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A better approach for solving a fuzzy multiobjective programming problem by level sets

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Abstract

In this paper we deal with the resolution of a fuzzy multiobjective programming problem using the level sets optimization. We compare it to other optimization strategies studied until now and we propose an algorithm to identify possible Pareto efficient optimal solutions.

Key words: Multiple Objective programming problems, Fuzzy numbers, Generalized Hukuhara differentiability.

1. Introduction

Lofti Zadeh said that fuzzy logic grew out of something as impractical as an idea, namely a reflection on the nature of precision: "As a systems engineer, I have always been a firm believer in the power of mathematics to solve all kinds of problems, but I also realized that such a precise conceptual framework imposed limitations. Solve all kinds of problems, but I also realized that such a precise conceptual framework imposed limitations. Psychology, linguistics, etc., are not precise. I have always wanted to reduce the gap between the real world, with all its inaccuracies, and classical mathematics. That was the origin of fuzzy sets and fuzzy logic."

Fuzzy set theory is the good framework to handle the imprecise information. An important and growing field is the Fuzzy Optimization. In 1978, Zimmerman [4] introduced the fuzziness in the multiobjective linear problem. Later, in 1994, with Delgado et al.[1] contributions' we can find the first papers on fuzzy optimization. The multiobjective programming problem comes up in some applications including water resources [2] or production planning problems [3].

A first and essential step is to establish the optimum notion in the fuzzy environment. In the literature we can find different strategies to define and find an optimum for a fuzzy function. One of them uses the approximation of a fuzzy set by a crisp number (defuzzification techniques). In [7] and [8] the author uses the embedding theorem and the scalarization in vector optimization to solve fuzzy multiobjective programming problem. However when we use a defuzzification operator which replaces a fuzzy set by a single number we generally loose too much important information.

In [9] an approximation of a fuzzy set by an interval is proposed. In this approach, a given fuzzy set is substituted by a crisp interval, which is -in some sense-close to the former one. A new interval approximation operator, the Nearest Interval Approximation Operator (NIA), which is the best one with respect to a certain measure of distance between fuzzy numbers is considered. He proposed an interval which minimize the distances between the initial fuzzy numbers and all its approximations.

In [10], the author explains what interesting is to use the Nearest Interval Approximation Operator to solve a multiobjective programming problem with fuzzy objective functions. They established a sufficient Karush-Kuhn-Tucker type of Pareto optimality conditions, using continuously gH-differentiable functions where the sum of the end-points functions is convex.

This can be summarized in the following table:

Methods	**Technique**	**Inconvenience**
Defuzzification [7] [8]	Replace a fuzzy set by a single number	Loose information
Nearest Interval Approximation [9]	Approximation of a fuzzy set by an interval	Depends on the distance

In 2017, Ren and Wang [5] given an approach for solving fuzzy bilevel programming problems through nearest interval approximation approach. In 2018, Sharma and Aggarwal [6] solving fully fuzzy multiobjective linear programming problems using nearest interval approximation of fuzzy number and interval programming.

Recently, Osuna-Gomez et al. [16] demonstrated optimality conditions for the case of a fuzzy problem with a single objective function.

The inspiration for this paper comes from the one published in 2009 by Wu [24] where sufficient optimality conditions were obtained to solve a fuzzy multiobjective problem but not necessary conditions. In this paper we face the challenge of searching for such necessary conditions and with more general hypotheses than those existing in the literature.

In this paper, we propose to use an *optimality concept by levels*, that

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means we are going to use the level sets that completely characterize the fuzzy set ([11]) to define the optimum. We demonstrate that this concept is more general that before mentioned. With no doubt the optimum notion should be determinated by the decisor's interest, so in some cases the crisp optima or interval-optima can be considered more satisfactory than the ones here showed. However, by including the latter to the previous, the search mechanisms proposed here will be valid whatever the concept used.

We consider that the optimization problem has multiple objectives, and propose an algorithm to examine all possible candidates to be Pareto optimal solutions under the only hypothesis that the fuzzy functions are gH-differentiable functions.

This paper is organized as follows. In section 2 we recall the order and arithmetic for using intervals and fuzzy numbers on α-level sets. In section 3 we put the focus on the multiobjective problem with fuzzy objective functions and the relationships between their satisficing solutions and Pareto solutions. In section 4 we present the gH-differentiable fuzzy function concept based on gH-difference of fuzzy numbers. It is the necessary tool to obtain optimality conditions. In section 5 we prove our main results. A necessary optimality condition for Pareto efficiency and an algorithm based on it are obtained. Finally, in section 6 we present the conclusion remarks.

2. Preliminaries

We recall the arithmetic and order for intervals we are going to use.

We denote by $\mathcal{K}_C$ the family of all bounded closed intervals in $\mathbb{R}$, i.e.,

$\mathcal{K}_C = \{[\underline{a}, \overline{a}] \ / \ \underline{a}, \overline{a} \in \mathbb{R} \text{ and } \underline{a} \leq \overline{a}\}$. If α, β are the end-points of the interval A (but it is not necessarily $\alpha \leq \beta$), we write $A = [\alpha \bigvee \beta]$.

For $A = [\underline{a}, \overline{a}]$, $B = \left[\underline{b}, \overline{b}\right] \in \mathcal{K}_C$ and $\nu \in \mathbb{R}$ we consider the following operations:

$$A+B = [\underline{a}, \overline{a}] + \left[\underline{b}, \overline{b}\right] = \left[\underline{a} + \underline{b}, \overline{a} + \overline{b}\right], \quad \nu A = \nu \left[\underline{a}, \overline{a}\right] = \begin{cases} [\nu\underline{a}, \nu\overline{a}] & \text{if} \quad \nu \geq 0, \\ [\nu\overline{a}, \nu\underline{a}] & \text{if} \quad \nu < 0, \end{cases} \tag{1}$$

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (a) \quad A = B + C, & \text{or} \\ (b) \quad B = A + (-1)C. \end{cases} \tag{2}$$

This difference (2), called generalized Hukuhara difference ($gH-$difference for short) has many interesting properties compared to other definitions (Minskowki, Hukuhara differences) for example $A \ominus_{gH} A = \{0\} = [0, 0]$. Also, the

gH-difference of two intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ always exists and it is equal to ([14])

$$A \ominus_{gH} B = \left[\min\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}, \max\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}\right].$$

Given two intervals we define the distance between A and B by $H(A, B) = \max\left\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\right\}$. It is well-known that $(\mathcal{K}_C, H)$ is a complete metric space.

We consider the following order relation in $\mathcal{K}_C$.

Definition 1. *Let $A, B \in \mathbb{R}$ be. It is said that*

- $A \underline{\preceq} B \Leftrightarrow \underline{a} \leq \underline{b}$ *and* $\bar{a} \leq \bar{b}$.
- $A \preceq B \Leftrightarrow A \underline{\preceq} B$ *and* $A \neq B$, *i.e.* $\underline{a} \leq \underline{b}$ *and* $\bar{a} \leq \bar{b}$, *with a strict inequality.*
- $A \prec B \Leftrightarrow \underline{a} < \underline{b}$ *and* $\bar{a} < \bar{b}$.

It is clear that $A \prec B \Rightarrow A \preceq B \Rightarrow A \underline{\preceq} B$.

A fuzzy set on $\mathbb{R}^n$ is a mapping $\tilde{u} : \mathbb{R}^n \longrightarrow [0, 1]$. For each fuzzy set $\tilde{u}$, we denote its α-level set as $[\tilde{u}]^\alpha = \{x \in \mathbb{R}^n \mid \tilde{u}(x) \geq \alpha\}$ for any $\alpha \in (0, 1]$. The support of $\tilde{u}$ is denoted by $supp(\tilde{u})$ where $supp(\tilde{u}) = \{x \in \mathbb{R}^n \mid \tilde{u}(x) > 0\}$. The closure of $supp(\tilde{u})$ is defined by the 0-level of u, .i.e. $[\tilde{u}]^0 = cl(supp(\tilde{u}))$ where $cl(M)$ means the closure of the subset $M \subset \mathbb{R}^n$. The core of $\tilde{u}$, $core(\tilde{u})$, is defined by $core(\tilde{u}) = \{x \in \mathbb{R}^n | \tilde{u}(x) = 1\}$.

Definition 2. *A fuzzy set on $\mathbb{R}$, $\tilde{u} : \mathbb{R} \longrightarrow [0, 1]$ is said to be a fuzzy interval or fuzzy number if the following properties are satisfied:*

1. *$\tilde{u}$ is normal, i.e. there exists $x_0 \in \mathbb{R}$ such that $\tilde{u}(x_0) = 1$;*
2. *$\tilde{u}$ is an upper semi-continuous function;*
3. $\tilde{u}(\lambda x + (1 - \lambda)y) \geq min\{\tilde{u}(x), \tilde{u}(y)\}$, $x, y \in \mathbb{R}$, $\lambda \in [0, 1]$;
4. $[\tilde{u}]^0$ *is compact.*

Let $\mathcal{F}_C$ denote the family of all fuzzy intervals on $\mathbb{R}$. So, for any $\tilde{u} \in \mathcal{F}_C$, we have that $[\tilde{u}]^\alpha \in \mathcal{K}_C$ for all $\alpha \in [0, 1]$ and we denote its α-levels by $[\tilde{u}]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, for all $\alpha \in [0, 1]$. A fuzzy interval is completely determined by $[\tilde{u}]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ satisfying certain conditions, [11].

Triangular fuzzy numbers are a special type of fuzzy numbers which are well determined by three real numbers $a \leq b \leq c$. We write $\tilde{b} = (a, b, c)$

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to denote the triangular fuzzy number $\tilde{b}$ with core or 1-level given by the singleton $\{b\}$ and whose α-levels sets are

$$\left[\tilde{b}\right]^{\alpha} = [a + (b - a)\alpha, c - (c - b)\alpha],$$

for all $\alpha \in [0, 1]$. A particular case of triangular fuzzy number (or fuzzy number) are the real numbers $a \in \mathbb{R}$ with membership function given by $\chi_{\{a\}}$, where χ_A denotes the characteristic function of the set A. Also considering the characteristic function we can see that any interval $A = [\underline{a}, \overline{a}]$ is a fuzzy interval, i.e. χ_A is a fuzzy interval (fuzzy number) such that $[\chi_A]^{\alpha} = A$, for all $\alpha \in [0, 1]$.

For fuzzy numbers, $\tilde{u}, \tilde{v} \in \mathcal{F}_C$, represented by $[\underline{u}_{\alpha}, \overline{u}_{\alpha}]$ and $[\underline{v}_{\alpha}, \overline{v}_{\alpha}]$ respectively, and for any real number λ, we define the addition $\tilde{u} + \tilde{v}$ and scalar multiplication $\lambda\tilde{u}$ as follows:

$$(\tilde{u} + \tilde{v})(x) = \sup_{y+z=x} \min\{\tilde{u}(y), \tilde{v}(z)\}, \quad (\lambda\tilde{u})(x) = \begin{cases} \tilde{u}\left(\frac{x}{\lambda}\right), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases}$$

It is well known that, for every $\alpha \in [0, 1]$,

$$[\tilde{u} + \tilde{v}]^{\alpha} = [(\underline{u + v})_{\alpha}, (\overline{u + v})_{\alpha}] = [\underline{u}_{\alpha} + \underline{v}_{\alpha}\ ,\ \overline{u}_{\alpha} + \overline{v}_{\alpha}], \tag{3}$$

$$[\lambda\tilde{u}]^{\alpha} = \left[(\underline{\lambda u})_{\alpha}, (\overline{\lambda u})_{\alpha}\right] = \lambda[\tilde{u}]^{\alpha} = \lambda\left[\underline{u}_{\alpha}, \overline{u}_{\alpha}\right] = \left[\min\{\lambda\underline{u}_{\alpha}, \lambda\overline{u}_{\alpha}\}, \max\{\lambda\underline{u}_{\alpha}, \lambda\overline{u}_{\alpha}\}\right]. \tag{4}$$

Definition 3. *([12]) Given two fuzzy intervals $\tilde{u}$ and $\tilde{v}$, the generalized Hukuhara difference (gH-difference for short) is the fuzzy interval $\tilde{w}$, if it exists, such that*

$$\tilde{u} \ominus_{gH} \tilde{v} = \tilde{w} \Leftrightarrow \begin{cases} & (i)\ \tilde{u} = \tilde{v} + \tilde{w}, \\ \text{or} & (ii)\ \tilde{v} = \tilde{u} + (-1)\tilde{w}. \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if w is a crisp number. Note that the case (i) coincides with Hukuhara difference (see [13]) and so the gH-difference concept is more general than the H-difference one.

If $\tilde{u} \ominus_{gH} \tilde{v}$ exists then, in terms of α-level sets, we have that

$$[\tilde{u} \ominus_{gH} \tilde{v}]^{\alpha} = [\tilde{u}]^{\alpha} \ominus_{gH} [\tilde{v}]^{\alpha} = [\min\{\underline{u}_{\alpha} - \underline{v}_{\alpha}\}, \max\{\overline{u}_{\alpha} - \overline{v}_{\alpha}\}], \tag{5}$$

for all $\alpha \in [0, 1]$, where $[u]^{\alpha} \ominus_{gH} [v]^{\alpha}$ denotes the gH-difference between two intervals (see [14, 12]).

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Given $\tilde{u}, \tilde{v} \in \mathcal{F}_C$, the distance between $\tilde{u}$ and $\tilde{v}$ is defined by

$$D(\tilde{u}, \tilde{v}) = \sup_{\alpha \in [0,1]} H\left([u]^\alpha, [v]^\alpha\right) = \sup_{\alpha \in [0,1]} \max\left\{\left|\underline{u}_\alpha - \underline{v}_\alpha\right|, \left|\overline{u}_\alpha - \overline{v}_\alpha\right|\right\}.$$

So $(\mathcal{F}_C, D)$ is a metric space [15].

Let us now consider the set of all vector fuzzy intervals, $\mathcal{F}_C^m$, i.e $\tilde{u} \in \mathcal{F}_C^m$ if $\tilde{u} = (\tilde{u}_1, ..., \tilde{u}_m)$ where each $\tilde{u}_i \in \mathcal{F}_C$. For any $\tilde{u} \in \mathcal{F}_C^m$ we have that

$$[\tilde{u}]^\alpha = \{x = (x_1, ..., x_m) \in \mathbb{R}^m | \tilde{u}_i(x_i) \geq \alpha, \quad i = 1, ..., m\}.$$

Geometrically it would be a cartesian product of closed intervals in $\mathbb{R}^m$

$$[\tilde{u}]^\alpha = \prod_{i=1}^m [\tilde{u}_i]^\alpha.$$

Lemma 1. *It is verified that* $[\tilde{u}]^\alpha \subseteq [u]^0$, *for all* $\alpha \in [0,1]$, $\forall \tilde{u} \in \mathcal{F}_C^m$.

Proof. If $x \in [\tilde{u}]^\alpha$ then $\tilde{u}_i(x_i) \geq \alpha \geq 0$, for all $i = 1, ...m$ and then $x \in [\tilde{u}]^0$.
■

We recall the usual order for fuzzy intervals based on α-level sets:

Definition 4. *For* $\tilde{u}, \tilde{v} \in \mathcal{F}_C$, *it is said that:*

- $\tilde{u} \preceq \tilde{v}$, *if for every* $\alpha \in [0,1]$, $[\tilde{u}]^\alpha \preceq [\tilde{v}]^\alpha$.
  If $\tilde{u} \preceq \tilde{v}$, $\tilde{v} \preceq \tilde{u}$ *then* $\tilde{u} = \tilde{v}$.
- $\tilde{u} \preceq \tilde{v}$ *if* $\tilde{u} \preceq \tilde{v}$ *and* $\exists \alpha_0 \in [0,1]$, *such that* $[\tilde{u}]^{\alpha_0} \preceq [\tilde{v}]^{\alpha_0}$.
- $\tilde{u} \prec \tilde{v}$ *if* $[\tilde{u}]^\alpha \prec [\tilde{v}]^\alpha$, $\forall \alpha \in [0,1]$.

For $\tilde{u}, \tilde{v} \in \mathcal{F}_C$ *if either* $\tilde{u} \preceq \tilde{v}$ *or* $\tilde{v} \preceq \tilde{u}$, *then it is said that* $\tilde{u}$ *and* $\tilde{v}$ *are comparable; otherwise they are incomparable.*

Note that $\preceq$ is a partial order relation on $\mathcal{F}_C$. So $\tilde{v} \succeq \tilde{u}$ instead of $\tilde{u} \preceq \tilde{v}$ can be written. We observe that if $\tilde{u} \prec \tilde{v}$ then $\tilde{u} \preceq \tilde{v}$ and then $\tilde{u} \preceq \tilde{v}$.

Henceforth, S denotes an open subset of $\mathbb{R}$. Let us consider $\tilde{f} : S \to \mathcal{F}_C$ a fuzzy function or fuzzy mapping. For each $\alpha \in [0,1]$, we associate with $\tilde{f}$ the interval-valued functions family $\tilde{f}_\alpha : S \to \mathcal{K}_C$ given by $\tilde{f}_\alpha(t) = [\tilde{f}(t)]^\alpha$. For any $\alpha \in [0,1]$, we denote

$$\tilde{f}_\alpha(t) = \left[\underline{f}_\alpha(t), \overline{f}_\alpha(t)\right] = \left[\underline{f}(\alpha, t), \overline{f}(\alpha, t)\right].$$

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Here, for each $\alpha \in [0,1]$ the real-valued endpoint functions $\underline{f}_\alpha, \overline{f}_\alpha : S \to \mathbb{R}$ are called lower and upper functions of $\tilde{f}$, respectively.

Once we have fixed the order we are going to use necessary to obtain optimal conditions, we will focus on our problem.

3. Multiobjective problem with fuzzy objective functions

In this paper, we consider the following multiobjective fuzzy mathematical programming problem:

$$\text{(P)} \quad \text{Min} \quad \left(\tilde{f}_1(x), ..., \tilde{f}_p(x)\right)$$

where $\tilde{f}_j : S \subseteq \mathbb{R} \to \mathcal{F}_C$, $j = 1, ..., p$ are fuzzy functions defined on S, an open non-empty subset in $\mathbb{R}$.

We need to interpret the meaning of "minimizer a vector fuzzy function". We are going to follow similar solution concept to non-dominated solution introduced by Pareto, and usually considered in real-valued multiobjective optimization.

Definition 5. *Let $\tilde{f} : S \to \mathcal{F}_C^p$ be a vector fuzzy function defined on S. It is said that $x^* \in S$ is an efficient solution or Pareto solution if there exists no $x \in S$ such that $\tilde{f}_j(x) \preceq \tilde{f}_j(x^*)$, $\forall j = 1, ..., p$ and $\exists k$ such that $\tilde{f}_k(x) \prec \tilde{f}_k(x^*)$.*

Notice that before definition coincides with the classic one when the functions are real-valued.

In [7] a *defuzzification function*, $\eta : \mathcal{F}_C \to \mathbb{R}$ and two order relations are defined

Definition 6. *Let $\tilde{u}$ and $\tilde{v}$ be fuzzy numbers. We write*

- *$\tilde{u} \preceq^1 \tilde{v}$ if the Hukuhara difference $\tilde{v} \ominus_H \tilde{u}$ exists and $\eta(\tilde{v} \ominus_H \tilde{u}) \geq 0$.*
- *$\tilde{u} \preceq^2 \tilde{v}$ if the Hukuhara difference $\tilde{v} \ominus_H \tilde{u}$ exists and $\tilde{v} \ominus_H \tilde{u}$ is nonnegative, where a fuzzy number $\tilde{a}$ is said nonnegative if $\underline{a}_\alpha \geq 0$ for all $\alpha \in [0,1]$.*

Notice that these definitions can only be applied to fuzzy numbers such that their Hukuhara difference exists, i.e. $\mu([\tilde{u}]^\alpha) \leq \mu([\tilde{v}]^\alpha)$, $\forall \alpha \in [0,1]$, where μ is the interval range.

Now we are going to relate those order relations on $\mathcal{F}_C$ with the one we propose to use "$\preceq$":

Proposition 1. *$\preceq^2$ and $\underline{\preceq}$ coincide when Hukuhara difference does exist.*

Proof. If $A, B \in \mathcal{K}_C$, $A \underline{\preceq} B$ if and only if $A \ominus_{gH} B \underline{\preceq} [0,0]$ because

$$A \ominus_{gH} B = \left[\min\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}, \max\left\{\underline{a} - \underline{b}, \overline{a} - \overline{b}\right\}\right].$$

If $A \underline{\preceq} B$ then $\underline{a} - \underline{b} \leq 0$ and $\overline{a} - \overline{b} \leq 0$. Thus $A \ominus_{gH} B \underline{\preceq} [0,0]$. And reciprocally.

In particular if the Hukuhara difference does exist then $\tilde{v} \ominus_H \tilde{u} = \tilde{v} \ominus_{gH} \tilde{u}$. If $\tilde{v} \ominus_{gH} \tilde{u}$ is nonnegative $[0,0] \underline{\preceq} [\tilde{v} \ominus_{gH} \tilde{u}]^\alpha$ for all $\alpha \in [0,1]$. From (5), $[0,0] \underline{\preceq} [\tilde{v}]^\alpha \ominus_{gH} [\tilde{u}]^\alpha$, therefore $[\tilde{u}]^\alpha \underline{\preceq} [\tilde{v}]^\alpha$ and $\tilde{u} \underline{\preceq} \tilde{v}$ from Definition 4. ∎

In [16] is shown that a minimum based on average index ordering relation is a minimum for α-level sets ordering relation when $\eta(\tilde{u}) = \int_Y \lambda \underline{u}_\alpha + (1 - \lambda)\overline{u}_\alpha dP(\alpha)$, where Y is a subset of the unit interval and P a probability distribution function on Y. In general:

Proposition 2. *If $\eta(\tilde{a}) \geq 0$ when $\tilde{a}$ is a nonnegative fuzzy number, then $\preceq^1$ is equivalent to $\underline{\preceq}$.*

Proof. Let us suppose that $\tilde{v} \ominus_H \tilde{u}$ exists, then $\tilde{u} \preceq^1 \tilde{v} \Leftrightarrow \eta(\tilde{v} \ominus_H \tilde{u}) \geq 0 \Leftrightarrow \eta(\tilde{v} \ominus_{gH} \tilde{u}) \geq 0 \Leftrightarrow \tilde{v} \ominus_{gH} \tilde{u}$ is nonnegative $\Leftrightarrow [0,0] \underline{\preceq} [\tilde{v} \ominus_{gH} \tilde{u}]^\alpha = [\tilde{v}]^\alpha \ominus_{gH} [\tilde{u}]^\alpha$ forall $\alpha \in [0,1] \Leftrightarrow \tilde{u} \underline{\preceq} \tilde{v}$. ∎

Definition 7. *An interval approximation of a fuzzy number is an operator $I : \mathcal{F}_C \to \mathcal{K}_C$ such that for $\tilde{u}, \tilde{v} \in \mathcal{F}_C$,*

(i) $I(\tilde{u}) \subset supp(\tilde{u})$,

(ii) $core(\tilde{u}) \subset I(\tilde{u})$,

(iii) $\forall \epsilon > 0, \exists \delta > 0 / D(\tilde{u}, \tilde{v}) < \delta \Rightarrow H(I(\tilde{u}), I(\tilde{v})) < \epsilon$.

Proposition 3. *[9] The interval*

$$N(\tilde{u}) = \left[\int_0^1 \underline{u}(x,\alpha)d\alpha, \int_0^1 \overline{u}(x,\alpha)d\alpha\right]$$

is an Interval Approximation of fuzzy number $\tilde{u}$ such that $N(\tilde{u})$ minimizes $D(\tilde{u}, I(\tilde{u}))$ for all I belonging to the space of interval approximation operators of fuzzy numbers. $N(\cdot)$ is called Nearest Interval Approximation (NIA).

Notice that $D(\tilde{u}, I(\tilde{u}))$ is the distance between fuzzy numbers, defined in $\mathcal{F}_C$ and, it can be considered here since each interval is also a fuzzy number with constat α-level sets for all $\alpha \in [0,1]$.

In [10] to find a solution of (P), the authors have been approximated it by the following interval multiobjective program:

$$\text{(P1)} \quad \text{Min} \quad (F_1(x), ..., F_p(x))$$

where $F_j(x)$ stands for the nearest interval approximation (NIA) of $\tilde{f}_j(x)$, i.e. $F_j(x) = \left[\int_0^1 \underline{f}_j(x,\alpha)d\alpha, \int_0^1 \overline{f}_j(x,\alpha)d\alpha\right]$, $x \in S$, that it is among interval approximation operators of a fuzzy number, the one that minimize the distance to the fuzzy number.

Definition 8. *[10] $x^* \in S$ is called a satisficing solution of (P) if it is a Pareto optimal solution of (P1), i.e., if there is no $x \in S$ such that $F_j(x) \preceq F_j(x^*)$ for all $j = 1, ..., p$ and $F(x) \neq F(x^*)$.*

Proposition 4. *If x^* is a satisficing solution of (P) then x^* is a Pareto solution for (P).*

Proof. Let us suppose that there exists $x \in S$ such that $\tilde{f}_j(x) \underline{\preceq} \tilde{f}_j(x^*)$. Then it follows that $\underline{f}_j(x,\alpha) \leq \underline{f}_j(x^*,\alpha)$ and $\overline{f}_j(x,\alpha) \leq \overline{f}_j(x^*,\alpha)$, $\forall\alpha$ and thus $F_j(x) \leq F_j(x^*)$ and besides there exists k such that $\underline{f}_k(x,\alpha) < \underline{f}_k(x^*,\alpha)$ and $\overline{f}_k(x,\alpha) < \overline{f}_k(x^*,\alpha)$, $\forall\alpha$ and thus $F_k(x) < F_k(x^*)$. It stands in contradiction to the hypothesis. ■

In the next section we give the convenient differentiability concept to get our objectives.

4. gH-Differentiable fuzzy functions

In Optimization Theory, when the functions are differentiable, their gradients allow us to find analytical optimality conditions. Different fuzzy differentiability concepts have been used in fuzzy optimization but they are more restrictive than the gH-differenciability. Notice that the gH-differentiability coincides with the H-differentiability [17], only when $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable and $(\underline{f}_\alpha)'(t) \leq (\overline{f}_\alpha)'(t)$ for all $\alpha \in [0,1]$. And G-differentiability implies gH-differentiability (see [19]).

Next, we present the gH-differentiable fuzzy function concept based on the gH-difference of fuzzy numbers.

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Definition 9. *([20]) The gH-derivative of a fuzzy function $\tilde{f} : S \to \mathcal{F}_C$ at $t_0 \in S$ is defined as*

$$\tilde{f}'(t_0) = \lim_{h \to 0} \frac{1}{h} \left[\tilde{f}(t_0 + h) \ominus_{gH} \tilde{f}(t_0) \right]. \tag{6}$$

If $\tilde{f}'(t_0) \in \mathcal{F}_C$ satisfying (6) exists, we say that $\tilde{f}$ is generalized Hukuhara differentiable (gH-differentiable, for short) at t_0.

The following results establish relationships between gH-differentiability of $\tilde{f}$ and the gH-differentiability of the interval-valued functions family $\tilde{f}_\alpha$, [22] and gH-differentiability of $\tilde{f}$ and the differentiability of its endpoint functions $\underline{f}_\alpha$ and $\overline{f}_\alpha$.

Theorem 1. *[22] If $\tilde{f} : S \to \mathcal{F}_C$ is gH-differentiable at $t_0 \in S$, then $\tilde{f}_\alpha$ is gH-differentiable at t_0 uniformly in $\alpha \in [0,1]$ and*

$$\tilde{f}'_\alpha(t_0) = [\tilde{f}'(t_0)]^\alpha,$$

for all $\alpha \in [0,1]$.

Theorem 2. *[23] Let $\tilde{f} : S \to \mathcal{F}_C$ be a fuzzy function. If $\tilde{f}$ is gH-differentiable at $t_0 \in S$ then the lateral derivatives of real-valued endpoints functions $(\underline{f}_\alpha)'_-(t_0)$, $(\underline{f}_\alpha)'_+(t_0)$, $(\overline{f}_\alpha)'_-(t_0)$ and $(\overline{f}_\alpha)'_+(t_0)$ exist, uniformly in $\alpha \in [0,1]$, and satisfy*

$$[\tilde{f}'(t_0)]^\alpha = \tilde{f}'_\alpha(t_0) = \left[(\underline{f}_\alpha)'_+(t_0) \bigvee (\overline{f}_\alpha)'_+(t_0) \right].$$

Remark 1. *As a consequence of before results we use the following notation:*

$$[\tilde{f}'(t_0)]^\alpha = \tilde{f}'_\alpha(t_0) = \left[(\tilde{f}'_\alpha(t_0))^L, (\tilde{f}'_\alpha(t_0))^U \right]$$

where

$$(\tilde{f}'_\alpha(t_0))^L = \min \left\{ (\underline{f}_\alpha)'_+(t_0), (\overline{f}_\alpha)'_+(t_0) \right\} = \min \left\{ (\underline{f}_\alpha)'_-(t_0), (\overline{f}_\alpha)'_-(t_0) \right\},$$

$$(\tilde{f}'_\alpha(t_0))^U = \max \left\{ (\underline{f}_\alpha)'_+(t_0), (\overline{f}_\alpha)'_+(t_0) \right\} = \max \left\{ (\underline{f}_\alpha)'_-(t_0), (\overline{f}_\alpha)'_-(t_0) \right\}.$$

Theorem 3. *[23] Let $\tilde{f} : S \to \mathcal{F}_C$ be a continuous fuzzy function in $(t_0 - \delta, t_0 + \delta) \subset S$ for some $\delta > 0$. Then $\tilde{f}$ is gH-differentiable at $t_0 \in S$ if and only if $(\underline{f}_\alpha)'_-(t_0)$, $(\underline{f}_\alpha)'_+(t_0)$, $(\overline{f}_\alpha)'_-(t_0)$ and $(\overline{f}_\alpha)'_+(t_0)$ exist, uniformly in $\alpha \in [0,1]$, and satisfy*

$$[\tilde{f}'(t_0)]^\alpha = \tilde{f}'_\alpha(t_0) = \left[(\underline{f}_\alpha)'_+(t_0) \bigvee (\overline{f}_\alpha)'_+(t_0)\right].$$

Definition 10. *Given a fuzzy vector function $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_p) \in \mathcal{F}_C^p$, we say that $\tilde{f}$ is a vector gH-differentiable fuzzy function at $t_0 \in S$ if and only if $\tilde{f}_j$ is gH-differentiable at t_0, for all $j = 1, ..., p$.*

We are now ready to proof the main results of this paper.

5. Necessary conditions for Pareto solutions

In this section we prove a necessary optimality condition for Pareto efficiency and give an algorithm based on it, in order to find the possible optima for our problem. With the necessary optimality conditions we can exclude feasible solutions that are not optimal. In Section 3 we have proved that the optimum concept used until now in the different strategies to solve a fuzzy multiobjective programming problem are more restrictive than the Pareto solution concept based on the α-level sets. So, the algorithm we develop here remains valid for the other optimum notions.

In [24] the author uses the same Pareto efficiency notion than in this paper and prove sufficient optimality conditions for it, but they use more restrictive hypotheses on the functions (level-wise differentiable functions) than the ones we suppose here and so, the necessary optimality condition presented here and the algorithm developed remain valid.

Remark 2. *For convenience, we introduce the following notations. Let $A = (A_1, ..., A_p)$ be with $A_j \in \mathcal{K}_C$. Let $\Lambda \in \mathcal{M}^{p\times 2}$, we denote by $\Lambda \times A$ the lineal combination*

$$\Lambda \times A = \sum_{j=1}^{p} \lambda_{j1}\underline{a}_j + \lambda_{j2}\overline{a}_j.$$

If $0 \in A_j$ for some $j = 1, ..., p$ then there exists $\Lambda \in \mathcal{M}^{p\times 2}$, $\lambda_{ji} \geq 0$ and not all zero, such that: $\Lambda \times A = 0$.

If $\Lambda \times A = 0$ with $\lambda_{ji} \geq 0$, but not all zero, there exist $\underline{\lambda}, \overline{\lambda}$, $(\underline{\lambda}, \overline{\lambda}) \geq 0$ such that

$$0 \in \left[\underline{\lambda}^T \underline{a}, \overline{\lambda}^T \overline{a} \right].$$

with $\underline{a} = (\underline{a}_1, ..., \underline{a}_p)$ and $\overline{a} = (\overline{a}_1, ..., \overline{a}_p)$.

Proposition 5. *Let $\tilde{f}$ be a vector gH-differentiable fuzzy function on S. If $x^* \in S$ is an efficient or Pareto solution for (P) then the following system does not have solution at $y \in \mathbb{R}$.*

$$y \left(\tilde{f}_j'(x^*) \right) \prec \tilde{0}^p \quad j = 1, ..., p. \tag{7}$$

Proof. Arguing by contradiction, let us suppose that $\exists y \in \mathbb{R}$ such that

$$y\tilde{f}_j'(x^*) \prec \tilde{0} \quad \forall j = 1, ...p.$$

Then,

$$y[\tilde{f}_j'(x^*)]^\alpha \prec [0,0] \quad \forall \alpha \in [0,1] \quad \forall j = 1, ..., p.$$

From Theorem 2, for each j there exist $(\underline{f_j}(\alpha, (x^*))'_+$ and $(\overline{f_j}(\alpha, x^*))'_+$, uniformly in $\alpha \in [0,1]$ and they satisfy

$$[\tilde{f}_j'(x^*)]^\alpha = \left[\min \left\{ (\underline{f_j}(\alpha, x^*))'_+, (\overline{f_j}(\alpha, x^*))'_+ \right\}, \max \left\{ (\underline{f_j}(\alpha, x^*))'_+, (\overline{f_j}(\alpha, x^*))'_+ \right\} \right]$$

Since

$$y \left[f_j'(x^*) \right]^\alpha \prec [0,0] \Leftrightarrow \begin{cases} y(\overline{f_j}(\alpha, x^*))'_+ < 0 \\ y(\underline{f_j}(\alpha, x^*))'_+ < 0 \end{cases}, \quad \forall \alpha \in [0,1].$$

From

$$y(\underline{f_j}(\alpha, x^*))'_+ = \lim_{t \to 0^+} \frac{1}{t} (\underline{f_j}(\alpha, x^* + yt) - \underline{f_j}(\alpha, x^*)) < 0$$

So it follows that there exist $\underline{\epsilon_j^+} > 0$, such that for all t, with $0 < t < \underline{\epsilon_j^+}$

$$\underline{f_j}(\alpha, x^* + yt) - \underline{f_j}(\alpha, x^*) < 0 \quad \forall \alpha \in [0,1],$$

$$\underline{f_j}(\alpha, x^*) - \underline{f_j}(\alpha, x^*) < 0, \quad \forall \alpha \in [0,1]. \tag{8}$$

And analogously there exists $\overline{\epsilon_j^+} > 0$ such that for all t, with $0 < t < \overline{\epsilon_j^+}$

$$\overline{f_j}(\alpha, x^* + yt) - \overline{f_j}(\alpha, x^*) < 0 \quad \forall \alpha \in [0,1],$$

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$$\overline{f_j}(\alpha, x^*) - \overline{f_j}(\alpha, x^*) < 0, \quad \forall \alpha \in [0,1]. \tag{9}$$

Taking $\epsilon = \min\{\underline{\epsilon_j^+}, \overline{\epsilon_j^+} : \quad j = 1, ..., p\}$, $t \in (0, \epsilon)$ and from (8) and (9):

$$\underline{f}(\alpha, x) - \underline{f}(\alpha, x^*) < 0 \quad \text{and} \quad \overline{f}(\alpha, x) - \overline{f}(\alpha, x^*) < 0, \quad \forall \alpha \in [0,1],$$

where we suppose that $x \in S$. Hence, $\exists x \in S$ and $\tilde{f}(x) \prec \tilde{f}(x^*)$, and this is a contradiction to x^* is an efficient solution for $\tilde{f}$. ■

Now, we prove the main result of this section.

Theorem 4. *Let $\tilde{f} : S \to \mathcal{F}_C^p$ be a vector gH-differentiable fuzzy function at $x^* \in S$. If x^* is an efficient or Pareto solution for (P), then there exists a non-negative matrix $\Lambda \in \mathcal{M}^{p\times 2}$ such that*

$$\Lambda \times \left[\tilde{f}'(x^*)\right]^0 = 0. \tag{10}$$

Proof. If x^* is a Pareto solution for $\tilde{f}$, then (7) does not have solution. From Lemma 1,

$$y\left[\tilde{f}'(x^*)\right]^\alpha \prec [0,0], \quad \forall \alpha \in [0,1] \quad \forall j = 1, ..., p \Leftrightarrow$$

$$y\left[\tilde{f}_j'(x^*)\right]^0 \prec [0,0] \quad \forall j = 1, ..., p.$$

Now, let us consider the following lineal system and let us see it has no solution:

$$\left.\begin{array}{l} yA < 0 \\ yB < 0 \end{array}\right\} \tag{11}$$

where $A, B \in \mathcal{M}^{p\times 1}$ and

$$A = \begin{pmatrix} \left((\tilde{f}_1')_0(x^*)\right)^L \\ \vdots \\ \left((\tilde{f}_p')_0(x^*)\right)^L \end{pmatrix}, \quad B = \begin{pmatrix} \left((\tilde{f}_1')_0(x^*)\right)^U \\ \vdots \\ \left((\tilde{f}_p')_0(x^*)\right)^U \end{pmatrix}.$$

If (11) has solution then the system (7) also would have solution and this is impossible by Proposition 5.

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Since (11) is a system of linear inequalities and it does not have solution , from Gordan's alternative theorem, there exist $\alpha, \beta \in \mathbb{R}^p$ with $\alpha \geqq 0$, $\beta \geqq 0$ but not all zero, such that

$$A^T\alpha + B^T\beta = 0 \Leftrightarrow \sum_{j=1}^{p} \alpha_j \left((\tilde{f}_j{}')_0(x^*) \right)^L + \beta_j \left((\tilde{f}_j')_0(x^*) \right)^U = 0. \qquad (12)$$

Redefining $\Lambda = (\alpha_j, \beta_j)$ we obtain that there exits $\Lambda \in \mathcal{M}^{p\times 2}$ such that

$$\Lambda \times \left[\tilde{f}'(x^*) \right]^0 = 0.$$

And the proof is completed. ■

Remark 3. *If there exists $j = 1, ..., p$ such that $0 \in [\tilde{f}_j'(x^*)]^0$ then (12) is verified, and so to identify possible candidates for Pareto solutions is reduced to identify those feasible solutions whose 0-level set of the derivative contain the zero element. It coincides with the result for an unique objective function given by Osuna-Gómez et al. [16].*

Remark 4. *Expression (12) is equivalent to the existence of a positive linear combination of*

$$\{ (\underline{f'}_j)_+(x^*, 0), (\overline{f'}_j)_+(x^*, 0), \quad j = 1, ..., p \}$$

equal to zero. That means that (12) is equivalent to the existence of $\underline{\lambda}, \overline{\lambda} \in \mathbb{R}^p$, $\underline{\lambda}, \overline{\lambda} \geqq 0$ not both zero such that

$$\underline{\lambda}^T (\tilde{f}')_0^L(x^*) + \overline{\lambda}^T (\tilde{f}')_0^U(x^*) = 0.$$

From Theorems 2, 4 and Remark 4, an algorithm to identify the Pareto solutions candidates can be design, in the same form than in classical Mathematical Programming we solve the equations where the gradients are equal to zero in order to identify the possible optima.

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Algorithm

Step 1. Start and input p (Objective function $(\tilde{f}_1(x), ..., \tilde{f}_p(x))$)
Step 2. Put $j = 1$, $P = \emptyset$, $P_+ = \emptyset$, $P_- = \emptyset$
Step 3. While $j \leq p$
  Step 3.1 $P_+(j) = P_-(j) = \emptyset$.
  Step 3.2 Calculate $\underline{f}'_j(x, 0)_+, \overline{f}'_j(x, 0)_+$
  Step 3.3 Define and find $l_j(x) = min\{\underline{f}'_j(x, 0)_+, \overline{f}'_j(x, 0)_+\}$,
    $u_j(x) = max\{\underline{f}'_j(x, 0)_+, \overline{f}'_j(x, 0)_+\}$
  Step 3.4 Put $[l_j(x), u_j(x)]$
  Step 3.5 For x such that $l_j(x) \leq 0 \leq u_j(x)$ then $x \in P$.
  Step 3.6 For x such that $l_j(x) \geq 0$ then $x \in P_+(j)$
  Step 3.7 For x such that $u_j(x) \leq 0$ then $x \in P_-(j)$
  Step 3.8 $j = j + 1$
Step 4. Set $P_+ = \bigcup_{j=1}^{p} P_+(j)$ and $P_- = \bigcup_{j=1}^{p} P_-(j)$. If $x \in P_+ \bigcap P_-$ then $x \in P$
Step 5. Print P= possible Pareto solutions set and stop.

Example 1. *We look for the Pareto optimal solutions for*

$$(P) \quad Min \quad \left(\tilde{f}_1(x), \tilde{f}_2(x)\right), x \in \mathbb{R}$$

- *In step 1, let us consider two fuzzy functions* $\tilde{f}_1, \tilde{f}_2 : \mathbb{R} \to \mathcal{F}_C$, *whose* α-*level sets are given by*

  $[\tilde{f}_1(x)]^\alpha = [(x - 2 + \alpha, (x - \alpha)^2], [\tilde{f}_2(x)]^\alpha = [1 + \alpha, 2(3 - \alpha)]x, \alpha \in [0, 1]$

  respectively because $\tilde{f}_2(x) = Cx$ *where* C *is a fuzzy interval.*

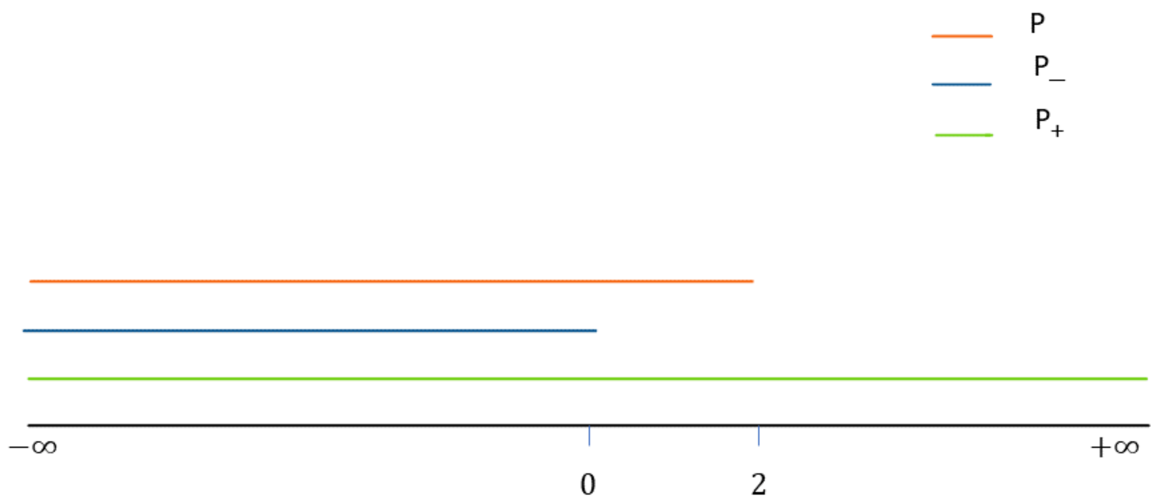
- *In step 2, for* $j = 1$, *we begin with* $P = \emptyset$, $P_+ = \emptyset$ *and* $P_- = \emptyset$.

- *In step 3, for* $j = 1$, $[\tilde{f}'_1(x)]^0 = [min\{2x - 4, 2x\}, max\{2x - 4, 2x\}]$.
  Then $l_1(x) = 2x - 4 \leq 0 \leq u_1(x) = 2x$ *so we get that* $P = [0, 2]$ *is a possible Pareto efficient solution for* (P), $P_+(1) = [2, +\infty)$ *and* $P_-(1) = (-\infty, 0]$

  For $j = 2$, $[\tilde{f}'_2(x)]^0 = [1, 6]$ *and so* $P = \emptyset, P_+(2) = \mathbb{R}$ *and* $P_-(2) = \emptyset$

- *In Step 4,* $P_+ = P_+(1) \bigcup P_+(2) = \mathbb{R}$ *and* $P_- = P_-(1) \bigcup P_-(2) = (-\infty, 0]$

- *In step 5, in summary, we get that* $x \in (-\infty, 2]$ *is a possible Pareto efficient solution for* (P).



6. Conclusions

In this paper we have addressed the resolution of a fuzzy multiobjective programming problem using the level sets optimization.

The results presented in this paper lead to the following conclusions:

- We have presented the gH-differentiable fuzzy function concept based on gH-difference of fuzzy numbers to fuzzy multiobjective optimization problem.
- We have obtained a necessary optimality condition for Pareto efficiency. We have completed and improved the results achieved by Wu [24] and Osuna-Gómez et al. [16].
- We have provided an algorithm and an example based on it.

In our opinion, future work will focus on algorithms or software that reflect the theoretical results achieved here, and identify further applications to real-world situations.

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