Exponentials and logarithms properties in an extended complex number field

Daniel Tischhauser

Independent researcher, Geneva, Switzerland

Email : dtischhauser.math@gmail.com

Abstract

In this study we demonstrate the complex logarithm and exponential multivalued results and identity failures are not induced by the exponentiation and logarithm operations, but are solely induced by the definition of complex numbers and exponentiation as in $\mathbb{C}$. We propose a new definition of the complex number set, in which the issues related to the identity failures and the multivalued results resolve. Furthermore the exponentiation is no longer defined by the logarithm, instead the complex logarithm formula can be deduced from the exponentiation. There is a cost as some algebraic properties of the addition and substraction will be diminished, though remaining valid to a certain extent. Finally we attempt a geometric and algebraic formalization of the new complex numbers set. It will appear clearly the new complex numbers system is a natural and harmonious complement to the $\mathbb{C}$ field.

Keywords: Complex number field; Complex exponentiation; Complex logarithm; Exponential and logarithm identities

1   Introduction

In 1749 L. Euler [1] solved a decades old controversy between G.W. Leibniz and J. Bernoulli regarding the appropriate definition for logarithms of negative and imaginary values, by producing the formula $\ln(z) = \ln(a + bi) = \ln|z| + \arg(z)i = \ln|z| + \theta i + 2k\pi i$, with $|z| = \sqrt{a^2 + b^2}$, θ the principal value of $\arg(z)$, $k \in \mathbb{Z}$.

The formula for complex exponentiation $z^w = (a + bi)^{m+ni} = x + yi$, where both $z, w \in \mathbb{C}$ was also given by L.Euler in 1749 [2] :

$$z^w = e^{w \ln z} = \left(e^{ln|z|+\theta i+2k\pi i}\right)^{m+ni} = e^{m \ln|z|-n\theta-n2k\pi} e^{(n \ln|z|+m\theta+m2k\pi)i} \tag{1.1}$$

$$x = |z|^m e^{-n\theta-n2k\pi} \cos(n \ln|z| + m\theta + m2k\pi) \tag{1.2}$$

$$y = |z|^m e^{-n\theta-n2k\pi} \sin(n \ln|z| + m\theta + m2k\pi) \tag{1.3}$$

The first complex logarithm formula $\log_z w = x + yi$, where both $z, w \in \mathbb{C}$, was given by M. Ohm in 1829 [3] :

$$\log_z w = \frac{\ln w}{\ln z} = \frac{\ln|w| + \theta_w i + 2k_w \pi i}{\ln|z| + \theta_z i + 2k_z \pi i} \tag{1.4}$$

$$x = \frac{\ln|w| \ln|z| + (\theta_w + 2k_w\pi)(\theta_z + 2k_z\pi)}{(\ln|z|)^2 + (\theta_z + 2k_z\pi)^2} \tag{1.5}$$

$$y = \frac{\ln|z|(\theta_w + 2k_w\pi) - \ln|w|(\theta_z + 2k_z\pi)}{(\ln|z|)^2 + (\theta_z + 2k_z\pi)^2} \tag{1.6}$$

The formula 1.4 is hardly mentionned in mathematic litterature. In 1921, F. Cajori in his History of exponentials and logarithms [4] expressed it this way : "The general logarithm system failed of

recognition as useful mathematical inventions".

Both general complex exponentiation and logarithm formulas are nevertheless used by complex number calculators, though usually only the principal value at $k = 0$ is returned. It's easily verified, by combining formulas 1.1 and 1.4, the complex logarithm is the reciprocal of the exponentiation within a particular branch :

$$
\begin{aligned}
\log_z(z^w) = \frac{\ln(z^w)}{\ln(z)} &= \frac{(m\ln|z| - n\theta_z - n2k\pi) + (n\ln|z| + m\theta_z + m2k\pi)i}{\ln|z| + \theta_z i + 2k_z\pi i} \\
&= \frac{(m+ni)\ln|z| + (m+ni)\theta_z i + (m+ni)2k\pi i}{\ln|z| + \theta_z i + 2k_z\pi i} \\
&= m + ni = w \quad (\text{when } k = k_z)
\end{aligned}
$$

In the same volume M. Ohm [3] studies the validity in $\mathbb{C}$ of the exponential and logarithm identities. He concludes the set of values on both sides of the identity equation can differ. As an example the left side of $(z^w)^v = z^{wv}$ will produce many more results than the right side, since exponentiation is performed twice. Furthermore many identities such as $\ln(wv) = \ln(w) + \ln(v)$ aren't always valid even when considering the principal value or any other branch.

List of exponential and logarithm identities valid when both operands $\in \mathbb{R}_+\backslash\{0\}$:

$$(xy)^a = x^a y^a \tag{1.7}$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} \tag{1.8}$$

$$x^a x^b = x^{a+b} \tag{1.9}$$

$$\frac{x^a}{x^b} = x^{a-b} \tag{1.10}$$

$$(x^a)^b = x^{ab} \tag{1.11}$$

$$\log_a(xy) = \log_a(x) + \log_a(y) \tag{1.12}$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y) \tag{1.13}$$

$$\log_a(x^y) = y\log_a(x) \tag{1.14}$$

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)} \tag{1.15}$$

Examples of identity failures in $\mathbb{C}$: (for clarity only the principal value at $k = 0$ is considered, the same outcome occurs on other values when $k \neq 0$)

$$(-1 \cdot -1)^{\frac{1}{2}} = (1)^{\frac{1}{2}} = 1 \neq (-1)^{\frac{1}{2}} \cdot (-1)^{\frac{1}{2}} = i \cdot i = -1$$

$$(i-1)^{2i} = \left(\sqrt{2}e^{\frac{3\pi}{4}i}\right)^{2i} = 2^i e^{-\frac{3\pi}{2}} \neq ((i-1)(i-1))^i = (-2i)^i = (2e^{-\frac{\pi}{2}i})^i = 2^i e^{\frac{\pi}{2}}$$

$$\ln\left((-i)^2\right) = \ln(-1) = \pi i \neq 2\ln(-i) = 2(-\tfrac{\pi}{2}i) = -\pi i$$

$$
\begin{aligned}
\log_{-2}((-2)^5) &= \log_{-2}(-32) = \frac{\ln(-32)}{\ln(-2)} = \frac{\ln(32)+\pi i}{\ln(2)+\pi i} = \frac{(\ln(32)+\pi i)(\ln(2)-\pi i)}{(\ln(2)+\pi i)(\ln(2)-\pi i)} \\
&= \frac{\ln(32)\ln(2) + (\ln(32)-\ln(2))\pi i + \pi^2}{(\ln(2))^2 + \pi^2} = 1.18568\ldots + 0.84157\ldots i \\
&\neq 5\log_{-2}(-2) = 5
\end{aligned}
$$

$$\ln(-1 \cdot i) = \ln(-i) = -\tfrac{\pi}{2}i \neq \ln(-1) + \ln(i) = \pi i + \tfrac{\pi}{2}i = \tfrac{3\pi}{2}i$$

The sections 2 and 3 are dedicated to the definition of a new complex number set, hereafter named $\mathbb{E}$, the equivalences between $\mathbb{C}$ and $\mathbb{E}$, and to the definition of complex operations $(+, -, \times, \div, \exp, \log)$ in $\mathbb{E}$. The exponentiation is no longer defined by the logarithm, instead the complex logarithm formula can be deduced from the exponentiation. Furthermore all operations return a single value result.

In section 4 we argue why the exponential and logarithm multivalued results and identity failures in $\mathbb{C}$ are not induced by the exponentiation and logarithm operations, but are induced by the definition of complex numbers in algebraic form $z = x + yi$ and by the definition of complex exponentiation $z^w = e^{w \ln z}$.

The section 5 includes all proofs regarding the validity of exponential and logarithm identities in $\mathbb{E}$, also including proofs of some related formulas.

The section 6 proposes a geometric representation of $\mathbb{E}$, of which the complex plane appears as an orthogonal projection. The complex exponentiation $z = z_1^{z_2}$ and logarithm $z = \log_{z_1}(z_2)$, with $z, z_1, z_2 \in \mathbb{E}$, can be simply represented as a mapping of the two operands elements to the result element.

The section 7 lists all algebraic properties of $\mathbb{E}$ and compares with the properties of the $\mathbb{R}$ and $\mathbb{C}$ fields.

2 Definition of complex numbers in complete form

By taking a broad definition of the complex number as a number composed of a real part and an imaginary part, the complete complex number set is defined as follow.

Definition 1 : Complex number in complete form

All numbers in the form $e^a e^{bi}$ $\cup$ $\{0\}$, where $a, b \in \mathbb{R}$ and $i^2 = -1$

The number set is hereafter named $\mathbb{E}$. The real part is defined as e^a and the imaginary part e^{bi}. The element 0 is included for compatibility with $\mathbb{C}$ and $\mathbb{R}$.

The exponential form of complex numbers $z = x + yi = |z|e^{\arg(z)i} = |z|e^{\theta i + 2k\pi i}$ has a similar definition, but remains explicitly linked to the algebraic form and must have a principal value θ of the argument $\arg(z)$ within the interval $]-\pi; \pi]$. The purpose of the integer k is precisely to link all values of the exponential form to their unique corresponding algebraic form. When a particular branch of the exponential form is considered, the imaginary part is bounded to a 2π interval.

In the complete form, the explicit link to the algebraic form and the constraint on the argument principal value are abolished. For example in $\mathbb{E}$ the numbers $e^0 e^{2\pi i}$ and $e^0 e^{4\pi i}$ aren't equal, each having distinct properties as it will be demonstrated in further sections. Within $\mathbb{C}$ the symbolic and geometric representation of both numbers are equally represented by 1 and by the coordinates $(x, y) = (1, 0)$ on the complex plane. Replacing $|z|$ by e^a allows the establishment of more elegant and symmetrical formulas.

Definition 2 : Set partitions of $\mathbb{E}$

Let the set $\mathbb{E}$ of complex numbers in complete form $e^a e^{bi}$ be partionned into $\mathbb{C}$ and $\mathbb{E} \backslash \mathbb{C}$ by restricting $\mathbb{C}$ to a 2π interval of the imaginary argument b, by convention the interval $b \in]-\pi; \pi]$. Every number $x + yi \in \mathbb{C}$ converted to it's unique corresponding complete form $e^a e^{bi}$ forms a distinct equivalence class together with numbers in the form

$e^a e^{(b+2k\pi)i} \in \mathbb{E}$ with $k \in \mathbb{Z}^*$.

The definition is equivalent as restricting $\mathbb{C}$ to the principal value of the exponential form of any complex number in algebraic form. Even with this restriction, the algebraic definition of $\mathbb{C}$ and the complex plane definition are not altered.

Definition 3 : Set precision and truncation

Let A be a set partitionned by an equivalence relation into two subsets A_1 and A_2, and let each element $a_1 \in A_1$ form a distinct equivalence class with an arbitrary number of elements $a_2 \in A_2$ such as every element a_1, a_2 are part of a unique given class. In such a set configuration, elements a_2 are defined as A precise, elements a_1 are defined as A_1 precise. Each element $a_2 \in A_2$ can be truncated to it's corresponding $a_1 \in A_1$ element, thus at a lower precision level.

As an example with the integer set $\mathbb{Z}$ partitionned into $\mathbb{N}$ and $\mathbb{Z}_-^*$, an integer is $\mathbb{Z}$ precise if negative, and is $\mathbb{N}$ precise if positive or zero. The abs function is the truncation function from $\mathbb{Z}$ to $\mathbb{N}$ precision level. It can be written $2 = |-2|_{\mathbb{N}}$.

The Euler formula $e^{bi} = \cos b + \sin b \; i$ is de facto the truncation function from $\mathbb{E}$ to $\mathbb{C}$ precision. The truncation can be noted $|z|_{\mathbb{C}} = |e^a e^{bi}|_{\mathbb{C}} = e^a \cos b + e^a \sin b \; i = e^a e^{|b|_c i}$, with the imaginary exponent truncated such as :

$$|b|_{\mathbb{C}} = \begin{cases} b \pmod{2\pi} & \text{if } b \pmod{2\pi} <= \pi \\ b \pmod{2\pi} - 2\pi & \text{if } b \pmod{2\pi} > \pi \end{cases}$$

Equalities such as $1 = e^{2k\pi i}$ no longer hold whenever $\mathbb{E}$ precision is required, the notation $|e^{2k\pi i}|_{\mathbb{C}} = e^{0i} = 1$ can be used to clearly indicate the truncation. The same notation can be used for any set combination providing an equivalence class and truncation function is defined, for example $1 = |\{1; -1\}|_{\mathbb{N}}$

Lemma 1 : Converting from complete form to algebraic form

$$z = e^a e^{bi} \implies e^a \cos b + e^a \sin b \; i = x + yi \tag{2.1}$$

$x = e^a \cos b$
$y = e^a \sin b$

The Euler formula used for the conversion isn't to be considered as an equality, from a $\mathbb{E}$ perspective an irreversible loss of information is induced when converting from complete to algebraic form.

Lemma 2 : Converting from algebraic form to complete form
(using the definition of complex number modulus and argument)

$$z = x + yi = |z|e^{\mathbf{Arg}(z)i} = e^{\ln|z|}e^{\theta_z i} \implies e^{\frac{1}{2}\ln(x^2+y^2)}e^{\mathbf{Atan}\left(\frac{y}{x}\right)i} = e^a e^{bi} \tag{2.2}$$

$a = \frac{1}{2}\ln(x^2 + y^2)$
$b = \mathbf{Atan}\left(\frac{y}{x}\right)$

By definition $z = 0$ is equivalent in $\mathbb{E}$ and $\mathbb{C}$.

The usage of ln, Arg and Atan functions must be clarified. The natural logarithm function is applied to the domain $\mathbb{R}_+^*$, hence is single valued. In the formula 2.2 only the principal value of the arg function is considered to remain consistent with definition 2. The limits of the traditional arctan function, with the result in the interval $]-\frac{\pi}{2}; \frac{\pi}{2}]$, requires the use of the atan2 function with 2 arguments whose result is included in the interval $]-\pi; \pi]$ without singularities. In this study the notation

Atan $\left(\frac{y}{x}\right)$ refers to the atan2 function where both arguments remain as the fraction numerator and denominator. This notation adjustment will ease the readability and handling of formulas, as obtained formulas always produce a fraction inside the Atan argument. The fraction can be simplified providing the numerator and denominator signum are preserved.

3   Definition of operations in complete form

Definition 4 : Fundamental complex operations in $\mathbb{E}$

The fundamental operations in $\mathbb{E}$ are defined as the 4 basic operations $+, -, \times, \div$ together with exponentiation and logarithm. All operands and results are expressed in complete form, thus $\in \mathbb{E}$.

Lemma 3 : Formulas for operations computation in $\mathbb{E}$

The operations computation use a combination of real functions and real operations $(+, -, \times, \div, \exp, \ln, \sin, \cos, \text{atan2})$. The formulas can be expressed in matricial form.

Let $z_1 = e^{a_1} e^{b_1 i} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $z_2 = e^{a_2} e^{b_2 i} = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$

$$z_1 \times z_2 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \times \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \end{pmatrix} \tag{3.1}$$

$$\frac{z_1}{z_2} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \div \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ b_1 - b_2 \end{pmatrix} \tag{3.2}$$

$$z_1^{z_2} = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \exp \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2) \\ e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2) \end{pmatrix} \tag{3.3}$$

$$\log_{z_1} z_2 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \log \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \ln \left(\frac{a_2{}^2 + b_2{}^2}{a_1{}^2 + b_1{}^2} \right) \\ \text{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} \right) \end{pmatrix} \tag{3.4}$$

$$z_1 + z_2 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} + \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \ln(e^{2a_1} + e^{2a_2} + 2e^{a_1+a_2} \cos(b_1 - b_2)) \\ \text{Atan}\left(\frac{e^{a_1} \sin b_1 + e^{a_2} \sin b_2}{e^{a_1} \cos b_1 + e^{a_2} \cos b_2} \right) \end{pmatrix} \tag{3.5}$$

$$z_1 - z_2 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} - \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \ln(e^{2a_1} + e^{2a_2} - 2e^{a_1+a_2} \cos(b_1 - b_2)) \\ \text{Atan}\left(\frac{e^{a_1} \sin b_1 - e^{a_2} \sin b_2}{e^{a_1} \cos b_1 - e^{a_2} \cos b_2} \right) \end{pmatrix} \tag{3.6}$$

Formulas are easier to handle when split between real and imaginary parts, let $z = e^a e^{bi}$:

$z = z_1 \times z_2$
$a = a_1 + a_2$
$b = b_1 + b_2$

$z = z_1 \div z_2$
$a = a_1 - b_1$
$b = b_1 - b_2$

$z = z_1^{z_2}$
$a = e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2)$

$z = \log_{z_1} z_2$
$a = \frac{1}{2} \ln \left(\frac{a_2{}^2 + b_2{}^2}{a_1{}^2 + b_1{}^2} \right)$

$$b = e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2) \qquad\qquad b = \text{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right)$$

$$z = z_1 + z_2 \qquad\qquad z = z_1 - z_2$$

$$a = \frac{1}{2}\ln(e^{2a_1} + e^{2a_2} + 2e^{a_1+a_2}\cos(b_1-b_2)) \qquad a = \frac{1}{2}\ln(e^{2a_1} + e^{2a_2} - 2e^{a_1+a_2}\cos(b_1-b_2))$$

$$b = \text{Atan}\left(\frac{e^{a_1}\sin b_1 + e^{a_2}\sin b_2}{e^{a_1}\cos b_1 + e^{a_2}\cos b_2}\right) \qquad b = \text{Atan}\left(\frac{e^{a_1}\sin b_1 - e^{a_2}\sin b_2}{e^{a_1}\cos b_1 - e^{a_2}\cos b_2}\right)$$

Proof of multiplication formula :
(using the identity $e^{w_1}\cdot e^{w_2} = e^{w_1+w_2}$ with $w_1, w_2 \in \mathbb{C}$)

$$z = z_1 \times z_2 = e^{a_1}e^{b_1 i}\cdot e^{a_2}e^{b_2 i} = e^{a_1}e^{a_2}e^{b_1 i}e^{b_2 i} = e^{a_1+a_2}e^{(b_1+b_2)i}$$
$$a = a_1 + a_2$$
$$b = b_1 + b_2$$

Proof of division formula :
(using the identity $e^{w_1}/e^{w_2} = e^{w_1-w_2}$ with $w_1, w_2 \in \mathbb{C}$)

$$z = z_1 \div z_2 = \frac{e^{a_1}e^{b_1 i}}{e^{a_2}e^{b_2 i}} = \frac{e^{a_1}}{e^{a_2}}\cdot\frac{e^{b_1 i}}{e^{b_2 i}} = e^{a_1-a_2}e^{(b_1-b_2)i}$$
$$a = a_1 - a_2$$
$$b = b_1 - b_2$$

Proof of exponentiation formula :

The formula $u^v = e^{v\ln u}$ with $u, v \in \mathbb{C}$ defines the complex exponentiation in $\mathbb{C}$, the formula is necessary given the base cannot be exploited directly in algebraic form, thus has to be converted into an infinity possible bases in the form $u = e^{\ln|u|+\theta i+2k\pi i}$. The exponent is then applied to the bases such as $u^v = \left(e^{\ln|u|+\theta i+2k\pi i}\right)^v = e^{v\ln|u|+v(\theta+2k\pi)i}$. The result is then reconverted into algebraic form. When calculated seperately for each integer k, the exponentiation can be defined as $(e^a e^{bi})^v = e^{av}e^{bvi}$ with a single-value result, the base and result being in complete form and the exponent being in algebraic form. For this reason the complex exponentiation will be easier to define in $\mathbb{E}$. Let $z_1 = e^{a_1}e^{b_1 i}$ and, using the conversion formula 2.1, let $z_2 = e^{a_2}e^{b_2 i} \implies e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i$.

$$z = z_1^{z_2} = \left(e^{a_1}e^{b_1 i}\right)^{(e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i)}$$
$$= \left(e^{a_1+b_1 i}\right)^{(e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i)}$$
$$= e^{(a_1+b_1 i)(e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i)}$$
$$= e^{(a_1 e^{a_2}\cos b_2 + a_1 e^{a_2}\sin b_2\, i + b_1 e^{a_2}\cos b_2\, i - b_1 e^{a_2}\sin b_2)}$$
$$= e^{e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2)}e^{e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2)i}$$
$$a = e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2)$$
$$b = e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2)$$

Proof of logarithm formula :

The logarithm formula can be directly reversed from the exponentiation formula 3.3. Counter to the definition of the complex logarithm in $\mathbb{C}$, both operands are here in $\mathbb{E}$ thus can be exploited directly in the formula without requiring any conversion. Let $z_1 = e^{a_1}e^{b_1 i}$ and $z_2 = e^{a_2}e^{b_2 i}$.

$$z = \log_{z_1}(z_2) \Longleftrightarrow (z_1)^z = z_2 \Longrightarrow a_2 = e^a(a_1 \cos b - b_1 \sin b) \text{ and } b_2 = e^a(b_1 \cos b + a_1 \sin b)$$

$$
\begin{aligned}
a_2{}^2 + b_2{}^2 &= e^{2a}(a_1 \cos b - b_1 \sin b)^2 + e^{2a}(b_1 \cos b + a_1 \sin b)^2 \\
&= e^{2a}(a_1{}^2 \cos^2 b - a_1 b_1 \cos b \sin b + b_1{}^2 \sin^2 b + b_1{}^2 \cos^2 b + a_1 b_1 \cos b \sin b + a_1{}^2 \sin^2 b) \\
&= e^{2a}(a_1{}^2 \cos^2 b + a_1{}^2 \sin^2 b + b_1{}^2 \cos^2 b + b_1{}^2 \sin^2 b) \\
&= e^{2a}(a_1{}^2 + b_1{}^2) \\
a &= \frac{1}{2} \ln\left(\frac{a_2{}^2 + b_2{}^2}{a_1{}^2 + b_1{}^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{a_2}{b_2} &= \frac{e^a(a_1 \cos b - b_1 \sin b)}{e^a(b_1 \cos b + a_1 \sin b)} \\
a_2(b_1 \cos b + a_1 \sin b) &= b_2(a_1 \cos b - b_1 \sin b) \\
\cos b(a_2 b_1 - a_1 b_2) &= -\sin b(a_1 a_2 + b_1 b_2) \\
\frac{\sin b}{\cos b} &= \frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} \\
b &= \mathrm{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right)
\end{aligned}
$$

Alternate proof of logarithm formula :

$$
\begin{aligned}
z = \log_{z_1}(z_2) &= \frac{\ln z_2}{\ln z_1} = \frac{a_2 + b_2 i}{a_1 + b_1 i} \\
&= \frac{(a_2 + b_2 i)(a_1 - b_1 i)}{(a_1 + b_1 i)(a_1 - b_1 i)} \\
&= \frac{a_1 a_2 + b_1 b_2 + a_1 b_2 i - a_2 b_1 i}{a_1{}^2 + b_1{}^2} \\
&= \frac{a_1 a_2 + b_1 b_2}{a_1{}^2 + b_1{}^2} + \frac{a_1 b_2 - a_2 b_1}{a_1{}^2 + b_1{}^2} i
\end{aligned}
$$

The result is in algebraic form and needs to be converted into complete form using conversion formula 2.2.

$$
\begin{aligned}
a &= \frac{1}{2} \ln \frac{(a_1 a_2 + b_1 b_2)^2 + (a_1 b_2 - a_2 b_1)^2}{(a_1{}^2 + b_1{}^2)^2} \\
&= \frac{1}{2} \ln \frac{a_1{}^2 a_2{}^2 + 2a_1 a_2 b_1 b_2 + b_1{}^2 b_2{}^2 + a_1{}^2 b_2{}^2 - 2a_1 a_2 b_1 b_2 + a_2{}^2 b_1{}^2}{(a_1{}^2 + b_1{}^2)^2} \\
&= \frac{1}{2} \ln \frac{a_1{}^2 a_2{}^2 + b_1{}^2 b_2{}^2 + a_1{}^2 b_2{}^2 + a_2{}^2 b_1{}^2}{(a_1{}^2 + b_1{}^2)^2} \\
&= \frac{1}{2} \ln \frac{a_1{}^2(a_2{}^2 + b_2{}^2) + b_1{}^2(a_2^2 + b_2^2)}{(a_1{}^2 + b_1{}^2)^2} \\
&= \frac{1}{2} \ln \frac{(a_2{}^2 + b_2{}^2)(a_1{}^2 + b_1{}^2)}{(a_1{}^2 + b_1{}^2)^2} \\
&= \frac{1}{2} \ln\left(\frac{a_2{}^2 + b_2{}^2}{a_1{}^2 + b_1{}^2}\right)
\end{aligned}
$$

$$
b = \mathrm{Atan}\left(\frac{\frac{a_1 b_2 - a_2 b_1}{a_1{}^2 + b_1{}^2}}{\frac{a_1 a_2 + b_1 b_2}{a_1{}^2 + b_1{}^2}}\right) = \mathrm{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right)
$$

Proof of addition and substraction formulas :

Both operands need to be converted into algebraic form using the formula 2.1, since no identity can be used directly in complete form for the operation. Let $z_1 = e^{a_1}\cos b_1 + e^{a_1}\sin b_1\, i$ and $z_2 = e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i$

$$z = z_1 \pm z_2 = (e^{a_1}\cos b_1 + e^{a_1}\sin b_1\, i) \pm (e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i)$$
$$= (e^{a_1}\cos b_1 \pm e^{a_2}\cos b_2) + (e^{a_1}\sin b_1 \pm e^{a_2}\sin b_2)i$$

The result is in algebraic form and needs to be converted into complete form using conversion formula 2.2.

$$a = \frac{1}{2}\ln((e^{a_1}\cos b_1 \pm e^{a_2}\cos b_2)^2 + (e^{a_1}\sin b_1 \pm e^{a_2}\sin b_2)^2)$$
$$= \frac{1}{2}\ln(e^{2a_1}\cos^2 b_1 \pm 2e^{a_1}e^{a_2}\cos b_1\cos b_2 + e^{2a_2}\cos^2 b_2 + e^{2a_1}\sin^2 b_1 \pm 2e^{a_1}e^{a_2}\sin b_1\sin b_2 + e^{2a_2}\sin^2 b_2)$$
$$= \frac{1}{2}\ln(e^{2a_1}(\cos^2 b_1 + \sin^2 b_1) + e^{2a_2}(\cos^2 b_2 + \sin^2 b_2) \pm 2e^{a_1}e^{a_2}(\cos b_1\cos b_2 + \sin b_1\sin b_2))$$
$$= \frac{1}{2}\ln(e^{2a_1} + e^{2a_2} \pm 2e^{a_1+a_2}\cos(b_1 - b_2))$$
$$b = \mathrm{Atan}\left(\frac{e^{a_1}\sin b_1 \pm e^{a_2}\sin b_2}{e^{a_1}\cos b_1 \pm e^{a_2}\cos b_2}\right)$$

Theorem 1 :

Multiplication, division and exponentiation results are $\mathbb{C}$ or $\mathbb{E}$ precise

One can easily deduce from formulas 3.1, 3.2 and 3.3 the result of the imaginary part isn't bounded by any limit and will be situated anywhere in $b \in \mathbb{R}$.

The exponentiation is more subtle since it requires mixed precision between the operands, the base in complete form is $\mathbb{E}$ or $\mathbb{C}$ precise, the exponent in algebraic form is $\mathbb{C}$ precise. Would $\mathbb{E}$ precise exponent be used, it would get truncated by the cosine and sine functions used in the formula 3.3

Multiplication and division operands and results are $\mathbb{E}$ or $\mathbb{C}$ precise, no truncation is performed by the formulas. One can notice even with $\mathbb{C}$ precise operands, the result may be $\mathbb{E}$ precise.

Theorem 2 :

Addition, substraction and logarithm results are only $\mathbb{C}$ precise

The formulas 3.4, 3.5 and 3.6 use the atan2 function in the imaginary part, thus the result will always be situated in the interval $b \in\,] -\pi; \pi]$, which is exactly the definition of the $\mathbb{C}$ precision.

The logarithm requires mixed precision, both operands requires the complete form which can therefore be $\mathbb{E}$ or $\mathbb{C}$ precise, but the result is always $\mathbb{C}$ precise.

The addition and substraction are the only operations not requiring the complete form hence no $\mathbb{E}$ precision, operands exceeding the required precision are truncated to $\mathbb{C}$ precision by formulas 3.5 and 3.6.

Theorem 3 :

All fundamental operations defined in $\mathbb{E}$ are monovalued

From the multiplication, division and exponentiation formulas 3.1, 3.2 and 3.3, one can deduce both the real and imaginary part will always give a single valued results, since no real multivalued function is used in the formulas.

Regarding the addition, substraction and logarithm formulas 3.5, 3.6 and 3.4, the real part will always be monovalued, the imaginary part using the atan2 function will return an imaginary argument within the interval $b \in\ ]-\pi;\pi]$, giving a single valued result element.

4   Comments on the exponentiation and logarithm definition in $\mathbb{C}$

The complex exponentiation in $\mathbb{C}$ is defined by the formula $z^w = e^{w \ln z} = e^{m \ln |z| - n\theta - n2k\pi} e^{(n \ln |z| + m\theta + m2k\pi)i}$, thus the primary result is in exponential form. To express the result in algebraic form a conversion is necessary. Using the Euler formula the sine and cosine functions truncate the result into algebraic form. During the conversion precision is lost and the result principal value may be shifted. For example $(-1)^3 = (e^{\ln(-1)})^3 = (e^{\pi i + 2k\pi i})^3 = e^{3\pi i + 6k\pi i}$. Converting those values into algebraic form returns $-1 = e^{\pi i + 2k\pi i}$, thus the principal value is reset to $e^{\pi i}$. Furthermore there is no possibility to convert expressions such as $e^{6k\pi i}$ without loss of precision. As another example $\left(i^{-5}\right)^i$, because of the reconversion of i^{-5} into algebraic form, the principal value of the final result is $\left(i^{-5}\right)^i = e^{\frac{\pi}{2}}$ instead of $e^{\frac{5\pi}{2}}$.

The multi valuation of the complex exponentiation isn't induced by the logarithm, but by the algebraic form of the base since no identity is available to exploit the base as such. The base has to be substituted by an infinity of bases in complete form, the so called exponential form, using the formula $z = |z|e^{\arg(z)i} = |z|e^{\theta i + 2k\pi i}$. In general the substitution is implicitly assumed, unless explicitly restricting the exponentiation to real positives with notations such as $|z|^\alpha$ or $\sqrt{a^2 + b^2}$ which both assume a single valued base $|z|e^{0i}$.

The complex logarithm as defined by L. Euler [1] is restricted to the base e. Euler himself doesn't mention a multivalued logarithm function, rather he speaks of each real or complex number having an infinite number of logarithms. Indeed, as for the exponentiation base, the logarithm operand cannot be exploited directly in algebraic form, thus has to be converted in exponential form, $\ln(z) = \ln(|z|e^{\arg(z)i}) = \ln(|z|e^{\theta i + 2k\pi i}) = \ln |z| + \theta i + 2k\pi i$. The results being in algebraic form, no conversion nor loss of precision is induced. The multi valuation is solely induced by the operand substitution, on the other hand $\ln |z|$ is assumed single valued as the operand is implicitly substituted by $|z|e^{0i}$. For example $\ln(1) = 2k\pi i, \ln(-1) = \pi i + 2k\pi i$. But expressions such as $\ln(e^{\pi i}) = \pi i$ are single valued, unless $e^{\pi i}$ is converted to algebraic form, in which case $\ln(e^{\pi i}) = \ln(-1) = \pi i + 2k\pi i$.

Notations such as $\ln, \log_2$ or $\log_{10}$ assume the logarithm base is in the form xe^{0i}. For bases the same logic applies as for the exponentiation, a base in algebraic form can be substitued by the equivalent exponential form, or by any particular value in complete form. For example $\log_{-1}$ can be interpreted as a single valued base $e^{\pi i}$.

It is clear there is only one unique exponentiation and logarithm operation, the different notation conventions and different assumptions regarding the operands substitutions are creating the confusion.

When dealing with exponentiation and logarithm in $\mathbb{C}$ the equality $e^{\alpha i} = e^{\alpha i + 2k\pi i}$ is automatically assumed, by an analogy with the trigonometric circle where an angle of α is equal to $\alpha + 2k\pi$. However from a $\mathbb{E}$ perspective only $|e^{\alpha i}|_{\mathbb{C}} = |e^{\alpha i + 2k\pi i}|_{\mathbb{C}}$ is valid. The formulas $\cos(\alpha) + i \sin(\alpha) = \cos(\alpha + 2k\pi) + i \sin(\alpha + 2k\pi) = \sum_{n=0}^{\infty} \frac{(\alpha i)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\alpha i + 2k\pi i)^n}{n!}$ are strictly equal, but are also equal in deconstructing the complete form and reconstructing a result in algebraic form, as such they literally truncate the precision of the complete form.

As it will be demonstrated in next section, the failures of exponential and logarithm identities in $\mathbb{C}$ are due to the unique reason the result of a multiplication, division or exponentiation cannot be held with a sufficient level of precision. During calculation steps, the result may be implicitely truncated to $\mathbb{C}$ precision level inducing an irremediable loss of information.

5 Exponentials and logarithms identities in $\mathbb{E}$

Theorem 4 : All exponentiation identities valid in $\mathbb{R}_+^*$ are valid in $\mathbb{E}^*$

The result is strictly identical on both sides of the identity when $z_1, z_2, z_3 \in \mathbb{E}^*$

$$(z_1 z_2)^{z_3} = z_1{}^{z_3} z_2{}^{z_3} \tag{5.1}$$

$$\left(\frac{z_1}{z_2}\right)^{z_3} = \frac{z_1{}^{z_3}}{z_2{}^{z_3}} \tag{5.2}$$

$$z_1{}^{z_2} z_1{}^{z_3} = z_1{}^{z_2+z_3} \tag{5.3}$$

$$\frac{z_1{}^{z_2}}{z_1{}^{z_3}} = z_1{}^{z_2-z_3} \tag{5.4}$$

$$(z_1{}^{z_2})^{z_3} = z_1{}^{z_2 z_3} \tag{5.5}$$

Theorem 5 : The product and quotient logarithm identities valid in $\mathbb{R}_+^*$ are valid in $\mathbb{E}^*$

The result is strictly identical on both sides of the identity when $z_1, z_2, z_3 \in \mathbb{E}^*$ and $z_1 \neq e^0 e^{0i}$

$$\log_{z_1}(z_2 z_3) = \log_{z_1}(z_2) + \log_{z_1}(z_3) \tag{5.6}$$

$$\log_{z_1}\left(\frac{z_2}{z_3}\right) = \log_{z_1}(z_2) - \log_{z_1}(z_3) \tag{5.7}$$

Theorem 6 : The power and base substitution logarithm identities valid in $\mathbb{R}_+^*$ are valid in $\mathbb{E}^*$ only at $\mathbb{C}$ precision level

The result truncated to $\mathbb{C}$ precision is strictly identical on both sides of the identity when $z_1, z_2, z_3, z_4 \in \mathbb{E}^*$ and $z_1, z_4 \neq e^0 e^{0i}$. The final operations on each side of the identity return different levels of precision, the identity cannot be a strict equality.

$$\log_{z_1}(z_2{}^{z_3}) = \left| z_3 \log_{z_1}(z_2) \right|_{\mathbb{C}} \tag{5.8}$$

$$\log_{z_1}(z_2) = \left| \frac{\log_{z_4}(z_1)}{\log_{z_4}(z_2)} \right|_{\mathbb{C}} \tag{5.9}$$

As demonstrated within the following proofs, the trivial cases of exponential and logarithm identity failures given in the introduction dissapear when both sides of the identity equation are calculated in $\mathbb{E}$, thus when the formulas 3.1 to 3.6 are used at every calculation step.

Proof of $(z_1 z_2)^{z_3} = z_1^{z_3} z_2^{z_3}$ is valid for all $z_1, z_2, z_3 \in \mathbb{E}^*$:

Combining the multiplication and exponentiation formulas 3.1 and 3.3, let $z_1 = e^{a_1} e^{b_1 i}$, $z_2 = e^{a_2} e^{b_2 i}$ and $z_3 = e^{a_3} e^{b_3 i}$

$$z = (z_1 z_2)^{z_3}$$

$$a = e^{a_3}((a_1 + a_2)\cos b_3 - (b_1 + b_2)\sin b_3)$$
$$b = e^{a_3}((b_1 + b_2)\cos b_3 + (a_1 + a_2)\sin b_3)$$

$$z = z_1^{z_3} z_2^{z_3}$$
$$a = e^{a_3}(a_1\cos b_3 - b_1\sin b_3) + e^{a_3}(a_2\cos b_3 - b_2\sin b_3)$$
$$\quad = e^{a_3}((a_1 + a_2)\cos b_3 - (b_1 + b_2)\sin b_3)$$
$$b = e^{a_3}(b_1\cos b_3 + a_1\sin b_3) + e^{a_3}(b_2\cos b_3 + a_2\sin b_3)$$
$$\quad = e^{a_3}((b_1 + b_2)\cos b_3 + (a_1 + a_2)\sin b_3)$$

Example (using the principal value at $k = 0$) :

$$(-1 \cdot -1)^{\frac{1}{2}} = (e^{\pi i}e^{\pi i})^{\frac{1}{2}} = (e^{2\pi i})^{\frac{1}{2}} = e^{\pi i} \implies -1$$

$$(-1)^{\frac{1}{2}} \cdot (-1)^{\frac{1}{2}} = (e^{\pi i})^{\frac{1}{2}}(e^{\pi i})^{\frac{1}{2}} = e^{\frac{\pi}{2}i}e^{\frac{\pi}{2}i} = e^{\pi i} \implies -1$$

When the first expression is evaluated in algebraic form in $\mathbb{C}$, the result is 1, the reason of the dissimilarity is because the result of the multiplication $-1 \cdot -1$ was implicitly truncated to a $\mathbb{C}$ precision level. In $\mathbb{E}$ equating $-1 \cdot -1 = 1$ is an over simplification : $e^{\pi i}e^{\pi i} = e^{2\pi i} \neq e^{0i}$, though in algebraic form the 2 values are indistinctive. This imprecision, invisible at first glance, is revealed when the exponent $\frac{1}{2}$ is applied on $e^{2\pi i}$ or e^{0i} giving different values, respectively -1 and 1. Similarly, $-i \cdot -i = e^{-\frac{\pi}{2}i}e^{-\frac{\pi}{2}i} = e^{-\pi i} \neq e^{\pi i}$ and $-1 \cdot i = e^{\pi i}e^{\frac{\pi}{2}i} = e^{\frac{3\pi}{2}i} \neq e^{-\frac{\pi}{2}i}$. On the other hand, $i \cdot i = -1$ and $i \cdot -i = 1$ are always valid.

Proof of $(z_1/z_2)^{z_3} = z_1^{z_3}/z_2^{z_3}$ is valid for all $z_1, z_2, z_3 \in \mathbb{E}^*$:

Combining the division and exponentiation formulas 3.2 and 3.3

$$z = \left(\frac{z_1}{z_2}\right)^{z_3}$$
$$a = e^{a_3}((a_1 - a_2)\cos b_3 - (b_1 - b_2)\sin b_3)$$
$$b = e^{a_3}((b_1 - b_2)\cos b_3 + (a_1 - a_2)\sin b_3)$$
$$z = \frac{z_1^{z_3}}{z_2^{z_3}}$$
$$a = e^{a_3}(a_1\cos b_3 - b_1\sin b_3) - e^{a_3}(a_2\cos b_3 - b_2\sin b_3)$$
$$\quad = e^{a_3}((a_1 - a_2)\cos b_3 - (b_1 - b_2)\sin b_3)$$
$$b = e^{a_3}(b_1\cos b_3 + a_1\sin b_3) - e^{a_3}(b_2\cos b_3 + a_2\sin b_3)$$
$$\quad = e^{a_3}((b_1 - b_2)\cos b_3 + (a_1 - a_2)\sin b_3)$$

Example :

$$\left(\frac{1}{-1}\right)^{\frac{1}{2}} = \left(\frac{e^{0i}}{e^{\pi i}}\right)^{\frac{1}{2}} = (e^{-\pi i})^{\frac{1}{2}} = e^{-\frac{\pi}{2}i} \implies -i$$

$$\frac{(1)^{\frac{1}{2}}}{(-1)^{\frac{1}{2}}} = \frac{(e^{0i})^{\frac{1}{2}}}{(e^{\pi i})^{\frac{1}{2}}} = \frac{e^{0i}}{e^{\frac{\pi}{2}i}} = e^{-\frac{\pi}{2}i} \implies -i$$

When the first expression is evaluated in algebraic form, the result is i. The error here is to consider $1/-1 = -1$ which is an implicit truncation at $\mathbb{C}$ precision level. In $\mathbb{E}$ $e^{0i}/e^{\pi i} = e^{-\pi i} \neq e^{\pi i}$. The exponent $\frac{1}{2}$ applied on $e^{\pi i}$ or $e^{-\pi i}$ giving different values in $\mathbb{C}$, respectively -i and i. Similarly, $-1/-i = e^{\pi i}/e^{-\frac{\pi}{2}i} = e^{\frac{3\pi}{2}i} \neq e^{-\frac{\pi}{2}i}$ and $-i/i = e^{-\frac{\pi}{2}i}/e^{\frac{\pi}{2}i} = e^{-\pi i} \neq e^{\pi i}$.

Proof of $\left(z_1^{z_2}\right)^{z_3} = z_1^{z_2 z_3}$ is valid for all $z_1, z_2, z_3 \in \mathbb{E}^*$:

Combining the multiplication and exponentiation formulas 3.1 and 3.3

$z = z_1^{z_2 z_3}$
$a = e^{a_2 + a_3}(a_1 \cos(b_2 + b_3) - b_1 \sin(b_2 + b_3))$
$b = e^{a_2 + a_3}(b_1 \cos(b_2 + b_3) + a_1 \sin(b_2 + b_3))$

$z = \left(z_1^{z_2}\right)^{z_3}$
$a = e^{a_3}\left(e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2)\cos b_3 - e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2)\sin b_3\right)$
$\quad = e^{a_2}e^{a_3}(a_1 \cos b_2 \cos b_3 - b_1 \sin b_2 \cos b_3 - b_1 \cos b_2 \sin b_3 - a_1 \sin b_2 \sin b_3)$
$\quad = e^{a_2 + a_3}(a_1(\cos b_2 \cos b_3 - \sin b_2 \sin b_3) - b_1(\sin b_2 \cos b_3 + \cos b_2 \sin b_3))$
$\quad = e^{a_2 + a_3}(a_1 \cos(b_2 + b_3) - b_1 \sin(b_2 + b_3))$
$b = e^{a_3}\left(e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2)\cos b_3 + e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2)\sin b_3\right)$
$\quad = e^{a_2}e^{a_3}(b_1 \cos b_2 \cos b_3 + a_1 \sin b_2 \cos b_3 + a_1 \cos b_2 \sin b_3 - b_1 \sin b_2 \sin b_3)$
$\quad = e^{a_2 + a_3}(b_1(\cos b_2 \cos b_3 - \sin b_2 \sin b_3) + a_1(\sin b_2 \cos b_3 + \cos b_2 \sin b_3))$
$\quad = e^{a_2 + a_3}(b_1 \cos(b_2 + b_3) + a_1 \sin(b_2 + b_3))$

$z = \left(z_1^{z_3}\right)^{z_2}$
$a = e^{a_2}\left(e^{a_3}(a_1 \cos b_3 - b_1 \sin b_3)\cos b_2 - e^{a_3}(b_1 \cos b_3 + a_1 \sin b_3)\sin b_2\right)$
$\quad = e^{a_2 + a_3}(a_1 \cos(b_2 + b_3) - b_1 \sin(b_2 + b_3))$
$b = e^{a_2}\left(e^{a_3}(b_1 \cos b_3 + a_1 \sin b_3)\cos b_2 + e^{a_3}(a_1 \cos b_3 - b_1 \sin b_3)\sin b_2\right)$
$\quad = e^{a_2 + a_3}(b_1 \cos(b_2 + b_3) + a_1 \sin(b_2 + b_3))$

Example 1 :

$$((i-1)^2)^i = ((e^{\frac{1}{2}\ln 2}e^{\frac{3\pi}{4}i})^2)^i = (e^{\ln 2}e^{\frac{3\pi}{2}i})^i = e^{i\ln 2}e^{-\frac{3\pi}{2}}$$

$$(i-1)^{2i} = (e^{\frac{1}{2}\ln 2}e^{\frac{3\pi}{4}})^{2i} = e^{i\ln 2}e^{-\frac{3\pi}{2}}$$

When $(i-1)^2$ is evaluated in algebraic form, the result obtained is $(i-1)(i-1) = -2i$ which is only true in $\mathbb{C}$. Some relevant precision for $\mathbb{E}$ has been lost during the evaluation : $(i-1)(i-1) = -2i = e^{\ln 2}e^{-\frac{\pi}{2}i} \neq (i-1)^2 = (e^{\frac{1}{2}\ln 2}e^{\frac{3\pi}{4}i})^2 = e^{\ln 2}e^{\frac{3\pi}{2}i}$.

Example 2 : (Clausen paradox [5])

$$(e^{1+2\pi ki})^{1+2\pi ki} = e^{(1+2\pi ki)^2} = e^{1+4\pi ki - 4\pi^2 k^2} = e^{1-4\pi^2 k^2}e^{4\pi ki}$$

$(e^{1+2\pi ki})^{1+2\pi ki} = e^{1+2\pi ki} = e \neq e^{1-4\pi^2 k^2}e^{4\pi ki}$, the equality holds only when $k = 0$.

In the first expression the exponentiation base is taken as multivalued $e^1 e^{2\pi ki}$, the exponent in algebraic form $1 + 2\pi ki$ is also multivalued, with both k synchronised. Nothing wrong here. The result of the exponentiation will obviously be multivalued, the first formula given is correct. In the second expression no exponentiation is performed, instead a double truncation from $\mathbb{E}$ to $\mathbb{C}$ precision. Equating $e^{1+2\pi ki} = e^1 e^{2\pi ki} = e \cdot 1 = e$ is imprecise, $|e^{1+2\pi ki}|_\mathbb{C} = e$ is correct. After the truncation only the value within the interval $b \in \,]-\pi; \pi]$ remains thus when $k = 0$.

Proof of $z_1{}^{z_2}z_1{}^{z_3} = z_1{}^{z_2 + z_3}$ is valid for all $z_1, z_2, z_3 \in \mathbb{E}^*$:

The identity is similar to the identity $e^{z_1}e^{z_2} = e^{z_1+z_2}$

$$e^{z_1+z_2} = e^{z_1}e^{z_2}$$
$$z^{z_1+z_2} = z^{z_1}z^{z_2}$$
$$(e^a e^{bi})^{z_1+z_2} = (e^a e^{bi})^{z_1}(e^a e^{bi})^{z_2}$$
$$(e^{a+bi})^{z_1+z_2} = (e^{a+bi})^{z_1}(e^{a+bi})^{z_2}$$
$$e^{(a+bi)(z_1+z_2)} = e^{(a+bi)z_1}e^{(a+bi)z_2} \quad \text{(all exponents can be reduced into the form } z = x + yi\text{)}$$
$$e^{(a+bi)(z_1+z_2)} = e^{(a+bi)z_1+(a+bi)z_2}$$
$$e^{(a+bi)(z_1+z_2)} = e^{(a+bi)(z_1+z_2)}$$

The identity can be verified using the multiplication 3.1 and exponentiation 3.3 formulas and the conversion formulas 2.1 and 2.2. Let $z_1 = e^{a_1}e^{b_1 i}, z_2 = e^{a_2}e^{b_2 i}$ and $z_3 = e^{a_3}e^{b_3 i}$

$$z = z_1^{z_2}z_1^{z_3}$$
$$a = e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2) + e^{a_3}(a_1 \cos b_3 - b_1 \sin b_3)$$
$$b = e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2) + e^{a_3}(b_1 \cos b_3 + a_1 \sin b_3)$$

$$z = z_1^{z_2+z_3}$$
$$= z_1^{(e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i) + (e^{a_3}\cos b_3 + e^{a_3}\sin b_3\, i)}$$
$$= z_1^{(e^{a_2}\cos b_2 + e^{a_3}\cos b_3) + (e^{a_2}\sin b_2 + e^{a_3}\sin b_3)i}$$
$$= z_1^{c+di}$$
$$a = e^{\frac{1}{2}\ln(c^2+d^2)}\left(a_1 \cos\left(\arctan\left(\tfrac{d}{c}\right)\right) - b_1 \sin\left(\arctan\left(\tfrac{d}{c}\right)\right)\right)$$
$$= \sqrt{c^2+d^2}\left(\frac{a_1}{\sqrt{1+\frac{d^2}{c^2}}} - \frac{\frac{b_1 d}{c}}{\sqrt{1+\frac{d^2}{c^2}}}\right)$$
$$= a_1 c - b_1 d$$
$$= a_1(e^{a_2}\cos b_2 + e^{a_3}\cos b_3) - b_1(e^{a_2}\sin b_2 + e^{a_3}\sin b_3)$$
$$= e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2) + e^{a_3}(a_1 \cos b_3 - b_1 \sin b_3)$$
$$b = e^{\frac{1}{2}\ln(c^2+d^2)}\left(b_1 \cos\left(\arctan\left(\tfrac{d}{c}\right)\right) + a_1 \sin\left(\arctan\left(\tfrac{d}{c}\right)\right)\right)$$
$$= \sqrt{c^2+d^2}\left(\frac{b_1}{\sqrt{1+\frac{d^2}{c^2}}} + \frac{\frac{a_1 d}{c}}{\sqrt{1+\frac{d^2}{c^2}}}\right)$$
$$= b_1 c + a_1 d$$
$$= b_1(e^{a_2}\cos b_2 + e^{a_3}\cos b_3) + a_1(e^{a_2}\sin b_2 + e^{a_3}\sin b_3)$$
$$= e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2) + e^{a_3}(b_1 \cos b_3 + a_1 \sin b_3)$$

Proof of $z_1{}^{z_2}/z_1{}^{z_3} = z_1{}^{z_2-z_3}$ is valid for all $z_1, z_2, z_3 \in \mathbb{E}^*$:

The identity is similar to the identity $e^{z_1}/e^{z_2} = e^{z_1-z_2}$

$$e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}$$
$$z^{z_1-z_2} = \frac{z^{z_1}}{z^{z_2}}$$
$$(e^a e^{bi})^{z_1-z_2} = \frac{(e^a e^{bi})^{z_1}}{(e^a e^{bi})^{z_2}}$$

$$\left(e^{a+bi}\right)^{z_1 - z_2} = \frac{\left(e^{a+bi}\right)^{z_1}}{\left(e^{a+bi}\right)^{z_2}}$$

$$e^{(a+bi)(z_1 - z_2)} = \frac{e^{(a+bi)z_1}}{e^{(a+bi)z_2}} \quad \text{(all exponents can be reduced into the form } z = x + yi)$$

$$e^{(a+bi)(z_1 - z_2)} = e^{(a+bi)z_1 - (a+bi)z_2}$$

$$e^{(a+bi)(z_1 - z_2)} = e^{(a+bi)(z_1 - z_2)}$$

Proof of $\log_{z_1}(z_2{}^{z_3}) = z_3 \log_{z_1}(z_2)$ is valid at $\mathbb{C}$ precision level for all $z_1, z_2, z_3 \in \mathbb{E}^*$:

Combining the multiplication and logarithm formulas 3.1 and 3.4

$$z = z_3 \log_{z_1}(z_2)$$

$$a = a_3 + \frac{1}{2} \ln\left(\frac{a_2{}^2 + b_2{}^2}{a_1{}^2 + b_1{}^2}\right)$$

$$b = b_3 + \text{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right) i$$

Combining the exponentiation and logarithm formulas 3.3 and 3.4

$$z = \log_{z_1}(z_2{}^{z_3})$$

$$a = \frac{1}{2} \ln\left(\frac{e^{2a_3}(a_2 \cos b_3 - b_2 \sin b_3)^2 + e^{2a_3}(b_2 \cos b_3 + a_2 \sin b_3)^2}{a_1{}^2 + b_1{}^2}\right)$$

$$= \frac{1}{2} \ln\left(\frac{e^{2a_3}(a_2{}^2 \cos^2 b_3 - a_2 b_2 \cos b_3 \sin b_3 + b_2{}^2 \sin^2 b_3 + b_2{}^2 \cos^2 b_3 + a_2 b_2 \cos b_3 \sin b_3 + b_2{}^2 \sin^2 b_3)}{a_1{}^2 + b_1{}^2}\right)$$

$$= \frac{1}{2} \ln\left(e^{2a_3} \cdot \frac{(a_2{}^2 + b_2{}^2)(\cos^2 b_3 + \sin^2 b_3)}{a_1{}^2 + b_1{}^2}\right)$$

$$= a_3 + \frac{1}{2} \ln\left(\frac{a_2{}^2 + b_2{}^2}{a_1{}^2 + b_1{}^2}\right)$$

$$b = \text{Atan}\left(\frac{a_1 e^{a_3}(b_2 \cos b_3 + a_2 \sin b_3) - e^{a_3}(a_2 \cos b_3 - b_2 \sin b_3)b_1}{a_1 e^{a_3}(a_2 \cos b_3 - b_2 \sin b_3) + b_1 e^{a_3}(b_2 \cos b_3 + a_2 \sin b_3)}\right)$$

$$= \text{Atan}\left(\frac{a_1 b_2 \cos b_3 + a_1 a_2 \sin b_3 - a_2 b_1 \cos b_3 + b_1 b_2 \sin b_3}{a_1 a_2 \cos b_3 - a_1 b_2 \sin b_3 + b_1 b_2 \cos b_3 + a_2 b_1 \sin b_3}\right)$$

$$= \text{Atan}\left(\frac{(a_1 b_2 - a_2 b_1) \cos b_3 + (a_1 a_2 + b_1 b_2) \sin b_3}{(a_1 a_2 + b_1 b_2) \cos b_3 - (a_1 b_2 - a_2 b_1) \sin b_3}\right)$$

$$= \text{Atan}\left(\frac{\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} + \frac{\sin b_3}{\cos b_3}}{1 - \frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} \cdot \frac{\sin b_3}{\cos b_3}}\right)$$

$$= \text{Atan}\left(\frac{\sin b_3}{\cos b_3}\right) + \text{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right)$$

$$= |b_3|_{\mathbb{C}} + \text{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right)$$

Example 1 :

$$\ln((-i)^2) = \ln\left(\left(e^{-\frac{\pi i}{2}}\right)^2\right) = \ln\left(e^{-\pi i}\right) = -\pi i$$

$$2 \ln(-i) = 2 \ln\left(e^{-\frac{\pi i}{2}}\right) = 2 \cdot \left(-\frac{\pi i}{2}\right) = -\pi i$$

Since $-i \cdot -i = e^{-\frac{\pi}{2} i} e^{-\frac{\pi}{2} i} = e^{-\pi i} \neq e^{\pi i}$

Example 2 :

In the following calculation done in $\mathbb{C}$ the exponentiation isn't applied to the imaginary part, giving a wrong result. There is anyway no place in $\mathbb{C}$ to hold the exact result of $(-2)^5$

$$\log_{-2}((-2)^5) = \log_{-2}(-32) = \frac{\ln(-32)}{\ln(-2)} = \frac{\ln(32)+\pi i}{\ln(2)+\pi i} = \frac{(\ln(32)+\pi i)(\ln(2)-\pi i)}{(\ln(2)+\pi i)(\ln(2)-\pi i)}$$
$$= \frac{\ln(32)\ln(2)+(\ln(32)-\ln(2))\pi i+\pi^2}{(\ln(2))^2+\pi^2} = 1.18568\ldots + 0.84157\ldots i$$

The same calculation in $\mathbb{E}$

$$z_1 = -2 = e^{\ln 2}e^{\pi i}$$
$$z_2 = (-2)^5 = (e^{\ln 2}e^{\pi i})^5 = e^{5\ln 2}e^{5\pi i}$$
$$z = \log_{z_1} z_2$$
$$a = \frac{1}{2}\ln\left(\frac{a_2{}^2+b_2{}^2}{a_1{}^2+b_1{}^2}\right) = \frac{1}{2}\ln\left(\frac{(5\ln 2)^2+(5\pi)^2}{(\ln 2)^2+\pi^2}\right) = \frac{1}{2}\ln(5^2) = \ln 5$$
$$b = \text{Atan}\left(\frac{a_1 b_2-b_1 a_2}{a_1 a_2+b_1 b_2}\right) = \text{Atan}\left(\frac{5\pi\ln 2-5\pi\ln 2}{2(\ln 2)^2+5\pi^2}\right) = \text{Atan}(0) = 0$$
$$z = e^{\ln 5}e^{0i} = 5$$
$$z = 5\log_{-2}(-2) = 5$$

Example 3 :

This example to show the identity can fail at $\mathbb{E}$ precision level

$$\ln\left((e^{\pi i})^{e^{32\pi i}}\right) = \ln\left(e^{\pi i}\right) = \pi i = e^{\frac{1}{2}\ln(\pi^2)}e^{\frac{\pi}{2}i}$$
$$e^{32\pi i}\ln\left(e^{\pi i}\right) = e^{32\pi i}e^{\frac{1}{2}\ln(\pi^2)}e^{\frac{\pi}{2}i} = e^{\frac{1}{2}\ln(\pi^2)}e^{\frac{65\pi}{2}i}$$

Proof of $\log_{z_1}(z_2) = \log_{z_3}(z_2)/\log_{z_3}(z_1)$ is valid at $\mathbb{C}$ precision for all $z_1, z_2, z_3 \in \mathbb{E}^*$:

$$z = \log_{z_1}(z_2)$$
$$a = \frac{1}{2}\ln\left(\frac{a_2{}^2+b_2{}^2}{a_1{}^2+b_1{}^2}\right)$$
$$b = \text{Atan}\left(\frac{a_1 b_2-a_2 b_1}{a_1 a_2+b_1 b_2}\right)$$

Combining the logarithm and division formulas 3.4 and 3.2

$$z = \frac{\log_{z_3}(z_2)}{\log_{z_3}(z_1)}$$
$$a = \frac{1}{2}\ln\left(\frac{a_2{}^2+b_2{}^2}{a_3{}^2+b_3{}^2}\right) - \frac{1}{2}\ln\left(\frac{a_1{}^2+b_1{}^2}{a_3{}^2+b_3{}^2}\right)$$
$$= \frac{1}{2}(\ln(a_2{}^2+b_2{}^2) - \ln(a_3{}^2+b_3{}^2) - \ln(a_1{}^2+b_1{}^2) + \ln(a_3{}^2+b_3{}^2))$$
$$= \frac{1}{2}(\ln(a_2{}^2+b_2{}^2) - \ln(a_1{}^2+b_1{}^2))$$
$$= \frac{1}{2}\ln\left(\frac{a_2{}^2+b_2{}^2}{a_1{}^2+b_1{}^2}\right)$$
$$b = \text{Atan}\left(\frac{a_3 b_2-a_2 b_3}{a_3 a_2+b_3 b_2}\right) - \text{Atan}\left(\frac{a_3 b_1-a_1 b_3}{a_3 a_1+b_3 b_1}\right)$$
$$= \text{Atan}\left(\frac{\frac{a_3 b_2-a_2 b_3}{a_3 a_2+b_3 b_2} - \frac{a_3 b_1-a_1 b_3}{a_3 a_1+b_3 b_1}}{1 + \frac{a_3 b_2-a_2 b_3}{a_3 a_2+b_3 b_2} \cdot \frac{a_3 b_1-a_1 b_3}{a_3 a_1+b_3 b_1}}\right)$$
$$= \text{Atan}\left(\frac{(a_3 b_2-a_2 b_3)(a_3 a_1+b_3 b_1) - (a_3 b_1-a_1 b_3)(a_3 a_2+b_3 b_2)}{(a_3 a_1+b_3 b_1)(a_3 a_2+b_3 b_2) + (a_3 b_2-a_2 b_3)(a_3 b_1-b_3 a_1)}\right)$$

$$= \mathrm{Atan}\left(\frac{a_1 a_3{}^2 b_2 + a_3 b_1 b_2 b_3 - a_1 a_2 a_3 b_3 - a_2 b_1 b_3{}^2 - a_2 a_3{}^2 b_1 - a_3 b_1 b_2 b_3 + a_1 a_2 a_3 b_3 + a_1 b_2 b_3{}^2}{a_1 a_2 a_3{}^2 + a_1 a_3 b_2 b_3 + a_2 a_3 b_1 b_3 + b_1 b_2 b_3{}^2 + a_3{}^2 b_1 b_2 - a_1 a_3 b_2 b_3 - a_2 a_3 b_1 b_3 + a_1 a_2 b_3{}^2} \right)$$

$$= \mathrm{Atan}\left(\frac{(a_3{}^2 + b_3{}^2)(a_1 b_2 - a_2 b_1)}{(a_3{}^2 + b_3{}^2)(a_1 a_2 + b_1 b_2)} \right)$$

$$= \mathrm{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} \right)$$

Example of identity failure at $\mathbb{E}$ precision level :

$$\log_{\frac{1}{4}}(4) = -1 = e^{\pi i} \qquad \frac{\log_2(4)}{\log_2\left(\frac{1}{4}\right)} = \frac{2}{-2} = e^{-\pi i}$$

Proof of $\log_{z_1}(z_2 z_3) = \log_{z_1} z_2 + \log_{z_1} z_3$ is valid for all $z_1, z_2, z_3 \in \mathbb{E}^*$:

Combining the multiplication and logarithm formulas 3.1 and 3.4

$$z = \log_{z_1}(z_2 z_3)$$

$$a = \frac{1}{2}\ln\left(\frac{(a_2 + a_3)^2 + (b_2 + b_3)^2}{a_1{}^2 + b_1{}^2} \right)$$

$$b = \mathrm{Atan}\left(\frac{a_1(b_2 + b_3) - b_1(a_2 + a_3)}{a_1(a_2 + a_3) + b_1(b_2 + b_3)} \right)$$

For simplicity, the algebraic form is used in the following equation, since neither the logarithm nor the addition require the complete form for the result representation

$$z = \log_{z_1}(z_2) + \log_{z_1}(z_3)$$

$$= \frac{a_1 a_2 + b_1 b_2}{a_1{}^2 + b_1{}^2} + \frac{a_1 b_2 - a_2 b_1}{a_1{}^2 + b_1{}^2} i + \frac{a_1 a_3 + b_1 b_3}{a_1{}^2 + b_1{}^2} + \frac{a_1 b_3 - a_3 b_1}{a_1{}^2 + b_1{}^2} i$$

$$= \frac{a_1 a_2 + b_1 b_2 + a_1 a_3 + b_1 b_3}{a_1{}^2 + b_1{}^2} + \frac{a_1 b_2 - a_2 b_1 + a_1 b_3 - a_3 b_1}{a_1{}^2 + b_1{}^2} i$$

The result in algebraic form needs to be converted into complete form using conversion formula 2.2

$$a = \frac{1}{2}\ln\left(\frac{(a_1 a_2 + b_1 b_2 + a_1 a_3 + b_1 b_3)^2 + (a_1 b_2 - a_2 b_1 + a_1 b_3 - a_3 b_1)^2}{(a_1{}^2 + b_1{}^2)^2} \right)$$

$$= \frac{1}{2}\ln\left(\frac{(a_1(a_2 + a_3) + b_1(b_2 + b_3))^2 + (a_1(b_2 + b_3) - b_1(a_2 + a_3))^2}{(a_1{}^2 + b_1{}^2)^2} \right)$$

$$= \frac{1}{2}\ln\left(\frac{a_1{}^2(a_2 + a_3)^2 + b_1{}^2(b_2 + b_3)^2 + a_1{}^2(b_2 + b_3)^2 + b_1{}^2(a_2 + a_3)^2}{(a_1{}^2 + b_1{}^2)^2} \right)$$

$$= \frac{1}{2}\ln\left(\frac{(a_1{}^2 + b_1{}^2)((a_2 + a_3)^2 + (b_2 + b_3)^2)}{(a_1{}^2 + b_1{}^2)^2} \right)$$

$$= \frac{1}{2}\ln\left(\frac{(a_2 + a_3)^2 + (b_2 + b_3)^2)}{a_1{}^2 + b_1{}^2} \right)$$

$$b = \mathrm{Atan}\left(\frac{a_1 b_2 - a_2 b_1 + a_1 b_3 - a_3 b_1}{a_1 a_2 + b_1 b_2 + a_1 a_3 + b_1 b_3} \right)$$

$$= \mathrm{Atan}\left(\frac{a_1(b_2 + b_3) - b_1(a_2 + a_3)}{a_1(a_2 + a_3) + b_1(b_2 + b_3)} \right)$$

Example :

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$$\ln(-1 \cdot -1) = \ln\left(e^{\pi i} e^{\pi i}\right) = \ln\left(e^{2\pi i}\right) = 2\pi i$$
$$\ln(-1) + \ln(-1) = \pi i + \pi i = 2\pi i$$

Proof of $\log_{z_1}(z_2/z_3) = \log_{z_1} z_2 - \log_{z_1} z_3$ is valid for all $z_1, z_2, z_3 \in \mathbb{E}^*$:

Combining the division and logarithm formulas 3.2 and 3.4

$$z = \log_{z_1}\left(\frac{z_2}{z_3}\right)$$

$$a = \frac{1}{2}\ln\left(\frac{(a_2 - a_3)^2 + (b_2 - b_3)^2}{a_1{}^2 + b_1{}^2}\right)$$

$$b = \mathrm{Atan}\left(\frac{a_1(b_2 - b_3) - b_1(a_2 - a_3)}{a_1(a_2 - a_3) + b_1(b_2 - b_3)}\right)$$

$$z = \log_{z_1}(z_2) - \log_{z_1}(z_3)$$
$$= \frac{a_1 a_2 + b_1 b_2}{a_1{}^2 + b_1{}^2} + \frac{a_1 b_2 - a_2 b_1}{a_1{}^2 + b_1{}^2}i - \frac{a_1 a_3 + b_1 b_3}{a_1{}^2 + b_1{}^2} - \frac{a_1 b_3 - a_3 b_1}{a_1{}^2 + b_1{}^2}i$$
$$= \frac{a_1 a_2 + b_1 b_2 - a_1 a_3 - b_1 b_3}{a_1{}^2 + b_1{}^2} + \frac{a_1 b_2 - a_2 b_1 - a_1 b_3 + a_3 b_1}{a_1{}^2 + b_1{}^2}i$$

$$a = \frac{1}{2}\ln\left(\frac{(a_1 a_2 + b_1 b_2 - a_1 a_3 - b_1 b_3)^2 + (a_1 b_2 - a_2 b_1 - a_1 b_3 + a_3 b_1)^2}{(a_1{}^2 + b_1{}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{(a_1(a_2 - a_3) + b_1(b_2 - b_3))^2 + (a_1(b_2 - b_3) - b_1(a_2 - a_3))^2}{(a_1{}^2 + b_1{}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{a_1{}^2(a_2 - a_3)^2 + b_1{}^2(b_2 - b_3)^2 + a_1{}^2(b_2 - b_3)^2 + b_1{}^2(a_2 - a_3)^2}{(a_1{}^2 + b_1{}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{(a_1{}^2 + b_1{}^2)((a_2 - a_3)^2 + (b_2 - b_3)^2)}{(a_1{}^2 + b_1{}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{(a_2 - a_3)^2 + (b_2 - b_3)^2)}{a_1{}^2 + b_1{}^2}\right)$$

$$b = \mathrm{Atan}\left(\frac{a_1 b_2 - a_2 b_1 - a_1 b_3 + a_3 b_1}{a_1 a_2 + b_1 b_2 - a_1 a_3 - b_1 b_3}\right)$$
$$= \mathrm{Atan}\left(\frac{a_1(b_2 - b_3) - b_1(a_2 - a_3)}{a_1(a_2 - a_3) + b_1(b_2 - b_3)}\right)$$

Miscellaneous formulas in $\mathbb{E}$:

The formulas 3.1 to 3.6 can be combined to obtain formulas linking the real and imaginary exponents of any expression combining the fundamental operations.

$z_2 = z_1{}^{\mathsf{w}} \cdot \mathsf{w}^{\alpha}$ **where** $\mathsf{w}, z_1, z_2 \in \mathbb{E}^*, z_1 \neq \mathsf{e}^0\mathsf{e}^{0\mathsf{i}}, \alpha \in \mathbb{R}$:

Explicit formulas can be obtained linking the real and imaginary exponents a_w, b_w of w.

$$z_2 = z_1{}^w \, w^{\alpha} \tag{5.10}$$
$$a_2 = e^{a_w}(a_1 \cos b_w - b_1 \sin b_w) + a_w \alpha \tag{5.11}$$

$$b_2 = e^{a_w}(b_1 \cos b_w + a_1 \sin b_w) + b_w \alpha \tag{5.12}$$

$$
\begin{aligned}
(a_2 - a_w \alpha)^2 &= e^{2a_w}(a_1 \cos b_w - b_1 \sin b_w)^2 \\
(b_2 - b_w \alpha)^2 &= e^{2a_w}(b_1 \cos b_w + a_1 \sin b_w)^2 \\
(a_2 - a_w \alpha)^2 + (b_2 - b_w \alpha)^2 &= e^{2a_w}((a_1 \cos b_w - b_1 \sin b_w)^2 + (b_1 \cos b_w + a_1 \sin b_w)^2) \\
&= e^{2a_w}(a_1^2 \cos^2 b_w - 2\cos b_w \sin b_w + b_1^2 \sin^2 b_w + b_1^2 \cos^2 b_w + 2\cos b_w \sin b_w + a_1^2 \sin^2 b_w) \\
&= e^{2a_w}(a_1^2(\cos^2 b_w + \sin^2 b_w) + b_1^2(\cos^2 b_w + \sin^2 b_w)) \\
&= e^{2a_w}(a_1^2 + b_1^2) \\
\frac{b_2 - b_w \alpha}{a_2 - a_w \alpha} &= \frac{a_1 \cos b_w - b_1 \sin b_w}{b_1 \cos b_w + a_1 \sin b_w}
\end{aligned}
$$

$$b_w = \frac{b_2 - \sqrt{e^{2a_w}(a_1^2 + b_1^2) - (a_2 - a_w \alpha)^2}}{\alpha} \tag{5.13}$$

$$a_w = \frac{a_2 - \dfrac{a_1 \cos b_w - b_1 \sin b_w}{b_1 \cos b_w + a_1 \sin b_w}(b_2 - b_w \alpha)}{\alpha} \tag{5.14}$$

$$a_w = \ln\left(\frac{b_2 - b_w \alpha}{b_1 \cos b_w + a_1 \sin b_w}\right) \tag{5.15}$$

$\mathsf{z_2 = \log_{z_1}(w) \cdot w^\alpha}$ **where** $\mathsf{w, z_1, z_2 \in \mathbb{E}^*, w, z_1 \neq e^0 e^{0i}, \alpha \in \mathbb{R}}$ **:**

Explicit formulas can be obtained linking the real and imaginary exponents a_w, b_w of w.

$$z_2 = \log_{z_1}(w)\, w^\alpha \tag{5.16}$$

$$a_2 = \frac{1}{2} \ln\left(\frac{a_w^2 + b_w^2}{a_1^2 + b_1^2}\right) + a_w \alpha \tag{5.17}$$

$$b_2 = \mathrm{Atan}\left(\frac{a_1 b_w - a_w b_1}{a_1 a_w + b_1 b_w}\right) + b_w \alpha = \mathrm{Atan}\left(\frac{b_w}{a_w}\right) - \mathrm{Atan}\left(\frac{b_1}{a_1}\right) \tag{5.18}$$

$$b_w = \sqrt{e^{2(a_2 - a_w \alpha)}(a_1^2 + b_1^2) - a_w^2} \tag{5.19}$$

$$a_w = b_w \cot\left(b_2 - b_w \alpha + \mathrm{Atan}\left(\frac{b_1}{a_1}\right)\right) \tag{5.20}$$

6   Geometric representation of $\mathbb{E}$

Logarithmic representation

The complex numbers in complete form are identified in a unique way by their real and imaginary exponents. To position the exponent coordinates on a Argand-Gauss diagram is a possible representation. One can notice $(a; bi)$ and $(a + bi)$ are equivalent notations for the coordinates, both are derived from the complete form $e^a e^{bi}$ or e^{a+bi}. Expressions at the exponent level only require $\mathbb{C}$ precision, thus all operations as defined in $\mathbb{C}$ can be used in an exponent. Expressions such as $-1 \cdot i = -i$ or $(-2)^2 = 4$, which both implicitly perfom a $\mathbb{C}$ truncation, can be used, the loss of precision will be without consequence. The number 0 is used in expressions as a normal number.
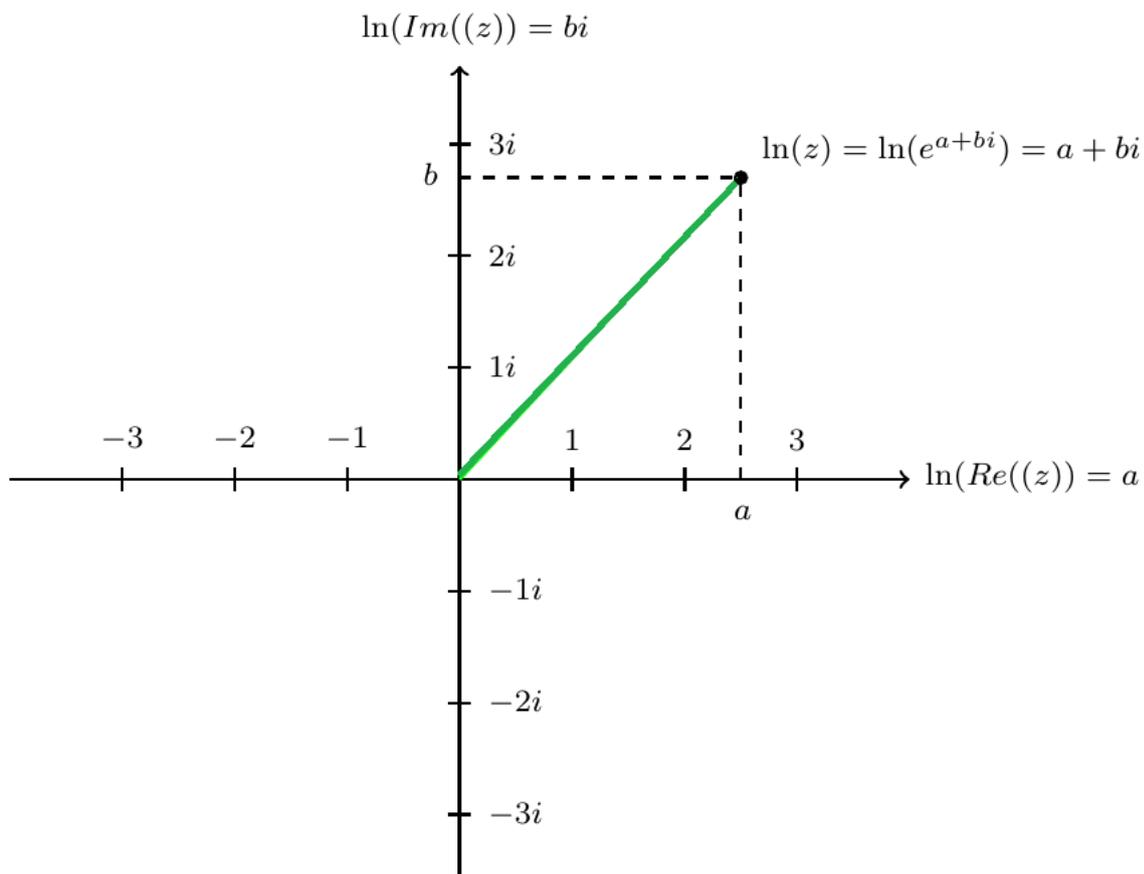
Figure 1: Logarithmic representation of $z = e^a e^{bi}$ on a Argand-Gauss diagram

The complex helicoid

The complex plane is clearly insufficient to represent $\mathbb{E}$ precise numbers, one can notice only $e^a e^{bi}$ with $b \in \,] - \pi; \pi]$ can be positioned in an unique way. The lack of "place" is solved by an additional axis, herafter named the i axis, on which the imaginary argument b can translate rectilinearly without any boundaries. The rotation of the imaginary argument b is maintained with a 2π period, giving a unique perpendicular half straight line for each b argument on which the real part e^a is positioned. Hereafter those half-lines are named "rays". Viewed in a three dimension euclidian space, with the origin situated at 0 on the i axis, every number $w = e^a e^{bi}$ can be given a unique orthogonal coordinate $(x, y, z) = (e^a \cos b, e^a \sin b, b)$. Thus the set $\mathbb{E}$ forms exactly an helicoid surface, herafter named the complex helicoid.
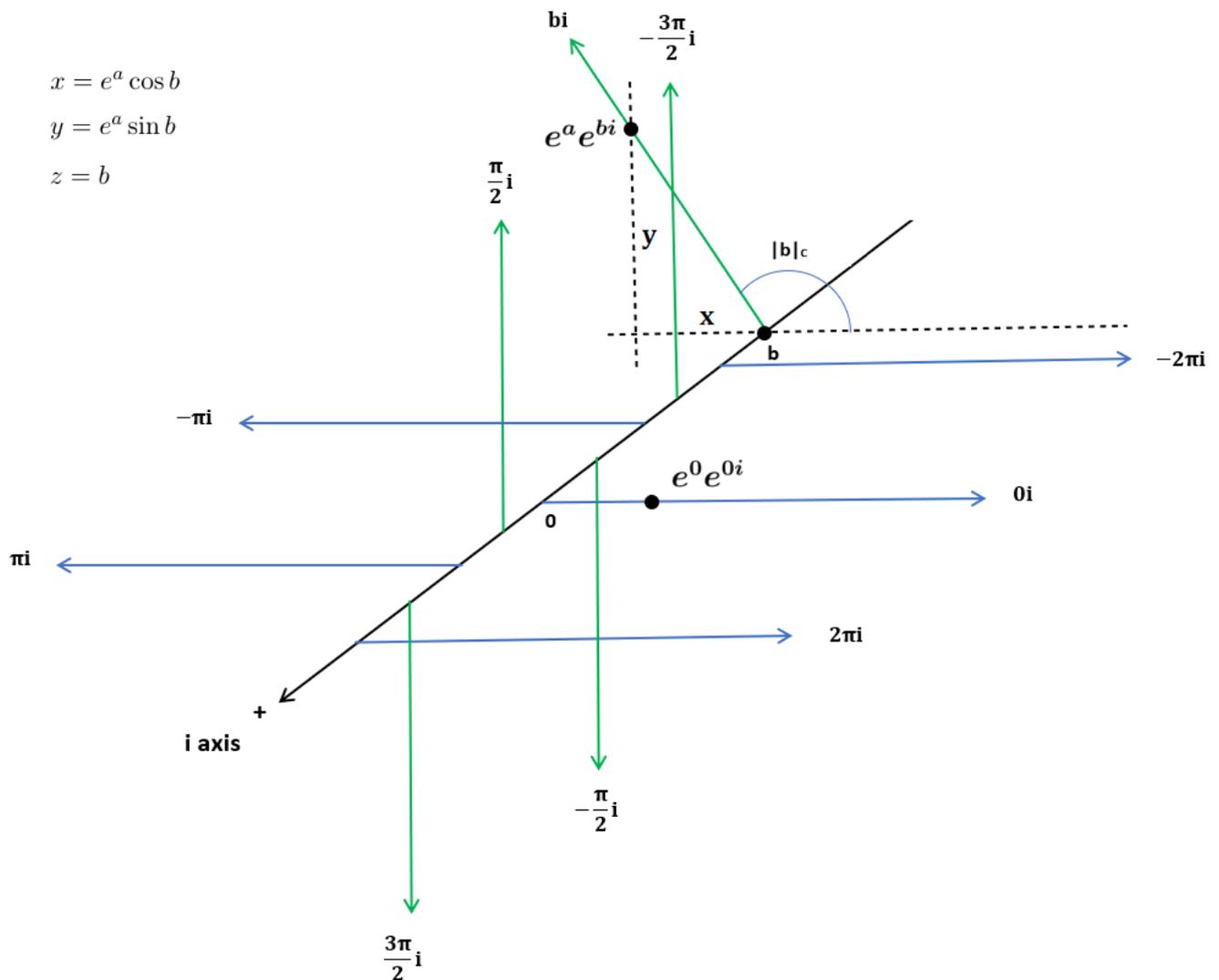

$$x = e^a \cos b$$
$$y = e^a \sin b$$
$$z = b$$

Figure 2: Representation of $e^a e^{bi}$ and $e^0 e^{0i}$ on the complex helicoid

The i axis is a singularity itself, on which only the value 0 can be positioned. The value 0 was included into the $\mathbb{E}$ set only for algebraic purpose. Zero and infinity are equally singularities since they don't have a unique position. The complex helicoid is far more than the Riemann surface of the logarithm function, it is the counterpart of the complex plane for $\mathbb{C}$ or the real axis for $\mathbb{R}$.

Constant functions representation on the complex helicoid

The constant function $w = e^a$ is the set of points situated at the position e^a of each ray. The function appears as an infinite helix surrounding the i axis. The multiplication and division operations such as $w = e^{a \pm a'}$ translate the position of the point on each ray, thus bring closer or further the helix to the i axis.

The constant function $w = e^{bi}$ is the set of points on a ray pointing in the direction given by b, excluding the 0 situated on the i axis. The multiplication and division operations such as $w = e^{(b \pm b')i}$ operate a rotation and a translation around and along the i axis.
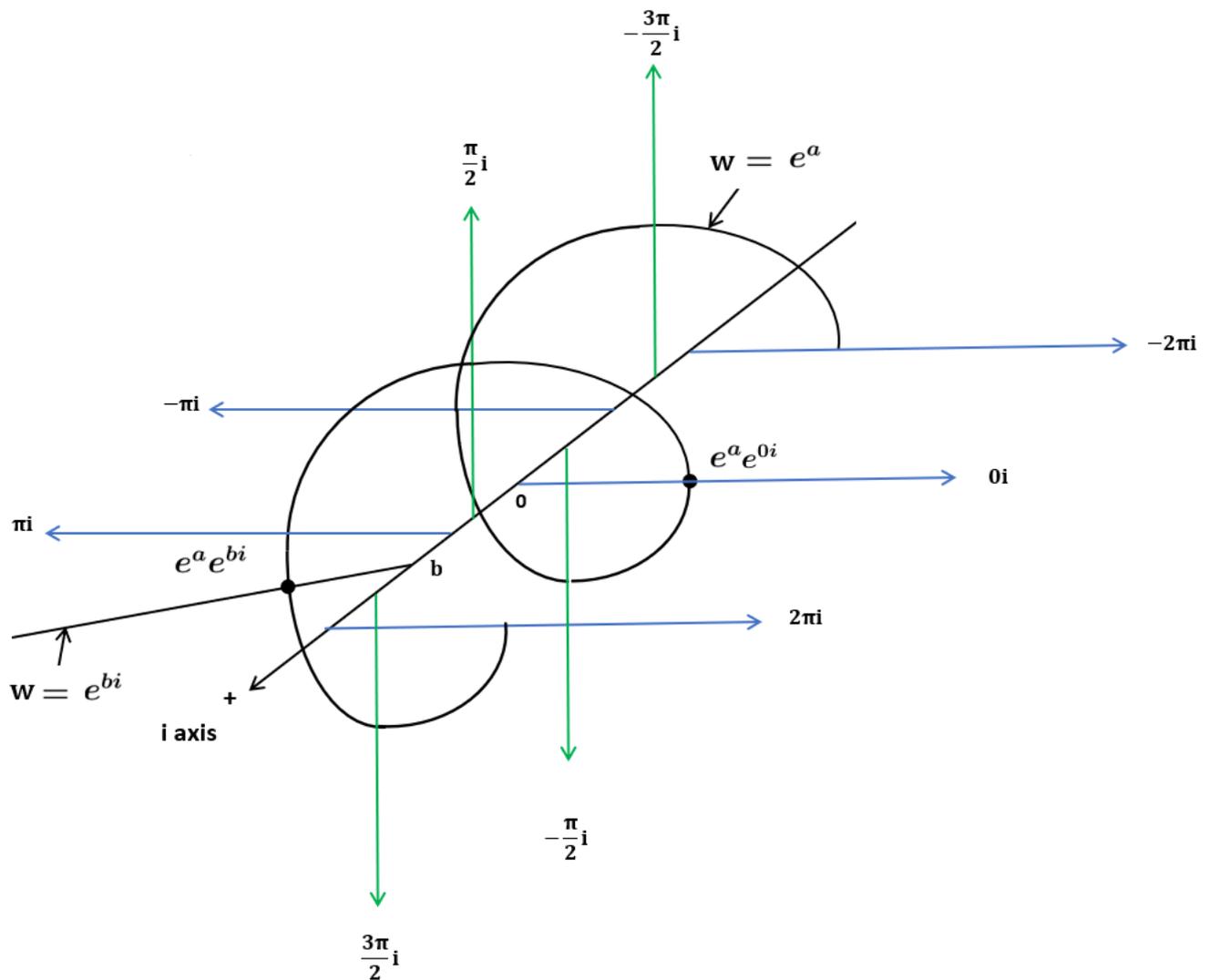

Figure 3: Constant functions $w = e^a$ and $w = e^{bi}$ representations on the complex helicoid

Complex helicoid projections on the plane

The orthogonal projection of the complex helicoid (x, y, z) to $(x, y, 0)$ represents the complex plane, through a totally new perspective. The projection corresponds exactly to a $\mathbb{C}$ truncation of $\mathbb{E}$ and can be noted as $P(w) = P(e^a e^{bi}) = P(e^a \cos b, e^a \sin b, b) = (e^a \cos b, e^a \sin b, 0)$ or as a truncation $|w|_{\mathbb{C}} = |e^a e^{bi}|_{\mathbb{C}} = e^a \cos b + e^a \sin b \, i$. The Re- axis cannot be passed without generating a singularity in $\mathbb{E}$. The singularity 0 is given the appearance of a normal point. The exponentials and logarithms identity failures in $\mathbb{C}$ represented on the complex plane are all due to a "careless" crossing of the Re-axis generating a $\mathbb{C}$ truncation. The projection shouldn't be confused with the logarithmic representation of $\mathbb{E}$, though both representations are graphicaly identical.
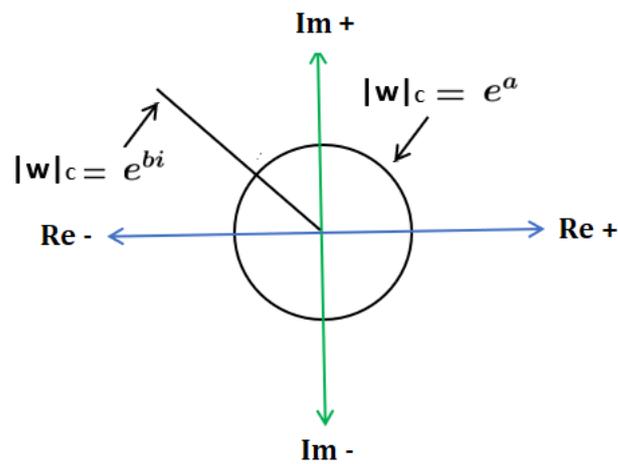

Figure 4: Projection of the complex helicoid (x, y, z) to $(x, y, 0)$

The orthogonal projection of the complex helicoid (x, y, z) to $(x, 0, z)$, transforms the constant helix into a cosine curve. The projection can be noted as $P(w) = P(e^a e^{bi}) = P(e^a \cos b, e^a \sin b, b) = (e^a \cos b, 0, b)$
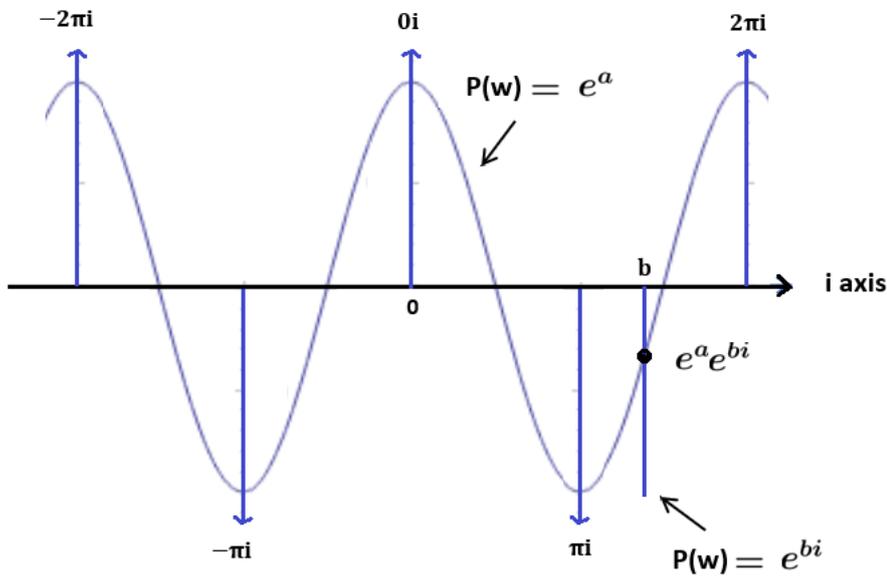

Figure 5: Projection of the complex helicoid (x, y, z) to $(x, 0, z)$

Similarly the orthogonal projection of the complex helicoid (x, y, z) to $(0, y, z)$, transforms the constant helix into a sine curve. The projection can be noted as $P(w) = P(e^a e^{bi}) = P(e^a \cos b, e^a \sin b, b) = (0, e^a \sin b, b)$
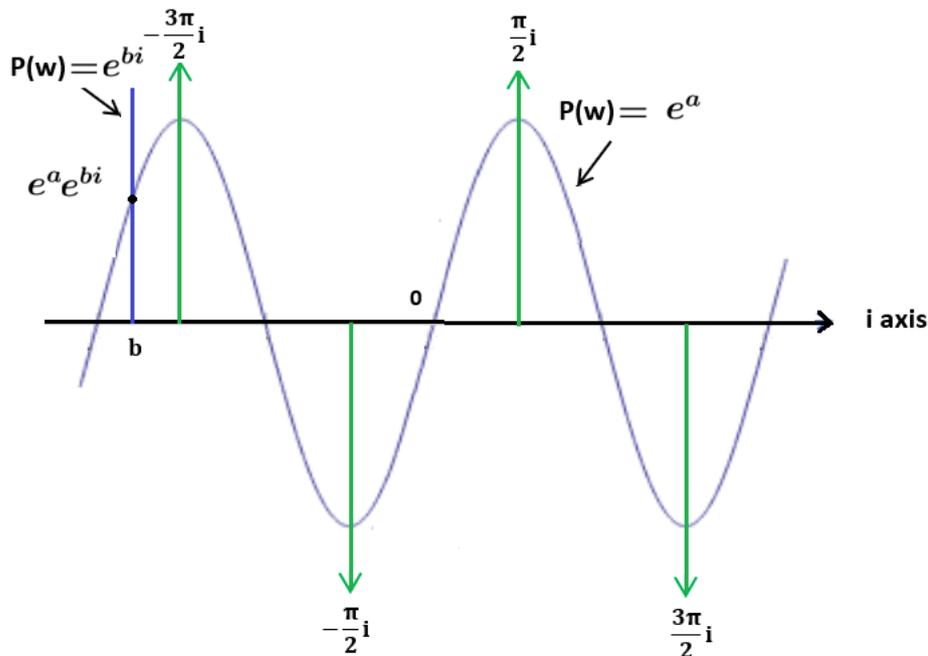

Figure 6: Projection of the complex helicoid (x, y, z) to $(0, y, z)$

Representation of the complex logarithm operation

Let $z_1, z_2 \in \mathbb{E}^*$ with $z_1, z_2 \neq e^0 e^{0i}$ be 2 points on the complex helicoid. The representation of the point $z = \log_{z_1}(z_2)$ finally reveals, under a new perspective, a similar formula as the division on the complex plane.

$$c_1 = \sqrt{a_1{}^2 + b_1{}^2} \quad c_2 = \sqrt{a_2{}^2 + b_2{}^2} \quad \alpha_1 = \mathrm{Atan}\,\frac{b_1}{a_1} \quad \alpha_2 = \mathrm{Atan}\,\frac{b_2}{a_2}$$

$$\boxed{z = \log_{z_1}(z_2) = e^{\frac{1}{2}\ln\left(\frac{a_2{}^2+b_2{}^2}{a_1{}^2+b_1{}^2}\right)} e^{\mathrm{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right)i} = \frac{\sqrt{a_2{}^2 + b_2{}^2}}{\sqrt{a_1{}^2 + b_1{}^2}}\, e^{\left(\mathrm{Atan}\,\frac{b_2}{a_2} - \mathrm{Atan}\,\frac{b_1}{a_1}\right)i} = \frac{c_2}{c_1}\, e^{(\alpha_2 - \alpha_1)i}}$$
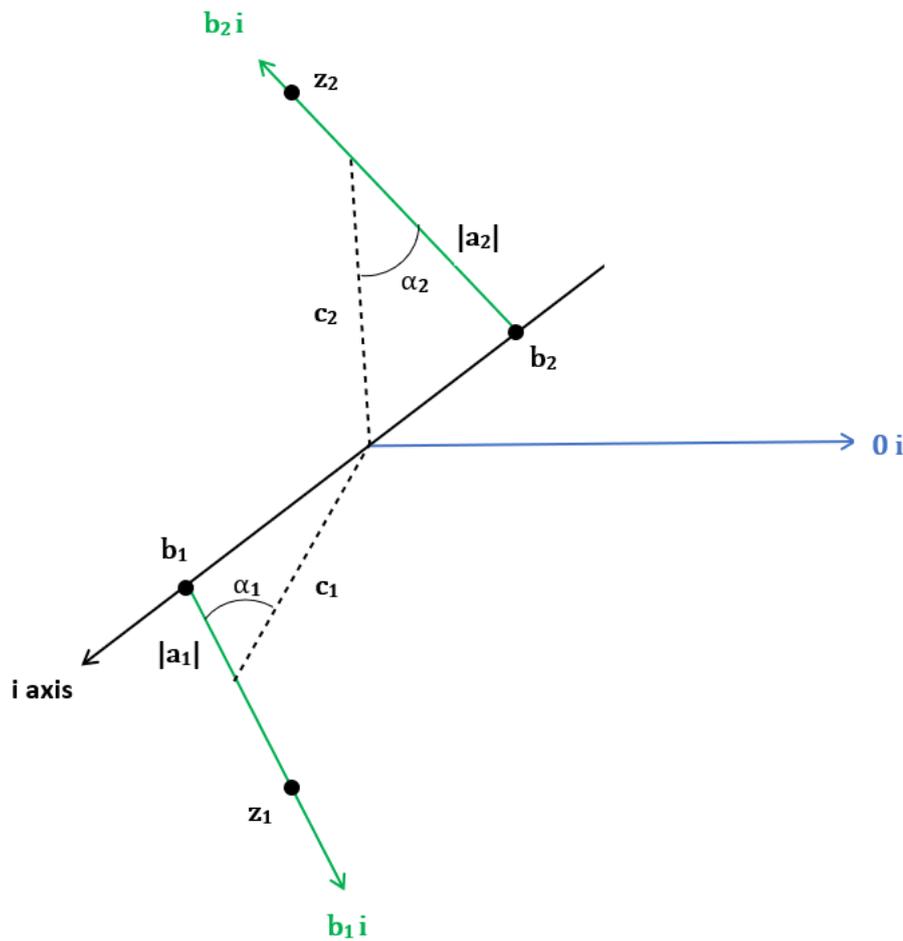

Figure 7: Logarithm operation representation on the complex helicoid

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Representation of the complex exponentiation operation

Let $z_1, z_2 \in \mathbb{E}^*$ with $z_1 \neq e^0 e^{0i}$ be 2 points on the complex helicoid. The representation of the point $z = z_1{}^{z_2}$ is best visualised by 2 formulas. The exponent z_2 only being used at $\mathbb{C}$ precision, quite obviously the full b_2 distance on the i axis isn't used in the formulas.

$$x_2 = e^{a_2} \cos b_2 \quad y_2 = e^{a_2} \sin b_2 \quad c_1 = \sqrt{a_1{}^2 + b_1{}^2} \quad \alpha_1 = \mathrm{Atan}\, \frac{b_1}{a_1} \quad \beta_1 = \mathrm{Atan}\, \frac{a_1}{b_1}$$

$$\boxed{z = z_1{}^{z_2} = e^{e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2)} e^{e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2)i} = e^{a_1 x_2 - b_1 y_2} e^{(b_1 x_2 + a_1 y_2)i}}$$

$$\boxed{z = z_1{}^{z_2} = e^{e^{a_2}\sqrt{a_1{}^2+b_1{}^2}\,\cos(b_2 + \mathrm{Atan}\,\frac{b_1}{a_1})} e^{e^{a_2}\sqrt{a_1{}^2+b_1{}^2}\,\cos(b_2 - \mathrm{Atan}\,\frac{a_1}{b_1})i} = e^{e^{a_2}c_1 \cos(b_2 + \alpha_1)} e^{e^{a_2}c_1 \cos(b_2 - \beta_1)i}}$$
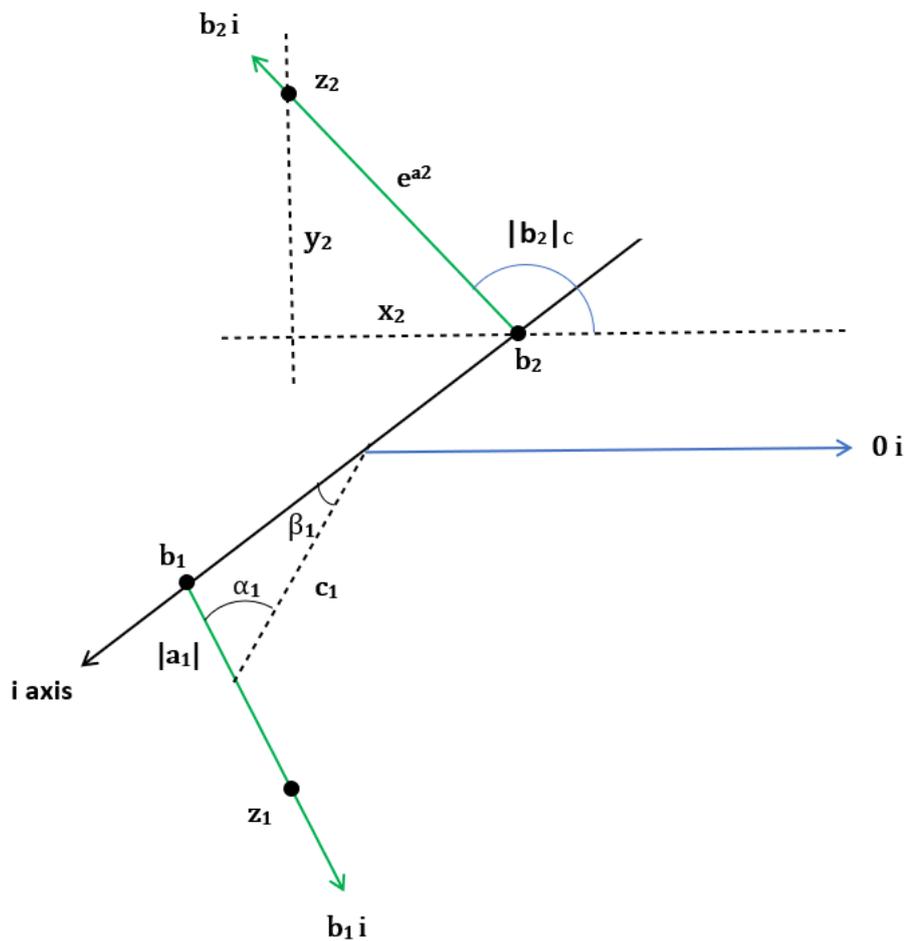

Figure 8: Exponentiation operation representation on the complex helicoid

Representation of the addition and substraction operations

The addition and substraction don't require any $\mathbb{E}$ precision, representing them on the complex helicoid is basically useless, a projection on the complex plane is sufficient. Let $z_1, z_2 \in \mathbb{E}^*$ by 2 points on the complex helicoid, with their corresponding projections $|z_1|_\mathbb{C} = x_1 + y_1 i$ and $|z_2|_\mathbb{C} = x_2 + y_2 i$ on the complex plane.

$$x_1 = e^{a_1} \cos b_1 \quad y_1 = e^{a_1} \sin b_1 \quad x_2 = e^{a_2} \cos b_2 \quad y_2 = e^{a_2} \sin b_2$$

$$z = z_1 \pm z_2 = e^{\frac{1}{2} \ln\left(e^{2a_1} + e^{2a_2} \pm 2e^{a_1+a_2} \cos(b_1-b_2)\right)} e^{\text{Atan}\left(\frac{e^{a_1} \sin b_1 \pm e^{a_2} \sin b_2}{e^{a_1} \cos b_1 \pm e^{a_2} \cos b_2}\right)i}$$

$$c = \sqrt{e^{2a_1} + e^{2a_2} - 2e^{a_1+a_2} \cos(b_1 - b_2)} \qquad d = \sqrt{e^{2a_1} + e^{2a_2} + 2e^{a_1+a_2} \cos(b_1 - b_2)}$$

$$z = z_1 - z_2 = c \cdot e^{\text{Atan}\left(\frac{y_1-y_2}{x_1-x_2}\right)i} \qquad\qquad\qquad z = z_1 + z_2 = d \cdot e^{\text{Atan}\left(\frac{y_1+y_2}{x_1+x_2}\right)i}$$
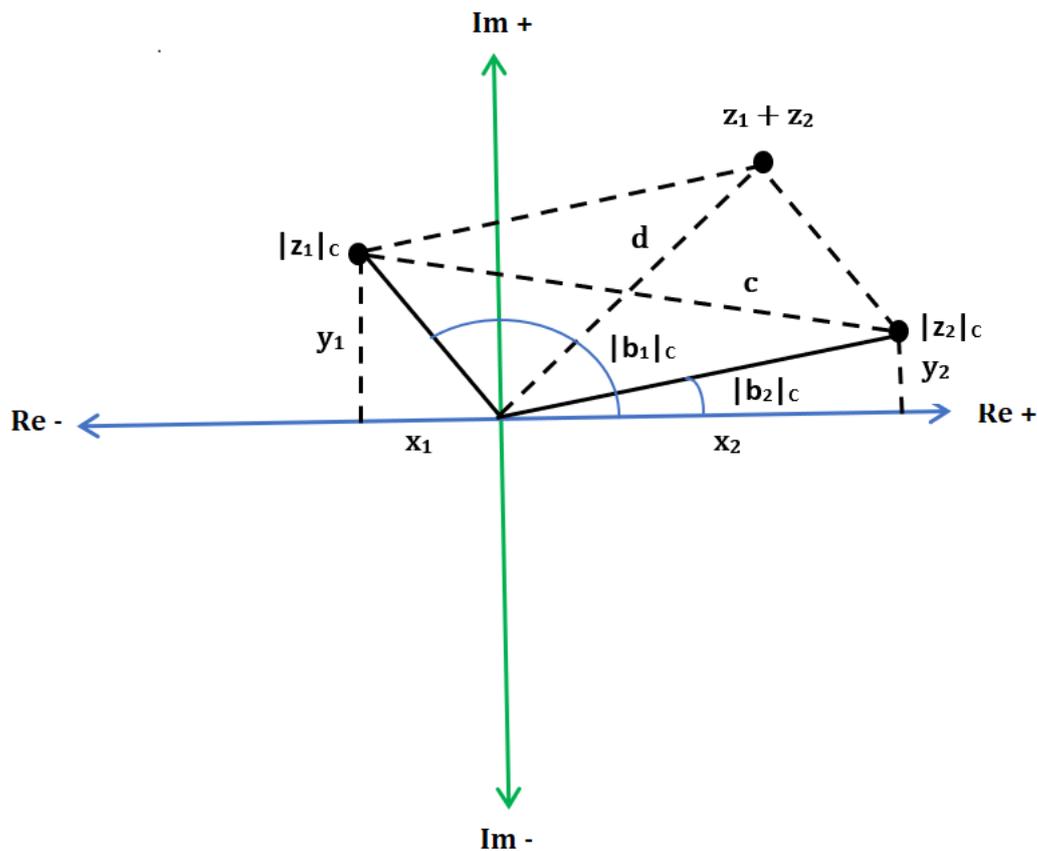

Figure 9: Addition and substraction representation on the complex plane

7   Algebraic properties of the $\mathbb{E}$ number set

From the properties of the real operations and functions composing a complex operation, it is possible to deduce the algebraic properties of $\mathbb{E}$. The properties of the value 0 are by convention inherited from $\mathbb{C}$ since the formulas 3.1 to 3.6 don't apply to that value.

Commutativity

The addition and the multiplication are the only commutative operations.

$$z_1 \times z_2 = z_2 \times z_1 \tag{7.1}$$
$$z_1 + z_2 = z_2 + z_1 \tag{7.2}$$

Associativity

The addition and the multiplication are the only associative operations.

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \tag{7.3}$$
$$(z_1 \times z_2) \times z_3 = z_1 \times (z_2 \times z_3) \tag{7.4}$$

Distributivity

The multiplication is distributive at $\mathbb{C}$ precision level over the addition and substraction, the division is right distributive at $\mathbb{C}$ precision level over the addition and substraction.

$$\mid z_1 \times (z_2 + z_3) \mid_{\mathbb{C}} = z_1 \times z_2 + z_1 \times z_3 \tag{7.5}$$
$$\mid z_1 \times (z_2 - z_3) \mid_{\mathbb{C}} = z_1 \times z_2 - z_1 \times z_3 \tag{7.6}$$
$$\left| \frac{z_2 + z_3}{z_1} \right|_{\mathbb{C}} = \frac{z_2}{z_1} + \frac{z_3}{z_1} \tag{7.7}$$
$$\left| \frac{z_2 - z_3}{z_1} \right|_{\mathbb{C}} = \frac{z_2}{z_1} - \frac{z_3}{z_1} \tag{7.8}$$

Identity element

The identity element of addition and multiplication :

$$z_1 \times e^0 e^{0i} = z_1 \tag{7.9}$$
$$z_1 + 0 = z_1 \tag{7.10}$$

The right identity element of division and substraction, exponentiation having an infinite set of right identities :

$$z_1 \ / \ e^0 e^{0i} = z_1 \tag{7.11}$$
$$z_1 - 0 = z_1 \tag{7.12}$$
$$z_1{}^{(e^0 e^{2k\pi i})} = z_1 \ \ (\text{with } k \in \mathbb{Z}) \tag{7.13}$$

Inverse

Multiplication, division and exponentiation are the exact reciprocal of their inverse operation :

$$\frac{z_1 \times z_2}{z_1} = z_2 \tag{7.14}$$

$$\frac{z_2}{z_1} \times z_1 = z_2 \tag{7.15}$$

$$z_1{}^{\log_{z1}(z_2)} = z_2 \tag{7.16}$$

Logarithm, addition and substraction are only the $\mathbb{C}$ precise reciprocal of their inverse operation:

$$\log_{z_1}(z_1{}^{z_2}) = |z_2|_{\mathbb{C}} \tag{7.17}$$

$$z_2 + z_1 - z_1 = |z_2|_{\mathbb{C}} \tag{7.18}$$

$$z_2 - z_1 + z_1 = |z_2|_{\mathbb{C}} \tag{7.19}$$

Symmetry

$$e^a e^{bi} \cdot e^{-a} e^{-bi} = e^0 e^{0i} \tag{7.20}$$

$$\frac{e^a e^{bi}}{e^a e^{bi}} = e^0 e^{0i} \tag{7.21}$$

$$e^a e^{bi} + e^a e^{bi+(2k+1)\pi i} = 0 \;\; (\text{with } k \in \mathbb{Z}) \tag{7.22}$$

$$e^a e^{bi} - e^a e^{bi+2k\pi i} = 0 \tag{7.23}$$

Singularities

At first let's consider the singularities of operations where both operands are in $\mathbb{E}\backslash\{0\}$.

From the logarithm formula 3.4, one can easily deduce logarithms have a singularity when $z_1 = e^0 e^{0i}$ and/or $z_2 = e^0 e^{0i}$ caused by the division by 0, the ln with argument 0 and the Atan with 0/0 argument. Interestingly the singularities vanish if the operands are in the form $e^0 e^{2k\pi i}$ with $k \neq 0$.

$$\log_{(e^0 e^{0i})}(z_2) = \infty \tag{7.24}$$

$$\log_{z_1}(e^0 e^{0i}) = 0 \tag{7.25}$$

$$\log_{(e^0 e^{0i})}(e^0 e^{0i}) = \text{undefined} \tag{7.26}$$

From the formulas 3.5 and 3.6, it's possible to deduce both addition and substraction have singularities caused by the ln with argument 0 and the arctan with 0/0 argument.

$$e^a e^{bi} + e^a e^{bi+(2k+1)\pi i} = 0 \;\; (with \; k \in \mathbb{Z}) \tag{7.27}$$

$$e^a e^{bi} - e^a e^{bi+2k\pi i} = 0 \tag{7.28}$$

The introduction of the element 0 allows to reduce some of the above singularities, but also adds new ones.

$$z_1 \cdot 0 = 0 \tag{7.29}$$

$$z_1 \,/\, 0 = \infty \tag{7.30}$$

$$0 \,/\, z_2 = 0 \tag{7.31}$$

$$0 \,/\, 0 = \text{undefined} \tag{7.32}$$

$$z_1 + 0 = z_1 \tag{7.33}$$

$$z_1 - 0 = z_1 \tag{7.34}$$

$$0 - z_2 = \left|z_2 \cdot e^{\pi i}\right|_{\mathbb{C}} \tag{7.35}$$

$$0 - 0 = 0 \tag{7.36}$$

$$z_1{}^0 = e^0 e^{0i} \tag{7.37}$$

$$0^{z_2} = 0 \tag{7.38}$$

$$0^0 = e^0 e^{0i} \tag{7.39}$$

$$\log_{z_1}(0) = \infty \tag{7.40}$$

$$\log_0(z_2) = 0 \tag{7.41}$$

$$\log_0(0) = \text{undefined} \tag{7.42}$$

In order to reduce the singularities, it is possible to include the infinite element such as $\mathbb{E}' = \mathbb{E} \cup \{\infty\}$, and define the results of operations using ∞. However it would lead to new singularities such as $\infty - \infty$, whatever definition of $\mathbb{E}'$ there will remain singularities that can ony be treated analytically.

Algebraic structure of $\mathbb{E}$

Conclusions can be made from formulas 3.1 to 3.6 and from the properties listed above :

- $\mathbb{E}$ together with the 6 fundamental operations has a closed algebraic structure, except for the singularities all results can be represented

- The mutiplication and division maintain all their intrinsic properties such as in $\mathbb{C}$

- The addition and substraction maintain all their intrinsic properties but only at $\mathbb{C}$ precision, since both operations don't require nor can provide any $\mathbb{E}$ precision

- The distributivity property generally only holds when both sides of the equation are converted to $\mathbb{C}$ precision, thus distributivity is only $\mathbb{C}$ precise

- The multiplication is clearly the defining operation and possess all the properties to constitute a multiplicative group $(\mathbb{E}^*, \cdot)$

- The field axioms aren't all verified, since the addition/substraction reciprocity and the distributivity don't hold exactly in $\mathbb{E}$

It would be a mistake to limit $\mathbb{E}$ to a multiplicative group, as many properties on the exponentiation and logarithm operations are added. Furthermore all properties hold to a certain extent, only limited by the operations maximum precision level.

Properties comparison between $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{E}$

Property	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{E}$
addition and substraction closure	yes	yes	yes
multiplication and division closure	yes	yes	yes
addition and substraction monovaluation	yes	yes	yes
multiplication and division monovaluation	yes	yes	yes
addition and multiplication commutativity	yes	yes	yes
addition and multiplication associativity	yes	yes	yes
multiplication distributivity over add/sub	yes	yes	only $\mathbb{C}$ precise
division right distributivity over add/sub	yes	yes	only $\mathbb{C}$ precise
identity element of add/sub	0	0	0
identity element of mult/div	1	1	$e^0 e^{0i} = 1$
addition/substraction inverse	yes	yes	only $\mathbb{C}$ precise
multiplication/division inverse	yes	yes	yes

Property	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{E}$
exponentiation closure $z_1^{z_2}$	only if $z_1 \in \mathbb{R}_+$	yes	yes
logarithm closure $\log_{z_1}(z_2)$	only if $z_1, z_2 \in \mathbb{R}_+$	yes	yes
exponentiation monovaluation	no	no	yes
logarithm monovaluation	yes	no	yes
exponentiation inverse of logarithm $z_1^{\log_{z_1}(z_2)} = z_2$	only if $z_1, z_2 \in \mathbb{R}_+$	only subset	yes
logarithm inverse of exponentiation $\log_{z_1}(z_1^{z_2}) = z_2$	only if $z_1 \in \mathbb{R}_+$	only subset	only $\mathbb{C}$ precise
exponential distributivity over mult/div $(z_1 z_2)^{z_3} = z_1^{z_3} z_2^{z_3} \quad \left(\frac{z_1}{z_2}\right)^{z_3} = \frac{z_1^{z_3}}{z_2^{z_3}}$	only if $z_1, z_2 \in \mathbb{R}_+$	no	yes
exponential product and quotient identity $z_1^{z_2} z_1^{z_3} = z_1^{z_2+z_3} \quad \frac{z_1^{z_2}}{z_1^{z_3}} = z_1^{z_2-z_3}$	only if $z_1 \in \mathbb{R}_+$	yes	yes
exponential power identity $(z_1^{z_2})^{z_3} = z_1^{z_2 z_3}$	only if $z_1 \in \mathbb{R}_+$	no	yes
logarithm product and quotient identity $\log_{z_1}(z_2 z_3) = \log_{z_1}(z_2) + \log_{z_1}(z_3)$ $\log_{z_1}\left(\frac{z_2}{z_3}\right) = \log_{z_1}(z_2) - \log_{z_1}(z_3)$	only if $z_1, z_2, z_3 \in \mathbb{R}_+$	no	yes
logarithm power identity $\log_{z_1}(z_2^{z_3}) = z_3 \log_{z_1}(z_2)$	only if $z_1, z_2 \in \mathbb{R}_+$	no	only $\mathbb{C}$ precise
logarithm base substitution $\log_{z_1}(z_2) = \frac{\log_{z_3}(z_1)}{\log_{z_3}(z_2)}$	only if $z_1, z_2, z_3 \in \mathbb{R}_+$	yes	only $\mathbb{C}$ precise

Proof of distributivity of multiplication over addition is $\mathbb{C}$ precise :

Combining the multiplication and addition formulas 3.1 and 3.5, let $z_1 = e^{a_1} e^{b_1 i}$, $z_2 = e^{a_2} e^{b_2 i}$ and $z_3 = e^{a_3} e^{b_3 i}$

$z = z_1 \cdot (z_2 + z_3)$

$a = a_1 + \frac{1}{2} \ln(e^{2a_2} + e^{2a_3} + 2e^{a_2+a_3} \cos(b_2 - b_3))$

$b = b_1 + \text{Atan}\left(\frac{e^{a_2} \sin b_2 + e^{a_3} \sin b_3}{e^{a_2} \cos b_2 + e^{a_3} \cos b_3}\right)$

$z = z_1 \cdot z_2 + z_1 \cdot z_3$

$a = \frac{1}{2} \ln(e^{2(a_1+a_2)} + e^{2(a_1+a_3)} + 2e^{(a_1+a_2)+(a_1+a_3)} \cos((b_1+b_2) - (b_1+b_3)))$

$\quad = \frac{1}{2} \ln(e^{2a_1}(e^{2a_2} + e^{2a_3} + 2e^{a_2+a_3} \cos(b_2 - b_3)))$

$\quad = a_1 + \frac{1}{2} \ln(e^{2a_2} + e^{2a_3} + 2e^{a_2+a_3} \cos(b_2 - b_3))$

$b = \text{Atan}\left(\frac{e^{a_1+a_2} \sin(b_1+b_2) + e^{a_1+a_3} \sin(b_1+b_3)}{e^{a_1+a_2} \cos(b_1+b_2) + e^{a_1+a_3} \cos(b_1+b_3)}\right)$

$\quad = \text{Atan}\left(\frac{e^{a_2}(\sin b_1 \cos b_2 + \cos b_1 \sin b_2) + e^{a_3}(\sin b_1 \cos b_3 + \cos b_1 \sin b_3)}{e^{a_2}(\cos b_1 \cos b_2 - \sin b_1 \sin b_2) + e^{a_3}(\cos b_1 \cos b_3 - \sin b_1 \sin b_3)}\right)$

$\quad = \text{Atan}\left(\frac{\cos b_1(e^{a_2} \sin b_2 + e^{a_3} \sin b_3) + \sin b_1(e^{a_2} \cos b_2 + e^{a_3} \cos b_3)}{\cos b_1(e^{a_2} \cos b_2 + e^{a_3} \cos b_3) - \sin b_1(e^{a_2} \sin b_2 + e^{a_3} \sin b_3)}\right)$

$$= \text{Atan}\left(\frac{\frac{e^{a_2}\sin b_2 + e^{a_3}\sin b_3}{e^{a_2}\cos b_2 + e^{a_3}\cos b_3} + \frac{\sin b_1}{\cos b_1}}{1 - \frac{e^{a_2}\sin b_2 + e^{a_3}\sin b_3}{e^{a_2}\cos b_2 + e^{a_3}\cos b_3} \cdot \frac{\sin b_1}{\cos b_1}} \right)$$

$$= \text{Atan}\left(\frac{\sin b_1}{\cos b_1} \right) + \text{Atan}\left(\frac{e^{a_2}\sin b_2 + e^{a_3}\sin b_3}{e^{a_2}\cos b_2 + e^{a_3}\cos b_3} \right)$$

$$= |b_1|_{\mathbb{C}} + \text{Atan}\left(\frac{e^{a_2}\sin b_2 + e^{a_3}\sin b_3}{e^{a_2}\cos b_2 + e^{a_3}\cos b_3} \right)$$

Example of distributivity failure at $\mathbb{E}$ precision level :

$$-2 \cdot (i+1) = e^{\ln 2}e^{\pi i} \cdot (e^{\frac{\pi}{2}i} + e^{0i}) = e^{\ln 2}e^{\pi i} \cdot e^{\frac{1}{2}\ln 2}e^{\frac{\pi}{4}i} = e^{\frac{3}{2}\ln 2}e^{\frac{5\pi}{4}i}$$

$$(-2 \cdot i) + (-2 \cdot 1) = e^{\ln 2}e^{\pi i} \cdot e^{\frac{\pi}{2}i} + e^{\ln 2}e^{\pi i} \cdot e^{0i} = e^{\ln 2}e^{\frac{3\pi}{2}i} + e^{\ln 2}e^{\pi i} = e^{\frac{3}{2}\ln 2}e^{-\frac{3\pi}{4}i}$$

In the first line the $\mathbb{E}$ precision is preserved because the final operation is a multiplication, in the second line the addition operates a $\mathbb{C}$ truncation, hence the results can be different.

Proof exponentiation is the exact inverse of logarithm :

Using the logarithm formula 3.4 converted into algebraic form, let $z_1 = e^{a_1}e^{b_1 i}$, $z_2 = e^{a_2}e^{b_2 i}$ and $\log_{z_1}(z_2) = \frac{a_1 a_2 + b_1 b_2}{a_1^2 + b_1^2} + \frac{a_1 b_2 - a_2 b_1}{a_1^2 + b_1^2}i$

$$z = z_1^{\log_{z_1}(z_2)} = z_2$$

$$a = \frac{a_1(a_1 a_2 + b_1 b_2) - b_1(a_1 b_2 - b_1 a_2)}{a_1^2 + b_1^2}$$

$$= \frac{a_1^2 a_2 + a_1 b_1 b_2 - b_1 a_1 b_2 + b_1^2 a_2}{a_1^2 + b_1^2}$$

$$= \frac{a_2(a_1^2 + b_1^2)}{a_1^2 + b_1^2} = a_2$$

$$b = \frac{b_1(a_1 a_2 + b_1 b_2) + a_1(a_1 b_2 - b_1 a_2)}{a_1^2 + b_1^2}$$

$$= \frac{b_1 a_1 a_2 + b_1^2 b_2 + a_1^2 b_2 - a_1 b_1 a_2}{a_1^2 + b_1^2}$$

$$= \frac{b_2(a_1^2 + b_1^2)}{a_1^2 + b_1^2} = b_2$$

8   Conclusion

The $\mathbb{E}$ set of complex numbers in complete form can be viewed as a "natural" extension of $\mathbb{C}$. Within the sequence $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{E}$ each set extends the capacity of the predecessor set by providing new elements, thus new symbolic representations of numbers. Each element in a given set is uniquely linked to a predecessor set element through an equivalence relation, therefore an element can always be truncated to the predecessor set precision level. Similarly the geometric representations are extended while preserving the predecessor sets representations. One can notice irrational, imaginary and complex numbers were all established to somehow achieve the closure of the exponentiation operation.

As demonstrated in this article, the complex exponentiation base and result cannot be represented precisely in algebraic form, in contrary to the exponent. For the complex logarithm only the result can be represented precisely in algebraic form. The establishment of the complete form of complex numbers is an attempt to solve all those "precision" issues, thus solving the identity failures and

multivalued results as in $\mathbb{C}$. We believe both complex exponentiation and logarithm are by far better defined and handled using the complete form.

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Author :

Daniel Tischhauser, Geneva, Switzerland
Email address: dtischhauser.math@gmail.com