New results on the (SSIE) with operator of the form $F_\Delta \subset \mathcal{E} + F'_x$ involving the spaces of strongly summable and convergent sequences by the Cesàro method

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Abstract

Given any sequence $a = (a_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_a for the set of all sequences $y = (y_n)_{n\geq 1}$ such that $y/a = (y_n/a_n)_{n\geq 1} \in E$. In this paper, we use the spaces w_∞, w_0 and w of strongly bounded, summable to zero and summable sequences, that are the sets of all sequences y such that $\left(n^{-1}\sum_{k=1}^{n}|y_k|\right)_n$ is bounded, tends to zero and such that $y - le \in w_0$, for some scalar l, respectively, (cf. [24, 22]). These sets where used in the statistical convergence, (cf. [17, Chapter 4]). Then we deal with the solvability of each of the (SSIE) $F_\Delta \subset \mathcal{E} + F'_x$ where $\mathcal{E}$ is a linear space of sequences, $F = c_0$, c, ℓ_∞, w_0, w, or w_∞ and $F' = c_0$, c, or ℓ_∞. For instance, the solvability of the (SSIE) $w_\Delta \subset w_0 + s_x^{(c)}$ consists in determining the set of all sequences $x = (x_n)_{n\geq 1} \in U^+$ that satisfy the following statement. For every sequence y that satisfy the condition $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}|y_k - y_{k-1} - l| = 0$, there are two sequences u and v, with $y = u + v$ such that $\lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}|u_k| = 0$ and $\lim_{n\to\infty}(v_n/x_n) = L$ for some scalars l and L. These results extend those stated in [11, 12, 10].

Key words: BK space, matrix transformations, multiplier of sequence spaces, sequence spaces inclusion equations.

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1 Introduction

We write ω for the set of all complex sequences $y = (y_k)_{k\geq 1}$, ℓ_∞, c and c_0 for the sets of all bounded, convergent and null sequences, respectively, also

$\ell^p = \{y \in \omega : \sum_{k=1}^{\infty} |y_k|^p < \infty\}$ for $1 \leq p < \infty$. If y, $z \in \omega$, then we write $yz = (y_n z_n)_{n \geq 1}$. Let $U = \{y \in \omega : y_n \neq 0\}$, $U^+ = \{y \in \omega : y_n > 0\}$. We write $z/u = (z_n/u_n)_{n \geq 1}$ for all $z \in \omega$ and all $u \in U$, in particular $1/u = e/u$, where e is the sequence with $e_n = 1$ for all n. Finally, if $a \in U^+$ and E is any subset of ω, then we put $E_a = (1/a)^{-1} * E = \{y \in \omega : y/a \in E\}$. Let E and F be subsets of ω. In [4], the sets s_a, s_a^0 and $s_a^{(c)}$ were defined for positive sequences a by $(1/a)^{-1} * E$ and $E = \ell_\infty$, c_0 and c, respectively. In [6] we defined the sum $E_a + F_b$ and the product $E_a * F_b$ were defined where E, F are any of the symbols s, s^0, or $s^{(c)}$. Recall that *the spaces* w_∞ *and* w_0 *of strongly bounded and summable to zero sequences by the Cesàro method,* are the sets of all y such that $\left(n^{-1} \sum_{k=1}^{n} |y_k|\right)_n$ is bounded and tend to zero respectively. In this way, Hardy and Littlewood [22], defined the set w *of strongly convergent sequences by the Cesàro method*, for real numbers as follows. A sequence y is said to be *strongly Cesàro convergent to* L, *if* $y - Le \in w_0$. These spaces were studied by Maddox [24], Malkowsky, Rakočević [27] and Malkowsky, Başar in [2]. In [13, 8, 21, 15] we gave some properties of well known operators defined by the sets $W_a = (1/a)^{-1} * w_\infty$ and $W_a^0 = (1/a)^{-1} * w_0$. In this paper, we deal with special *sequence spaces inclusion equations (SSIE),* which are determined by an inclusion, for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $\left(E_{f(x)}\right)_T$ where f maps U^+ to itself, E is any linear space of sequences and T is a triangle, (cf. [12, 11, 10, 17]). Some results on the (SSIE) were stated in [16, 11, 12, 10], the results stated in [11], [12] and [10], were put together in [17]. In [11] we dealt with the class of (SSIE) of the form $F \subset E_a + F'_x$ where $F \in \{c_0, \ell^p, w_0, w_\infty\}$ and E, F' are any of the sets c_0, c, s_1, ℓ^p, w_0, or w_∞ with $p \geq 1$. Then we stated some results on the solvability of the corresponding (SSIE) in the particular case when $a = (r^n)_n$ and we dealt with the case when $F = F'$. In [12] we dealt with the (SSIE) of the form $F \subset E_a + F'_x$ with $e \in F$ and we determined the solutions of these (SSIE) when $a = (r^n)_{n \geq 1}$, F is either c, or s_1 and E, F' are any of the sets c_0, c, s_1, ℓ^p, w_0, or w_∞ with $p \geq 1$. Then we solved each of the (SSIE) $c \subset D_r * E_\Delta + c_x$, with $E \in \{c_0, c, s_1\}$, and the (SSIE) $s_1 \subset D_r * (s_1)_\Delta + s_x$. We also studied the (SSIE) $c \subset D_r * E_{C_1} + s_x^{(c)}$ with $E \in \{c, s_1\}$ and $s_1 \subset D_r * (s_1)_{C_1} + s_x$ where C_1 is the Cesàro operator defined by $(C_1)_n y = n^{-1} \sum_{k=1}^{n} y_k$ for all y, and we dealt with the solvability of the (SSE) associated with the previous (SSIE) and defined by $D_r * E_{C_1} + s_x^{(c)} = c$ with $E \in \{c_0, c, s_1\}$ and $D_r * E_{C_1} + s_x = s_1$ with $E \in \{c, s_1\}$. In [10] we dealt with the solvability of the (SSIE) of the form $\ell_\infty \subset \mathcal{E} + F'_x$ where $\mathcal{E}$ is a given linear space of sequences and F' is either c_0, or ℓ_∞. Then, for given linear space $\mathcal{E}$ of sequences, we solved each of the (SSIE) $c_0 \subset \mathcal{E} + s_x$ and $c \subset \mathcal{E} + s_x^{(c)}$ and the (SSE) $\mathcal{E} + s_x^{(c)} = c$.

In this paper, we use the difference sequence spaces $(c_0)_\Delta$, c_Δ and $(\ell_\infty)_\Delta$ introduced by Kizmaz, (cf. [23]) and we deal with the solvability of each of the (SSIE)

$$F_\Delta \subset \mathcal{E} + F'_x,$$

where $F = c_0$, c, ℓ_∞, w_0, w_∞, or w, and $F' = c_0$, c, or ℓ_∞ and $\mathcal{E}$ is a linear space of sequences.

This paper is organized as follows. In Section 2, we recall some well known results on sequence spaces and matrix transformations. In Section 3, we recall some results on the multipliers of some sets. In Section 4, we recall some results used for the solvability of the (SSIE). In Section 5, we deal with the solvability of the (SSIE) with operator to solve each of the (SSIE) of the form $c_\Delta \subset \mathcal{E} + F'_x$, $(c_0)_\Delta \subset \mathcal{E} + F'_x$ and $(\ell_\infty)_\Delta \subset \mathcal{E} + F'_x$ with $F' = c_0$, c, or ℓ_∞. In Section 6, we study each of the (SSIE) $(w_\infty)_\Delta \subset \mathcal{E} + F'_x$ where $F' = c_0$, c, or ℓ_∞. Finally, in Section 7, we study the solvability of the (SSIE) $F_\Delta \subset \mathcal{E} + F'_x$ where $F = w_0$, or w and $F' = c_0$, c, or ℓ_∞.

2 Preliminaries and notations

An FK space is a *complete metric space*, for which convergence implies *coordinatewise convergence*. A *BK space* is a Banach space of sequences that is, an *FK space*. A BK space E *is said to have AK if for every sequence* $y = (y_k)_{k\geq 1} \in E$, *then* $y = \lim_{p\to\infty} \sum_{k=1}^{p} y_k e^{(k)}$, *where* $e^{(k)} = (0, ..., 1, ...)$, 1 *being in the* $k-th$ *position.*

For a given infinite matrix $A = (\mathbf{a}_{nk})_{n,k\geq 1}$ we define the operators $A_n = (\mathbf{a}_{nk})_{k\geq 1}$ for any integer $n \geq 1$, by $A_n y = \sum_{k=1}^{\infty} \mathbf{a}_{nk} y_k$, where $y = (y_k)_{k\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator A defined by $Ay = (A_n y)_{n\geq 1}$ mapping between sequence spaces. When A *maps* E *into* F, *where* E *and* F *are subsets of* ω, *we write* $A \in (E, F)$, (cf. [24, 3, 30, 26]). It is well known that if E has AK then, the set $\mathcal{B}(E)$ of all *bounded linear operators* L mapping in E, with norm $\|L\| = \sup_{y\neq 0} \left(\|L(y)\|_E / \|y\|_E\right)$ satisfies the identity $\mathcal{B}(E) = (E, E)$. We denote by ω, c_0, c, ℓ_∞ the sets of all sequences, the sets of null, convergent and bounded sequences. For any subset F of ω, we write $F_A = \{y \in \omega \ : \ Ay \in F\}$ and for any subset E of ω we write $AE = \{y \in \omega \ : \text{there is } x \in E \text{ such that } y = Ax\}$. Then, for given sequence $u \in \omega$ we define the diagonal matrix D_u by $[D_u]_{nn} = u_n$ for all n. It is interesting to rewrite the set E_u using a diagonal matrix. Let E be any subset of ω and $u \in U^+$ we have $E_u = D_u * E = \{y = (y_n)_{n\geq 1} \in \omega \ : y/u \in E\}$. We use the sets s_a^0, $s_a^{(c)}$ and s_a defined as follows (cf. [4, 5]). For given $a \in U^+$ we put $D_a * c_0 = s_a^0$, $D_a * c = s_a^{(c)}$ and $D_a * \ell_\infty = s_a$. We frequently write c_a instead of $s_a^{(c)}$ to simplify. Each of the spaces $D_a * E$, where $E \in \{c_0, c, \ell_\infty\}$ is a *BK space normed* by $\|y\|_{s_a} = \sup_{n\geq 1} (|y_n| / a_n)$ and s_a^0 *has AK*. If $a = (R^n)_{n\geq 1}$ with $R > 0$, then we write s_R, s_R^0 and $s_R^{(c)}$, for the sets s_a, s_a^0 and $s_a^{(c)}$, respectively. We also write D_R for $D_{(R^n)_{n\geq 1}}$. When $R = 1$, we obtain $s_1 = \ell_\infty$, $s_1^0 = c_0$ and $s_1^{(c)} = c$. Recall that $S_1 = (s_1, s_1)$ is a *Banach algebra* and $(c_0, s_1) = (c, \ell_\infty) = (s_1, s_1) = S_1$. We have $A \in S_1$ if and only if $\sup_n \left(\sum_{k=1}^{\infty} |\mathbf{a}_{nk}|\right) < \infty$. Recall the Schur's result (cf. [30, Theorem 1.17.8, p. 15]) on the class (s_1, c). We have $A \in (s_1, c)$ if and only if $\lim_{n\to\infty} \mathbf{a}_{nk} = l_k$ for some scalar l_k, $k = 1$, 2,..., and $\lim_{n\to\infty} \sum_{k=1}^{\infty} |\mathbf{a}_{nk}| = \sum_{k=1}^{\infty} |l_k|$, where the series $\sum_{k=1}^{\infty} |l_k|$ is convergent.

We also use the following two lemmas, where the infinite matrix $\mathcal{T}$ is said to be a triangle if $\mathcal{T}_{nk} = 0$ for $k > n$ and $\mathcal{T}_{nn} \neq 0$ for all n.

Lemma 1 *[7, Lemma 9, p. 45] Let $\mathcal{T}'$ and $\mathcal{T}''$ be any given triangles and let E, $F \subset \omega$. Then, for any given operator $\mathcal{T}$ represented by a triangle we have $\mathcal{T} \in (E_{\mathcal{T}'}, F_{\mathcal{T}''})$ if and only if $\mathcal{T}''\mathcal{T}\mathcal{T}'^{-1} \in (E, F)$.*

Taking $T' = D_{1/a}$ and $\mathcal{T}'' = D_b$ for $a, b \in U^+$ we obtain the next well-known result.

Lemma 2 *Let a, $b \in U^+$ and let E, $F \subset \omega$ be any linear spaces. We have $A \in (E_a, F_b)$ if and only if $D_{1/b}AD_a \in (E, F)$.*

3 On the triangle $C(\lambda)$ and on the multipliers of special sets

In this section, we define the spaces of *strongly bounded and summable sequences by the Cesàro method*. Then we recall some results on the multipliers of sequence spaces involving the previous spaces.

3.1 On the triangles $C(\lambda)$ and $\Delta(\lambda)$ and the sets w_0, w and w_∞.

For $\lambda \in U$ the infinite matrices $C(\lambda)$ and $\Delta(\lambda)$ are triangles defined as follows. We have $[C(\lambda)]_{nk} = 1/\lambda_n$ for $k \leq n$, this triangle was used, for instance in [20, 18], see also the Rhaly matrix studied by [28, 29]). Then, the nonzero entries of $\Delta(\lambda)$ are determined by $[\Delta(\lambda)]_{nn} = \lambda_n$ for all n, and $[\Delta(\lambda)]_{n,n-1} = -\lambda_{n-1}$ for all $n \geq 2$. It can be shown that the matrix $\Delta(\lambda)$ is the inverse of $C(\lambda)$, that is, $C(\lambda)(\Delta(\lambda)y) = \Delta(\lambda)(C(\lambda)y) = y$ for all $y \in \omega$. If $\lambda = e$ we obtain the well known operator of the first difference represented by $\Delta(e) = \Delta$. We then have $\Delta_n y = y_n - y_{n-1}$ for all $n \geq 1$, with the convention $y_0 = 0$. We have $\Sigma = C(e)$ and then, we may write $C(\lambda) = D_{1/\lambda}\Sigma$. Note that $\Delta = \Sigma^{-1}$. The *Cesàro operator* is defined by $C_1 = C\left((n)_{n\geq 1}\right)$. In the following, we use the inverse of C_1 defined by $C_1^{-1} = \Delta(\lambda)$ where $\lambda = (n)_{n\geq 1}$. We use the set of sequences that are *$a-$strongly bounded and $a-$strongly convergent to zero,* defined for $a \in U^+$ by $W_a = \left\{y \in \omega : \sup_n \left(n^{-1}\sum_{k=1}^n |y_k|/a_k\right) < \infty\right\}$, and $W_a^0 = \left\{y \in \omega : \lim_{n\to\infty}\left(n^{-1}\sum_{k=1}^n |y_k|/a_k\right) = 0\right\}$, (cf. [21, 15]). For $a = (r^n)_{n\geq 1}$ the set W_a and W_a^0 are denoted by W_r and W_r^0. For $r = 1$ we obtain the well-known *spaces* w_∞ and w_0 *of strongly bounded and strongly null sequences by the Cesàro method* (cf. [25]).

3.2 On the multipliers of some sets.

First, we need to recall some well known results. Let y and z be sequences and let E and F be two subsets of ω, we then write $M(E, F) = \{y \in \omega : \; yz \in F \text{ for all } z \in E\}$, the set $M(E,F)$ is called the *multiplier space of* E *and* F. We will use the next lemmas.

Lemma 3 *Let* E, $\widetilde{E}$, F *and* $\widetilde{F}$ *be arbitrary subsets of* ω. *Then (i)* $M(E,F) \subset M\left(\widetilde{E},F\right)$ *for all* $\widetilde{E} \subset E$. *(ii)* $M(E,F) \subset M\left(E,\widetilde{F}\right)$ *for all* $F \subset \widetilde{F}$.

Lemma 4 *Let* a, $b \in U^+$ *and let* E *and* F *be two subsets of* ω. *Then we have* $D_a * E \subset D_b * F$ *if and only if* $a/b \in M(E,F)$.

From Lemma 2 we obtain the next result.

Lemma 5 *[**??**, Corollary 3.2, p. 4] Let* a, $b \in U^+$. *Then we have: (i)* $M\left(s_a^0, \chi_b'\right) = s_{b/a}$ *where* χ' *is any of the symbols* s^0, $s^{(c)}$, *or* s. *(ii)* $M(\chi_a, s_b) = s_{b/a}$ *where* χ *is any of the symbols* $s^{(c)}$, *or* s. *(iii)* $M\left(s_a, s_b^{(c)}\right) = M\left(s_a, s_b^0\right) = s_{b/a}^0$ *and* $M\left(s_a^{(c)}, s_b^{(c)}\right) = s_{b/a}^{(c)}$.

In the following, we use the results stated below, (cf. [11, Lemma 6, pp. 214-215]).

Lemma 6 *We have: (i) (a)* $M(c,c_0) = M(\ell_\infty,c) = M(\ell_\infty,c_0) = c_0$ *and* $M(c,c) = c$. *(b)* $M(E,\ell_\infty) = M(c_0,F) = \ell_\infty$ *for* E, $F = c_0$, c, *or* ℓ_∞. *(ii) (a)* $M(w_\infty,\ell_\infty) = M(w_0,F) = s_{(1/n)_{n\geq 1}}$ *for* $F = c_0$, c, *or* ℓ_∞. *(b)* $M(w_\infty,c_0) = M(w_\infty,c) = s^0_{(1/n)_{n\geq 1}}$. *(c)* $M(E,w_0) = w_0$ *for* $E = s_1$, *or* c. *(d)* $M(E,w_\infty) = w_\infty$ *for* $E = c_0$, s_1, *or* c.

To state results on the multipliers involving the set w, we need the next elementary lemmas.

Lemma 7 *We have* $w \subset s^0_{(n)_{n\geq 1}}$.

Proof. Let $y \in w$. Then, by the inequality $n^{-1}|y_n - l| \leq n^{-1}\sum_{k=1}^n |y_k - l|$ for some scalar l and for all n, we deduce $n^{-1}|y_n - l| \to 0$ $(n \to \infty)$, and since $n^{-1}|y_n| \leq n^{-1}|y_n - l| + n^{-1}|l|$ we conclude $y \in s^0_{(n)_{n\geq 1}}$ and $w \subset s^0_{(n)_{n\geq 1}}$. ■

Lemma 8 *We have* $M(w,\ell_\infty) = M(w,c) = M(w,c_0) = s_{(1/n)_{n\geq 1}}$.

Proof. By Lemma 7, we have $M\left(s^0_{(n)_{n\geq 1}}, c_0\right) \subset M(w,c_0)$ and by Part (i) of Lemma 5 we have $s_{(1/n)_{n\geq 1}} = M\left(s^0_{(n)_{n\geq 1}}, c_0\right) \subset M(w,c_0)$. Then, using Part (ii) (a) of Lemma 6, we conclude

$$s_{(1/n)_{n\geq 1}} \subset M(w,c_0) \subset M(w,c) \subset M(w,\ell_\infty) \subset M(w_0,\ell_\infty) = s_{(1/n)_{n\geq 1}},$$

This completes the proof. ■

Remark 9 *By [14, Remark 3.4] we have* $M(w_0,w_\infty) = M(w_\infty,w_\infty) = \ell_\infty$.

4 On the sequence spaces inclusions.

In this section, we are interested in the study of the set of all positive sequences x that satisfy the inclusion $F \subset \mathcal{E} + F'_x$ where $\mathcal{E}$, F and F' are linear spaces of sequences. We may consider this problem as a *perturbation problem*. If we know the set $M(F, F')$, then the solutions of the *elementary inclusion* $F'_x \supset F$ are determined by $1/x \in M(F, F')$. Now, the question is: let $\mathcal{E}$ be a linear space of sequences. What are the solutions of the *perturbed* inclusion $F'_x + \mathcal{E} \supset F$? An additionnal question may be the following one: what are the conditions on $\mathcal{E}$ under which the solutions of the elementary and the perturbed inclusions are the same ?

4.1 Some definitions and results used for the solvability of some (SSIE).

In the following, we use the notation $\mathcal{I}(\mathcal{E}, F, F') = \{x \in U^+ : F \subset \mathcal{E} + F'_x\}$, where $\mathcal{E}$, F and F' are linear spaces of sequences and $a \in U^+$. We can state the next elementary results.

Lemma 10 *Let $\mathcal{E}$, $\mathcal{E}_1$, F, F', $\mathcal{F}$ and F'' be linear spaces of sequences. Then we have: (i) If $\mathcal{E}_1 \subset \mathcal{E}$, then $\mathcal{I}(\mathcal{E}_1, F, F') \subset \mathcal{I}(\mathcal{E}, F, F')$. (ii) If $\mathcal{F} \subset F$, then $\mathcal{I}(\mathcal{E}, F, F') \subset \mathcal{I}(\mathcal{E}, \mathcal{F}, F')$. (iii) If $F' \subset F''$, then $\mathcal{I}(\mathcal{E}, F, F') \subset \mathcal{I}(\mathcal{E}, F, F'')$.*

For any set χ of sequences we let $\overline{\chi} = \{x \in U^+ : 1/x \in \chi\}$. Then we write $\Phi = \{c_0, c, \ell_\infty, w_0, w, w_\infty\}$. By $c(1)$ we define the set of all sequences $\alpha \in U^+$ that satisfy the condition $\lim_{n\to\infty} \alpha_n = 1$. Then we consider the condition

$$G \subset G_{1/\alpha} \text{ for all } \alpha \in c(1), \tag{1}$$

for any given linear space G of sequences. Notice that condition (1) is satisfied for all $G \in \Phi$. Then we denote by U_1^+ the set of all sequences α with $0 < \alpha_n \leq 1$ for all n. We consider the condition

$$G \subset G_{1/\alpha} \text{ for all } \alpha \in U_1^+. \tag{2}$$

for any given linear space G of sequences. To show some results on the (SSIE), we introduce a linear space of sequences H which contains the spaces E and F' and we will use the fact that if H satisfies the condition in (2) then we have $H_a + H_b = H_{a+b}$ for all a, $b \in U^+$ (cf. [14, Proposition 5.1, pp. 599-600]). Notice that c does not satisfy this condition, but each of the sets c_0, ℓ_∞, ℓ^p, w_0 and w_∞ satisfies the condition in (2). So we have for instance $s_a^0 + s_b^0 = s_{a+b}^0$ and $W_a + W_b = W_{a+b}$.

4.2 Some properties of the set $\mathcal{I}(\mathcal{E}, F, F')$.

We need the next lemma involving the multiplier of F and F', which is an extension of Lemma 10.

Lemma 11 *Let $\mathcal{E}$, $\mathcal{E}_0$, F, $\mathcal{F}$ and F' be linear spaces of sequences. Then we have: (i) $\overline{M(F,F')} \subset \mathcal{I}(\mathcal{E},F,F')$. (ii) If $\mathcal{I}(\mathcal{E}_0,F,F') \subset \overline{M(F,F')}$, for any linear space of sequences $\mathcal{E} \subset \mathcal{E}_0$, then $\mathcal{I}(\mathcal{E},F,F') = \overline{M(F,F')}$. (iii) If $\mathcal{I}(\mathcal{E},\mathcal{F},F') \subset \overline{M(F,F')}$, for some linear space of sequences $\mathcal{F} \subset F$, then $\mathcal{I}(\mathcal{E},F,F') = \overline{M(F,F')}$.*

Proof. (i) Let $x \in \overline{M(F,F')}$. Then, we successively obtain $1/x \in M(F,F')$, $F \subset F'_x$, $F \subset \mathcal{E} + F'_x$ and $x \in \mathcal{I}(\mathcal{E},F,F')$. This implies $\overline{M(F,F')} \subset \mathcal{I}(\mathcal{E},F,F')$ and (i) holds. (ii) We have $\mathcal{I}(\mathcal{E},F,F') \subset \mathcal{I}(\mathcal{E}_0,F,F') \subset \overline{M(F,F')}$ and we conclude by (i) that $\mathcal{I}(\mathcal{E},F,F') = \overline{M(F,F')}$. (iii) follows from the inclusions $\overline{M(F,F')} \subset \mathcal{I}(\mathcal{E},F,F') \subset \mathcal{I}(\mathcal{E},\mathcal{F},F') \subset \overline{M(F,F')}$. ■

5 On the solvability of the (SSIE) with operator of the form $F_\Delta \subset \mathcal{E}+F'_x$, where F, $F' \in \{c_0, c, \ell_\infty\}$

In this section, we determine multipliers involving some difference sequence spaces. Then we state a general result on the solvability of the (SSIE) with operator $F_\Delta \subset \mathcal{E} + F'_x$ with $e \in F$. Then we apply these results to solve each of the (SSIE) $c_\Delta \subset \mathcal{E} + F'_x$ and $(c_0)_\Delta \subset \mathcal{E} + F'_x$ and $(\ell_\infty)_\Delta \subset \mathcal{E} + F'_x$ with $F' = c_0$, c, or ℓ_∞.

5.1 On the multipliers of the form $M(X_\Delta, Y)$ where X, $Y \in \{c_0, c, \ell_\infty\}$.

In all that follows, for $a \in U^+$, we use the triangle $D_a\Sigma$, whose the nonzero entries are defined by $(D_a\Sigma)_{nk} = a_n$ for $k \le n$. We have $(D_a\Sigma)_n y = a_n \sum_{k=1}^n y_k$ for all $y \in \omega$ and for all n. This triangle is also called the Rhally matrix, (cf. [28, 29]). We obtain some results on the multipliers involving the sets of the difference sequence spaces $(c_0)_\Delta$, c_Δ and $(\ell_\infty)_\Delta$ introduced by Kizmaz, (cf. [23], see also [1]), and stated in the next lemma.

Lemma 12 *(i) $M((c_0)_\Delta, Y) = s_{(1/n)_{n\ge1}}$ where $Y = c_0$, c or ℓ_∞. (ii) $M(c_\Delta, c_0) = s^0_{(1/n)_{n\ge1}}$, $M(c_\Delta, c) = s^{(c)}_{(1/n)_{n\ge1}}$ and $M(c_\Delta, \ell_\infty) = s_{(1/n)_{n\ge1}}$. (iii) $M((\ell_\infty)_\Delta, c_0) = M((\ell_\infty)_\Delta, c) = s^0_{(1/n)_{n\ge1}}$ and $M((\ell_\infty)_\Delta, \ell_\infty) = s_{(1/n)_{n\ge1}}$.*

Proof. (i) follows from the proof of [9, Proposition 7.1 p. 95]. (ii) We have $a \in M(c_\Delta, c_0)$ if and only if $D_a\Sigma \in (c, c_0)$ and by the characterization of (c, c_0) we have $na_n \to 0$ $(n \to \infty)$ and $a \in s^0_{(1/n)_{n\ge1}}$. In the same way, we have $a \in M(c_\Delta, c)$ if and only if $D_a\Sigma \in (c, c)$ and by the characterization of (c, c) we obtain $a \in s^{(c)}_{(1/n)_{n\ge1}}$. The identity $M(c_\Delta, \ell_\infty) = s_{(1/n)_{n\ge1}}$ can be obtain using similar arguments. (iii) We show $M((\ell_\infty)_\Delta, c) \subset s^0_{(1/n)_{n\ge1}}$. For this, let $a \in M((\ell_\infty)_\Delta, c)$. Then we have $D_a\Sigma \in (\ell_\infty, c)$ which implies $D_a\Sigma \in (c, c)$ and $(na_n)_{n\ge1} \in c$. This implies $\lim_{n\to\infty} a_n = 0$ and by the

Schur theorem we obtain $\lim_{n\to\infty}\left(|a_n|\sum_{k=1}^{n}1\right)=0$ and $a\in s^0_{(1/n)_{n\geq 1}}$. So we have shown the inclusion $M\left((\ell_\infty)_\Delta, c\right)\subset s^0_{(1/n)_{n\geq 1}}$. Now, it can easily be seen that $D_{(1/n)_{n\geq 1}}\Sigma\in(\ell_\infty,\ell_\infty)$ which implies $(\ell_\infty)_\Delta\subset s_{(n)_{n\geq 1}}$ and using Lemma 5, we obtain $s^0_{(1/n)_{n\geq 1}}=M\left(s_{(n)_{n\geq 1}},c_0\right)\subset M\left((\ell_\infty)_\Delta,c_0\right)$. So we have shown the inclusions $s^0_{(1/n)_{n\geq 1}}\subset M\left((\ell_\infty)_\Delta,c_0\right)\subset M\left((\ell_\infty)_\Delta,c\right)\subset s^0_{(1/n)_{n\geq 1}}$ and we conclude $M\left((\ell_\infty)_\Delta,c_0\right)=M\left((\ell_\infty)_\Delta,c\right)=s^0_{(1/n)_{n\geq 1}}$. Using (ii) and the inclusion $(\ell_\infty)_\Delta\subset s_{(n)_{n\geq 1}}$ we obtain

$$s_{(1/n)_{n\geq 1}}=M\left(s_{(n)_{n\geq 1}},\ell_\infty\right)\subset M\left((\ell_\infty)_\Delta,\ell_\infty\right)\subset M\left(c_\Delta,\ell_\infty\right)=s_{(1/n)_{n\geq 1}}$$

and the identity $M\left((\ell_\infty)_\Delta,\ell_\infty\right)=s_{(1/n)_{n\geq 1}}$ holds. This completes the proof. ■

5.2 General result on the solvability of the (SSIE) with operator $F_\Delta\subset\mathcal{E}+F'_x$ with $e\in F$.

In the following, we use the next result.

Theorem 13 *Let F, F' and $\mathcal{E}$ be linear spaces of sequences. Assume $e\in F$, $\mathcal{E}\subset s^0_{(n)_{n\geq 1}}$ and that F' satisfies the condition in (1). Then, the set $\mathcal{I}\left(\mathcal{E},F_\Delta,F'\right)$ of all the positive solutions $x=(x_n)_{n\geq 1}$ of the (SSIE) $F_\Delta\subset\mathcal{E}+F'_x$ satisfies the inclusion $\mathcal{I}\left(\mathcal{E},F_\Delta,F'\right)\subset\overline{F'_{(1/n)_{n\geq 1}}}$. Moreover, if $F'_{(1/n)_{n\geq 1}}\subset M\left(F_\Delta,F'\right)$ then*

$$\mathcal{I}\left(\mathcal{E},F_\Delta,F'\right)=\overline{F'_{(1/n)_{n\geq 1}}}. \tag{3}$$

Proof. Let $x\in\mathcal{I}\left(\mathcal{E},F_\Delta,F'\right)$. Then we have $F_\Delta\subset\mathcal{E}+F'_x$ and since $e\in F$ we have $(n)_{n\geq 1}\in F_\Delta$ and there are $\alpha\in\mathcal{E}$ and $\varphi\in F'$ such that $n=\alpha_n+x_n\varphi_n$ for all n. Then we have

$$\frac{n}{x_n}\left(1-\frac{\alpha_n}{n}\right)=\varphi_n \text{ for all } n,$$

and the condition $\mathcal{E}\subset s^0_{(n)_{n\geq 1}}$ implies $\lim_{n\to\infty}\alpha_n/n=0$. Since F' satisfies the condition in (1) we obtain $(n/x_n)_{n\geq 1}\in F'$ and $x\in\overline{F'_{(1/n)_{n\geq 1}}}$. So we have shown the inclusion $\mathcal{I}\left(\mathcal{E},F_\Delta,F'\right)\subset\overline{F'_{(1/n)_{n\geq 1}}}$. Using Part (i) of Lemma 11, where $\overline{M\left(F_\Delta,F'\right)}\subset\mathcal{I}\left(\mathcal{E},F_\Delta,F'\right)$, we conclude $\overline{F'_{(1/n)_{n\geq 1}}}\subset\mathcal{I}\left(\mathcal{E},F_\Delta,F'\right)$. This completes the proof. ■

5.3 Solvability of the (SSIE) $c_\Delta\subset\mathcal{E}+F'_x$ where $F'=c_0$, c or ℓ_∞.

As a direct consequence of Theorem 13 and Lemma 12, we obtain the following result on the sets of all positive sequences $x=(x_n)_{n\geq 1}$ that satisfy each of the (SSIE) with operator $c_\Delta\subset\mathcal{E}+F'_x$ with $F'=c_0$, c or ℓ_∞.

Theorem 14 *Let $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$ be a linear space of sequences. We have*

$$\mathcal{I}\left(\mathcal{E}, c_\Delta, F'\right) = \begin{cases} \overline{s^0_{(1/n)_{n\geq1}}} & \text{for } F' = c_0, \\ \overline{s^{(c)}_{(1/n)_{n\geq1}}} & \text{for } F' = c, \\ \overline{s_{(1/n)_{n\geq1}}} & \text{for } F' = \ell_\infty. \end{cases}$$

Proof. The result follows from Part (ii) of Lemma 12 and Theorem 13, where $F = c$, and $F' = c_0$, c and ℓ_∞ respectively. ■

We may state some immediate applications of Theorem 14.

Example 15 *Using Lemma 11 and Theorem 14, it can easily be seen that the sets of the positive solutions $x = (x_n)_{n\geq1}$ of each of the (SSIE) with operator, $c_\Delta \subset \ell_\infty + s^{(c)}_x$ and $c_\Delta \subset c + s^{(c)}_x$ and $c_\Delta \subset (c_0)_\Delta + s^{(c)}_x$ are determined by $(n/x_n)_{n\geq1} \in c$. Then, the solutions of each of the (SSIE) $c_\Delta \subset (c_0)_\Delta + s^0_x$, $c_\Delta \subset \ell_\infty + s^0_x$ and $c_\Delta \subset c + s^0_x$ are determined by $n/x_n \to 0$ $(n \to \infty)$. In a similar way, the solutions of each of the (SSIE) $c_\Delta \subset (c_0)_\Delta + s_x$, $c_\Delta \subset \ell_\infty + s_x$ and $c_\Delta \subset c + s_x$ are determined by $(n/x_n)_{n\geq1} \in \ell_\infty$.*

Example 16 *It can easily be seen that $w_0 \subset s^0_{(n)_{n\geq1}}$. This implies that the set of all sequences $x = (x_n)_{n\geq1} \in U^+$ that satisfy the (SSIE) with operator $c_\Delta \subset w_0 + s^0_x$ is determined by $n/x_n \to 0$ $(n \to \infty)$.*

Example 17 *The set of all positive sequences that satisfy the (SSIE) $c_\Delta \subset c_{C_1} + s^0_x$ is determined by $\mathcal{I}\left(c_{C_1}, c_\Delta, c_0\right) = \overline{s^0_{(1/n)_{n\geq1}}}$. Then, the set of all positive sequences that satisfy the (SSIE) $c_\Delta \subset c_{C_1} + s_x$ is determined by $\mathcal{I}\left(c_{C_1}, c_\Delta, \ell_\infty\right) = \overline{s_{(1/n)_{n\geq1}}}$.*

5.4 Solvability of the (SSIE) of the form $(c_0)_\Delta \subset \mathcal{E} + F'_x$.

In this part, Theorem 13 cannot be applied since $e \notin c_0$. So we need to use some results stated in Section 4.

Theorem 18 *Let $\mathcal{E} \subset s_\theta$ for some $\theta \in s^0_{(n)_{n\geq1}}$, be a linear space of sequences and let $F' = c_0$, c or ℓ_∞. Then, the set of all the solutions of the (SSIE) $(c_0)_\Delta \subset \mathcal{E} + F'_x$ is determined by $\mathcal{I}\left(\mathcal{E}, (c_0)_\Delta, F'\right) = \overline{s_{(1/n)_{n\geq1}}}$.*

Proof. Let $x \in \mathcal{I}\left(\mathcal{E}, (c_0)_\Delta, F'\right)$ where $F' = c_0$, c or ℓ_∞. Then we have $(c_0)_\Delta \subset \mathcal{E} + F'_x$ and since $F' \subset s_1$ and s_1 satisfies the condition in (2), we obtain $\mathcal{E} + F'_x \subset s_\theta + s_x = s_{\theta+x}$ and $(c_0)_\Delta \subset s_{\theta+x}$. Then we have $D_{1/(\theta+x)}\Sigma \in (c_0, s_1)$ and by the characterization of (c_0, s_1) we have $n/\left(\theta_n + x_n\right) = O(1)$ $(n \to \infty)$. Using the inclusion $\mathcal{E} \subset s_\theta$ with $\theta \in s^0_{(n)_{n\geq1}}$ we have $n/x_n = O(1)$ $(n \to \infty)$, that is, $x \in \overline{s_{(1/n)_{n\geq1}}}$. We conclude $\mathcal{I}\left(\mathcal{E}, (c_0)_\Delta, F'\right) \subset \overline{s_{(1/n)_{n\geq1}}}$. The converse follows from Theorem 13 and Part (i) of Lemma 12, where $M\left((c_0)_\Delta, s_1\right) = s_{(1/n)_{n\geq1}}$. ■

Example 19 *By Theorem 18 with $\theta = e$, we deduce that the set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the (SSIE) $(c_0)_\Delta \subset \ell_\infty + F'_x$ is determined by $\mathcal{I}(\ell_\infty, (c_0)_\Delta, F') = \overline{s_{(1/n)_{n\geq 1}}}$ for $F' = c_0$, c or ℓ_∞.*

We consider another example, where $bv_p = \ell^p_\Delta$ with $p > 1$ is the set of $p-$bounded variations, (cf. [1]).

Example 20 *Let $p > 1$. The set $bv_p = \ell^p_\Delta$ satisfies the inclusion $bv_p \subset s_\theta$ if and only if $D_{1/\theta}\Sigma \in (\ell^p, s_1)$. By the characterization of (ℓ^p, s_1), (cf. [27, Theorem 1.37, p. 161]) we obtain $(n/\theta^q_n)_{n\geq 1} \in \ell_\infty$. We may take $\theta_n = n^{1/q}$ with $q = p/(p-1)$, which implies $\theta \in s^0_{(n)_{n\geq 1}}$, and by Theorem 18 we conclude that the set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the (SSIE) $(c_0)_\Delta \subset bv_p + F'_x$ is determined by $\mathcal{I}(bv_p, (c_0)_\Delta, F') = \overline{s_{(1/n)_{n\geq 1}}}$ for $F' = c_0$, c or ℓ_∞.*

5.5 Solvability of the (SSIE) of the form $bv_\infty \subset \mathcal{E} + F'_x$.

In this part, we use the notation bv_∞ for the difference sequence space $(\ell_\infty)_\Delta$, (cf. [1]) and we study each of the (SSIE) $bv_\infty \subset \mathcal{E} + F'_x$, where $F' \in \{c_0, c, \ell_\infty\}$.

Theorem 21 *Let $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$ be a linear space of sequences. Then, the sets of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy each of the (SSIE) $bv_\infty \subset \mathcal{E} + s_x$, $bv_\infty \subset \mathcal{E} + s^0_x$ and $bv_\infty \subset \mathcal{E} + s^{(c)}_x$ are determined by*

$$\mathcal{I}(\mathcal{E}, bv_\infty, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}} \text{ and } \mathcal{I}(\mathcal{E}, bv_\infty, c_0) = \mathcal{I}(\mathcal{E}, bv_\infty, c) = \overline{s^0_{(1/n)_{n\geq 1}}}.$$

Proof. First, we show the identities $\mathcal{I}(\mathcal{E}, bv_\infty, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}}$ and $\mathcal{I}(\mathcal{E}, bv_\infty, c_0) = \overline{s^0_{(1/n)_{n\geq 1}}}$. From Theorem 13, where $\mathcal{E} = s^0_{(n)_{n\geq 1}}$, $F = \ell_\infty$ and $F' = \ell_\infty$, and c_0 respectively, we obtain $\mathcal{I}(\mathcal{E}, bv_\infty, \ell_\infty) \subset \overline{s_{(1/n)_{n\geq 1}}}$ and $\mathcal{I}(\mathcal{E}, bv_\infty, c_0) \subset \overline{s^0_{(1/n)_{n\geq 1}}}$. Then, by Part (iii) of Lemma 12, we have $M(bv_\infty, \ell_\infty) = s_{(1/n)_{n\geq 1}}$ and $M(bv_\infty, c_0) = s^0_{(1/n)_{n\geq 1}}$ and we conclude by Part (iii) of Lemma 11. Now we show the identity $\mathcal{I}(\mathcal{E}, bv_\infty, c) = \overline{s^0_{(1/n)_{n\geq 1}}}$. For this, we let $x \in \mathcal{I}(\mathcal{E}, (\ell_\infty)_\Delta, c)$. Then we have $(\ell_\infty)_\Delta \subset s^0_{(n)_{n\geq 1}} + s^{(c)}_x$, and by Theorem 13, where $\mathcal{E} = s^0_{(n)_{n\geq 1}}$, $F = \ell_\infty$ and $F' = c$, we have $\mathcal{I}(\mathcal{E}, (\ell_\infty)_\Delta, c) \subset s^{(c)}_{(1/n)_{n\geq 1}}$ and $(n/x_n)_{n\geq 1} \in c$. Now, we show the inclusion $(\ell_\infty)_\Delta \subset s^{(c)}_{(n+x_n)_{n\geq 1}}$. We have $s^0_{(n)_{n\geq 1}} \subset s^{(c)}_{(n+x_n)_{n\geq 1}}$ since $n/(n+x_n) = O(1)$ $(n \to \infty)$. Then we have

$$\frac{x_n}{n + x_n} = \frac{1}{\frac{n}{x_n} + 1} \text{ for all } n,$$

and as we have just seen, we have $\lim_{n\to\infty} n/x_n = l$ for some scalar l and

$$\lim_{n\to\infty} \frac{1}{\dfrac{n}{x_n} + 1} = \frac{1}{l+1} > 0.$$

Thus, we have shown the inclusion $s_x^{(c)} \subset s_{(n+x_n)_{n\geq 1}}^{(c)}$. These statements imply the inclusions $(\ell_\infty)_\Delta \subset s_{(n)_{n\geq 1}}^0 + s_x^{(c)} \subset s_{(n+x_n)_{n\geq 1}}^{(c)}$ and since $M((\ell_\infty)_\Delta, c) = s_{(1/n)_{n\geq 1}}^0$ we obtain $(1/(n+x_n))_{n\geq 1} \in s_{(1/n)_{n\geq 1}}^0$. Then we have $n/(n+x_n) \to 0$ $(n \to \infty)$ and $(n/x_n)_{n\geq 1} \in c_0$, and we have shown the inclusion $\mathcal{I}(\mathcal{E}, (\ell_\infty)_\Delta, c) \subset s_{(1/n)_{n\geq 1}}^0$. Finally, since $M((\ell_\infty)_\Delta, c) = s_{(1/n)_{n\geq 1}}^0$, by Part (i) of Lemma 11, we conclude $\mathcal{I}(\mathcal{E}, (\ell_\infty)_\Delta, c) = \overline{s_{(1/n)_{n\geq 1}}^0}$. This completes the proof. ■

We obtain the following result, where $bs = (\ell_\infty)_\Sigma$ is the set of all bounded series.

Example 22 *The solutions of each of the (SSIE) $bv_\infty \subset \ell_\infty + s_x^{(c)}$ and $bv_\infty \subset bs + s_x^{(c)}$ are determined by $\mathcal{I}(\ell_\infty, bv_\infty, c) = \mathcal{I}(bs, bv_\infty, c) = \overline{s_{(1/n)_{n\geq 1}}^0}$.*

Using similar arguments as in Example 20, we obtain the following result.

Corollary 23 *Let $p \geq 1$. The solutions of the (SSIE) $bv_\infty \subset bv_p + s_x^{(c)}$ are determined by $\mathcal{I}(bv_p, bv_\infty, c) = \overline{s_{(1/n)_{n\geq 1}}^0}$.*

6 Solvability of the (SSIE) of the form $(w_\infty)_\Delta \subset \mathcal{E} + F'_x$

In this part, we deal with each of the (SSIE) with operator of the form $(w_\infty)_\Delta \subset \mathcal{E} + s_x^0$, $(w_\infty)_\Delta \subset \mathcal{E} + s_x$ and $(w_\infty)_\Delta \subset \mathcal{E} + s_x^{(c)}$. For instance, the solvability of the (SSIE) $(w_\infty)_\Delta \subset s_{(n)_{n\geq 1}}^0 + s_x$ consists in determining the set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the next statement. For every y such that $n^{-1}\sum_{k=1}^n |y_k - y_{k-1}| = O(1)$ there are two sequences u and v with $y = u + v$ where $\lim_{n\to\infty} u_n/n = 0$ and $v_n/x_n = O(1)$ $(n \to \infty)$.

6.1 Determination of the sets $M((w_\infty)_\Delta, Y)$ with $Y \in \{c_0, c, \ell_\infty\}$

We state the next Lemma.

Lemma 24 *We have (i) $M((w_\infty)_\Delta, s_1) = s_{(1/n)_{n\geq 1}}$ and (ii) $M((w_\infty)_\Delta, c_0) = M((w_\infty)_\Delta, c) = s_{(1/n)_{n\geq 1}}^0$.*

Proof. (i) We have $\Delta \in (w_\infty, w_\infty)$ which implies $w_\infty \subset (w_\infty)_\Delta$ and $M((w_\infty)_\Delta, s_1) \subset M(w_\infty, s_1) = s_{(1/n)_{n\geq 1}}$. Then we have $w_\infty \subset (\ell_\infty)_{C_1}$ and $(w_\infty)_\Delta \subset [(\ell_\infty)_{C_1}]_\Delta$ and since $C_1\Delta = D_{(1/n)_{n\geq 1}}\Sigma\Delta = D_{(1/n)_{n\geq 1}}I = D_{(1/n)_{n\geq 1}}$ we obtain $(w_\infty)_\Delta \subset (\ell_\infty)_{D_{(1/n)_{n\geq 1}}} = s_{(n)_{n\geq 1}}$. Then, by Part (ii) of Lemma 5, we obtain $s_{(1/n)_{n\geq 1}} = M\left(s_{(n)_{n\geq 1}}, s_1\right) \subset M((w_\infty)_\Delta, s_1)$. So we have shown the identity $M((w_\infty)_\Delta, s_1) = s_{(1/n)_{n\geq 1}}$. (ii) First, we show the inclusion $s_{(1/n)_{n\geq 1}}^0 \subset M((w_\infty)_\Delta, c_0)$. As we have just seen, we have $(w_\infty)_\Delta \subset s_{(n)_{n\geq 1}}$ and $s^0_{(1/n)_{n\geq 1}} = M\left(s_{(n)_{n\geq 1}}, c_0\right) \subset M\left((w_\infty)_\Delta, c_0\right)$. Then, by the inclusion $w_\infty \subset (w_\infty)_\Delta$ we deduce $M\left((w_\infty)_\Delta, c_0\right) \subset M\left(w_\infty, c_0\right) = s^0_{(1/n)_{n\geq 1}}$ and we conclude $M\left((w_\infty)_\Delta, c_0\right) = s^0_{(1/n)_{n\geq 1}}$. Now, we show the identity $M\left((w_\infty)_\Delta, c\right) = s^0_{(1/n)_{n\geq 1}}$. As above, the inclusion $w_\infty \subset (w_\infty)_\Delta$ implies $M\left((w_\infty)_\Delta, c\right) \subset M\left(w_\infty, c\right)$. Then, by Part (ii) (b) of Lemma 6, we have $M\left(w_\infty, c\right) = s^0_{(1/n)_{n\geq 1}}$ and we obtain $M\left((w_\infty)_\Delta, c\right) \subset s^0_{(1/n)_{n\geq 1}}$. Using the identity $M\left((w_\infty)_\Delta, c_0\right) = s^0_{(1/n)_{n\geq 1}}$ and the inclusion $M\left((w_\infty)_\Delta, c_0\right) \subset M\left((w_\infty)_\Delta, c\right)$, we obtain $M\left((w_\infty)_\Delta, c_0\right) = M\left((w_\infty)_\Delta, c\right) = s^0_{(1/n)_{n\geq 1}}$. This completes the proof. ∎

6.2 Application to the solvability of the (SSIE) of the form $(w_\infty)_\Delta \subset \mathcal{E} + F'_x$.

In the following theorem, we solve each of the (SSIE) $(w_\infty)_\Delta \subset \mathcal{E} + F'_x$, where $F' \in \{c_0, c, \ell_\infty\}$.

Theorem 25 *Let $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$ be a linear space of sequences. Then,*

(i) The set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the (SSIE) $(w_\infty)_\Delta \subset \mathcal{E} + s_x$ is determined by $\mathcal{I}\left(\mathcal{E}, (w_\infty)_\Delta, s_1\right) = \overline{s_{(1/n)_{n\geq 1}}}$.

(ii) The sets of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy each of the (SSIE) $(w_\infty)_\Delta \subset \mathcal{E} + s^0_x$ and $(w_\infty)_\Delta \subset \mathcal{E} + s^{(c)}_x$ are determined by

$$\mathcal{I}\left(\mathcal{E}, (w_\infty)_\Delta, c_0\right) = \mathcal{I}\left(\mathcal{E}, (w_\infty)_\Delta, c\right) = \overline{s^0_{(1/n)_{n\geq 1}}}. \quad (4)$$

Proof. (i) By Part (i) of Theorem 21 and since $(\ell_\infty)_\Delta \subset (w_\infty)_\Delta$ we have $\mathcal{I}\left(\mathcal{E}, (w_\infty)_\Delta, s_1\right) \subset \mathcal{I}\left(\mathcal{E}, (\ell_\infty)_\Delta, s_1\right) = \overline{s_{(1/n)_{n\geq 1}}}$. Then, by Lemma 12 and Lemma 24, we have $M\left((w_\infty)_\Delta, s_1\right) = M\left((\ell_\infty)_\Delta, s_1\right) = s_{(1/n)_{n\geq 1}}$. We conclude by Part (i) of Lemma 11, that $\mathcal{I}\left(\mathcal{E}, (w_\infty)_\Delta, s_1\right) = \overline{s_{(1/n)_{n\geq 1}}}$. (ii) From Part (ii) of Theorem 21 and Lemma 24, we obtain the next two statements, $\overline{s^0_{(1/n)_{n\geq 1}}} = \overline{M\left((w_\infty)_\Delta, c_0\right)} \subset \mathcal{I}\left(\mathcal{E}, (w_\infty)_\Delta, c_0\right)$ and $\mathcal{I}\left(\mathcal{E}, (w_\infty)_\Delta, c_0\right) \subset \mathcal{I}\left(\mathcal{E}, (w_\infty)_\Delta, c\right) \subset \mathcal{I}\left(\mathcal{E}, (\ell_\infty)_\Delta, c\right) = \overline{s^0_{(1/n)_{n\geq 1}}}$. This implies the identities in (4) and completes the proof. ∎

Example 26 *Since $w_0 \subset s^0_{(n)_{n\geq 1}}$, the set of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the (SSIE) $(w_\infty)_\Delta \subset w_0 + s_x$ is determined by $x_n \geq Kn$ for all n and for some $K > 0$. Similarly, the sets of all positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the (SSIE) $(w_\infty)_\Delta \subset w_0 + s^0_x$ is determined by $\lim_{n\to\infty} x_n/n = \infty$.*

Example 27 *By the characterization of (c, c_0), we can see that $D_{(1/n)_{n\geq 1}} C_1^{-1} \in (c, c_0)$ which implies the inclusion $c_{C_1} \subset s^0_{(n)_{n\geq 1}}$. This implies that the solutions of the (SSIE) $(w_\infty)_\Delta \subset c_{C_1} + s^0_x$ are determined by $\lim_{n\to\infty} x_n/n = \infty$.*

In the following, we solve the (SSIE) $(w_\infty)_\Delta \subset W_r^0 + s_x^{(c)}$, where $W_r^0 = D_r w_0$ for $r > 0$. This solvability consists in determining the set of all sequences $x = (x_n)_{n\geq 1} \in U^+$ that satisfy the following statement. For every sequence $y = (y_n)_{n\geq 1}$ for which $n^{-1}\sum_{k=1}^{n} |y_k - y_{k-1}| \leq K$ for some $K > 0$ and for all n, there are two sequences u and v, with $y = u+v$ such that $n^{-1}\sum_{k=1}^{n} |u_k| / r^k \to 0$ $(n \to \infty)$ and $\lim_{n\to\infty} (v_n/x_n) = L$ for some scalar L.

Corollary 28 *Let $r > 0$. The set $\mathcal{I}_r^w$ of all the positive sequences $x = (x_n)_{n\geq 1}$ that satisfy the (SSIE) $(w_\infty)_\Delta \subset W_r^0 + s_x^{(c)}$ is determined by* $\mathcal{I}_r^w = \begin{cases} \overline{s^0_{(1/n)_{n\geq 1}}} & \text{if } r \leq 1, \\ U^+ & \text{if } r > 1. \end{cases}$

Proof. The inclusion $W_r^0 \subset s^0_{(n)_{n\geq 1}}$ holds if and only if $(r^n/n)_{n\geq 1} \in M(w_0, c_0)$, and from the identity $M(w_0, c_0) = s_{(1/n)_{n\geq 1}}$ this inclusion holds for all $r \leq 1$. Thus, by Theorem 25 we have $\mathcal{I}_r^w = \overline{s^0_{(1/n)_{n\geq 1}}}$ for all $r \leq 1$. Let $r > 1$. Then we have $r^{-n}\sum_{k=1}^{n} k = o(1)$ $(n \to \infty)$ and $D_{1/r}\Sigma \in \left(s_{(n)_{n\geq 1}}, c_0\right)$. Since $\left(s_{(n)_{n\geq 1}}, c_0\right) \subset (w_\infty, w_0)$ this implies $D_{1/r}\Sigma \in (w_\infty, w_0)$ and the inclusion $(w_\infty)_\Delta \subset W_r^0$ holds for all $r > 1$. This completes the proof. ■

7 On the solvability of the (SSIE) of the form $F_\Delta \subset \mathcal{E} + F'_x$ involving the sets w_0, or w.

In this section, we determine the multipliers $M(w_\Delta, Y)$ and $M((w_0)_\Delta, Y)$ where $Y = c_0$, c, or ℓ_∞. Then we apply these results to the solvability of the (SSIE) with operator $F_\Delta \subset \mathcal{E} + F'_x$ where $F = w_0$, or w and $F' = c_0$, c, or ℓ_∞.

7.1 On the multipliers of the form $M(w_\Delta, Y)$ and $M((w_0)_\Delta, Y)$

In this part, we determine the multipliers $M(w_\Delta, Y)$ and $M((w_0)_\Delta, Y)$ where $Y = c_0$, c, or ℓ_∞.

Lemma 29 *(i) $M((w_0)_\Delta, Y) = s_{(1/n)_{n\geq 1}}$ for $Y = c_0$, c, or ℓ_∞. (ii) (a) $M(w_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$, (b) $M(w_\Delta, c) = s^{(c)}_{(1/n)_{n\geq 1}}$ and (c) $M(w_\Delta, \ell_\infty) = s_{(1/n)_{n\geq 1}}$.*

Proof. (i) follows from the proof of [9, Proposition 7.3, p. 98]. (ii) (a) We show $M(w_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$. Since $c \subset w$, then $c_\Delta \subset w_\Delta$ and by Part (i) we obtain $M(w_\Delta, c_0) \subset M(c_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$. Then, by Part (ii) of Lemma 24, we have $M((w_\infty)_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$ and by Part (iii) of Lemma 5, the inclusion $w_\Delta \subset (w_\infty)_\Delta$ implies $s^0_{(1/n)_{n\geq 1}} = M((w_\infty)_\Delta, c_0) \subset M(w_\Delta, c_0)$. So we have shown $M(w_\Delta, c_0) = s^0_{(1/n)_{n\geq 1}}$. (ii) (b) We show $M(w_\Delta, c) = s^{(c)}_{(1/n)_{n\geq 1}}$. We have $c_\Delta \subset w_\Delta$ and by Part (ii) of Lemma 12, we obtain $M(w_\Delta, c) \subset M(c_\Delta, c) = s^{(c)}_{(1/n)_{n\geq 1}}$. Then we show the inclusion $s^{(c)}_{(1/n)_{n\geq 1}} \subset M(w_\Delta, c)$. We have $w \subset c_{C_1}$ and $w_\Delta \subset (c_{C_1})_\Delta$, and since $C_1\Delta = D_{(1/n)_{n\geq 1}}$ we obtain $(c_{C_1})_\Delta = s^{(c)}_{(n)_{n\geq 1}}$ and we conclude $w_\Delta \subset c_{D_{(1/n)_{n\geq 1}}} = s^{(c)}_{(n)_{n\geq 1}}$. Then, by Part (iii) of Lemma 5, we have $s^{(c)}_{(1/n)_{n\geq 1}} = M\left(s^{(c)}_{(n)_{n\geq 1}}, c\right) \subset M(w_\Delta, c)$ and we have shown the identity $M(w_\Delta, c) = s^{(c)}_{(1/n)_{n\geq 1}}$. (ii) (c) From Part (i) and Remark 6.1, we obtain

$$s_{(1/n)_{n\geq 1}} = M\left((w_\infty)_\Delta, \ell_\infty\right) \subset M(w_\Delta, \ell_\infty) \subset M\left((w_0)_\Delta, \ell_\infty\right) = s_{(1/n)_{n\geq 1}}.$$

This shows the identity $M(w_\Delta, \ell_\infty) = s_{(1/n)_{n\geq 1}}$. This completes the proof. ∎

7.2 Application to the solvability of the (SSIE) $F_\Delta \subset \mathcal{E} + F'_x$ where $F = w_0$, or w and $F' = c_0$, c, or ℓ_∞.

In this part, under some conditions on $\mathcal{E}$ we solve each of the (SSIE) with operator (1) $(w_0)_\Delta \subset \mathcal{E} + s^0_x$, (2) $(w_0)_\Delta \subset \mathcal{E} + s^{(c)}_x$, (3) $(w_0)_\Delta \subset \mathcal{E} + s_x$ and (1') $w_\Delta \subset \mathcal{E} + s^0_x$, (2') $w_\Delta \subset \mathcal{E} + s^{(c)}_x$, (3') $w_\Delta \subset \mathcal{E} + s_x$.

We can state the following theorem.

Theorem 30 *Let $\mathcal{E}$ be a linear space of sequences. Then we have:*

(i) Assume $\mathcal{E} \subset s_\theta$ for some $\theta \in s^0_{(n)_{n\geq 1}}$. Then $\mathcal{I}(\mathcal{E}, (w_0)_\Delta, F') = \overline{s_{(1/n)_{n\geq 1}}}$ for $F' = c_0$, c, or ℓ_∞.

(ii) Assume $\mathcal{E} \subset s^0_{(n)_{n\geq 1}}$. Then (a) $\mathcal{I}(\mathcal{E}, w_\Delta, c_0) = \overline{s^0_{(1/n)_{n\geq 1}}}$, (b) $\mathcal{I}(\mathcal{E}, w_\Delta, c) = \overline{s^{(c)}_{(1/n)_{n\geq 1}}}$ and (c) $\mathcal{I}(\mathcal{E}, w_\Delta, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}}$.

Proof. (i) By Part (i) of Lemma 29 we have $s_{(1/n)_{n\geq 1}} = M((w_0)_\Delta, c_0)$, and by Part (i) of Lemma 11 we have $\overline{s_{(1/n)_{n\geq 1}}} \subset \mathcal{I}(\mathcal{E}, (w_0)_\Delta, c_0)$. Then, by the inclusion $(c_0)_\Delta \subset (w_0)_\Delta$ and using Theorem 18, we have $\mathcal{I}(\mathcal{E}, (w_0)_\Delta, \ell_\infty) \subset \mathcal{I}(\mathcal{E}, (c_0)_\Delta, \ell_\infty) = \overline{s_{(1/n)_{n\geq 1}}}$. We conclude

$$\overline{s_{(1/n)_{n\geq 1}}} \subset \mathcal{I}(\mathcal{E}, (w_0)_\Delta, c_0) \subset \mathcal{I}(\mathcal{E}, (w_0)_\Delta, c) \subset \mathcal{I}(\mathcal{E}, (w_0)_\Delta, \ell_\infty) \subset \overline{s_{(1/n)_{n\geq 1}}}$$

and we have shown (i). (ii) follows from the inclusions $\overline{M(w_\Delta, F')} \subset \mathcal{I}(\mathcal{E}, w_\Delta, F') \subset \mathcal{I}(\mathcal{E}, c_\Delta, F') = \overline{M(c_\Delta, F')}$, and from Part (ii) of Lemma 29 and Part (ii) of Lemma 12, where we have $M(w_\Delta, F') = M(c_\Delta, F')$ for $F' = c_0$, c, or ℓ_∞. ∎

Example 31 *By Part (ii) of Theorem 30, the solutions of the (SSIE) $w_\Delta \subset w_0 + s^{(c)}_x$ are determined by $(n/x_n)_{n\geq 1} \in c$. As we have seen in Example 27, we have the inclusion $c_{C_1} \subset s^0_{(n)_{n\geq 1}}$, and by Part (ii) (b) of Theorem 30, the solutions of the (SSIE) $w_\Delta \subset c_{C_1} + s^{(c)}_x$ are determined by $(n/x_n)_{n\geq 1} \in c$.*

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