Exponentials and Logarithms Properties in an Extended Complex Number Field

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Abstract

It is well established the complex exponential and logarithm are multivalued functions, both failing to maintain most identities originally valid over the positive integers domain. Moreover the general case of complex logarithm, with a complex base, is hardly mentionned in mathematic litterature. We study the exponentiation and logarithm as binary operations where all operands are complex. In a redefined complex number system using an extension of the $\mathbb{C}$ field, hereafter named $\mathbb{E}$, we prove both operations always produce single value results and maintain the validity of identities such as $\log_u(wv) = \log_u(w) + \log_u(v)$ where $u, v, w \in \mathbb{E}$. There is a cost as some algebraic properties of the addition and subtraction will be diminished, though remaining valid to a certain extent. In order to handle formulas in a $\mathbb{C}$ and $\mathbb{E}$ dual number system, we introduce the notion of set precision and set truncation. We show the complex numbers as defined in $\mathbb{C}$ are insufficiently precise to grasp all subtleties of some complex operations, resulting in multivaluation, identity failures and, in specific cases, wrong results. A geometric representation of the new complex number system is proposed, in which the complex plane appears as an orthogonal projection, and where the representation of the exponentiation and logarithm turns out to be trivial. Finally we attempt an algebraic formalization of $\mathbb{E}$.

Keywords: Complex number field; Complex exponentiation; Complex logarithm; Exponential and logarithm identities

1 Introduction

In 1749 L. Euler [1] solved a decades old controversy between G.W. Leibniz and J. Bernoulli over the appropriate definition for logarithms of negative and imaginary values, by producing the formula $\ln(z) = \ln(a + bi) = \ln|z| + \arg(z)i = \ln|z| + \theta i + 2k\pi i$, where $|z| = \sqrt{a^2 + b^2}$, θ the principal value of $\arg(z)$, $k \in \mathbb{Z}$.

The formula for complex exponentiation $z^w = (a+bi)^{m+ni} = x+yi$, where both $z, w \in \mathbb{C}$, was also given the same year by L.Euler in another study [2].

$$z^w = e^{w\ln z} = \left(e^{ln|z|+\theta i+2k\pi i}\right)^{m+ni} = e^{m\ln|z|-n\theta-n2k\pi}e^{(n\ln|z|+m\theta+m2k\pi)i} \tag{1.1}$$

$$x = |z|^m e^{-n\theta-n2k\pi}\cos(n\ln|z| + m\theta + m2k\pi) \tag{1.2}$$

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$$y = |z|^m e^{-n\theta - n2k\pi} \sin(n \ln|z| + m\theta + m2k\pi) \tag{1.3}$$

The formula 1.1 produces an infinite number of results, depending on the value of the k integer. Due to the periodicity of the sine and cosine functions, the results reduce to a finite number of results if the exponent $w \in \mathbb{Q}$.

The first complex logarithm formula $\log_z w = x + yi$, where both $z, w \in \mathbb{C}$, was given by M. Ohm in 1829 [3].

$$\log_z w = \frac{\ln w}{\ln z} = \frac{\ln|w| + \theta_w i + 2k_w\pi i}{\ln|z| + \theta_z i + 2k_z\pi i} \tag{1.4}$$

$$x = \frac{\ln|w| \ln|z| + (\theta_w + 2k_w\pi)(\theta_z + 2k_z\pi)}{(\ln|z|)^2 + (\theta_z + 2k_z\pi)^2} \tag{1.5}$$

$$y = \frac{\ln|z|(\theta_w + 2k_w\pi) - \ln|w|(\theta_z + 2k_z\pi)}{(\ln|z|)^2 + (\theta_z + 2k_z\pi)^2} \tag{1.6}$$

The formula 1.4 produces an infinite number of results, depending on both k_z and k_w integers. The formula is hardly mentionned in mathematic literature. In 1921, F. Cajori in his History of exponentials and logarithms [4] expressed it this way :

> The general logarithm system failed of recognition as useful mathematical inventions.

Both general complex exponentiation and logarithm formulas are nevertheless used by complex number calculators, though usually only the principal value at $k = k_z = k_w = 0$ is returned. The multivaluation of formulas 1.1 and 1.4 can be attributed to the multi-valued complex logarithm function $z \mapsto \ln(z)$, each k integer corresponding to a branch of the logarithm.

In the same volume M. Ohm [3] studies the validity of the exponential and logarithm identities in $\mathbb{C}$. He concludes the set of values on both sides of the identity equation can differ. As an example the left side of $(z^w)^v = z^{wv}$ will produce many more results than the right side, since exponentiation is performed twice. He differentiates "complete" identities producing the same set of results on both sides of the equation, versus "incomplete" identities in which the results differ.

The formulas of Euler and Ohm show that all results of exponentiation and logarithm can be expressed in the form $x + yi$. Thus both operations are algebraically closed in $\mathbb{C}$ and can be defined either as multivalued functions or, when considering a particular branch, as ordinary functions $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. However the closure has come at a cost, firstly most identities equations valid in $\mathbb{R}_+$ can fail in $\mathbb{C}$, secondly the multivaluation forces an arbitrary selection of a branch prior to any result evaluation. Furthermore one could consider the uselessness and geometric meaningless of the general complex logarithm as an abnormality.

In tables 1 and 2 we summarize the validity in $\mathbb{R}_+$ and $\mathbb{C}$ of the exponentiation and logarithm main properties and identities.

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Table 1: Exponentiation and logarithm properties.

Property	**Validity in $\mathbb{R}_+$**	**Validity in $\mathbb{C}$**
Exponentiation $z_1^{z_2}$ closure	yes	yes
Logarithm $\log_{z_1}(z_2)$ closure	no	yes
Exponentiation monovaluation	yes	no
Logarithm monovaluation	yes	no
Exponentiation inverse of logarithm $z_1{}^{\log_{z_1}(z_2)} = z_2$	yes	yes, subset [1]
Logarithm inverse of exponentiation $\log_{z_1}(z_1{}^{z_2}) = z_2$	yes	yes, subset [1]

[1] The left side of the equation produces many more results, of which only a subset is equal to the right side. The equation always holds at principal value

Table 2: Exponentiation and logarithm identities.

Identity	**Validity in $\mathbb{R}_+$**	**Validity in $\mathbb{C}$**
Exponent distributivity over multiplication $(z_1z_2)^{z_3} = z_1{}^{z_3}z_2{}^{z_3}$	yes	no [2]
Exponent distributivity over division $\left(\frac{z_1}{z_2}\right)^{z_3} = \frac{z_1{}^{z_3}}{z_2{}^{z_3}}$	yes	no [2]
Exponential product $z_1{}^{z_2}z_1{}^{z_3} = z_1{}^{z_2+z_3}$	yes	yes, subset [1]
Exponential quotient $\frac{z_1{}^{z_2}}{z_1{}^{z_3}} = z_1{}^{z_2-z_3}$	yes	yes, subset [1]
Exponential power $(z_1{}^{z_2})^{z_3} = z_1{}^{z_2z_3}$	yes	no [2]
Logarithm product $\log_{z_1}(z_2z_3) = \log_{z_1}(z_2) + \log_{z_1}(z_3)$	yes	no [2]
Logarithm quotient $\log_{z_1}\left(\frac{z_2}{z_3}\right) = \log_{z_1}(z_2) - \log_{z_1}(z_3)$	yes	no [2]
Logarithm power $\log_{z_1}(z_2{}^{z_3}) = z_3\log_{z_1}(z_2)$	yes	no [2]
Logarithm base substitution $\log_{z_1}(z_2) = \frac{\log_{z_3}(z_1)}{\log_{z_3}(z_2)}$	yes	yes, if $z_3 \in \mathbb{R}_+$

[1] The left side of the equation produces many more results, of which only a subset is equal to the right side. The equation always holds at principal value
[2] Both sides of identities equations produce a different set of results, the equation not being necessarily valid at the principal value. In further sections we give the exact conditions for the identity validity

Trivial examples of identity failures in $\mathbb{C}$:

For clarity only the principal value at $k = 0$ is considered, the same outcome occurs on other branches.

$$(-1 \cdot -1)^{\frac{1}{2}} = (1)^{\frac{1}{2}} = 1 \neq (-1)^{\frac{1}{2}} \cdot (-1)^{\frac{1}{2}} = i \cdot i = -1$$

$(i-1)^{2i} = \left(\sqrt{2}e^{\frac{3\pi}{4}i}\right)^{2i} = 2^i e^{-\frac{3\pi}{2}} \neq ((i-1)(i-1))^i = (-2i)^i = (2e^{-\frac{\pi}{2}i})^i = 2^i e^{\frac{\pi}{2}}$

$\ln\left((-i)^2\right) = \ln(-1) = \pi i \neq 2\ln(-i) = 2(-\frac{\pi}{2}i) = -\pi i$

$\log_{-2}((-2)^5) = \log_{-2}(-32) = \frac{\ln(-32)}{\ln(-2)} = \frac{\ln(32)+\pi i}{\ln(2)+\pi i} = \frac{(\ln(32)+\pi i)(\ln(2)-\pi i)}{(\ln(2)+\pi i)(\ln(2)-\pi i)}$
$= \frac{\ln(32)\ln(2)+(\ln(32)-\ln(2))\pi i+\pi^2}{(\ln(2))^2+\pi^2} = 1.18568\ldots + 0.84157\ldots i$
$\neq 5\log_{-2}(-2) = 5$

$\ln(-1\cdot i) = \ln(-i) = -\frac{\pi}{2}i \neq \ln(-1) + \ln(i) = \pi i + \frac{\pi}{2}i = \frac{3\pi}{2}i$

In this article we study the complex exponentiation and logarithm as binary operations, only the general case where all operands are complex is considered. The aim is to propose a redefinition of the complex number set in which the issues described above resolve. The idea is to introduce a new form of complex number, derived from the exponential form $z = |z|e^{\arg(z)i} = |z|e^{\theta i + 2k\pi i}$, that extends the possibilities of the algebraic form $z = x + yi$. This new form is hereafter named the complete form. It will become clear the complete form is necessary to grasp all the subtleties of the exponentiation and logarithm operations, and that a strict equality cannot be maintained between the complete form and algebraic form. The properties of the basic operations $(+, -, \times, \div)$ will be impacted by the redefinition, though most properties such as commutativity and associativity remain valid.

The sections 2 and 3 are dedicated to the definition of the set of complex numbers in complete form, hereafter named $\mathbb{E}$, the equivalences between $\mathbb{C}$ and $\mathbb{E}$, and to the definition of complex operations $(+, -, \times, \div, \exp, \log)$ in $\mathbb{E}$. The exponentiation is no longer defined by the logarithm, instead the complex logarithm formula can be deduced from the exponentiation. Moreover all operations produce a single value result. In order to handle formulas in a $\mathbb{C}$ and $\mathbb{E}$ dual number system, we introduce here the notion of set precision and set truncation.

The section 4 includes all proofs and some examples over the validity of the exponential and logarithm identities in $\mathbb{E}$. All the trivial identity failure cases given above resolve.

In the section 5 we show how to obtain explicit formulas in $\mathbb{E}$ linking the real and imaginary parts of some transcendental equations solutions.

The section 6 proposes a geometric representation of $\mathbb{E}$, of which the complex plane appears as an orthogonal projection. The complex exponentiation $z = {z_1}^{z_2}$ and logarithm $z = \log_{z_1}(z_2)$, where $z, z_1, z_2 \in \mathbb{E}$, can simply be represented, in a similar way as the multiplication and division in $\mathbb{C}$.

The section 7 lists all algebraic properties of $\mathbb{E}$ and compares them with the properties of the $\mathbb{R}$ and $\mathbb{C}$ fields.

In section 8 we argue why the exponentiation and logarithm multivalued results and identity failures in $\mathbb{C}$ are not induced by the operations, but are induced by an intrinsic limitation of the complex numbers algebraic form $z = x + yi$.

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2 Complex numbers in complete form

Definition 1. *Set of complex number in complete form*

By taking a broad definition of the complex number as a number composed of a real part and an imaginary part, the complete complex number set is defined as all numbers in the form $e^a e^{bi}$,$\cup$ $\{0\}$, where $a, b \in \mathbb{R}$ and $i^2 = -1$

The number set is hereafter named $\mathbb{E}$. The real part (or real value) is defined as e^a and the imaginary part e^{bi}, where a is the real argument and b the imaginary argument. The element 0 is included for compatibility with $\mathbb{C}$ and $\mathbb{R}$.

Remark. *Equivalence with the exponential form*

The exponential form of complex numbers $z = x + yi = |z|e^{\arg(z)i} = |z|e^{\theta i+2k\pi i}$ has a similar but not identical definition. It remains explicitly linked to the algebraic form and must have a principal value θ of the argument $\arg(z)$ within the interval $]-\pi;\pi]$. The purpose of the integer k is precisely to link all values of the exponential form to their unique corresponding algebraic form. Geometrically, the 2π periodicity of the imaginary argument is purposely maintaining the correlation with the complex plane.

In the complete form, the explicit link to the algebraic form and the constraint on the argument principal value are abolished. For example in $\mathbb{E}$ the numbers $e^0e^{2\pi i}$ and $e^0e^{4\pi i}$ are not equal, each having distinct properties as it will be demonstrated in further sections. Within $\mathbb{C}$ the symbolic and geometric representation of both numbers are equally represented by 1 and by the coordinates $(x, y) = (1, 0)$ on the complex plane.

Replacing $|z|$ by e^a allows the establishment of more elegant and symmetrical formulas. We use the new denomination complete form to avoid any ambiguity.

It is tempting to equate the complete form and the complex exponential function $z \mapsto e^z$ in $\mathbb{C}$. However we will demonstrate why the complete form is necessary in calculations involving complex numbers, and thus cannot be reduced to the algebraic form.

Definition 2. *Equivalence between $\mathbb{C}$ and $\mathbb{E}$ sets*

Let the set $\mathbb{E}$ of complex numbers in complete form $e^a e^{bi}$ be partitionned into $\mathbb{C}$ and $\mathbb{E}\backslash\mathbb{C}$ by restricting $\mathbb{C}$ to a 2π interval of the imaginary argument b, by convention the interval $b \in]-\pi;\pi]$. Each number $x + yi \in \mathbb{C}$, converted into its unique corresponding complete form $e^a e^{bi}$, forms then a distinct equivalence class together with numbers in the form $e^a e^{(b+2k\pi)i} \in \mathbb{E}$ with $k \in \mathbb{Z}^*$.

The definition is equivalent as restricting $\mathbb{C}$ to the principal value of the exponential form of complex numbers. Even with this restriction, the algebraic definition of $\mathbb{C}$ and the complete plane definition are not altered.

Definition 3. *Set precision and truncation*

Let A be a set partitionned by an equivalence relation into two subsets A_1 and A_2, and let each element $a_1 \in A_1$ form a distinct equivalence class with an arbitrary number of elements $a_2 \in A_2$ such as each element a is part of a unique given class. In such a set configuration, elements a_2 are defined as A precise, elements a_1 are defined as A_1 precise. Each element $a_2 \in A_2$ can be truncated to its unique corresponding $a_1 \in A_1$ element, thus at a lower precision level. The truncation is noted $a_1 = \mid a_2 \mid_{A_1}$.

Example 1. $\mathbb{Z}$ *and* $\mathbb{N}$ *precision*

Let the integer set $\mathbb{Z}$ be partitionned into $\mathbb{N}$ and $\mathbb{Z}_{<0}$, an integer is $\mathbb{Z}$ precise if negative, and is $\mathbb{N}$ precise if positive or zero. The abs function is the truncation function from $\mathbb{Z}$ to $\mathbb{N}$ precision level.

Example 2. $\mathbb{E}$ *and* $\mathbb{C}$ *precision*

The Euler formula $e^{bi} = \cos b + \sin b \; i$ is de facto the truncation function from $\mathbb{E}$ to $\mathbb{C}$ precision. The truncation can be noted $|z|_{\mathbb{C}} = |e^a e^{bi}|_{\mathbb{C}} = e^a \cos b + e^a \sin b \; i = e^a e^{|b|_{\mathbb{C}} i}$, with the imaginary argument truncated such as :

$$|b|_{\mathbb{C}} = \begin{cases} b \pmod{2\pi} & \text{if } b \pmod{2\pi} <= \pi \\ b \pmod{2\pi} - 2\pi & \text{if } b \pmod{2\pi} > \pi \end{cases}$$

Equalities such as $1 = e^{4\pi i}$ or $1 = e^{2k\pi i}$ no longer hold whenever $\mathbb{E}$ precision is required, the notation $|e^{2k\pi i}|_{\mathbb{C}} = e^{0i} = 1$ or $|e^{(2k+1)\pi i}|_{\mathbb{C}} = e^{\pi i} = -1$ can be used to clearly indicate the truncation. Whenever the imaginary argument is inside the interval $b \in]-\pi;\pi]$, the complete or algebraic form can be used indifferently.

Remark.

The $\mathbb{E}$ set of complex numbers can be viewed as a "natural" extension of $\mathbb{C}$. Within the set sequence $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{E}$ each element in a given set is uniquely linked to a predecessor set element through an equivalence relation, therefore an element can always be truncated to the predecessor set precision level.

Lemma 1. *Converting from complete form to algebraic form*

$$z = e^a e^{bi} \implies e^a \cos b + e^a \sin b \; i = x + yi \tag{2.1}$$
$$x = e^a \cos b \tag{2.2}$$
$$y = e^a \sin b \tag{2.3}$$

The Euler formula used for the conversion is not to be considered as an equality. From a $\mathbb{E}$ perspective an irreversible loss of information is induced when converting from complete to algebraic form if the imaginary argument is outside the interval $]-\pi;\pi]$.

Lemma 2. *Converting from algebraic form to complete form*

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Using the definition of complex number modulus and argument. By definition $z = 0$ is equivalent in $\mathbb{E}$ and $\mathbb{C}$.

$$z = x + yi = |z|e^{\mathrm{Arg}(z)i} = e^{\ln|z|}e^{\theta_z i} \Longrightarrow e^{\frac{1}{2}\ln(x^2+y^2)}e^{\mathrm{Atan}\left(\frac{y}{x}\right)i} = e^a e^{bi} \tag{2.4}$$

$$a = \frac{1}{2}\ln(x^2 + y^2) \tag{2.5}$$

$$b = \mathrm{Atan}\left(\frac{y}{x}\right) \tag{2.6}$$

Remark. *Usage of* ln*,* Arg *and* Atan *functions*

The natural logarithm function is applied to the domain $\mathbb{R}_{>0}$, hence is single valued. In the formula 2.4 only the principal value of the arg function is considered to remain consistent with definition 2. The limits of the traditional arctan function, with the result in the interval $]-\frac{\pi}{2};\frac{\pi}{2}]$, requires the use of the atan2 function with 2 arguments whose result is included in the interval $]-\pi;\pi]$ without singularities. In this study the notation Atan $\left(\frac{y}{x}\right)$ always refers to the atan2 function where both arguments remain as the fraction numerator and denominator. This notation adjustment will ease the readibility and handling of formulas, as obtained formulas always produce a fraction inside the Atan argument. The fraction can be simplified providing the numerator and denominator signum are preserved.

Definition 4. *Norm and phase of complex numbers in* $\mathbb{E}$

The norm c and phase ϕ of $z = e^a e^{bi}$ are defined as follow :

$$c = \sqrt{a^2 + b^2} \tag{2.7}$$

$$\phi = \mathrm{Atan}\left(\frac{a}{b}\right) \tag{2.8}$$

Remark.

The definition is similar to the complex modulus and argument as defined in $\mathbb{C}$, though in $\mathbb{E}$ the geometric meaning is quite different and explains why the fraction is inversed.

3 Binary operations in complete form

Definition 5. *Complex binary operations in* $\mathbb{E}$

The operations in $\mathbb{E}$ are defined as the 4 basic operations $+, -, \times, \div$ with exponentiation and logarithm. All operands and results are expressed in complete form.

Lemma 3. *Formulas for operations computation in* $\mathbb{E}$

Similarly as in $\mathbb{C}$, the operations computation use a combination of real functions and real operations $(+, -, \times, \div, \exp, \ln, \sin, \cos, \mathrm{Atan})$. Let $z_1 = e^{a_1}e^{b_1 i}$ and $z_2 = e^{a_2}e^{b_2 i}$

$$\mathsf{z_1 \times z_2 = e^{a_1+a_2}\ e^{(b_1+b_2)i}} \tag{3.1}$$

$$\mathsf{z_1 \div z_2 = e^{a_1-a_2}\ e^{(b_1-b_2)i}} \tag{3.2}$$

$$\mathtt{z}_1^{\mathtt{z}_2} = \mathtt{e}^{\left(\mathtt{e}^{\mathtt{a}_2}(\mathtt{a}_1 \cos \mathtt{b}_2 - \mathtt{b}_1 \sin \mathtt{b}_2)\right)}\ \mathtt{e}^{\left(\mathtt{e}^{\mathtt{a}_2}(\mathtt{b}_1 \cos \mathtt{b}_2 + \mathtt{a}_1 \sin \mathtt{b}_2)\right)\mathtt{i}} \tag{3.3}$$

$$\log_{\mathtt{z}_1} \mathtt{z}_2 = \mathtt{e}^{\frac{1}{2} \ln \frac{\mathtt{a}_2{}^2 + \mathtt{b}_2{}^2}{\mathtt{a}_1{}^2 + \mathtt{b}_1{}^2}}\ \mathtt{e}^{\mathrm{Atan}\, \frac{\mathtt{a}_1 \mathtt{b}_2 - \mathtt{a}_2 \mathtt{b}_1}{\mathtt{a}_1 \mathtt{a}_2 + \mathtt{b}_1 \mathtt{b}_2}\ \mathtt{i}} \tag{3.4}$$

$$\mathtt{z}_1 + \mathtt{z}_2 = \mathtt{e}^{\frac{1}{2} \ln\left(\mathtt{e}^{2\mathtt{a}_1} + \mathtt{e}^{2\mathtt{a}_2} + 2\mathtt{e}^{\mathtt{a}_1 + \mathtt{a}_2} \cos(\mathtt{b}_1 - \mathtt{b}_2)\right)}\ \mathtt{e}^{\mathrm{Atan}\, \frac{\mathtt{e}^{\mathtt{a}_1} \sin \mathtt{b}_1 + \mathtt{e}^{\mathtt{a}_2} \sin \mathtt{b}_2}{\mathtt{e}^{\mathtt{a}_1} \cos \mathtt{b}_1 + \mathtt{e}^{\mathtt{a}_2} \cos \mathtt{b}_2}\ \mathtt{i}} \tag{3.5}$$

$$\mathtt{z}_1 - \mathtt{z}_2 = \mathtt{e}^{\frac{1}{2} \ln\left(\mathtt{e}^{2\mathtt{a}_1} + \mathtt{e}^{2\mathtt{a}_2} - 2\mathtt{e}^{\mathtt{a}_1 + \mathtt{a}_2} \cos(\mathtt{b}_1 - \mathtt{b}_2)\right)}\ \mathtt{e}^{\mathrm{Atan}\, \frac{\mathtt{e}^{\mathtt{a}_1} \sin \mathtt{b}_1 - \mathtt{e}^{\mathtt{a}_2} \sin \mathtt{b}_2}{\mathtt{e}^{\mathtt{a}_1} \cos \mathtt{b}_1 - \mathtt{e}^{\mathtt{a}_2} \cos \mathtt{b}_2}\ \mathtt{i}} \tag{3.6}$$

Formulas are easier to handle when split between real and imaginary parts, in this study we mostly use the split notation. Let $z = e^a e^{bi}$:

$$z = z_1 \times z_2 \qquad a = a_1 + a_2 \qquad b = b_1 + b_2$$

$$z = z_1 \div z_2 \qquad a = a_1 - b_1 \qquad b = b_1 - b_2$$

$$z = z_1^{z_2}$$
$$a = e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2)$$
$$b = e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2)$$

$$z = \log_{z_1} z_2$$
$$a = \frac{1}{2} \ln\left(\frac{a_2{}^2 + b_2{}^2}{a_1{}^2 + b_1{}^2}\right)$$
$$b = \mathrm{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right)$$

$$z = z_1 + z_2$$
$$a = \frac{1}{2} \ln(e^{2a_1} + e^{2a_2} + 2e^{a_1 + a_2} \cos(b_1 - b_2))$$
$$b = \mathrm{Atan}\left(\frac{e^{a_1} \sin b_1 + e^{a_2} \sin b_2}{e^{a_1} \cos b_1 + e^{a_2} \cos b_2}\right)$$

$$z = z_1 - z_2$$
$$a = \frac{1}{2} \ln(e^{2a_1} + e^{2a_2} - 2e^{a_1 + a_2} \cos(b_1 - b_2))$$
$$b = \mathrm{Atan}\left(\frac{e^{a_1} \sin b_1 - e^{a_2} \sin b_2}{e^{a_1} \cos b_1 - e^{a_2} \cos b_2}\right)$$

Lemma 4. *Formulas for operations computation in $\mathbb{E}$ involving the norm and phase*

The formulas 3.3 and 3.4 can be simplified using the norm and phase. Let $z = e^a e^{bi}$, $z_1 = e^{a_1} e^{b_1 i}$ and $z_2 = e^{a_2} e^{b_2 i}$ with their corresponding norms and phases (c, ϕ), (c_1, ϕ_1) and (c_2, ϕ_2)

$$z = z_1^{z_2} = e^a e^{bi} = e^{c \sin \phi}\ e^{(c \cos \phi) i}$$

$$c = c_1 e^{a_2} \tag{3.7}$$
$$\phi = |\phi_1 - b_2|_{\mathbb{C}} \tag{3.8}$$

$$z = \log_{z_1} z_2 = e^a e^{bi}$$

$$a = \ln\left(\frac{c_2}{c_1}\right) \tag{3.9}$$
$$b = |\phi_1 - \phi_2|_{\mathbb{C}} \tag{3.10}$$

Proof. *Multiplication formula*

Using the identity $e^{w_1} \cdot e^{w_2} = e^{w_1+w_2}$ where $w_1, w_2 \in \mathbb{C}$ [6]

$$z = z_1 \times z_2 = e^{a_1}e^{b_1 i} \cdot e^{a_2}e^{b_2 i} = e^{a_1}e^{a_2}e^{b_1 i}e^{b_2 i} = e^{a_1+a_2}e^{(b_1+b_2)i}$$
$$a = a_1 + a_2$$
$$b = b_1 + b_2$$

Proof. *Division formula*

Using the identity $e^{w_1} / e^{w_2} = e^{w_1-w_2}$ where $w_1, w_2 \in \mathbb{C}$ [6]

$$z = z_1 \div z_2 = \frac{e^{a_1}e^{b_1 i}}{e^{a_2}e^{b_2 i}} = \frac{e^{a_1}}{e^{a_2}} \cdot \frac{e^{b_1 i}}{e^{b_2 i}} = e^{a_1-a_2}e^{(b_1-b_2)i}$$
$$a = a_1 - a_2$$
$$b = b_1 - b_2$$

Proof. *Exponentiation formula*

The formula $u^w = e^{w \ln u}$ with $w, u \in \mathbb{C}$ defines the complex exponentiation in $\mathbb{C}$, the formula is necessary given the base cannot be exploited directly in algebraic form. The formula is equivalent as converting the base into an infinity of bases in the form $u = e^{\ln|u|+\theta i+2k\pi i}$. The exponent is then applied to the bases such as $u^w = \left(e^{\ln|u|+\theta i+2k\pi i}\right)^w = e^{w\ln|u|+w(\theta+2k\pi)i}$. The result is then reconverted into algebraic form. When calculated separately for each integer k, the exponentiation can be defined as $(e^a e^{bi})^w = e^{aw}e^{bwi}$ with a single valued result, the base and result being in complete form and the exponent in algebraic form. Let $z_1 = e^{a_1}e^{b_1 i}$ and, using the conversion formula 2.1, let $z_2 = e^{a_2}e^{b_2 i} \implies e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i$.

$$\begin{aligned}
z = z_1^{z_2} &= \left(e^{a_1}e^{b_1 i}\right)^{(e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i)} \\
&= \left(e^{a_1+b_1 i}\right)^{(e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i)} \\
&= e^{(a_1+b_1 i)(e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i)} \\
&= e^{(a_1 e^{a_2}\cos b_2 + a_1 e^{a_2}\sin b_2\, i + b_1 e^{a_2}\cos b_2\, i - b_1 e^{a_2}\sin b_2)} \\
&= e^{e^{a_2}(a_1\cos b_2 - b_1\sin b_2)}e^{e^{a_2}(b_1\cos b_2 + a_1\sin b_2)i} \\
a &= e^{a_2}(a_1\cos b_2 - b_1\sin b_2) \\
b &= e^{a_2}(b_1\cos b_2 + a_1\sin b_2)
\end{aligned}$$

$$\begin{aligned}
c^2 &= a^2 + b^2 = e^{2a_2}(a_1\cos b_2 - b_1\sin b_2)^2 + e^{2a_2}(b_1\cos b_2 + a_1\sin b_2)^2 \\
&= e^{2a_2}({a_1}^2\cos^2 b - a_1 b_1\cos b\sin b + {b_1}^2\sin^2 b + {b_1}^2\cos^2 b + a_1 b_1\cos b\sin b + {a_1}^2\sin^2 b) \\
&= e^{2a_2}({a_1}^2\cos^2 b + {a_1}^2\sin^2 b + {b_1}^2\cos^2 b + {b_1}^2\sin^2 b) \\
&= e^{2a_2}({a_1}^2 + {b_1}^2) \\
c &= c_1 e^{a_2} \\
\phi &= \operatorname{Atan}\left(\frac{a}{b}\right) = \operatorname{Atan}\left(\frac{e^{a_2}(a_1\cos b_2 - b_1\sin b_2)}{e^{a_2}(b_1\cos b_2 + a_1\sin b_2)}\right) = \operatorname{Atan}\left(\frac{a_1\cos b_2 - b_1\sin b_2}{b_1\cos b_2 + a_1\sin b_2}\right) \\
&= \operatorname{Atan}\left(\frac{\frac{a_1}{a_1} - \frac{\sin b_2}{\cos b_2}}{1 + \frac{a_1}{b_1}\frac{\sin b_2}{\cos b_2}}\right)
\end{aligned}$$

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$$= \operatorname{Atan}\left(\frac{b_1}{a_1}\right) - \operatorname{Atan}\left(\frac{\sin b_2}{\cos b_2}\right) = |\phi_1 - b_2|_{\mathbb{C}}$$

Proof. *Logarithm formula*

The logarithm formula can be directly reversed from the exponentiation formula 3.3. Counter to the definition of the complex logarithm in $\mathbb{C}$, both operands are here in $\mathbb{E}$ thus can be exploited directly in the formula without requiring any conversion. Let $z_1 = e^{a_1}e^{b_1 i}$ and $z_2 = e^{a_2}e^{b_2 i}$.

$z = \log_{z_1}(z_2) \iff (z_1)^z = z_2$, thus $a_2 = e^a(a_1 \cos b - b_1 \sin b)$ and $b_2 = e^a(b_1 \cos b + a_1 \sin b)$

$$\begin{aligned}
{a_2}^2 + {b_2}^2 &= e^{2a}(a_1 \cos b - b_1 \sin b)^2 + e^{2a}(b_1 \cos b + a_1 \sin b)^2 \\
&= e^{2a}({a_1}^2 \cos^2 b - a_1 b_1 \cos b \sin b + {b_1}^2 \sin^2 b + {b_1}^2 \cos^2 b + a_1 b_1 \cos b \sin b + {a_1}^2 \sin^2 b) \\
&= e^{2a}({a_1}^2 \cos^2 b + {a_1}^2 \sin^2 b + {b_1}^2 \cos^2 b + {b_1}^2 \sin^2 b) \\
&= e^{2a}({a_1}^2 + {b_1}^2) \\
a &= \frac{1}{2} \ln\left(\frac{{a_2}^2 + {b_2}^2}{{a_1}^2 + {b_1}^2}\right) = \ln\left(\frac{c_2}{c_1}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{a_2}{b_2} &= \frac{e^a(a_1 \cos b - b_1 \sin b)}{e^a(b_1 \cos b + a_1 \sin b)} \\
a_2(b_1 \cos b + a_1 \sin b) &= b_2(a_1 \cos b - b_1 \sin b) \\
\cos b(a_2 b_1 - a_1 b_2) &= -\sin b(a_1 a_2 + b_1 b_2) \\
\frac{\sin b}{\cos b} &= \frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} \\
b &= \operatorname{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right) = \operatorname{Atan}\left(\frac{\frac{a_1}{b_1} - \frac{a_2}{b_2}}{1 + \frac{a_1}{b_1}\frac{a_2}{b_2}}\right) \\
&= \operatorname{Atan}\left(\frac{a_1}{b_1}\right) - \operatorname{Atan}\left(\frac{a_2}{b_2}\right) = |\phi_1 - \phi_2|_{\mathbb{C}}
\end{aligned}$$

Proof. *Alternate proof of logarithm formula*

$$\begin{aligned}
z &= \log_{z_1}(z_2) = \frac{\ln z_2}{\ln z_1} = \frac{a_2 + b_2 i}{a_1 + b_1 i} \\
&= \frac{(a_2 + b_2 i)(a_1 - b_1 i)}{(a_1 + b_1 i)(a_1 - b_1 i)} \\
&= \frac{a_1 a_2 + b_1 b_2 + a_1 b_2 i - a_2 b_1 i}{{a_1}^2 + {b_1}^2} \\
&= \frac{a_1 a_2 + b_1 b_2}{{a_1}^2 + {b_1}^2} + \frac{a_1 b_2 - a_2 b_1}{{a_1}^2 + {b_1}^2} i
\end{aligned}$$

The result is in algebraic form and needs to be converted into complete form using conversion formula 2.4.

$$a = \frac{1}{2} \ln \frac{(a_1 a_2 + b_1 b_2)^2 + (a_1 b_2 - a_2 b_1)^2}{({a_1}^2 + {b_1}^2)^2}$$

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$$
\begin{aligned}
&= \frac{1}{2}\ln\frac{{a_1}^2{a_2}^2 + 2a_1a_2b_1b_2 + {b_1}^2{b_2}^2 + {a_1}^2{b_2}^2 - 2a_1a_2b_1b_2 + {a_2}^2{b_1}^2}{({a_1}^2+{b_1}^2)^2}\\
&= \frac{1}{2}\ln\frac{{a_1}^2{a_2}^2 + {b_1}^2{b_2}^2 + {a_1}^2{b_2}^2 + {a_2}^2{b_1}^2}{({a_1}^2+{b_1}^2)^2}\\
&= \frac{1}{2}\ln\frac{{a_1}^2({a_2}^2+{b_2}^2) + {b_1}^2(a_2^2+{b_2}^2)}{({a_1}^2+{b_1}^2)^2}\\
&= \frac{1}{2}\ln\frac{({a_2}^2+{b_2}^2)({a_1}^2+{b_1}^2)}{({a_1}^2+{b_1}^2)^2}\\
&= \frac{1}{2}\ln\left(\frac{{a_2}^2+{b_2}^2}{{a_1}^2+{b_1}^2}\right)
\end{aligned}
$$

$$
b = \text{Atan}\left(\frac{\frac{a_1b_2-a_2b_1}{{a_1}^2+{b_1}^2}}{\frac{a_1a_2+b_1b_2}{{a_1}^2+{b_1}^2}}\right) = \text{Atan}\left(\frac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}\right)
$$

Proof. *Addition and subtraction formulas*

Both operands need to be converted into algebraic form using the formula 2.1, since no identity can be used directly in complete form. Let $z_1 = e^{a_1}\cos b_1 + e^{a_1}\sin b_1\, i$ and $z_2 = e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i$

$$
\begin{aligned}
z &= z_1 \pm z_2 = (e^{a_1}\cos b_1 + e^{a_1}\sin b_1\ i) \pm (e^{a_2}\cos b_2 + e^{a_2}\sin b_2\ i)\\
&= (e^{a_1}\cos b_1 \pm e^{a_2}\cos b_2) + (e^{a_1}\sin b_1 \pm e^{a_2}\sin b_2)i
\end{aligned}
$$

The result is in algebraic form and needs to be converted into complete form using conversion formula 2.4.

$$
\begin{aligned}
a &= \frac{1}{2}\ln((e^{a_1}\cos b_1 \pm e^{a_2}\cos b_2)^2 + (e^{a_1}\sin b_1 \pm e^{a_2}\sin b_2)^2)\\
&= \frac{1}{2}\ln(e^{2a_1}\cos^2 b_1 \pm 2e^{a_1}e^{a_2}\cos b_1\cos b_2 + e^{2a_2}\cos^2 b_2 + e^{2a_1}\sin^2 b_1 \pm 2e^{a_1}e^{a_2}\sin b_1\sin b_2 +\\
&\quad e^{2a_2}\sin^2 b_2)\\
&= \frac{1}{2}\ln(e^{2a_1}(\cos^2 b_1 + \sin^2 b_1) + e^{2a_2}(\cos^2 b_2 + \sin^2 b_2) \pm 2e^{a_1}e^{a_2}(\cos b_1\cos b_2 + \sin b_1\sin b_2))\\
&= \frac{1}{2}\ln(e^{2a_1} + e^{2a_2} \pm 2e^{a_1+a_2}\cos(b_1-b_2))\\
b &= \text{Atan}\left(\frac{e^{a_1}\sin b_1 \pm e^{a_2}\sin b_2}{e^{a_1}\cos b_1 \pm e^{a_2}\cos b_2}\right)
\end{aligned}
$$

Theorem 1. *Within the $\mathbb{E}$ number set, $\mathbb{E}$ precision is the highest possible precision level obtained as result of a multiplication, division or exponentiation operation*

From formulas 3.1, 3.2 and 3.3 we can easily deduce the result of the imaginary argument is not bounded by any limit and will be situated anywhere in $b \in \mathbb{R}$.

Remark.

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The operations can be defined as functions $f : \mathbb{E} \times \mathbb{E} \to \mathbb{E}$, giving exactly four single variable continuous functions : $z \mapsto w \cdot z$; $z \mapsto w/z$; power function $z \mapsto z^w$; exponential function $z \mapsto w^z$.

The complex exponentiation operation is more subtle since the exponent gets truncated to $\mathbb{C}$ precision by the cosine and sine functions used in the formula 3.3. On the other hand, the base and result require $\mathbb{E}$ precision.

Multiplication and division operands and results are at maximum $\mathbb{E}$ precise, no truncation is performed by the formulas 3.1 and 3.2. One can notice even with $\mathbb{C}$ precise operands, the result may be $\mathbb{E}$ precise.

Theorem 2. *Within the $\mathbb{E}$ number set, $\mathbb{C}$ precision is the highest possible precision level obtained as result of a logarithm, addition or subtraction operation*

The formulas 3.4, 3.5 and 3.6 use the Atan function in the imaginary part, thus the result will always be situated inside the interval $b \in]-\pi;\pi]$, which is exactly the definition of the $\mathbb{C}$ precision. The domain of the corresponding functions is therefore $f : \mathbb{E} \times \mathbb{E} \to \mathbb{C}$.

Remark.

Exactly four single variable continuous functions can be obtained: $z \mapsto w+z$; $z \mapsto w-z$; logarithm function $z \mapsto \log_w z$; logarithm base function $z \mapsto \log_z w$. The singularities induced by the values 0 and $e^0 e^{0i} = 1$ are studied in a further section.

The complex logarithm operation requires mixed precision, both operands require the complete form which can therefore be at maximum $\mathbb{E}$ precise, but the result is always at maximum $\mathbb{C}$ precise.

The addition and subtraction are the only operations not requiring the complete form hence no $\mathbb{E}$ precision, operands exceeding the required precision are truncated to $\mathbb{C}$ precision by formulas 3.5 and 3.6.

Theorem 3. *All binary complex operations defined in $\mathbb{E}$ are monovalued*

From the formulas 3.1 to 3.6, we can deduce that both the real and imaginary part will always give a single valued results, since no real multivalued function is used in the formulas.

Remark.

The Atan function as defined in this study is monovalued. An alternate definition with a multivalued result of periodicity 2π is possible and would imply the logarithm, addition and subtraction are multivalued in $\mathbb{E}$. Though a matter of definition, the single valuation arctangent is far more consistent algebraically and also geometrically as it will be seen in further sections. The logarithm, addition and subtraction results are intrinsically limited to $\mathbb{C}$ precision, it makes no sense to map the result on all numbers in $\mathbb{E}$ belonging to the same equivalence class. In the same way a function defined as $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ returns one positive integer, not the equivalent negative integer.

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4 Exponentials and logarithms identities in $\mathbb{E}$

Theorem 4. *All exponentiation identities valid in $\mathbb{R}_+^*$ are valid in $\mathbb{E}^*$*

The result is strictly identical on both sides of the identity when $z_1, z_2, z_3 \in \mathbb{E}^*$

$$(z_1 z_2)^{z_3} = {z_1}^{z_3} {z_2}^{z_3} \tag{4.1}$$
$$\left(\frac{z_1}{z_2}\right)^{z_3} = \frac{{z_1}^{z_3}}{{z_2}^{z_3}} \tag{4.2}$$
$${z_1}^{z_2} {z_1}^{z_3} = {z_1}^{z_2+z_3} \tag{4.3}$$
$$\frac{{z_1}^{z_2}}{{z_1}^{z_3}} = {z_1}^{z_2-z_3} \tag{4.4}$$
$$({z_1}^{z_2})^{z_3} = {z_1}^{z_2 z_3} \tag{4.5}$$

Theorem 5. *The product and quotient logarithm identities valid in $\mathbb{R}_+^*$ are valid in $\mathbb{E}^*$*

The result is strictly identical on both sides of the identity when $z_1, z_2, z_3 \in \mathbb{E}^*$ and $z_1 \neq e^0 e^{0i}$

$$\log_{z_1}(z_2 z_3) = \log_{z_1}(z_2) + \log_{z_1}(z_3) \tag{4.6}$$
$$\log_{z_1}\left(\frac{z_2}{z_3}\right) = \log_{z_1}(z_2) - \log_{z_1}(z_3) \tag{4.7}$$

Theorem 6. *The power and base substitution logarithm identities valid in $\mathbb{R}_+^*$ are valid in $\mathbb{E}^*$ only at $\mathbb{C}$ precision level*

The result truncated to $\mathbb{C}$ precision is strictly identical on both sides of the identity when $z_1, z_2, z_3, z_4 \in \mathbb{E}^*$ and $z_1, z_4 \neq e^0 e^{0i}$. The final operations on each side of the identity return different levels of precision, the identity cannot be a strict equality.

$$\log_{z_1}({z_2}^{z_3}) = \left| z_3 \log_{z_1}(z_2) \right|_{\mathbb{C}} \tag{4.8}$$
$$\log_{z_1}(z_2) = \left| \frac{\log_{z_4}(z_1)}{\log_{z_4}(z_2)} \right|_{\mathbb{C}} \tag{4.9}$$

As demonstrated within the following proofs, the trivial cases of exponential and logarithm identity failures given in the introduction dissapear when both sides of the identity equation are calculated in $\mathbb{E}$, thus when the formulas 3.1 to 3.6 are used at every calculation step.

Proof. $(z_1 z_2)^{z_3} = z_1^{z_3} z_2^{z_3}$ *is valid for all $z_1, z_2, z_3 \in \mathbb{E}^*$*

Combining the multiplication and exponentiation formulas 3.1 and 3.3, let $z_1 = e^{a_1} e^{b_1 i}$, $z_2 = e^{a_2} e^{b_2 i}$ and $z_3 = e^{a_3} e^{b_3 i}$

$$z = (z_1 z_2)^{z_3}$$
$$a = e^{a_3}((a_1 + a_2)\cos b_3 - (b_1 + b_2)\sin b_3)$$
$$b = e^{a_3}((b_1 + b_2)\cos b_3 + (a_1 + a_2)\sin b_3)$$

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$$
\begin{aligned}
z &= z_1^{z_3} z_2^{z_3} \\
a &= e^{a_3}(a_1 \cos b_3 - b_1 \sin b_3) + e^{a_3}(a_2 \cos b_3 - b_2 \sin b_3) \\
&= e^{a_3}((a_1 + a_2) \cos b_3 - (b_1 + b_2) \sin b_3) \\
b &= e^{a_3}(b_1 \cos b_3 + a_1 \sin b_3) + e^{a_3}(b_2 \cos b_3 + a_2 \sin b_3) \\
&= e^{a_3}((b_1 + b_2) \cos b_3 + (a_1 + a_2) \sin b_3)
\end{aligned}
$$

Example 3. $(-1 \cdot -1)^{\frac{1}{2}} \neq (-1)^{\frac{1}{2}} \cdot (-1)^{\frac{1}{2}}$

$$(-1 \cdot -1)^{\frac{1}{2}} \implies (e^{\pi i} e^{\pi i})^{\frac{1}{2}} = (e^{2\pi i})^{\frac{1}{2}} = e^{\pi i} = -1$$

$$(-1)^{\frac{1}{2}} \cdot (-1)^{\frac{1}{2}} \implies (e^{\pi i})^{\frac{1}{2}} (e^{\pi i})^{\frac{1}{2}} = e^{\frac{\pi}{2} i} e^{\frac{\pi}{2} i} = e^{\pi i} = -1$$

When the first expression is evaluated in algebraic form in $\mathbb{C}$, the primary result is 1, the reason of the dissimilarity is because the result of the multiplication $-1 \cdot -1$ was implicitly truncated to a $\mathbb{C}$ precision level. In $\mathbb{E}$ equating $-1 \cdot -1 = 1$ is an over simplification : $e^{\pi i} e^{\pi i} = e^{2\pi i} \neq e^{0i}$, though in algebraic form the 2 values are indistinctive. This imprecision, invisible at first glance, is revealed when the exponent $\frac{1}{2}$ is applied on $e^{2\pi i}$ or e^{0i} giving different values, respectively -1 and 1. Similarly, $-i \cdot -i = e^{-\frac{\pi}{2} i} e^{-\frac{\pi}{2} i} = e^{-\pi i} \neq e^{\pi i}$ and $-1 \cdot i = e^{\pi i} e^{\frac{\pi}{2} i} = e^{\frac{3\pi}{2} i} \neq e^{-\frac{\pi}{2} i}$. On the other hand, $i \cdot i = -1$ and $i \cdot -i = 1$ are always valid.

Proof. $(z_1/z_2)^{z_3} = z_1^{z_3}/z_2^{z_3}$ *is valid for all* $z_1, z_2, z_3 \in \mathbb{E}^*$

Combining the division and exponentiation formulas 3.2 and 3.3

$$
\begin{aligned}
z &= \left(\frac{z_1}{z_2}\right)^{z_3} \\
a &= e^{a_3}((a_1 - a_2) \cos b_3 - (b_1 - b_2) \sin b_3) \\
b &= e^{a_3}((b_1 - b_2) \cos b_3 + (a_1 - a_2) \sin b_3) \\
z &= \frac{z_1^{z_3}}{z_2^{z_3}} \\
a &= e^{a_3}(a_1 \cos b_3 - b_1 \sin b_3) - e^{a_3}(a_2 \cos b_3 - b_2 \sin b_3) \\
&= e^{a_3}((a_1 - a_2) \cos b_3 - (b_1 - b_2) \sin b_3) \\
b &= e^{a_3}(b_1 \cos b_3 + a_1 \sin b_3) - e^{a_3}(b_2 \cos b_3 + a_2 \sin b_3) \\
&= e^{a_3}((b_1 - b_2) \cos b_3 + (a_1 - a_2) \sin b_3)
\end{aligned}
$$

Example 4. $(1 \,/ -1)^{\frac{1}{2}} \neq (1)^{\frac{1}{2}}/(-1)^{\frac{1}{2}}$

$$\left(\frac{1}{-1}\right)^{\frac{1}{2}} \implies \left(\frac{e^{0i}}{e^{\pi i}}\right)^{\frac{1}{2}} = (e^{-\pi i})^{\frac{1}{2}} = e^{-\frac{\pi}{2} i} = -i$$

$$\frac{(1)^{\frac{1}{2}}}{(-1)^{\frac{1}{2}}} \implies \frac{(e^{0i})^{\frac{1}{2}}}{(e^{\pi i})^{\frac{1}{2}}} = \frac{e^{0i}}{e^{\frac{\pi}{2} i}} = e^{-\frac{\pi}{2} i} = -i$$

When the first expression is evaluated in algebraic form, the primary result is i. The error here is to consider $1/-1 = -1$ which is an implicit truncation at $\mathbb{C}$ precision level. In $\mathbb{E}$ $e^{0i}/e^{\pi i} = e^{-\pi i} \neq e^{\pi i}$. The exponent $\frac{1}{2}$ applied on $e^{\pi i}$ or $e^{-\pi i}$ giving different values in $\mathbb{C}$, respectively -i and i. Similarly, $-1/-i = e^{\pi i}/e^{-\frac{\pi}{2} i} = e^{\frac{3\pi}{2} i} \neq e^{-\frac{\pi}{2} i}$ and $-i/i = e^{-\frac{\pi}{2} i}/e^{\frac{\pi}{2} i} = e^{-\pi i} \neq e^{\pi i}$.

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Proof. $(z_1^{z_2})^{z_3} = z_1^{z_2 z_3}$ *is valid for all* $z_1, z_2, z_3 \in \mathbb{E}^*$

Combining the multiplication and exponentiation formulas 3.1 and 3.3

$$z = z_1^{z_2 z_3}$$
$$a = e^{a_2+a_3}(a_1 \cos(b_2+b_3) - b_1 \sin(b_2+b_3))$$
$$b = e^{a_2+a_3}(b_1 \cos(b_2+b_3) + a_1 \sin(b_2+b_3))$$

$$z = (z_1^{z_2})^{z_3}$$
$$\begin{aligned} a &= e^{a_3}(e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2) \cos b_3 - e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2) \sin b_3) \\ &= e^{a_2}e^{a_3}(a_1 \cos b_2 \cos b_3 - b_1 \sin b_2 \cos b_3 - b_1 \cos b_2 \sin b_3 - a_1 \sin b_2 \sin b_3) \\ &= e^{a_2+a_3}(a_1(\cos b_2 \cos b_3 - \sin b_2 \sin b_3) - b_1(\sin b_2 \cos b_3 + \cos b_2 \sin b_3) \\ &= e^{a_2+a_3}(a_1 \cos(b_2+b_3) - b_1 \sin(b_2+b_3)) \\ b &= e^{a_3}(e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2) \cos b_3 + e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2) \sin b_3) \\ &= e^{a_2}e^{a_3}(b_1 \cos b_2 \cos b_3 + a_1 \sin b_2 \cos b_3 + a_1 \cos b_2 \sin b_3 - b_1 \sin b_2 \sin b_3) \\ &= e^{a_2+a_3}(b_1(\cos b_2 \cos b_3 - \sin b_2 \sin b_3) + a_1(\sin b_2 \cos b_3 + \cos b_2 \sin b_3) \\ &= e^{a_2+a_3}(b_1 \cos(b_2+b_3) + a_1 \sin(b_2+b_3)) \end{aligned}$$

$$z = (z_1^{z_3})^{z_2}$$
$$\begin{aligned} a &= e^{a_2}(e^{a_3}(a_1 \cos b_3 - b_1 \sin b_3) \cos b_2 - e^{a_3}(b_1 \cos b_3 + a_1 \sin b_3) \sin b_2) \\ &= e^{a_2+a_3}(a_1 \cos(b_2+b_3) - b_1 \sin(b_2+b_3)) \\ b &= e^{a_2}(e^{a_3}(b_1 \cos b_3 + a_1 \sin b_3) \cos b_2 + e^{a_3}(a_1 \cos b_3 - b_1 \sin b_3) \sin b_2) \\ &= e^{a_2+a_3}(b_1 \cos(b_2+b_3) + a_1 \sin(b_2+b_3)) \end{aligned}$$

Example 5. $((i-1)^2)^i \neq (i-1)^{2i}$

$$((i-1)^2)^i \implies ((e^{\frac{1}{2}\ln 2} e^{\frac{3\pi}{4}i})^2)^i = (e^{\ln 2} e^{\frac{3\pi}{2}i})^i = e^{i \ln 2} e^{-\frac{3\pi}{2}}$$

$$(i-1)^{2i} \implies (e^{\frac{1}{2}\ln 2} e^{\frac{3\pi}{4}})^{2i} = e^{i \ln 2} e^{-\frac{3\pi}{2}}$$

When $(i-1)^2$ is evaluated in algebraic form, the result obtained is $(i-1)(i-1) = -2i$ which is only true in $\mathbb{C}$. Some relevant precision for $\mathbb{E}$ has been lost during the evaluation : $(i-1)(i-1) = -2i = e^{\ln 2} e^{-\frac{\pi}{2}i} \neq (i-1)^2 = (e^{\frac{1}{2}\ln 2} e^{\frac{3\pi}{4}i})^2 = e^{\ln 2} e^{\frac{3\pi}{2}i}$.

Example 6. *Clausen paradox [5] [7]*

$$(e^{1+2\pi k i})^{1+2\pi k i} = e^{(1+2\pi k i)^2} = e^{1+4\pi k i - 4\pi^2 k^2} = e^{1-4\pi^2 k^2} e^{4\pi k i}$$

$(e^{1+2\pi k i})^{1+2\pi k i} = e^{1+2\pi k i} = e \neq e^{1-4\pi^2 k^2} e^{4\pi k i}$, the equality holds only when $k = 0$.

In the first expression the exponentiation base is taken as multivalued $e^1 e^{2\pi k i}$, the exponent in algebraic form $1 + 2\pi k i$ is also multivalued, with both k synchronised. Nothing wrong here. The result of the exponentiation will obviously be multivalued, the first formula given is correct assuming computation is done in $\mathbb{E}$. In the second expression no

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exponentiation is performed, instead a double truncation from $\mathbb{E}$ to $\mathbb{C}$ precision. Equating $e^{1+2\pi ki} = e^1 e^{2\pi ki} = e \cdot 1 = e$ is imprecise, $|e^{1+2\pi ki}|_{\mathbb{C}} = e$ is correct. After the truncation only the value within the interval $b \in]-\pi;\pi]$ remains thus when $k = 0$.

Proof. *$z_1{}^{z_2} z_1{}^{z_3} = z_1{}^{z_2+z_3}$ is valid for all $z_1, z_2, z_3 \in \mathbb{E}^*$*

The identity is similar to the identity $e^{w_1} e^{w_2} = e^{w_1+w_2}$, with $w_1, w_2 \in \mathbb{C}$

$$\begin{aligned}
z^{z_1+z_2} &= z^{z_1} z^{z_2} \\
(e^a e^{bi})^{z_1+z_2} &= (e^a e^{bi})^{z_1} (e^a e^{bi})^{z_2} \\
(e^{a+bi})^{z_1+z_2} &= (e^{a+bi})^{z_1} (e^{a+bi})^{z_2} \\
e^{(a+bi)(z_1+z_2)} &= e^{(a+bi)z_1} e^{(a+bi)z_2} \quad \text{(all exponents can be reduced into the form } z = x + yi) \\
e^{(a+bi)(z_1+z_2)} &= e^{(a+bi)z_1+(a+bi)z_2} \\
e^{(a+bi)(z_1+z_2)} &= e^{(a+bi)(z_1+z_2)}
\end{aligned}$$

The identity can be verified using the multiplication 3.1 and exponentiation 3.3 formulas and the conversion formulas 2.1 and 2.4. Let $z_1 = e^{a_1} e^{b_1 i}$, $z_2 = e^{a_2} e^{b_2 i}$ and $z_3 = e^{a_3} e^{b_3 i}$

$$\begin{aligned}
z &= z_1{}^{z_2} z_1{}^{z_3} \\
a &= e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2) + e^{a_3}(a_1 \cos b_3 - b_1 \sin b_3) \\
b &= e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2) + e^{a_3}(b_1 \cos b_3 + a_1 \sin b_3)
\end{aligned}$$

$$\begin{aligned}
z &= z_1{}^{z_2+z_3} \\
&= z_1{}^{(e^{a_2}\cos b_2 + e^{a_2}\sin b_2\, i) + (e^{a_3}\cos b_3 + e^{a_3}\sin b_3\, i)} \\
&= z_1{}^{(e^{a_2}\cos b_2 + e^{a_3}\cos b_3) + (e^{a_2}\sin b_2 + e^{a_3}\sin b_3)i} \\
&= z_1{}^{c+di} \\
a &= e^{\frac{1}{2}\ln(c^2+d^2)}(a_1 \cos(\text{Atan}\left(\tfrac{d}{c}\right)) - b_1 \sin(\text{Atan}\left(\tfrac{d}{c}\right))) \\
&= \sqrt{c^2+d^2}\left(\frac{a_1}{\sqrt{1+\frac{d^2}{c^2}}} - \frac{\frac{b_1 d}{c}}{\sqrt{1+\frac{d^2}{c^2}}}\right) \\
&= a_1 c - b_1 d \\
&= a_1(e^{a_2}\cos b_2 + e^{a_3}\cos b_3) - b_1(e^{a_2}\sin b_2 + e^{a_3}\sin b_3) \\
&= e^{a_2}(a_1 \cos b_2 - b_1 \sin b_2) + e^{a_3}(a_1 \cos b_3 - b_1 \sin b_3) \\
b &= e^{\frac{1}{2}\ln(c^2+d^2)}(b_1 \cos(\text{Atan}\left(\tfrac{d}{c}\right)) + a_1 \sin(\text{Atan}\left(\tfrac{d}{c}\right))) \\
&= \sqrt{c^2+d^2}\left(\frac{b_1}{\sqrt{1+\frac{d^2}{c^2}}} + \frac{\frac{a_1 d}{c}}{\sqrt{1+\frac{d^2}{c^2}}}\right) \\
&= b_1 c + a_1 d \\
&= b_1(e^{a_2}\cos b_2 + e^{a_3}\cos b_3) + a_1(e^{a_2}\sin b_2 + e^{a_3}\sin b_3) \\
&= e^{a_2}(b_1 \cos b_2 + a_1 \sin b_2) + e^{a_3}(b_1 \cos b_3 + a_1 \sin b_3)
\end{aligned}$$

Proof. *$z_1{}^{z_2} / z_1{}^{z_3} = z_1{}^{z_2-z_3}$ is valid for all $z_1, z_2, z_3 \in \mathbb{E}^*$*

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The identity is similar to the identity $e^{w_1}/e^{w_2} = e^{w_1 - w_2}$, with $w_1, w_2 \in \mathbb{C}$

$$\begin{aligned}
z^{z_1 - z_2} &= \frac{z^{z_1}}{z^{z_2}} \\
(e^a e^{bi})^{z_1 - z_2} &= \frac{(e^a e^{bi})^{z_1}}{(e^a e^{bi})^{z_2}} \\
(e^{a+bi})^{z_1 - z_2} &= \frac{(e^{a+bi})^{z_1}}{(e^{a+bi})^{z_2}} \\
e^{(a+bi)(z_1 - z_2)} &= \frac{e^{(a+bi)z_1}}{e^{(a+bi)z_2}} \quad \text{(all exponents can be reduced into the form } z = x + yi) \\
e^{(a+bi)(z_1 - z_2)} &= e^{(a+bi)z_1 - (a+bi)z_2} \\
e^{(a+bi)(z_1 - z_2)} &= e^{(a+bi)(z_1 - z_2)}
\end{aligned}$$

Proof. $\log_{z_1}({z_2}^{z_3}) = z_3 \log_{z_1}(z_2)$ *is valid at* $\mathbb{C}$ *precision level for all* $z_1, z_2, z_3 \in \mathbb{E}^*$

Combining the multiplication and logarithm formulas 3.1 and 3.4

$$\begin{aligned}
z &= z_3 \log_{z_1}(z_2) \\
a &= a_3 + \frac{1}{2} \ln\left(\frac{{a_2}^2 + {b_2}^2}{{a_1}^2 + {b_1}^2}\right) \\
b &= b_3 + \text{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right) i
\end{aligned}$$

Combining the exponentiation and logarithm formulas 3.3 and 3.4

$$\begin{aligned}
z &= \log_{z_1}({z_2}^{z_3}) \\
a &= \frac{1}{2} \ln\left(\frac{e^{2a_3}(a_2 \cos b_3 - b_2 \sin b_3)^2 + e^{2a_3}(b_2 \cos b_3 + a_2 \sin b_3)^2}{{a_1}^2 + {b_1}^2}\right) \\
&= \frac{1}{2} \ln\left(\frac{e^{2a_3}({a_2}^2 \cos^2 b_3 + {b_2}^2 \sin^2 b_3 + {b_2}^2 \cos^2 b_3 + {b_2}^2 \sin^2 b_3 \pm 2a_2 b_2 \cos b_3 \sin b_3)}{{a_1}^2 + {b_1}^2}\right) \\
&= \frac{1}{2} \ln\left(e^{2a_3} \cdot \frac{({a_2}^2 + {b_2}^2)(\cos^2 b_3 + \sin^2 b_3)}{{a_1}^2 + {b_1}^2}\right) \\
&= a_3 + \frac{1}{2} \ln\left(\frac{{a_2}^2 + {b_2}^2}{{a_1}^2 + {b_1}^2}\right)
\end{aligned}$$

$$\begin{aligned}
b &= \text{Atan}\left(\frac{a_1 e^{a_3}(b_2 \cos b_3 + a_2 \sin b_3) - e^{a_3}(a_2 \cos b_3 - b_2 \sin b_3) b_1}{a_1 e^{a_3}(a_2 \cos b_3 - b_2 \sin b_3) + b_1 e^{a_3}(b_2 \cos b_3 + a_2 \sin b_3)}\right) \\
&= \text{Atan}\left(\frac{a_1 b_2 \cos b_3 + a_1 a_2 \sin b_3 - a_2 b_1 \cos b_3 + b_1 b_2 \sin b_3}{a_1 a_2 \cos b_3 - a_1 b_2 \sin b_3 + b_1 b_2 \cos b_3 + a_2 b_1 \sin b_3}\right) \\
&= \text{Atan}\left(\frac{(a_1 b_2 - a_2 b_1) \cos b_3 + (a_1 a_2 + b_1 b_2) \sin b_3}{(a_1 a_2 + b_1 b_2) \cos b_3 - (a_1 b_2 - a_2 b_1) \sin b_3}\right) \\
&= \text{Atan}\left(\frac{\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} + \frac{\sin b_3}{\cos b_3}}{1 - \frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2} \cdot \frac{\sin b_3}{\cos b_3}}\right) \\
&= \text{Atan}\left(\frac{\sin b_3}{\cos b_3}\right) + \text{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right) \\
&= |b_3|_{\mathbb{C}} + \text{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right)
\end{aligned}$$

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Example 7. $\ln((-i)^2) \neq 2\ln(-i)$

$\ln((-i)^2) \implies \ln\left(\left(e^{-\frac{\pi i}{2}}\right)^2\right) = \ln\left(e^{-\pi i}\right) = -\pi i$

$2\ln(-i) \implies 2\ln\left(e^{-\frac{\pi i}{2}}\right) = 2 \cdot \left(-\frac{\pi i}{2}\right) = -\pi i$

The identity fails in $\mathbb{C}$ since $-i \cdot -i = e^{-\frac{\pi}{2}i}e^{-\frac{\pi}{2}i} = e^{-\pi i} \neq e^{\pi i}$

Example 8. $\log_{-2}((-2)^5) \neq 5\log_{-2}(-2)$

In the following calculation done in $\mathbb{C}$ the exponentiation is not applied to the imaginary part, giving a wrong result. There is anyway no place in $\mathbb{C}$ to hold the exact result of $(-2)^5$

$\log_{-2}((-2)^5) = \log_{-2}(-32) = \frac{\ln(-32)}{\ln(-2)} = \frac{\ln(32)+\pi i}{\ln(2)+\pi i} = \frac{(\ln(32)+\pi i)(\ln(2)-\pi i)}{(\ln(2)+\pi i)(\ln(2)-\pi i)}$
$= \frac{\ln(32)\ln(2)+(\ln(32)-\ln(2))\pi i+\pi^2}{(\ln(2))^2+\pi^2} = 1.18568\ldots + 0.84157\ldots i$

The same calculation in $\mathbb{E}$

$z_1 = -2 \implies e^{\ln 2}e^{\pi i}$
$z_2 = (-2)^5 \implies (e^{\ln 2}e^{\pi i})^5 = e^{5\ln 2}e^{5\pi i}$
$z = \log_{z_1} z_2$
$a = \frac{1}{2}\ln\left(\frac{{a_2}^2+{b_2}^2}{{a_1}^2+{b_1}^2}\right) = \frac{1}{2}\ln\left(\frac{(5\ln 2)^2+(5\pi)^2}{(\ln 2)^2+\pi^2}\right) = \frac{1}{2}\ln(5^2) = \ln 5$
$b = \operatorname{Atan}\left(\frac{a_1b_2-b_1a_2}{a_1a_2+b_1b_2}\right) = \operatorname{Atan}\left(\frac{5\pi\ln 2-5\pi\ln 2}{2(\ln 2)^2+5\pi^2}\right) = \operatorname{Atan}(0) = 0$
$z = e^{\ln 5}e^{0i} = 5$

Example 9. *Identity failure at $\mathbb{E}$ precision level*

$\ln\left((e^{\pi i})^{e^{32\pi i}}\right) = \ln\left(e^{\pi i}\right) = \pi i = e^{\frac{1}{2}\ln(\pi^2)}e^{\frac{\pi}{2}i}$
$e^{32\pi i}\ln\left(e^{\pi i}\right) = e^{32\pi i}e^{\frac{1}{2}\ln(\pi^2)}e^{\frac{\pi}{2}i} = e^{\frac{1}{2}\ln(\pi^2)}e^{\frac{65\pi}{2}i}$

Proof. $\log_{z_1}(z_2) = \log_{z_3}(z_2)/\log_{z_3}(z_1)$ *is valid at $\mathbb{C}$ precision for all $z_1, z_2, z_3 \in \mathbb{E}^*$*

$$z = \log_{z_1}(z_2)$$
$$a = \frac{1}{2}\ln\left(\frac{{a_2}^2+{b_2}^2}{{a_1}^2+{b_1}^2}\right)$$
$$b = \operatorname{Atan}\left(\frac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}\right)$$

Combining the logarithm and division formulas 3.4 and 3.2

$$z = \frac{\log_{z_3}(z_2)}{\log_{z_3}(z_1)}$$

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$$\begin{aligned}
a &= \frac{1}{2}\ln\left(\frac{{a_2}^2+{b_2}^2}{{a_3}^2+{b_3}^2}\right) - \frac{1}{2}\ln\left(\frac{{a_1}^2+{b_1}^2}{{a_3}^2+{b_3}^2}\right)\\
&= \frac{1}{2}(\ln({a_2}^2+{b_2}^2) - \ln({a_3}^2+{b_3}^2) - \ln({a_1}^2+{b_1}^2) + \ln({a_3}^2+{b_3}^2))\\
&= \frac{1}{2}(\ln({a_2}^2+{b_2}^2) - \ln({a_1}^2+{b_1}^2))\\
&= \frac{1}{2}\ln\left(\frac{{a_2}^2+{b_2}^2}{{a_1}^2+{b_1}^2}\right)\\
b &= \text{Atan}\left(\frac{a_3b_2-a_2b_3}{a_3a_2+b_3b_2}\right) - \text{Atan}\left(\frac{a_3b_1-a_1b_3}{a_3a_1+b_3b_1}\right)\\
&= \text{Atan}\left(\frac{\frac{a_3b_2-a_2b_3}{a_3a_2+b_3b_2} - \frac{a_3b_1-a_1b_3}{a_3a_1+b_3b_1}}{1+\frac{a_3b_2-a_2b_3}{a_3a_2+b_3b_2}\cdot\frac{a_3b_1-a_1b_3}{a_3a_1+b_3b_1}}\right)\\
&= \text{Atan}\left(\frac{(a_3b_2-a_2b_3)(a_3a_1+b_3b_1)-(a_3b_1-a_1b_3)(a_3a_2+b_3b_2)}{(a_3a_1+b_3b_1)(a_3a_2+b_3b_2)+(a_3b_2-a_2b_3)(a_3b_1-b_3a_1)}\right)\\
&= \text{Atan}\left(\frac{a_1{a_3}^2b_2+a_3b_1b_2b_3-a_1a_2a_3b_3-a_2b_1{b_3}^2-a_2{a_3}^2b_1-a_3b_1b_2b_3+a_1a_2a_3b_3+a_1b_2{b_3}^2}{a_1a_2{a_3}^2+a_1a_3b_2b_3+a_2a_3b_1b_3+b_1b_2{b_3}^2+{a_3}^2b_1b_2-a_1a_3b_2b_3-a_2a_3b_1b_3+a_1a_2{b_3}^2}\right)\\
&= \text{Atan}\left(\frac{({a_3}^2+{b_3}^2)(a_1b_2-a_2b_1)}{({a_3}^2+{b_3}^2)(a_1a_2+b_1b_2)}\right)\\
&= \text{Atan}\left(\frac{a_1b_2-a_2b_1}{a_1a_2+b_1b_2}\right)
\end{aligned}$$

Example 10. *Identity failure at $\mathbb{E}$ precision level*

$$\log_{\frac{1}{4}}(4) = -1 = e^{\pi i} \qquad \frac{\log_2(4)}{\log_2\left(\frac{1}{4}\right)} = \frac{2}{-2} = \frac{e^{\ln 2}e^{0i}}{e^{\ln 2}e^{\pi i}} = e^{-\pi i}$$

Proof. $\log_{z_1}(z_2z_3) = \log_{z_1} z_2 + \log_{z_1} z_3$ *is valid for all* $z_1, z_2, z_3 \in \mathbb{E}^*$

Combining the multiplication and logarithm formulas 3.1 and 3.4

$$\begin{aligned}
z &= \log_{z_1}(z_2z_3)\\
a &= \frac{1}{2}\ln\left(\frac{(a_2+a_3)^2+(b_2+b_3)^2}{{a_1}^2+{b_1}^2}\right)\\
b &= \text{Atan}\left(\frac{a_1(b_2+b_3)-b_1(a_2+a_3)}{a_1(a_2+a_3)+b_1(b_2+b_3)}\right)
\end{aligned}$$

For simplicity, the algebraic form is used in the following equation, since neither the logarithm nor the addition require the complete form for the result representation

$$\begin{aligned}
z &= \log_{z_1}(z_2) + \log_{z_1}(z_3)\\
&= \frac{a_1a_2+b_1b_2}{{a_1}^2+{b_1}^2} + \frac{a_1b_2-a_2b_1}{{a_1}^2+{b_1}^2}i + \frac{a_1a_3+b_1b_3}{{a_1}^2+{b_1}^2} + \frac{a_1b_3-a_3b_1}{{a_1}^2+{b_1}^2}i\\
&= \frac{a_1a_2+b_1b_2+a_1a_3+b_1b_3}{{a_1}^2+{b_1}^2} + \frac{a_1b_2-a_2b_1+a_1b_3-a_3b_1}{{a_1}^2+{b_1}^2}i
\end{aligned}$$

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The result in algebraic form needs to be converted into complete form using conversion formula 2.4

$$a = \frac{1}{2}\ln\left(\frac{(a_1a_2+b_1b_2+a_1a_3+b_1b_3)^2+(a_1b_2-a_2b_1+a_1b_3-a_3b_1)^2}{({a_1}^2+{b_1}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{(a_1(a_2+a_3)+b_1(b_2+b_3))^2+(a_1(b_2+b_3)-b_1(a_2+a_3))^2}{({a_1}^2+{b_1}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{{a_1}^2(a_2+a_3)^2+{b_1}^2(b_2+b_3)^2+{a_1}^2(b_2+b_3)^2+{b_1}^2(a_2+a_3)^2}{({a_1}^2+{b_1}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{({a_1}^2+{b_1}^2)((a_2+a_3)^2+(b_2+b_3)^2)}{({a_1}^2+{b_1}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{(a_2+a_3)^2+(b_2+b_3)^2)}{{a_1}^2+{b_1}^2}\right)$$

$$b = \operatorname{Atan}\left(\frac{a_1b_2-a_2b_1+a_1b_3-a_3b_1}{a_1a_2+b_1b_2+a_1a_3+b_1b_3}\right)$$
$$= \operatorname{Atan}\left(\frac{a_1(b_2+b_3)-b_1(a_2+a_3)}{a_1(a_2+a_3)+b_1(b_2+b_3)}\right)$$

Example 11. $\ln(-1\cdot -1) \neq \ln(-1)+\ln(-1)$

$\ln(-1\cdot -1) = \ln\left(e^{\pi i}e^{\pi i}\right) = \ln\left(e^{2\pi i}\right) = 2\pi i$
$\ln(-1)+\ln(-1) = \pi i + \pi i = 2\pi i$

Proof. $\log_{z_1}(z_2/z_3) = \log_{z_1} z_2 - \log_{z_1} z_3$ *is valid for all* $\mathsf{z}_1, \mathsf{z}_2, \mathsf{z}_3 \in \mathbb{E}^*$

Combining the division and logarithm formulas 3.2 and 3.4

$$z = \log_{z_1}\left(\frac{z_2}{z_3}\right)$$
$$a = \frac{1}{2}\ln\left(\frac{(a_2-a_3)^2+(b_2-b_3)^2}{{a_1}^2+{b_1}^2}\right)$$
$$b = \operatorname{Atan}\left(\frac{a_1(b_2-b_3)-b_1(a_2-a_3)}{a_1(a_2-a_3)+b_1(b_2-b_3)}\right)$$

$$z = \log_{z_1}(z_2) - \log_{z_1}(z_3)$$
$$= \frac{a_1a_2+b_1b_2}{{a_1}^2+{b_1}^2} + \frac{a_1b_2-a_2b_1}{{a_1}^2+{b_1}^2}i - \frac{a_1a_3+b_1b_3}{{a_1}^2+{b_1}^2} - \frac{a_1b_3-a_3b_1}{{a_1}^2+{b_1}^2}i$$
$$= \frac{a_1a_2+b_1b_2-a_1a_3-b_1b_3}{{a_1}^2+{b_1}^2} + \frac{a_1b_2-a_2b_1-a_1b_3+a_3b_1}{{a_1}^2+{b_1}^2}i$$

$$a = \frac{1}{2}\ln\left(\frac{(a_1a_2+b_1b_2-a_1a_3-b_1b_3)^2+(a_1b_2-a_2b_1-a_1b_3+a_3b_1)^2}{({a_1}^2+{b_1}^2)^2}\right)$$

$$= \frac{1}{2}\ln\left(\frac{(a_1(a_2-a_3)+b_1(b_2-b_3))^2+(a_1(b_2-b_3)-b_1(a_2-a_3))^2}{({a_1}^2+{b_1}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{{a_1}^2(a_2-a_3)^2+{b_1}^2(b_2-b_3)^2+{a_1}^2(b_2-b_3)^2+{b_1}^2(a_2-a_3)^2}{({a_1}^2+{b_1}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{({a_1}^2+{b_1}^2)((a_2-a_3)^2+(b_2-b_3)^2)}{({a_1}^2+{b_1}^2)^2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{(a_2-a_3)^2+(b_2-b_3)^2)}{{a_1}^2+{b_1}^2}\right)$$

$$b = \text{Atan}\left(\frac{a_1b_2-a_2b_1-a_1b_3+a_3b_1}{a_1a_2+b_1b_2-a_1a_3-b_1b_3}\right)$$
$$= \text{Atan}\left(\frac{a_1(b_2-b_3)-b_1(a_2-a_3)}{a_1(a_2-a_3)+b_1(b_2-b_3)}\right)$$

5 Formulas for transcendental equations

The formulas 3.1 to 3.6 can be combined to obtain formulas linking the real and imaginary arguments of expressions using the complex operations.

Example 12. $z_2 = {z_1}^w \cdot w^\alpha$ *where* $w, z_1, z_2 \in \mathbb{E}^*, z_1 \neq e^0e^{0i}, \alpha \in \mathbb{R}$

Explicit formulas linking the real and imaginary arguments a_w, b_w of w can be obtained.

$$z_2 = {z_1}^w\ w^\alpha$$
$$a_2 = e^{a_w}(a_1\cos b_w - b_1\sin b_w) + a_w\alpha$$
$$b_2 = e^{a_w}(b_1\cos b_w + a_1\sin b_w) + b_w\alpha$$

$$(a_2-a_w\alpha)^2 = e^{2a_w}(a_1\cos b_w - b_1\sin b_w)^2$$
$$(b_2-b_w\alpha)^2 = e^{2a_w}(b_1\cos b_w + a_1\sin b_w)^2$$
$$(a_2-a_w\alpha)^2+(b_2-b_w\alpha)^2 = e^{2a_w}((a_1\cos b_w - b_1\sin b_w)^2+(b_1\cos b_w + a_1\sin b_w)^2)$$
$$= e^{2a_w}(a_1^2\cos^2 b_w + b_1^2\sin^2 b_w + b_1^2\cos^2 b_w + a_1^2\sin^2 b_w \pm 2a_1b_1\cos b_w\sin b_w)$$
$$= e^{2a_w}({a_1}^2(\cos^2 b_w+\sin^2 b_w)+{b_1}^2(\cos^2 b_w+\sin^2 b_w))$$
$$= e^{2a_w}({a_1}^2+{b_1}^2)$$
$$\frac{b_2-b_w\alpha}{a_2-a_w\alpha} = \frac{a_1\cos b_w - b_1\sin b_w}{b_1\cos b_w + a_1\sin b_w}$$

From which the final formulas are obtained :

$$e^{2a_w}({a_1}^2+{b_1}^2) = (a_2-a_w\alpha)^2+(b_2-b_w\alpha)^2 \tag{5.1}$$
$$b_w = \frac{b_2-\sqrt{e^{2a_w}({a_1}^2+{b_1}^2)-(a_2-a_w\alpha)^2}}{\alpha} \tag{5.2}$$

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$$a_w = \frac{a_2 - \dfrac{a_1 \cos b_w - b_1 \sin b_w}{b_1 \cos b_w + a_1 \sin b_w}(b_2 - b_w\alpha)}{\alpha} \tag{5.3}$$

$$a_w = \ln\left(\frac{b_2 - b_w\alpha}{b_1 \cos b_w + a_1 \sin b_w}\right) \tag{5.4}$$

Example 13. $z_2 = \log_{z_1}(w) \cdot w^\alpha$ *where* $w, z_1, z_2 \in \mathbb{E}^*, w, z_1 \neq e^0 e^{0i}, \alpha \in \mathbb{R}$

Explicit formulas linking the real and imaginary arguments a_w, b_w of w can be obtained.

$$z_2 = \log_{z_1}(w)\ w^\alpha$$

$$a_2 = \frac{1}{2}\ln\left(\frac{{a_w}^2 + {b_w}^2}{{a_1}^2 + {b_1}^2}\right) + a_w\alpha$$

$$b_2 = \text{Atan}\left(\frac{a_1 b_w - a_w b_1}{a_1 a_w + b_1 b_w}\right) + b_w\alpha = \text{Atan}\left(\frac{b_w}{a_w}\right) - \text{Atan}\left(\frac{b_1}{a_1}\right) + b_w\alpha$$

From which the final formulas are obtained :

$$b_w = \sqrt{e^{2(a_2 - a_w\alpha)}({a_1}^2 + {b_1}^2) - {a_w}^2} \tag{5.5}$$

$$a_w = b_w \cot\left(b_2 - b_w\alpha + \text{Atan}\left(\frac{b_1}{a_1}\right)\right) \tag{5.6}$$

6 Geometric representation of $\mathbb{E}$

6.1 The complex helicoid

Definition 6. *Geometric representation of* $\mathbb{E}$ *: the complex helicoid*

The complex plane is clearly insufficient to represent $\mathbb{E}$ precise numbers, one can notice only $e^a e^{bi}$ with $b \in]-\pi;\pi]$ can be positioned in a unique way. The lack of "space" is solved by an additional axis, hereafter named the i axis, on which the imaginary argument b can translate rectilinearly without any boundaries. The rotation of the imaginary argument b is maintained with a 2π period, giving a unique perpendicular half straight line for each b argument on which the real part e^a is positioned. Hereafter those half-lines are named "rays". Viewed in a three dimension euclidian space, with the origin situated at 0 on the i axis, every number $w = e^a e^{bi}$ can be given a unique orthogonal coordinate $(x, y, z) = (e^a \cos b, e^a \sin b, b)$. Thus the set $\mathbb{E}$ forms exactly an helicoid surface, hereafter named the complex helicoid.

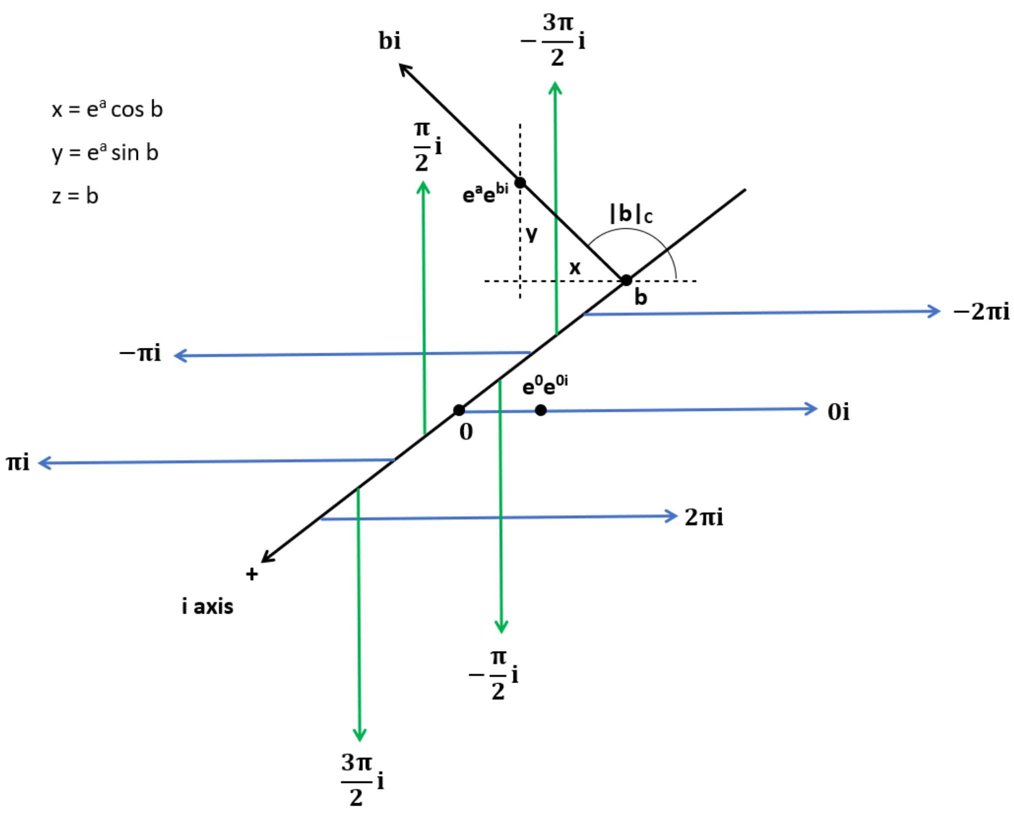

Figure 1: Representation of $e^a e^{bi}$ and $e^0 e^{0i}$ on the complex helicoid

Remark.

The i axis is a singularity itself, on which only the value 0 can be positioned, the value 0 was included into the $\mathbb{E}$ set only for algebraic purpose. The representation is similar as the Riemann surface of the complex logarithm function in $\mathbb{C}$, but with a different meaning and purpose. From a $\mathbb{E}$ perspective the complex helicoid is the counterpart of the complex plane for $\mathbb{C}$ and the real axis for $\mathbb{R}$, on which all numbers $e^a e^{bi}$ are connected without any discontinuity. In this sense the complex helicoid is not to be considered as a layering of

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n complex plane sheets. In a similar way the complex plane is not usually considered as the gluing of n real axis.

6.2 Constant functions representation on the complex helicoid

The constant function $w = e^a$ is the set of points situated at the position e^a on each ray. The function appears as an infinite helix surrounding the i axis. The multiplication and division operations such as $w = e^{a \pm a'}$ translate the position of the point on each ray, thus bring closer or further the helix to the i axis.

The constant function $w = e^{bi}$ is the set of points on a ray pointing in the direction given by b, excluding the 0 situated on the i axis. The multiplication and division operations such as $w = e^{(b \pm b')i}$ operate a rotation and a translation around and along the i axis.

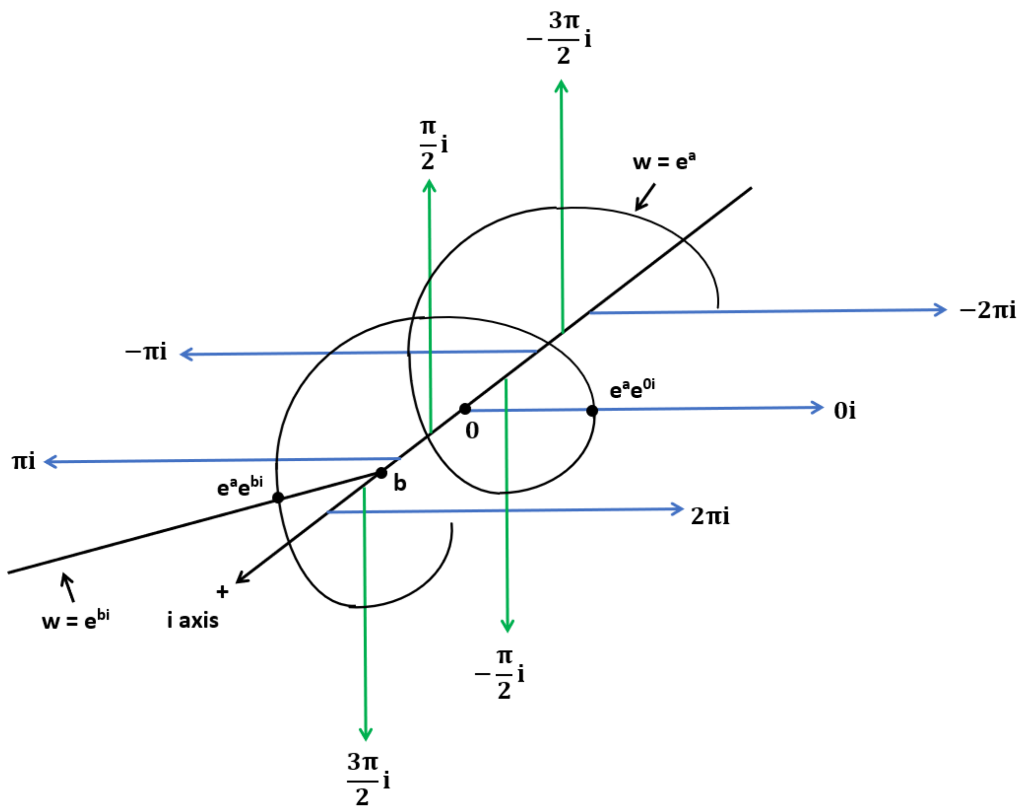

Figure 2: Constant functions $w = e^a$ and $w = e^{bi}$ representations on the complex helicoid

The constant functions underline the geometrical difference between real and imaginary values translations in $\mathbb{E}$. When tied to a fixed imaginary part, the real part e^a translates rectilinearly on a half-straight line. With a fixed real part, the imaginary part e^{bi} spirals on an unbounded helix.

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6.3 Representation of the complex logarithm operation

Let $z_1, z_2 \in \mathbb{E}^*$ with $z_1, z_2 \neq e^0 e^{0i}$ be 2 points on the complex helicoid. The representation of the point $z = \log_{z_1}(z_2)$ reveals, under a new perspective, a similar formula as the division on the complex plane.

$$c_1 = \sqrt{{a_1}^2 + {b_1}^2} \quad c_2 = \sqrt{{a_2}^2 + {b_2}^2} \quad \phi_1 = \text{Atan}\,\frac{a_1}{b_1} \quad \phi_2 = \text{Atan}\,\frac{a_2}{b_2}$$

Since $z = \log_{z_1}(z_2) = e^{\frac{1}{2}\ln\left(\frac{{a_2}^2+{b_2}^2}{{a_1}^2+{b_1}^2}\right)} e^{\text{Atan}\left(\frac{a_1 b_2 - a_2 b_1}{a_1 a_2 + b_1 b_2}\right)i} = \frac{\sqrt{{a_2}^2+{b_2}^2}}{\sqrt{{a_1}^2+{b_1}^2}}\, e^{\left(\text{Atan}\,\frac{a_1}{b_1} - \text{Atan}\,\frac{a_2}{b_2}\right)i}$

$$\boxed{z = \log_{z_1}(z_2) = \frac{c_2}{c_1}\, e^{(|\phi_1 - \phi_2|_{\mathbb{C}})i}}$$

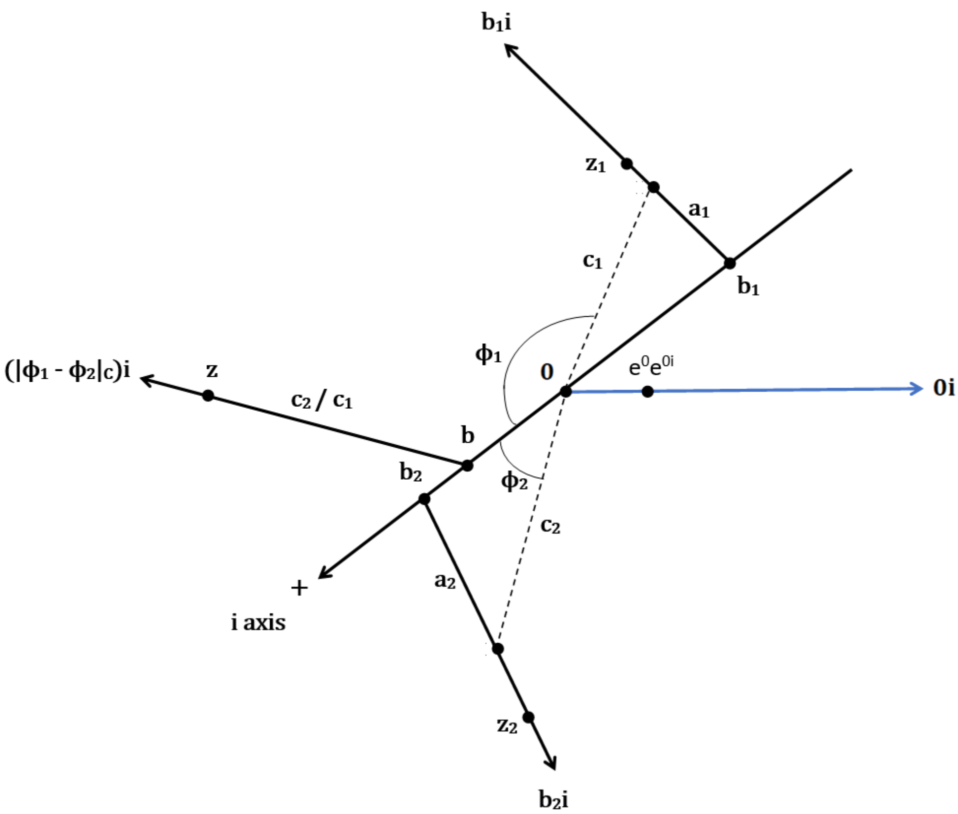

Figure 3: Logarithm operation representation on the complex helicoid

In this representation it is assumed a_1 and a_2 are positive. In case of a negative real argument a_i, the corresponding phase ϕ_i is negative.

6.4 Representation of the complex exponentiation operation

Let $z_1, z_2 \in \mathbb{E}^*$ with $z_1 \neq e^0 e^{0i}$ be 2 points on the complex helicoid. The representation of the point $z = {z_1}^{z_2}$ is best visualised by 2 formulas. The exponent z_2 only being used at $\mathbb{C}$ precision, quite obviously the full b_2 distance on the i axis is not used in the exponentiation formulas.

$$x_2 = e^{a_2}\cos b_2 \quad y_2 = e^{a_2}\sin b_2 \quad c_1 = \sqrt{{a_1}^2 + {b_1}^2} \quad \phi_1 = \text{Atan}\,\frac{a_1}{b_1}$$

Since $z = {z_1}^{z_2} = e^{e^{a_2}(a_1\cos b_2 - b_1 \sin b_2)} e^{e^{a_2}(b_1\cos b_2 + a_1\sin b_2)i}$

$$\boxed{z = {z_1}^{z_2} = e^{a_1 x_2 - b_1 y_2} e^{(b_1 x_2 + a_1 y_2)i}}$$

or using the norm and phase

$$\boxed{z = {z_1}^{z_2} = e^{e^{a_2} c_1 \sin(b_2 - \phi_1)} e^{e^{a_2} c_1 \cos(b_2-\phi_1)i}}$$

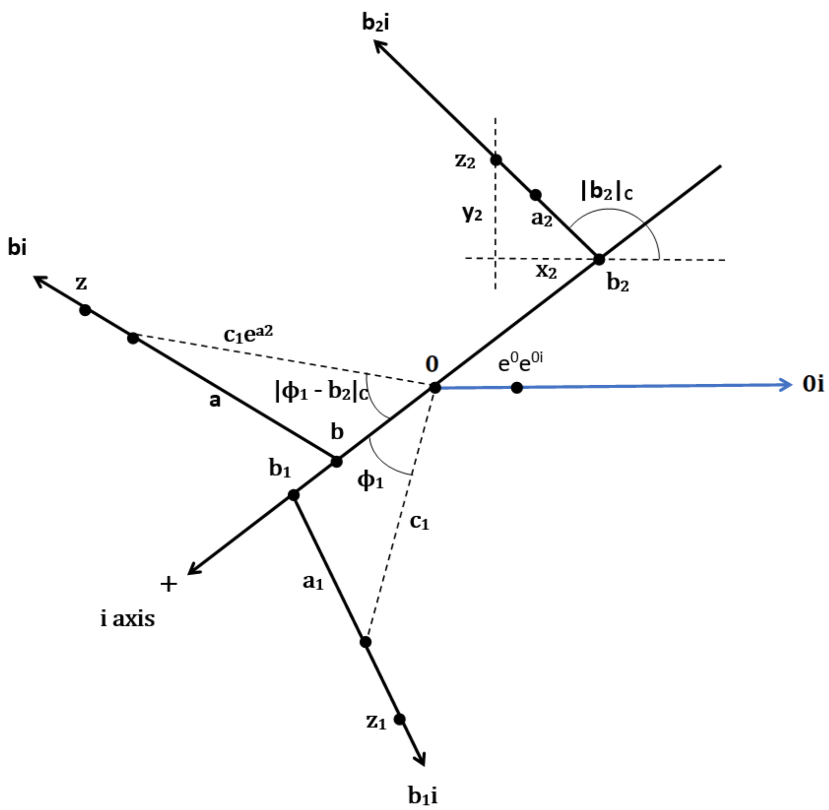

Figure 4: Exponentiation operation representation on the complex helicoid

In this representation it is assumed a_1, a_2 and a are positive.

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6.5 Complex helicoid projections on the plane

The orthogonal projection of the complex helicoid (x, y, z) to $(x, y, 0)$ represents the complex plane, through a new perspective. The projection corresponds exactly to a $\mathbb{C}$ truncation of $\mathbb{E}$ and can be noted as $P(w) = P(e^a e^{bi}) = P(e^a \cos b, e^a \sin b, b) = (e^a \cos b, e^a \sin b, 0)$ or as a truncation $|w|_{\mathbb{C}} = |e^a e^{bi}|_{\mathbb{C}} = e^a \cos b + e^a \sin b\ i$. The singularity 0 is given the appearance of a normal point. The exponentials and logarithms identity failures in $\mathbb{C}$ represented on the complex plane are all due to a "careless" crossing of the Re- axis generating a $\mathbb{C}$ truncation. The projection should not be confused with the logarithmic representation of $\mathbb{E}$ which will be seen further, though both representations are graphicaly identical.

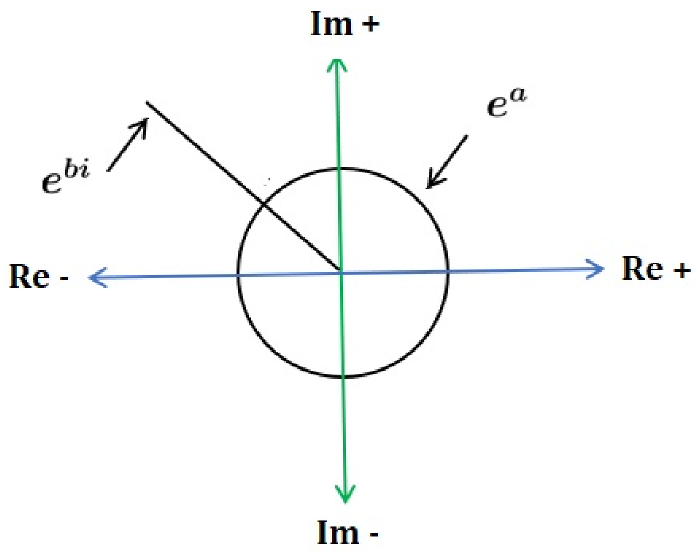

Figure 5: Projection of the complex helicoid (x, y, z) to $(x, y, 0)$

Similarly, the orthogonal projections of the complex helicoid (x, y, z) to $(x, 0, z)$ and (x, y, z) to $(0, y, z)$, maps the constant helix into a cosine and sine curve.

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6.6 Logarithmic representation

The complex numbers in complete form are identified in a unique way by their real and imaginary arguments. Positioning the arguments coordinates on a Wessel-Argand-Gauss diagram is therefore a logarithmic representation of $\mathbb{E}$. One can notice $(a; bi)$ and $(a + bi)$ are equivalent notations for the coordinates. Expressions at the exponent level only require $\mathbb{C}$ precision, thus all operations as defined in $\mathbb{C}$ can be used in an exponent. For example $-1 \cdot i = -i$ or $(-2)^2 = 4$, which both implicitly perfom a $\mathbb{C}$ truncation, can be used, the loss of precision will be without consequence. The number 0 is used in expressions as a normal number.

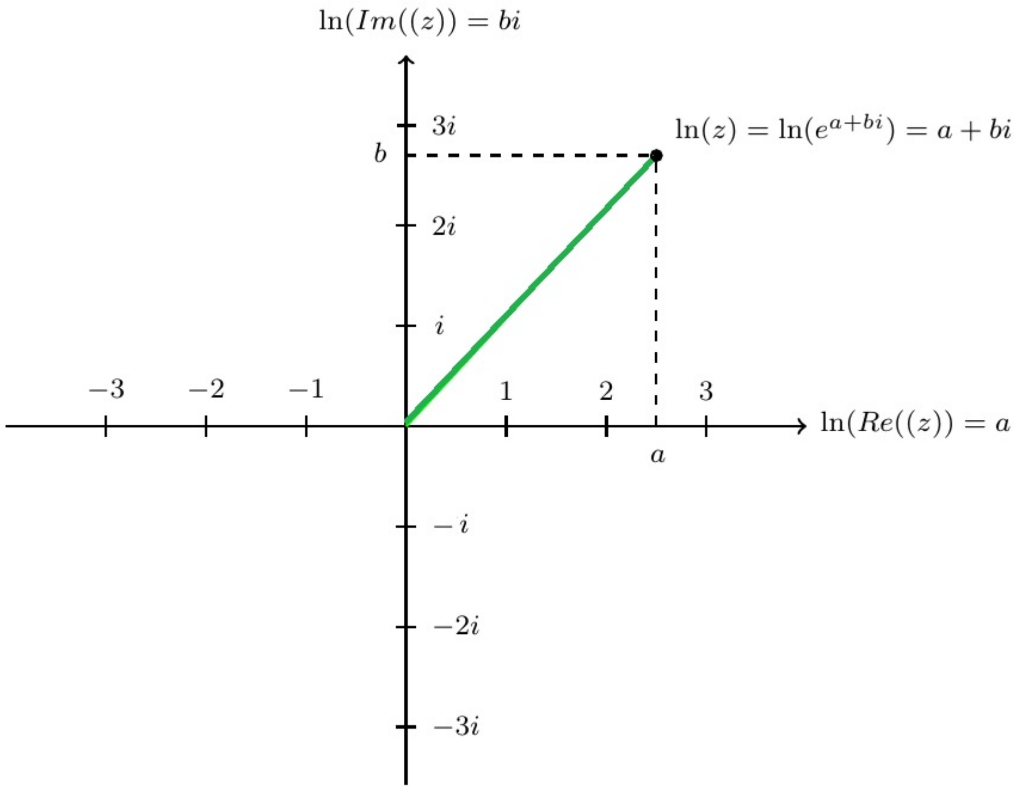

Figure 6: Logarithmic representation of $z = e^a e^{bi}$ on a Wessel-Argand-Gauss diagram

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6.7 Representation of the addition and subtraction operations

The addition and subtraction do not require any $\mathbb{E}$ precision, representing them on the complex helicoid is basically useless, a projection on the complex plane is sufficient. Let $z_1, z_2 \in \mathbb{E}^*$ by 2 points on the complex helicoid, with their corresponding projections $|z_1|_{\mathbb{C}} = x_1 + y_1 i$ and $|z_2|_{\mathbb{C}} = x_2 + y_2 i$ on the complex plane.

$$x_1 = e^{a_1} \cos b_1 \quad y_1 = e^{a_1} \sin b_1 \quad x_2 = e^{a_2} \cos b_2 \quad y_2 = e^{a_2} \sin b_2$$

$$z = z_1 \pm z_2 = e^{\frac{1}{2} \ln\left(e^{2a_1} + e^{2a_2} \pm 2e^{a_1+a_2} \cos(b_1 - b_2)\right)} e^{\text{Atan}\left(\frac{e^{a_1} \sin b_1 \pm e^{a_2} \sin b_2}{e^{a_1} \cos b_1 \pm e^{a_2} \cos b_2}\right) i}$$

$$c = \sqrt{e^{2a_1} + e^{2a_2} - 2e^{a_1+a_2} \cos(b_1 - b_2)} \qquad d = \sqrt{e^{2a_1} + e^{2a_2} + 2e^{a_1+a_2} \cos(b_1 - b_2)}$$

$$z = z_1 - z_2 = c \cdot e^{\text{Atan}\left(\frac{y_1 - y_2}{x_1 - x_2}\right) i} \qquad z = z_1 + z_2 = d \cdot e^{\text{Atan}\left(\frac{y_1 + y_2}{x_1 + x_2}\right) i}$$

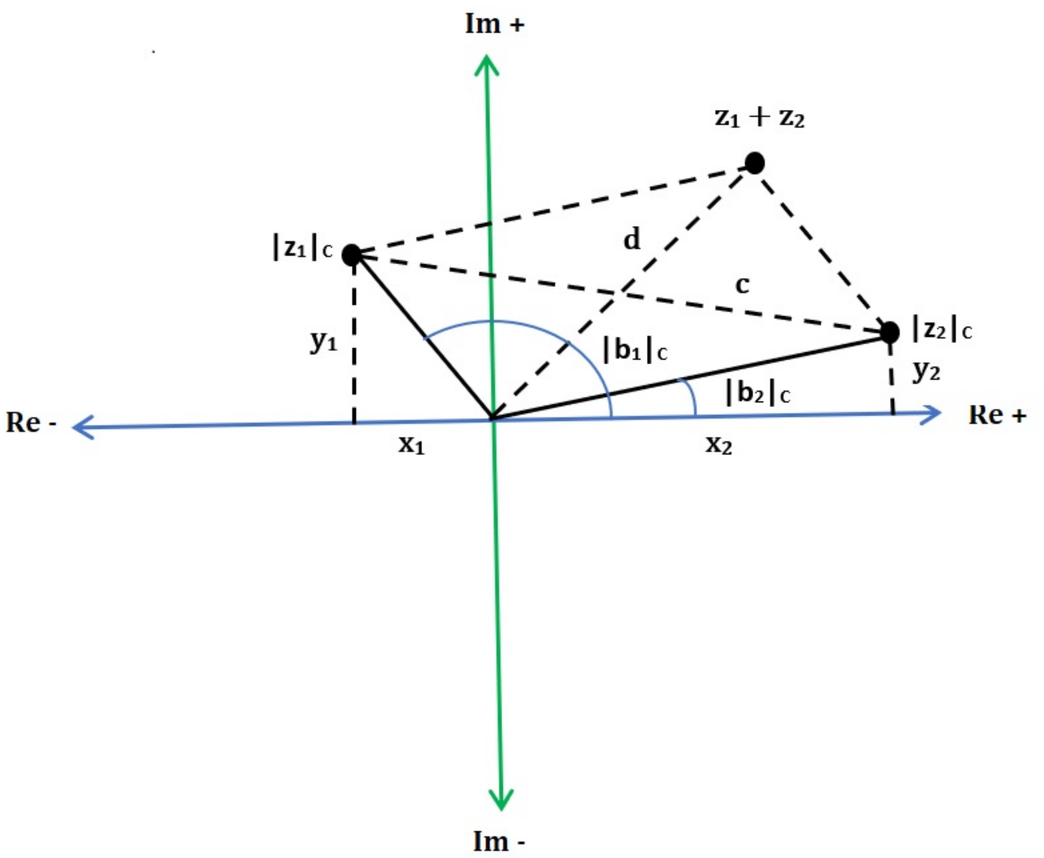

Figure 7: Addition and subtraction representation on the complex plane

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7 Algebraic properties of the $\mathbb{E}$ number set

From the properties of the real operations and functions composing a complex operation, it is possible to deduce the algebraic properties of $\mathbb{E}$. The properties of the value 0 are by convention inherited from $\mathbb{C}$ since the formulas 3.1 to 3.6 do not apply to that value.

7.1 Commutativity

The addition and the multiplication are the only commutative operations.

$$z_1 \times z_2 = z_2 \times z_1 \tag{7.1}$$
$$z_1 + z_2 = z_2 + z_1 \tag{7.2}$$

7.2 Associativity

The addition and the multiplication are the only associative operations.

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \tag{7.3}$$
$$(z_1 \times z_2) \times z_3 = z_1 \times (z_2 \times z_3) \tag{7.4}$$

7.3 Distributivity

The multiplication is distributive at $\mathbb{C}$ precision level over the addition and subtraction, the division is right distributive at $\mathbb{C}$ precision level over the addition and subtraction.

$$\mid z_1 \times (z_2 + z_3) \mid_{\mathbb{C}} = z_1 \times z_2 + z_1 \times z_3 \tag{7.5}$$
$$\mid z_1 \times (z_2 - z_3) \mid_{\mathbb{C}} = z_1 \times z_2 - z_1 \times z_3 \tag{7.6}$$
$$\left| \frac{z_2 + z_3}{z_1} \right|_{\mathbb{C}} = \frac{z_2}{z_1} + \frac{z_3}{z_1} \tag{7.7}$$
$$\left| \frac{z_2 - z_3}{z_1} \right|_{\mathbb{C}} = \frac{z_2}{z_1} - \frac{z_3}{z_1} \tag{7.8}$$

Proof. *Distributivity of multiplication over addition is $\mathbb{C}$ precise*

Combining the multiplication and addition formulas 3.1 and 3.5, let $z_1 = e^{a_1}e^{b_1 i}$, $z_2 = e^{a_2}e^{b_2 i}$ and $z_3 = e^{a_3}e^{b_3 i}$

$$z = z_1 \cdot (z_2 + z_3)$$
$$a = a_1 + \frac{1}{2}\ln(e^{2a_2} + e^{2a_3} + 2e^{a_2+a_3}\cos(b_2 - b_3))$$
$$b = b_1 + \text{Atan}\left(\frac{e^{a_2}\sin b_2 + e^{a_3}\sin b_3}{e^{a_2}\cos b_2 + e^{a_3}\cos b_3}\right)$$
$$z = z_1 \cdot z_2 + z_1 \cdot z_3$$
$$a = \frac{1}{2}\ln(e^{2(a_1+a_2)} + e^{2(a_1+a_3)} + 2e^{(a_1+a_2)+(a_1+a_3)}\cos((b_1 + b_2) - (b_1 + b_3)))$$
$$= \frac{1}{2}\ln(e^{2a_1}(e^{2a_2} + e^{2a_3} + 2e^{a_2+a_3}\cos(b_2 - b_3)))$$

$$= a_1 + \frac{1}{2}\ln(e^{2a_2} + e^{2a_3} + 2e^{a_2+a_3}\cos(b_2 - b_3))$$

$$
\begin{aligned}
b &= \text{Atan}\left(\frac{e^{a_1+a_2}\sin(b_1+b_2)+e^{a_1+a_3}\sin(b_1+b_3)}{e^{a_1+a_2}\cos(b_1+b_2)+e^{a_1+a_3}\cos(b_1+b_3)}\right)\\
&= \text{Atan}\left(\frac{e^{a_2}(\sin b_1\cos b_2+\cos b_1\sin b_2)+e^{a_3}(\sin b_1\cos b_3+\cos b_1\sin b_3)}{e^{a_2}(\cos b_1\cos b_2-\sin b_1\sin b_2)+e^{a_3}(\cos b_1\cos b_3-\sin b_1\sin b_3)}\right)\\
&= \text{Atan}\left(\frac{\cos b_1(e^{a_2}\sin b_2+e^{a_3}\sin b_3)+\sin b_1(e^{a_2}\cos b_2+e^{a_3}\cos b_3)}{\cos b_1(e^{a_2}\cos b_2+e^{a_3}\cos b_3)-\sin b_1(e^{a_2}\sin b_2+e^{a_3}\sin b_3)}\right)\\
&= \text{Atan}\left(\frac{\frac{e^{a_2}\sin b_2+e^{a_3}\sin b_3}{e^{a_2}\cos b_2+e^{a_3}\cos b_3}+\frac{\sin b_1}{\cos b_1}}{1-\frac{e^{a_2}\sin b_2+e^{a_3}\sin b_3}{e^{a_2}\cos b_2+e^{a_3}\cos b_3}\cdot\frac{\sin b_1}{\cos b_1}}\right)\\
&= \text{Atan}\left(\frac{\sin b_1}{\cos b_1}\right)+\text{Atan}\left(\frac{e^{a_2}\sin b_2+e^{a_3}\sin b_3}{e^{a_2}\cos b_2+e^{a_3}\cos b_3}\right)\\
&= |b_1|_{\mathbb{C}}+\text{Atan}\left(\frac{e^{a_2}\sin b_2+e^{a_3}\sin b_3}{e^{a_2}\cos b_2+e^{a_3}\cos b_3}\right)
\end{aligned}
$$

Example 14. *Distributivity failure at $\mathbb{E}$ precision level*

$$-2\cdot(i+1)\implies e^{\ln 2}e^{\pi i}\cdot(e^{\frac{\pi}{2}i}+e^{0i})=e^{\ln 2}e^{\pi i}\cdot e^{\frac{1}{2}\ln 2}e^{\frac{\pi}{4}i}=e^{\frac{3}{2}\ln 2}e^{\frac{5\pi}{4}i}$$
$$(-2\cdot i)+(-2\cdot 1)\implies e^{\ln 2}e^{\pi i}\cdot e^{\frac{\pi}{2}i}+e^{\ln 2}e^{\pi i}\cdot e^{0i}=e^{\ln 2}e^{\frac{3\pi}{2}i}+e^{\ln 2}e^{\pi i}=e^{\frac{3}{2}\ln 2}e^{-\frac{3\pi}{4}i}$$

In the first line the $\mathbb{E}$ precision is preserved because the final operation is a multiplication, in the second line the addition operates a $\mathbb{C}$ truncation, hence the results can be different.

7.4 Identity element

The identity element of addition and multiplication :

$$z_1\times e^0e^{0i}=z_1 \tag{7.9}$$
$$z_1+0=z_1 \tag{7.10}$$

The right identity element of division and subtraction, exponentiation having an infinite set of right identities :

$$z_1\,/\,e^0e^{0i}=z_1 \tag{7.11}$$
$$z_1-0=z_1 \tag{7.12}$$
$$z_1^{(e^0e^{2k\pi i})}=z_1\quad(\text{with } k\in\mathbb{Z}) \tag{7.13}$$

7.5 Inverse

Multiplication, division and exponentiation are the exact reciprocal of their inverse operation :

$$\frac{z_1\times z_2}{z_1}=z_2 \tag{7.14}$$
$$\frac{z_2}{z_1}\times z_1=z_2 \tag{7.15}$$

$$z_1{}^{\log_{z1}(z_2)} = z_2 \tag{7.16}$$

Logarithm, addition and subtraction are only the $\mathbb{C}$ precise reciprocal of their inverse operation:

$$\log_{z_1}(z_1{}^{z_2}) = |z_2|_{\mathbb{C}} \tag{7.17}$$
$$z_2 + z_1 - z_1 = |z_2|_{\mathbb{C}} \tag{7.18}$$
$$z_2 - z_1 + z_1 = |z_2|_{\mathbb{C}} \tag{7.19}$$

Proof. *Exponentiation is the exact inverse of logarithm*

Using the logarithm formula 3.4 converted into algebraic form, let $z_1 = e^{a_1}e^{b_1 i}$, $z_2 = e^{a_2}e^{b_2 i}$ and $\log_{z_1}(z_2) = \frac{a_1a_2+b_1b_2}{a_1{}^2+b_1{}^2} + \frac{a_1b_2-a_2b_1}{a_1{}^2+b_1{}^2}i$

$$z = z_1{}^{\log_{z_1}(z_2)} = z_2$$
$$a = \frac{a_1(a_1a_2+b_1b_2) - b_1(a_1b_2 - b_1a_2)}{a_1{}^2+b_1{}^2}$$
$$= \frac{a_1{}^2a_2 + a_1b_1b_2 - b_1a_1b_2 + b_1{}^2a_2}{a_1{}^2+b_1{}^2}$$
$$= \frac{a_2(a_1{}^2+b_1{}^2)}{a_1{}^2+b_1{}^2} = a_2$$
$$b = \frac{b_1(a_1a_2+b_1b_2) + a_1(a_1b_2 - b_1a_2)}{a_1{}^2+b_1{}^2}$$
$$= \frac{b_1a_1a_2 + b_1{}^2b_2 + a_1{}^2b_2 - a_1b_1a_2}{a_1{}^2+b_1{}^2}$$
$$= \frac{b_2(a_1{}^2+b_1{}^2)}{a_1{}^2+b_1{}^2} = b_2$$

7.6 Symmetry

$$e^ae^{bi} \cdot e^{-a}e^{-bi} = e^0e^{0i} \tag{7.20}$$
$$\frac{e^ae^{bi}}{e^ae^{bi}} = e^0e^{0i} \tag{7.21}$$
$$e^ae^{bi} + e^ae^{bi+(2k+1)\pi i} = 0 \ \ (\text{with } k \in \mathbb{Z}) \tag{7.22}$$
$$e^ae^{bi} - e^ae^{bi+2k\pi i} = 0 \tag{7.23}$$

7.7 Singularities

At first we consider the singularities of operations where both operands are in $\mathbb{E}\backslash\{0\}$.

From the logarithm formula 3.4, one can easily deduce logarithms have a singularity when $z_1 = e^0e^{0i}$ and/or $z_2 = e^0e^{0i}$ caused by the division by 0, the ln with operand 0 and the Atan with 0/0 argument. Interestingly the singularities vanish if the operands are in the form $e^0e^{2k\pi i}$ with $k \neq 0$.

$$\log_{(e^0e^{0i})}(z_2) = \infty \tag{7.24}$$

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$$\log_{z_1}(e^0 e^{0i}) = 0 \quad (7.25)$$
$$\log_{(e^0 e^{0i})}(e^0 e^{0i}) = \text{undefined} \quad (7.26)$$

From the formulas 3.5 and 3.6, it is possible to deduce both addition and subtraction have singularities caused by the ln with operand 0 and the Atan with 0/0 argument.

$$e^a e^{bi} + e^a e^{bi+(2k+1)\pi i} = 0 \;\; (with\ k \in \mathbb{Z}) \quad (7.27)$$
$$e^a e^{bi} - e^a e^{bi+2k\pi i} = 0 \quad (7.28)$$

The introduction of the element 0 allows to reduce some of the above singularities, but also adds new ones.

$$z_1 \cdot 0 = 0 \quad (7.29)$$
$$z_1 \;/\; 0 = \infty \quad (7.30)$$
$$0 \;/\; z_2 = 0 \quad (7.31)$$
$$0 \;/\; 0 = \text{undefined} \quad (7.32)$$
$$z_1 + 0 = z_1 \quad (7.33)$$
$$z_1 - 0 = z_1 \quad (7.34)$$
$$0 - z_2 = \left| z_2 \cdot e^{\pi i} \right|_{\mathbb{C}} \quad (7.35)$$
$$0 - 0 = 0 \quad (7.36)$$
$${z_1}^0 = e^0 e^{0i} \quad (7.37)$$
$$0^{z_2} = 0 \quad (7.38)$$
$$0^0 = e^0 e^{0i} \quad (7.39)$$
$$\log_{z_1}(0) = \infty \quad (7.40)$$
$$\log_0(z_2) = 0 \quad (7.41)$$
$$\log_0(0) = \text{undefined} \quad (7.42)$$

In order to reduce the singularities, it is possible to include the infinite element such as $\mathbb{E}' = \mathbb{E} \cup \{\infty\}$, and define the results of operations using ∞. However it would lead to new singularities such as $\infty - \infty$, whatever definition of $\mathbb{E}'$ there will remain singularities that can ony be treated analytically.

7.8 Algebraic structure of $\mathbb{E}$

Conclusions can be made from formulas 3.1 to 3.6 and from the properties listed above :

- For each of the 6 complex operations, $\mathbb{E}$ has a closed algebraic structure, except for the singularities all results can be represented
- The multiplication and division maintain all their intrinsic properties such as in $\mathbb{C}$
- The addition and subtraction maintain all their intrinsic properties but only at $\mathbb{C}$ precision, since both operations do not require nor can provide any $\mathbb{E}$ precision
- The distributivity property generally only holds when the left side is truncated to $\mathbb{C}$ precision, thus distributivity is only $\mathbb{C}$ precise

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- The multiplication is clearly the defining operation and possess all the properties to constitute a multiplicative group $(\mathbb{E}^*, \cdot)$
- The field axioms are not all verified, since the addition/subtraction reciprocity and the distributivity do not hold exactly in $\mathbb{E}$

It would be a mistake to limit $\mathbb{E}$ to a multiplicative group, as many properties over the exponentiation and logarithm operations are added. All properties and identities hold to a certain extent, only limited by the operations maximum precision level. $\mathbb{E}$ is more to be considered as a complete number system.

7.9 Properties comparison between $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{E}$

Table 3: Basic operations properties.

Property	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{E}$
Addition and subtraction closure	yes	yes	yes
Multiplication and division closure	yes	yes	yes
Addition and subtraction monovaluation	yes	yes	yes
Multiplication and division monovaluation	yes	yes	yes
Addition and multiplication commutativity	yes	yes	yes
Addition and multiplication associativity	yes	yes	yes
Multiplication distributivity over add/sub	yes	yes	$\mathbb{C}$ precise
Division right distributivity over add/sub	yes	yes	$\mathbb{C}$ precise
Identity element of add/sub	0	0	0
Identity element of mult/div	1	1	$e^0 e^{0i} = 1$
Addition/subtraction inverse	yes	yes	$\mathbb{C}$ precise
Multiplication/division inverse	yes	yes	yes

Table 4: Exponentiation and logarithm properties.

Property	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{E}$
Exponentiation $z_1^{z_2}$ closure	if $z_1 \in \mathbb{R}_+$	yes	yes
Logarithm $\log_{z_1}(z_2)$ closure	if $z_1, z_2 \in \mathbb{R}_+$	yes	yes
Exponentiation monovaluation	no	no	yes
Logarithm monovaluation	yes	no	yes
Exponentiation inverse of logarithm			
${z_1}^{\log_{z_1}(z_2)} = z_2$	if $z_1, z_2 \in \mathbb{R}_+$	yes, subset [1]	yes
Logarithm inverse of exponentiation			
$\log_{z_1}({z_1}^{z_2}) = z_2$	if $z_1 \in \mathbb{R}_+$	yes, subset [1]	$\mathbb{C}$ precise

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[1] The left side of the equation produces many more results, of which only a subset is equal to the right side. The equation always holds at principal value

Table 5: Exponentiation and logarithm identities.

Property	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{E}$
Exponent distributivity over multiplic. $(z_1 z_2)^{z_3} = {z_1}^{z_3} {z_2}^{z_3}$	if $z_1, z_2 \in \mathbb{R}_+$	no [2]	yes
Exponent distributivity over division $\left(\frac{z_1}{z_2}\right)^{z_3} = \frac{{z_1}^{z_3}}{{z_2}^{z_3}}$	if $z_1, z_2 \in \mathbb{R}_+$	no [2]	yes
Exponential product ${z_1}^{z_2} {z_1}^{z_3} = {z_1}^{z_2+z_3}$	if $z_1 \in \mathbb{R}_+$	yes, subset [1]	yes
Exponential quotient $\frac{{z_1}^{z_2}}{{z_1}^{z_3}} = {z_1}^{z_2-z_3}$	if $z_1 \in \mathbb{R}_+$	yes, subset [1]	yes
Exponential power $({z_1}^{z_2})^{z_3} = {z_1}^{z_2 z_3}$	if $z_1 \in \mathbb{R}_+$	no [2]	yes
Logarithm product $\log_{z_1}(z_2 z_3) = \log_{z_1}(z_2) + \log_{z_1}(z_3)$	if $z_1, z_2, z_3 \in \mathbb{R}_+$	no [2]	yes
Logarithm quotient $\log_{z_1}\left(\frac{z_2}{z_3}\right) = \log_{z_1}(z_2) - \log_{z_1}(z_3)$	if $z_1, z_2, z_3 \in \mathbb{R}_+$	no [2]	yes
Logarithm power $\log_{z_1}({z_2}^{z_3}) = z_3 \log_{z_1}(z_2)$	if $z_1, z_2 \in \mathbb{R}_+$	no [2]	$\mathbb{C}$ precise
Logarithm base substitution $\log_{z_1}(z_2) = \frac{\log_{z_3}(z_1)}{\log_{z_3}(z_2)}$	if $z_1, z_2, z_3 \in \mathbb{R}_+$	if $z_3 \in \mathbb{R}_+$	$\mathbb{C}$ precise

[1] The left side of the equation produces many more results, of which only a subset is equal to the right side. The equation always holds at principal value

[2] Both sides of identities equations produce a different set of results, the equation not being necessarily valid at the principal value

The exponential identities ${z_1}^{z_2} {z_1}^{z_3} = {z_1}^{z_2+z_3}$ and $\frac{{z_1}^{z_2}}{{z_1}^{z_3}} = {z_1}^{z_2-z_3}$ remain valid in $\mathbb{C}$ at the principal or at any other branch. The reason is clear, both identities never alter the exponentiation base, only the exponents are altered, as they do not require $\mathbb{E}$ precision no $\mathbb{C}$ truncation occurs. All other identities alter the exponentiation base or logarithm base or operand, thus any value outside the $]-\pi;\pi]$ interval of the imaginary argument will trigger an identity failure.

8 Comments on the exponentiation and logarithm definition in $\mathbb{C}$

The complex exponentiation in $\mathbb{C}$ is defined by the formula 1.1, thus the primary multivalued result is in the form $e^{a+\alpha k\pi} e^{(b+\beta k\pi)i}$, where $a, \alpha, b, \beta \in \mathbb{R}$ and $k \in \mathbb{Z}$. To express the result in algebraic form a conversion is necessary, using the Euler formula the sine and cosine functions convert the result into algebraic form. However, during the conversion, precision is lost and the result principal value may be shifted. For example $(-1)^3 = (e^{\ln(-1)})^3 = (e^{\pi i+2k\pi i})^3 = e^{3\pi i+6k\pi i}$. Converting those values into algebraic form returns $-1 = e^{\pi i+2k\pi i}$, thus the principal value is reset to $e^{\pi i}$. Moreover there is no possibility to convert expressions such as $e^{6k\pi i}$ without loss of information. In the example $(i^{-5})^i$, when the result of i^{-5} is reconverted into algebraic form the principal value result is shifted from $e^{-\frac{5\pi i}{2}}$ to $e^{-\frac{\pi i}{2}}$, the result of $(i^{-5})^i$ becoming $e^{\frac{\pi}{2}}$ instead of $e^{\frac{5\pi}{2}}$.

The multivaluation of the complex exponentiation is not induced by the logarithm, but by the algebraic form of the base. Since no identity is available to exploit the base as such, the formula 1.1 is equivalent as substituting the base by an infinity of bases, the so-called exponential form, using the formula $z = |z|e^{\arg(z)i} = |z|e^{\theta i+2k\pi i}$. In general the multivaluation is assumed, unless explicitly restricting to real positives with notations such as $|z|^\alpha$ or $\sqrt{a^2+b^2}$ which both assume a single valued real positive base.

The complex logarithm as defined by L. Euler [1] is restricted to the base e or at least to real positive values. Euler himself does not mention a multivalued logarithm function, rather he speaks of each real or complex number having an infinite number of logarithms. Indeed, as for the exponentiation base, the logarithm operand cannot be exploited directly in algebraic form, thus has to be converted into exponential form, $\ln(z) = \ln(|z|e^{\arg(z)i}) = \ln(|z|e^{\theta i+2k\pi i}) = \ln|z| + \theta i + 2k\pi i$. The primary result being in algebraic form, no conversion is required nor any loss of precision is induced. The multivaluation is solely induced by the operand substitution, for example $\ln(1) = 2k\pi i$ and $\ln(-1) = \pi i + 2k\pi i$. On the other hand $\ln|z|$ is assumed single valued as the operand is implicitly substituted by xe^{0i}.

Notations such as $\ln, \log_2$ or $\log_{10}$ assume the logarithm base is in the form xe^{0i}. For bases the same logic applies as for the exponentiation, a base in algebraic form can be substituted by the equivalent exponential form, or by any particular value in complete form. As an example, for $\log_{-1}$ the base can be assumed as monovalued $e^{\pi i}$ or multivalued $e^{\pi i+2k\pi i}$.

It is clear there is only one unique exponentiation and one unique logarithm complex operation. The different notations conventions and different assumptions regarding the operands substitutions are creating some confusion, which can be blamed on the lack of precision of the algebraic form. In complete form, real positive numbers are not fundamentally different, all operands are in the form $e^a e^{bi}$, moreover the concepts of principal value and branches are no longer necessary. Expressions such as $(e^{\pi i})^{\frac{1}{3}}$ or $\ln(e^{\pi i})$ return a single valued result in $\mathbb{E}$. The same expressions in $\mathbb{C}$ are multivalued because $e^{\pi i}$ is converted into algebraic form, assuming $e^{\pi i} = -1 = e^{\pi i+2k\pi i}$

In general, when dealing with exponentiation and logarithm in $\mathbb{C}$ the equality $e^{\alpha i} = e^{\alpha i+2k\pi i}$ is automatically assumed, by an analogy with the trigonometric circle where an angle of α is equal to $\alpha + 2k\pi$. It turns out this assumption is responsible for the exponentiation and logarithm multivaluation. From a $\mathbb{E}$ perspective only $|e^{\alpha i}|_{\mathbb{C}} = |e^{\alpha i+2k\pi i}|_{\mathbb{C}}$ is valid. The formulas $\cos(\alpha) + i\sin(\alpha) = \cos(\alpha+2k\pi) + i\sin(\alpha+2k\pi) = \sum_{n=0}^{\infty}\frac{(\alpha i)^n}{n!} = \sum_{n=0}^{\infty}\frac{(\alpha i+2k\pi i)^n}{n!}$ are strictly equal, but are also equal in deconstructing the complete form and reconstructing a result in algebraic form, as such they literally truncate the precision of the complete form.

9 Conclusion

As demonstrated in this article, the complex exponentiation base and result, the complex logarithm base and operand cannot be represented precisely in algebraic form. The same observation holds for the multiplication and division results when used in combination with an exponentiation or logarithm. For this reason alone, multivalued results, identity failures and even wrong results are obtained when computing exclusively in $\mathbb{C}$.

The establishment of the complete form is an attempt to restore the properties of exponentiation and logarithm, and to ease the conceptualization and handling of both operations when all operands are complex. Moreover the $\mathbb{E}$ set of complex numbers in complete form can be viewed as a "natural" extension of $\mathbb{C}$. Within the sequence $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R} \subset \mathbb{C} \subset \mathbb{E}$ each set extends the capacity of the predecessor set by providing new elements, thus new symbolic representations of numbers. Each element in a given set is uniquely linked to a predecessor set element through an equivalence relation, therefore an element can always be truncated to the predecessor set precision level. Similarly the geometric representations are extended while preserving the predecessor sets representations.

Labelling expressions such as $e^a e^{bi}$ as numbers might seem strange, though we believe it is totally justified by the extra precision and possibilities they introduce, as they overcome some limitations encountered in $\mathbb{C}$ with the algebraic form. As we have frequently illustrated with examples, it remains possible to combine the algebraic and complete form inside expressions and formulas. Within the $]-\pi;\pi]$ boundary of the imaginary argument both forms can be used indifferently, but outside that interval, the complete form and formulas 3.1 to 3.6 should be used in complex number calculators.

We do not consider the $\mathbb{C}$ precision limitation on some properties and identities as an insurmountable issue. Expressions involving additions and subtractions, such as polynomials, do not require $\mathbb{E}$ precision, the algebraic form and the basic operations $(+, -, \times, \div)$ as defined in $\mathbb{C}$ are always sufficient. The required precision is more to be considered as a matter of choice depending on the context where the complex operations are used.

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