\sqrt{2t}),\ \forall t \in \left(0,\frac{1}{2}\right] \text{ and } C \ge 0. \tag{135}$$
Now, $\beta(t)$ also satisfy assumption P_1. Similarly, $\beta\left(\frac{1}{2}\right) = \beta'\left(\frac{1}{2}\right)$ implies $A=2$. So, $\beta(t) \le 0$ is also fulfilled. Therefore, we have
$$\beta''(t) - \frac{\beta^2(t)}{8t^2} - \frac{\lambda}{2} \tag{136}$$
$$= \frac{3C}{2\sqrt{2t}} - \frac{C^2t^2\left(2-\sqrt{2t}\right)^2}{8t^2} - \frac{\lambda}{2}, \tag{137}$$
$$= \frac{3C}{4\sqrt{2t}}\left(2 - \frac{2\lambda\sqrt{2t}}{3C}\right) - \frac{C^2\left(2-\sqrt{2t}\right)^2}{8}, \tag{138}$$
$$\ge \frac{3C}{4\sqrt{2t}}\left(2-\sqrt{2t}\right) - \frac{C^2\left(2-\sqrt{2t}\right)^2}{8}, \text{ since } \lambda \le \tfrac{3C}{2} \tag{139}$$
$$= \frac{C^2}{8\sqrt{2t}}\left(2-\sqrt{2t}\right)\left(\left(\sqrt{2t}-1\right)^2 + \frac{-C+6}{C}\right), \tag{140}$$
$$\ge 0,\ \forall t \in \left(0,\frac{1}{2}\right], \text{ since } C \le 6. \tag{141}$$
Hence, the proof is complete. □

Theorem 3.1. *Let* $\lambda_0 \in \mathbb{R}^+$. *If* $0 \le \lambda < \lambda_0$, *then the equation (10) corresponding to different types of boundary condition are solvable. Also there is no solution of these problems if* $\lambda > \lambda_0$. *Furthermore, every solutions* $w(r)$ *of governing equation corresponding to these three types of boundary conditions satisfy*
$$w(r) \le 0,\ r \in (0,1] \text{ and } \lim_{r\to 0^+} w(r) = 0. \tag{142}$$

Proof. The proof of this can be deduced from Lemma 3.1, Lemma 3.2, Lemma 1.1, Lemma 1.2, Lemma 1.3 and Lemma 1.5. □

Proposition 3.1. *Corresponding to equations (6) and (7) the value of* λ_0 *admits the estimates*
$$144 \le \lambda_0 \le 307. \tag{143}$$

Proof. From Lemma 7.7 in [10] and Lemma 3.5, we get the equation (143). □

Proposition 3.2. *Corresponding to equations (6) and (9) the value of* λ_0 *admits the estimates*
$$\frac{256}{9} \le \lambda_0 \le \frac{384}{11}. \tag{144}$$

Proof. From Lemma 3.3 and Lemma 3.4, we have the equation (144). □

Proposition 3.3. *Corresponding to equations (6) and (8) the value of λ_0 admits the estimates*

$$9 \leq \lambda_0 \leq 11.63. \tag{145}$$

Proof. By using the Lemma 7.6 in [10] and Lemma 3.6, we have the equation (145). □

4. Numerical results and discussion

Here, we prove numerical data to validate our derived theoretical results. In subsection 4.1 we derive the numerical estimation of the bounds computed by ADM. In subsection 4.2, numerically we show the existence of at least one solution.

4.1. ADM

To find the approximate solutions, we develop the iterative numerical schemes with the help of the Fredholm integral equations (15), (16) and (17) respectively. Now, we decompose the solution $u(t)$ of the form $u(t) = \sum_{i=0}^{\infty} u_i(t)$, and approximate the nonlinear term in terms of Adomian's polynomials ([13]) which is given by

$$N(u(t)) = -\frac{1}{2}u^2(t) = \sum_{i=0}^{\infty} A_i(u_0, u_1, \cdots, u_i), \tag{146}$$

where

$$A_i = \frac{1}{i!}\frac{d^i}{d\beta^i} N\left(\sum_{j=0}^{i} \beta^j u_j\right)\Bigg|_{\beta=0}, \; i = 0, 1, 2, \cdots . \tag{147}$$

Therefore from integral equation (15), we define

$$\text{Scheme of Problem 1} = \begin{cases} u_0(t) = -ct - \frac{\lambda}{4}t\left(\frac{1}{2} - t\right), \\ \vdots \\ u_{n+1}(t) = \int_0^t \left(\frac{s}{2} - \frac{t}{2}\right)\frac{A_n}{2s^2}ds, \\ \vdots, \\ \text{and } c = -\int_0^{\frac{1}{2}} \sum_{i=0}^{n} \frac{A_i}{2s^2}\left(\frac{1}{2} - s\right)ds, \end{cases} \tag{148}$$

We compute the arbitary constant c by using a Mathematica program. For better understanding, we present below the algorithm of our proposed technique corresponding to equation (15).

Algorithm:

Step 1. Convert Fredholm integral equation (15) into Voltera integral equation.

Step 2. Identify the constant term, and approximate the nonlinear term by equation (146).

Step 3. Consider $u_0(t)$ as in (148), and obtain $u_i(t)$ for $i = 1, 2, \cdots, n + 1$.

Step 4. Approximate the term $\dfrac{u^2}{4s^2}$ by $-\sum_{i=0}^{n}\dfrac{A_i}{2s^2}$ in the equation $c = \int_0^{\frac{1}{2}} \dfrac{u^2}{4s^2}\left(\dfrac{1}{2} - s\right)ds.$

Step

tions $u(t) = \sum_{i=0}^{n+1} u_i.$

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Again, we apply the above algorithm on equations (16) and (17), and we define the following iterative schemes:

$$\text{Scheme of Problem 2} = \begin{cases} u_0(t) = -ct - \frac{\lambda}{4}t(1-t), \\ \vdots \\ u_{n+1}(t) = \int_0^t (s-t)\frac{A_n}{4s^2}ds, \\ \vdots, \\ \text{and } c = -\int_0^{\frac{1}{2}} \sum_{i=0}^n \frac{A_i}{4s^2}ds, \end{cases} \tag{149}$$

$$\text{and Scheme of Problem 3} = \begin{cases} u_0(t) = -ct - \frac{\lambda}{4}t(\frac{3}{2}-t), \\ \vdots \\ u_{n+1}(t) = \int_0^t (\frac{s}{2}-\frac{t}{2})\frac{A_n}{2s^2}ds, \\ \vdots, \\ \text{and } c = -\int_0^{\frac{1}{2}} \sum_{i=0}^n \left(\frac{1}{2}+s\right)\frac{A_i}{2s^2}ds. \end{cases} \tag{150}$$

Approximate solutions of equations (16) and (17) can be written as $u(t) = \sum_{i=0}^{n+1} u_i(t)$, provided the series is convergent for $n \to \infty$. Recently, the convergence of ADM was established by Verma et al. in [28]. Now by using the transformation $t = \frac{r^2}{2}$, $u(t) = w(r)$, $w(r) = r\phi'(r)$ and $\phi(1) = 0$, we get the solutions of equation (6). We arrive at two cases:

Case (a): $\lambda \geq 0$.

For $\lambda = 0$, we get one trivial and one non trivial solutions. For $0 < \lambda \leq \lambda_{\text{critical}}$, we always find two non-trivial solutions. We may refer them as upper and lower solution respectively. Corresponding to equations (9), (8) and (7), we find the critical value of λ, i.e. $\lambda_{\text{critical}}$, is to be 31.94, 11.34 and 168.76 respectively. For $\lambda > \lambda_{\text{critical}}$, we do not find any numerical solutions as the value of c become imaginary. In subsection 4.1.1, we tabulate residual errors of the approximate solutions corresponding to some λ.

Case (b): $\lambda < 0$.

In this case, we always have two nontrivial numerical solutions corresponding to three types of boundary conditions. One solution is negative (namely the negative solution) and the other solution is positive (namely the positive solution). We do not find any negative critical λ. We place residue errors subsection 4.1.1.

4.1.1. Tables

Here, we have placed below some numerical data of approximate solutions of $\phi(r)$ corresponding to different types of boundary conditions. If we are increasing the value of λ, we see that the residue error of the lower solution is increasing and the residue error of the upper solution is decreasing ([see: table 1]). Similarly, if we are decreasing the value of negative λ, we see that the residue error of both positive and negative solutions are decreasing ([see: table 2]). Same remarks are made to tables 3, 4, 5 and 6.

Table 1: Maximum absolute residue error of approximate solutions $\phi(r)$ corresponding to boundary conditions (8):

	Lower solution		Upper solution	
λ	0	31.94	0	31.94
	0	1.95399E-14	8.3347E-07	1.86517E-14

Table 2: Maximum absolute residue error of approximate solutions $\phi(r)$ corresponding to boundary conditions (8):

	Positive solution		Negative solution	
λ	-1	-15	-1	-15
	1.42341E-06	0.001199979	1.37856E-16	3.66374E-15

Table 3: Maximum absolute residue error of approximate solutions $\phi(r)$ corresponding to boundary conditions (9):

	Lower solution		Upper solution	
λ	0	11.34	0	11.34
	0	3.55271E-15	3.55271E-15	2.66454E-15

Table 4: Maximum absolute residue error of approximate solutions $\phi(r)$ corresponding to boundary conditions (9):

	Positive solution		Negative solution	
λ	-1	-15	-1	-15
	6.83897E-14	4.59188E-13	2.54571E-16	7.77156E-16

Table 5: Maximum absolute residue error of approximate solutions $\phi(r)$ corresponding to boundary conditions (7):

	Lower solution		Upper solution	
λ	0	168.76	0	168.76
	0	6.1668E-11	0.000557377	1.70296E-10

Table 6: Maximum absolute residue error of approximate solutions $\phi(r)$ corresponding to boundary conditions (7):

	Positive solution		Negative solution	
λ	-1	-15	-1	-15
	0.000557832	0.000562261	7.29668E-17	6.2797E-16

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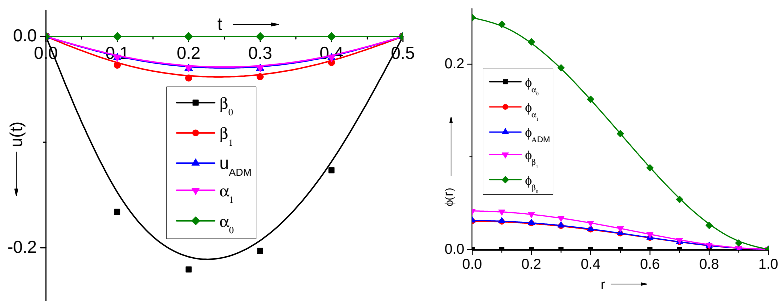
4.2. Monotone iterative method

Here we compute the monotone iterations by using equations (52), (53), (54) and (55) corresponding to three types of boundary condition.

Corresponding to problem (12): By using Lemma 3.5, we chose the lower and upper iterations

$$\beta_0(t) = -3t\left(1-\sqrt{2t}\right) \ \& \ \alpha_0(t) = 0, \ \forall t \in [0,1]. \tag{151}$$

Therefore, it is easy to show that α_0 and β_0 both are satisfying the inequalities (50), (51), (56) and (57) such that $\beta_0 \leq \alpha_0$. We consider $k = -1$. Hence by using the Theorem 2.1 we have monotonically as well as uniformly convergent sequence $\{\beta_n\}$ and $\{\alpha_n\}$ which are converging to the solution v and u of the problem (12). We denote u_{ADM} be the approximation of the solution of (12) computed by ADM. By using the transformation $t = \frac{r^2}{2}$, $w(r) = u(t)$ and $w(r) = r\phi'(r)$ we have the fourth order iterations corresponding to the monotone iteration α_i and β_i. ϕ_{α_i} and ϕ_{β_i} be the fourth order solutions corresponding to the monotone iteration α_i and β_i for $i = 0, 1, \cdots$, respectively. In figure 4.1 we have shown the numerical results corresponding to $\lambda = 2$.



(a) $\lambda = -1$ (b) $\lambda = -15$

Figure 4.1. Monotone iterations of (52), (53), (54) and (55) corresponding to problem (12) for $k = -1$ and $\lambda = 2$.

Corresponding to problem (13): From the Lemma 3.4, here we chose the initial monotone iterations is as follows

$$\beta_0(t) = -t\left(\frac{3}{2}-\sqrt{2t}\right) \ \& \ \alpha_0(t) = 0, \ \forall t \in [0,1]. \tag{152}$$

Same remarks follows as we have discussed in above. In figure 4.2, we have placed the the numerical results for $k = -1$ and $\lambda = 2$.

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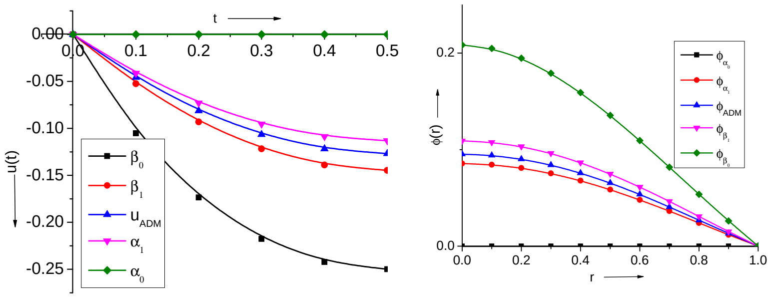


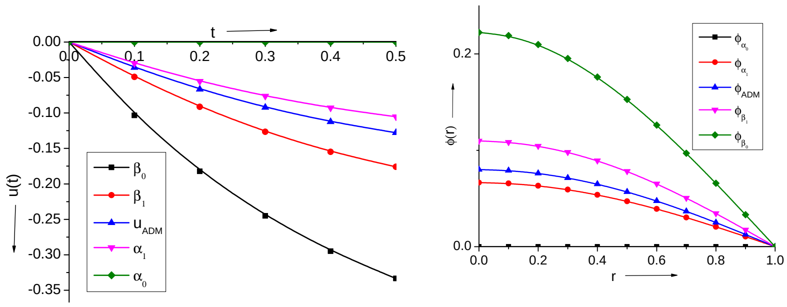
(a) $\lambda = -1$ (b) $\lambda = -15$

Figure 4.2. Monotone iterations of (52), (53), (54) and (55) corresponding to problem (13) for $k = -1$ and $\lambda = 2$.

Corresponding to problem (14): Here, we consider the the initial monotone iterations

$$\beta_0(t) = -\frac{2}{3}t\left(2 - \sqrt{2t}\right) \;\&\; \alpha_0(t) = 0, \; \forall t \in [0,1]. \tag{153}$$

We have also made same remarks as in above. Monotone lower and upper iterations corresponding to second order as well as fourth order differential equation is plotted in figure 4.3.



(a) $\lambda = -1$ (b) $\lambda = -15$

Figure 4.3. Monotone iterations of (52), (53), (54) and (55) corresponding to problem (14) for $k = -1$ and $\lambda = 1$.

5. Conclusions

In this work, we derived some qualitative properties of the singular boundary value problems which arise in the theory of epitaxial growth. Also, we proved the existence of solution and find out a range of parameter k for which the nonlinear problem has multiple solutions in the region D_0. We established the bounds of the parameter λ, from which we concluded about the nonexistence of solutions. Also the boundary value problems have multiple solutions, therefore it is challenging for researchers to get an suitable scheme to capture both solutions with desired acuracy. But, here we successfully developed the iterative schemes, and captured both solutions together with high acuracy. From tables 1- 4, we saw that the approximate solutions computed by our proposed method converge to the exact solutions very fast. But, corresponding to boundary conditions (7), we notice that, positive approximate solution converge to exact positive solution very slowly ([See: table 6]). We verified that our numerical results are well matched with our theoretical results as well as existing numerical results ([28]). Among all point of view, we conclude that, our proposed technique is quit powerful and efficient. Furthermore, this technique will be an effective tool to solve BVPs, which have multiple solutions.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism

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