The Order of Euler's Totient Function

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Abstract

The Möbius function is commonly used to define Euler's totient function and the Mangoldt function. Similarly, the summatory Möbius function (the Mertens function) is used to define the summatory totient function and the summatory Mangoldt function (the second Chebyshev function). Analogues of the product formula for the totient function are introduced. An analogue of the summatory totient function with many additive properties is introduced.

1   Introduction

If $n \geq 1$ the Euler totient function $\varphi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n. If $n > 1$, then $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where $p_1, p_2, \ldots, p_k$ are primes. The Möbius function $\mu(n)$ is defined as follows: $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $a_1 = a_2 = \cdots = a_k = 1$, or $\mu(n) = 0$ otherwise. Note that $\mu(n) = 0$ if and only if n has a square factor > 1. The Euler totient function is related to the Möbius function through the following formula:

(1) $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$

See Theorem 2.3 of Apostol's [1] book. The Mangoldt function $\Lambda(n)$ is defined to be $\log(p)$ if $n = p^m$ for some prime p and some $m \geq 1$, or 0 otherwise. $\Lambda(n)$ is expressed in terms of the logarithm as follows:

(2) $\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log(d)$

See Theorem 2.11 of Apostol's book. The second Chebyshev function $\psi(x)$ is defined to be $\sum_{n \leq x} \Lambda(n)$. The average order of $\varphi(n)$ is:

(3) $\sum_{n \leq x} \varphi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$ so the average order of $\varphi(n)$ is $3n/\pi^2$.

See Theorem 3.7 of Apostol's book. A function used to prove $\lim_{x \to \infty} \frac{M(x)}{x} = 0$ where $M(x)$ is the Mertens function is also generalized. This definition is:

(4) If $x \geq 1$ we define $H(x) = \sum_{n \leq x} \mu(n) \log n$.

See pg. 91 of Apostol's book. The definition of derivatives of arithmetical functions is:

(5) For any arithmetical function f we define its derivative f' to be the arithmetical function given by the equation $f'(n) = f(n) \log n$ for $n \geq 1$.

See section 2.18 of Apostol's book. The derivative concept can be used to derive a formula of Selberg which can be used as the starting point of an elementary proof of the prime number theorem. This formula is:

(6) The Selberg identity. For $n \geq 1$ we have $\Lambda(n) \log n + \sum_{d|n} \Lambda(d) \Lambda(\frac{n}{d}) = \sum_{d|n} \mu(d) \log^2 \frac{n}{d}$.

See Theorem 2.27 of Apostol's book. This identity will also be generalized. Let $\Phi(x)$ denote $\sum_{n=1}^{x} \varphi(n)$. Let $M(x)$ denote $\sum_{n=1}^{x} \mu(n)$.

2   The Redheffer Matrix

The squarefree divisors of $n = 2, 3, 4, 5, 6, \ldots$ are 2, 3, 2, 5, (2, 3, 6), 7, 2, 3, (2, 5, 10), 11, (2, 3, 6),.... For example, for $n \leq 12$ there are $\lfloor 12/2 \rfloor$ 2's, $\lfloor 12/3 \rfloor$ 3's, $\lfloor 12/5 \rfloor$ 5's, $\lfloor 12/6 \rfloor$ 6's, $\lfloor 12/7 \rfloor$ 7's, $\lfloor 12/10 \rfloor$ 10's, and $\lfloor 12/11 \rfloor$ 11's.

Mikolás [2] proved that $\sum_{i=1}^{x} M(\lfloor x/i \rfloor) = 1$. In general, $\sum_{i=1}^{x} M(\lfloor x/(in) \rfloor) = 1$, $n = 1, 2, 3, ..., x$ (since $\lfloor \lfloor x/n \rfloor / i \rfloor = \lfloor x/(in) \rfloor$). Let R'_x denote a square matrix where element (i, j) equals 1 if j divides i or 0 otherwise. (In a Redheffer matrix, element (i, j) equals 1 if i divides j or if $j = 1$. Redheffer [3] proved that the determinant of such a x by x matrix equals $M(x)$.) Let T_x denote the matrix obtained from R'_x by element-by-element multiplication of the columns by $M(\lfloor x/1 \rfloor), M(\lfloor x/2 \rfloor), M(\lfloor x/3 \rfloor), ..., M(\lfloor x/x \rfloor)$. For example, the T_x matrix for $x = 12$ is

$$
\begin{array}{ccccccccccc}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
$$

Let U_x denote the matrix obtained from T_x by element-by-element multiplication of the columns by $\varphi(j)$. The sum of the sums of the columns of U_x then equals $\Phi(x)$. $i = \sum_{d|i} \varphi(d)$, so $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i$ (the sum of the sums of the rows of U_x) equals $\Phi(x)$. This relationship was proved by Cox [4] (and previously by Mikolás).

(7) $\sum_{i=1}^{x} M(\lfloor x/i \rfloor)i = \Phi(x)$

A plot of the sums of the rows for $x = 8000$ is given in Figure 1.
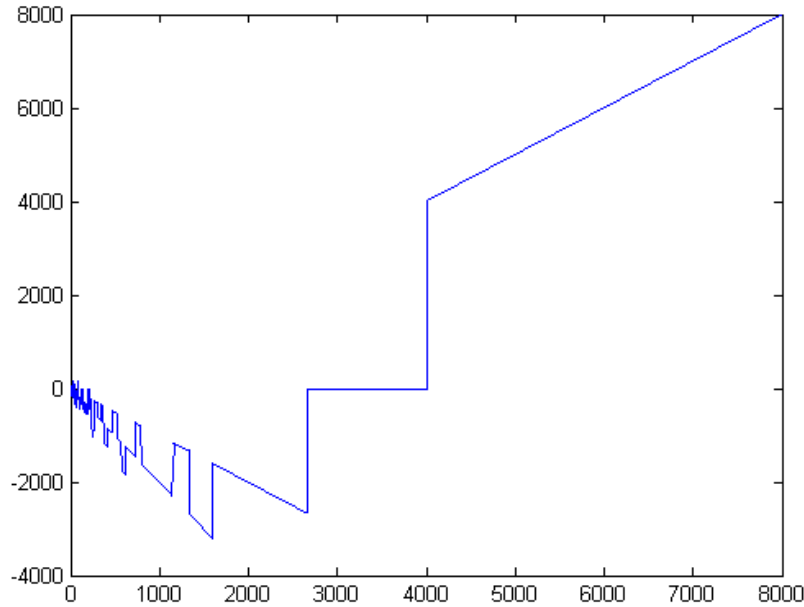
Figure 1: Plot of the sums of the rows for $x = 8000$

This plot indicates that $\Phi(x)$ equals a series involving the squarefree numbers 1, 2, 3, 5, 6, 7, 10,…. For $x = 200$, the values from $\lfloor x/2 \rfloor + 1$ to $\lfloor x/1 \rfloor$ are 101, 102, 103,…,200 (the initial value is $\lfloor x/2 \rfloor + 1$). The values from $\lfloor x/3 \rfloor + 1$ to $\lfloor x/2 \rfloor$ are 0. The values from $\lfloor x/5 \rfloor + 1$ to $\lfloor x/3 \rfloor$ are $-41, -42, -43,\ldots - 66$. The values from $\lfloor x/6 \rfloor + 1$ to $\lfloor x/5 \rfloor$ are $-68, -70, -80,\ldots - 66$ (the initial value is the negative of twice $\lfloor x/6 \rfloor + 1$). The process continues with larger squarefree divisors. Several squarefree divisors may map to the same term in the series - the smallest is selected. There are other multiples of the initial values and other multiples of 0 can occur. For $x = 200$, the squarefree numbers and corresponding multiples are (2, 1), (3, 0), (5, −1), (6, −2), (7, −1), (10, −2), (11, −1), (13, −2), (14, −3), (15, −2), (17, −1), (19, −2), (21, −3), (23, −1), (26, −2), (29, −1), (34, −3), (41, 0), (51, −3), (67, −1), (101, 1), and (201, −8). The multiple for a squarefree number q is $M(q − 1)$. Each term in the series is the sum of an arithmetic progression and is thus quadratic in terms of x. About 0.81 times the first term equals $\Phi(x)$.

If $q < \sqrt{x}$, $\lfloor x/q \rfloor \neq \lfloor x/r \rfloor$ where r is the next squarefree number. The asymptotic density of the squarefree numbers is $\frac{6}{\pi^2}$. (Even for x as small as 10, this gives 6.08. The actual number is 6. For $x < 50$, this gives 30.40. The actual number is 30.) The sum of the terms corresponding to the first $\lceil \sqrt{\frac{6x}{\pi^2}} \rceil$ squarefree numbers is an approximation of $\Phi(x)$ typically accurate to less than 0.5% relative error for fairly large x values (say greater than 300). A plot of $\Phi(x)$ and this approximation for $x = 40, 41, 42, \ldots, 110$ is given in Figure 2.
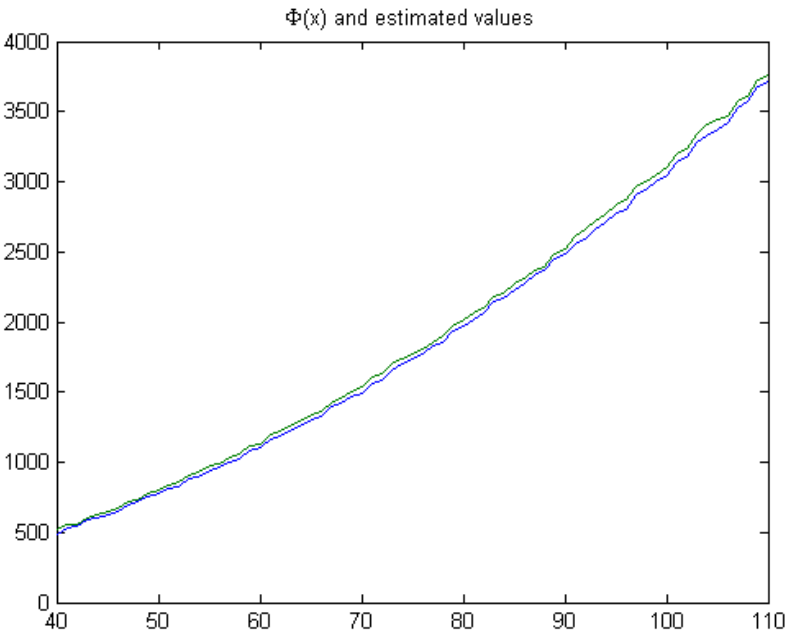
Figure 2: Plot of $\Phi(x)$ and this approximation for $x = 40, 41, 42, \ldots, 110$

For $x = 40$ to 41, the squarefree numbers used are 2, 3, 5, 6, and 7. For $x = 42$ to 59, the squarefree numbers used are 2, 3, 5, 6, 7, and 10. For $x = 60$ to 80, the squarefree numbers used are 2, 3, 5, 6, 7, 10, and 11 (with 7 exceptions). For $x = 81$ to 105, the squarefree numbers used are 2, 3, 5, 6, 7, 10, 11, and 13 (with 3 exceptions). For $x = 106$ to 110, the squarefree numbers used are 2, 3, 5, 6, 7, 10, 11, 13, and 14. For the first exceptions, there are no 11's so 13's are used. For the second exceptions there are no 11's or 14's so 13's and 15's are used. A plot of the $\Phi(x)$ values is given in Figure 3.
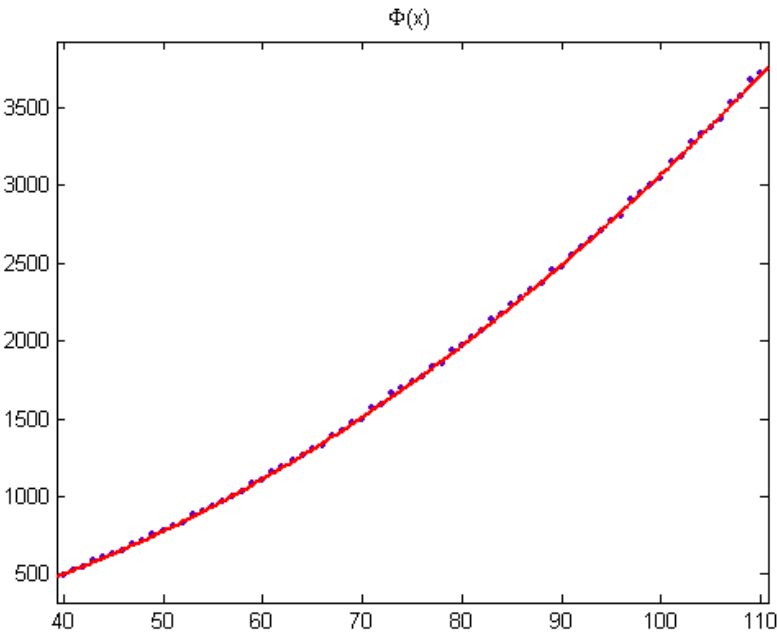
Figure 3: Plot of the $\Phi(x)$ values

For a quadratic least-squares fit of the data, $p_1 = 0.3057$ with a 95% confidence interval of $(0.2982, 0.3132)$, $p_2 = 0.0276$ with a 95% confidence interval of $(-1.103, 1.158)$, $p_3 = 10.79$ with a 95% confidence interval of $(-29.57, 51.15)$, SSE=9565, R-squared=0.9998, and RMSE=11.86. For a quadratic least-squares fit of the approximation, SSE=$1.311 \cdot 10^4$, R-squared=0.9998, and RMSE=13.88. (SSE denotes sum-squared error and RMSE denotes root-mean-square error.) A plot of the relative errors is given in Figure 4.
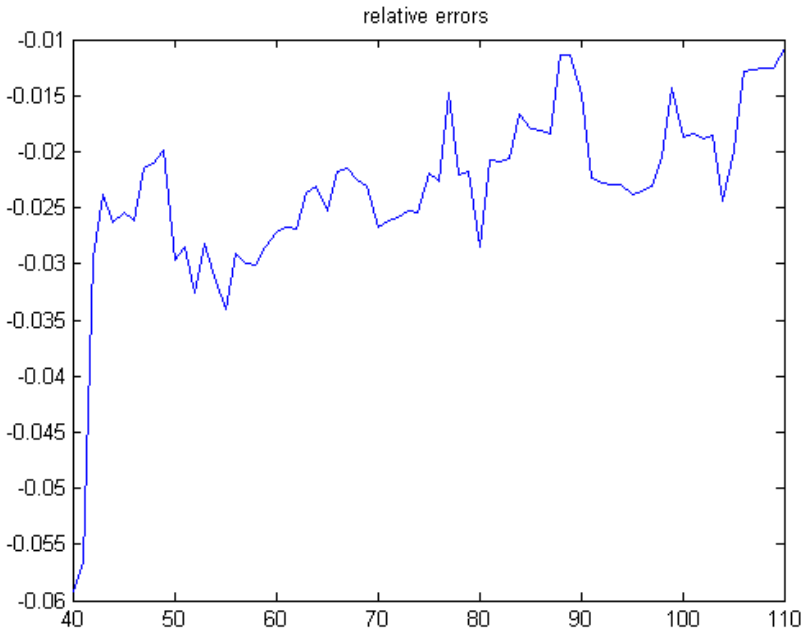
Figure 4: Plot of the relative errors

3 The Second Chebyshev Function

Let U_x denote the matrix obtained from T_x by element-by-element multiplication of the columns by $\Lambda(j)$. A plot of the sums of the columns for $x = 30$ is is given in Figure 5.
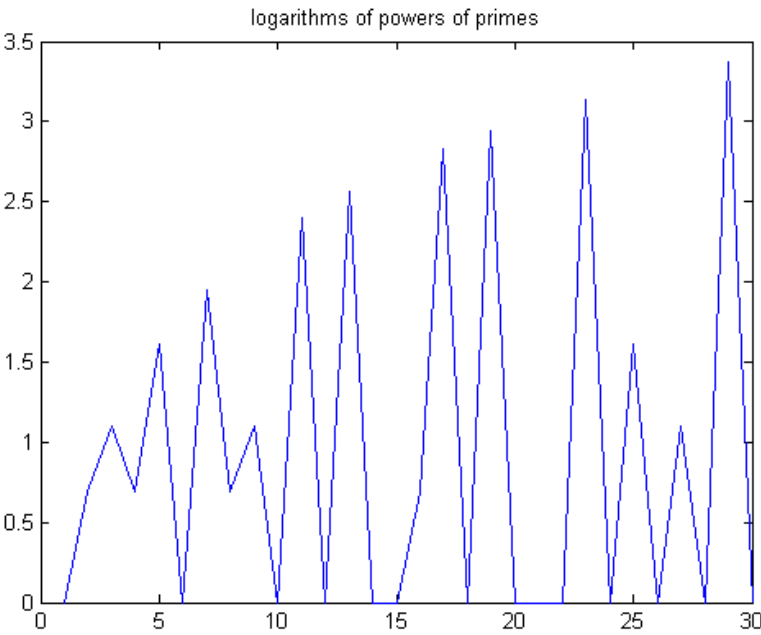
Figure 5: Plot of the sums of the columns for $x = 30$

The sum of the sums of the columns is then $\psi(x)$. Let $r_n, n = 1, 2, 3, \ldots, x$, denote the sums of the rows. A plot of r_n for $n = 1, 2, 3, \ldots 200$ is given in Figure 6.
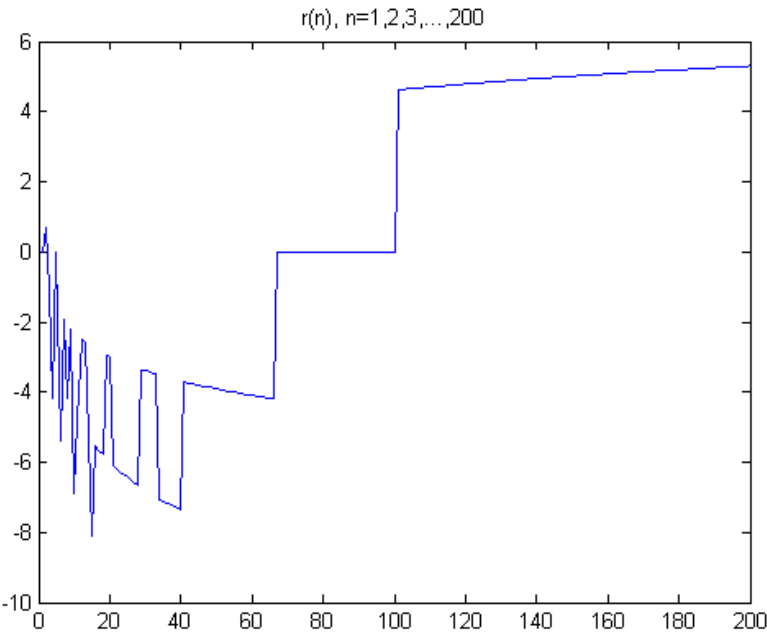
Figure 6: Plot of r_n for $n = 1, 2, 3, \ldots 200$

This plot indicates that $\psi(x)$ equals a series involving the squarefree numbers 1, 2, 3, 5, 6, 7, 10,.... Each of the "line segments" in the above plot is approximately quadratic. For $x = 200$, the r_n values from $n = \lfloor x/2 \rfloor + 1$ to $\lfloor x/1 \rfloor$ (101, 102, 103,...,200) are log(101), log(102), log(103),...,log(200). A plot of these values is given in Figure 7.
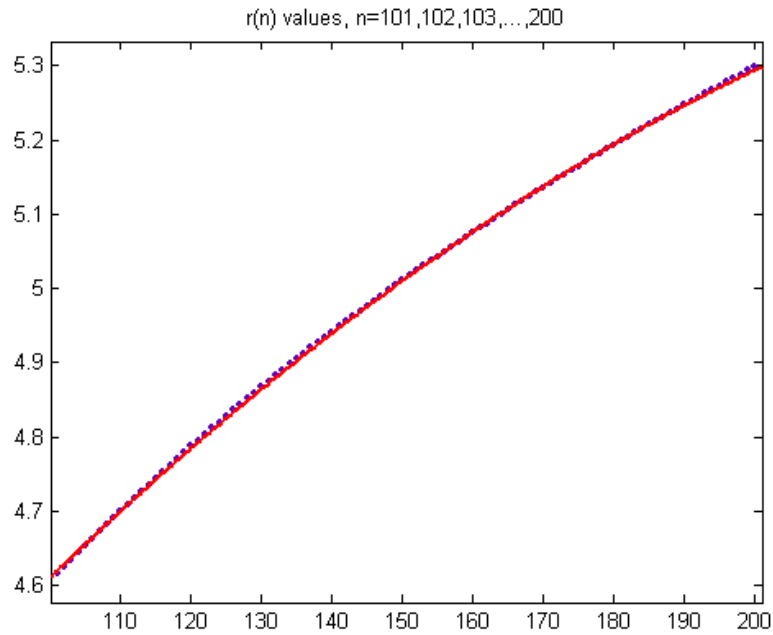
Figure 7: Plot of values from $n = \lfloor x/2 \rfloor + 1$ to $\lfloor x/1 \rfloor$ (101, 102, 103,...,200)

For a quadratic least-squares fit of the data, $p_1 = -2.319 \cdot 10^{-5}$ with a 95% confidence interval of $(-2.373 \cdot 10^{-5}, -2.264 \cdot 10^{-5})$, $p_2 = 0.01378$ with a 95% confidence interval of $(0.01361, 0.01394)$, $p_3 = 3.466$ with a 95% confidence interval of $(3.454, 3.478)$, SSE=0.0004042, R-squared=0.9999, and RMSE=0.002041.

The r_n values from $n = \lfloor x/3 \rfloor + 1$ to $\lfloor x/2 \rfloor$ (67, 69, 70, ...,100) are 0. The r_n values from $n = \lfloor x/5 \rfloor + 1$ to $\lfloor x/3 \rfloor$ (41, 42, 43,...,66) are $-\log(41)$, $-\log(42)$, $-\log(43)$,...,$-\log(66)$. A plot of these values is given in Figure 8.
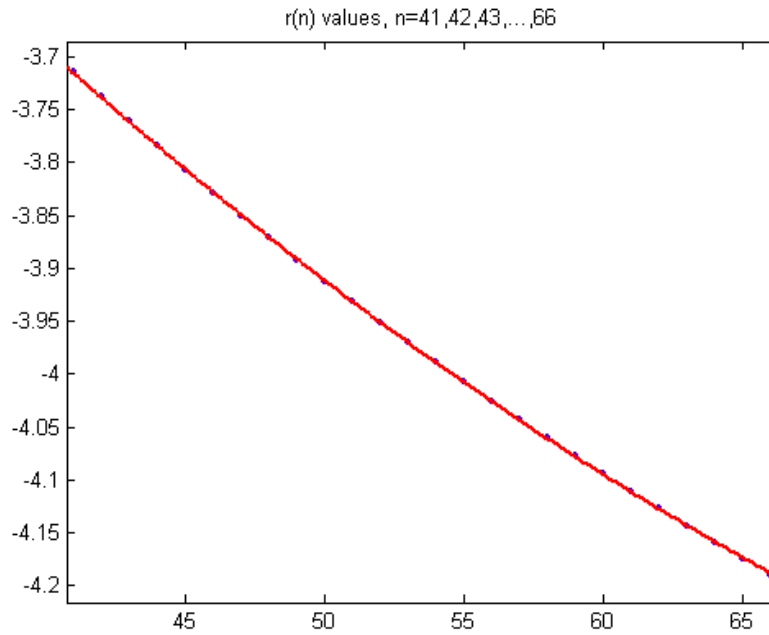
Figure 8: Plot of r_n values from $n = \lfloor x/5 \rfloor + 1$ to $\lfloor x/3 \rfloor$ (41, 42, 43,...,66)

For a quadratic least-squares fit of the data, $p_1 = 0.0001795$ with a 95% confidence interval of $(0.0001728, 0.0001857)$, $p_2 = -0.03809$ with a 95% confidence interval of $(-0.03878, -0.03741)$, $p_3 = -0.2455$ with a 95% confidence interval of $(-2.473, -2.437)$, SSE=$1.454 \cdot 10^{-5}$, R-squared=1, and RMSE=0.000795.

The r_n values from $n = \lfloor x/6 \rfloor + 1$ to $\lfloor x/5 \rfloor$ (34, 35, 36,...,40) are $-2\log(34)$, $-2\log(35)$, $-2\log(36)$,...,$-2\log(40)$. A plot of these values is given in Figure 9.
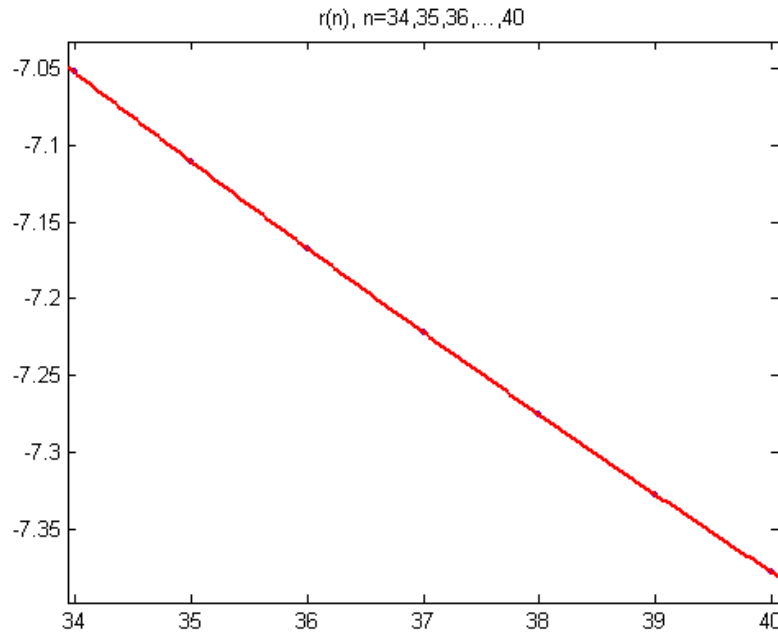
Figure 9: Plot of r_n values from $n = \lfloor x/6 \rfloor + 1$ to $\lfloor x/5 \rfloor$ (34, 35, 36,...,40)

For a quadratic least-squares fit of the data, $p_1 = 0.000733$ with a 95% confidence interval of $(0.0007035, 0.0007625)$, $p_2 = -0.1084$ with a 95% confidence interval of $(-0.1106, -0.1062)$, $p_3 = -4.215$ with a 95% confidence interval of $(-4.255, -4.175)$, SSE=$3.795 \cdot 10^{-8}$, R-squared=1, and RMSE=$9.75 \cdot 10^{-5}$.

As with $\Phi(x)$, the process continues with larger squarefree divisors. Again, the multiple for a squarefree number q is $M(q-1)$. A plot of the partial sum of the Chebyshev function corresponding to the squarefree numbers 2, 3, 5, 6, 7, and 10 is given in Figure 10.
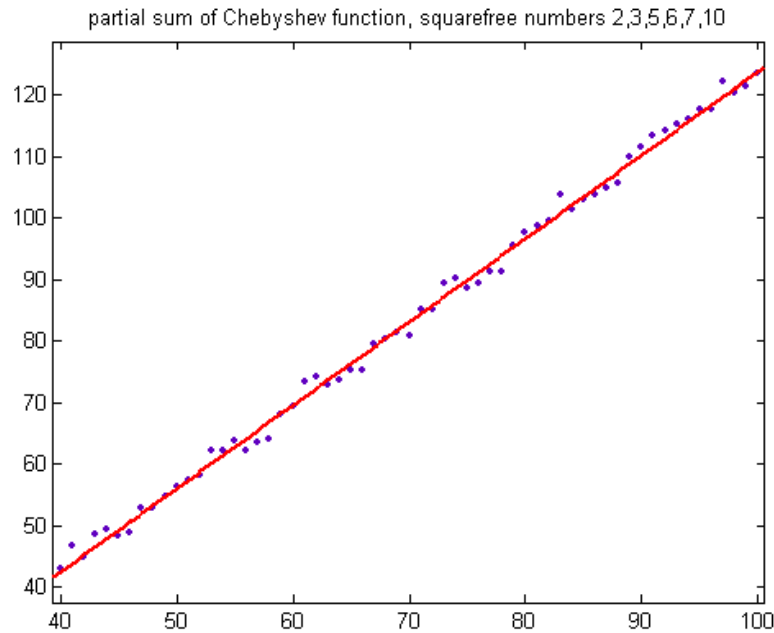
Figure 10: Plot of the partial sum of the Chebyshev function corresponding to the squarefree numbers 2, 3, 5, 6, 7, and 10

For a linear least-squares fit of the data, $p_1 = 1.353$ with a 95% confidence interval of $(.1331, 1.374)$, $p_2 = -11.53$ with a 95% confidence interval of $(-13.09, -10.02)$, SSE=125.3, R-squared=0.9964, and RMSE=1.457. The slope is greater than 1 and the SSE and RMSE are less than that for $\psi(x)$. A form of the prime number theorem is

(8) $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$

See Theorem 4.4 of Apostol's book.

For $x = 20$, $\psi(x) = 19.26566$. The next x value is not a prime power, so the value does not change. This is effected in the above by the logarithms of the factors of 21 canceling out - $\log(21) - \log(7) - \log(3) = 0$. Similarly, the logarithms of the factors of 22 cancel out - $\log(22) - \log(11) - \log(2) = 0$. The next x value is prime. For $x = 23$, $\psi(x) = 22.40115$, an increase of $\log(23)$. The next x value is not a prime power. The logarithms of the factors of 24 cancel out - $\log(24) - \log(12) - \log(8) + \log(6) + \log(4) - \log(3) - \log(2) = 0$.

4   The Function $H(x)$

Let S_x denote the matrix obtained from T_x by element-by-element multiplication of the columns by $\log(j)$. Let r_n, $n = 1,2,3,\ldots,x$ denote the sums of the rows of S_x. A plot of r_n for $x = 500$ is given in Figure 11.
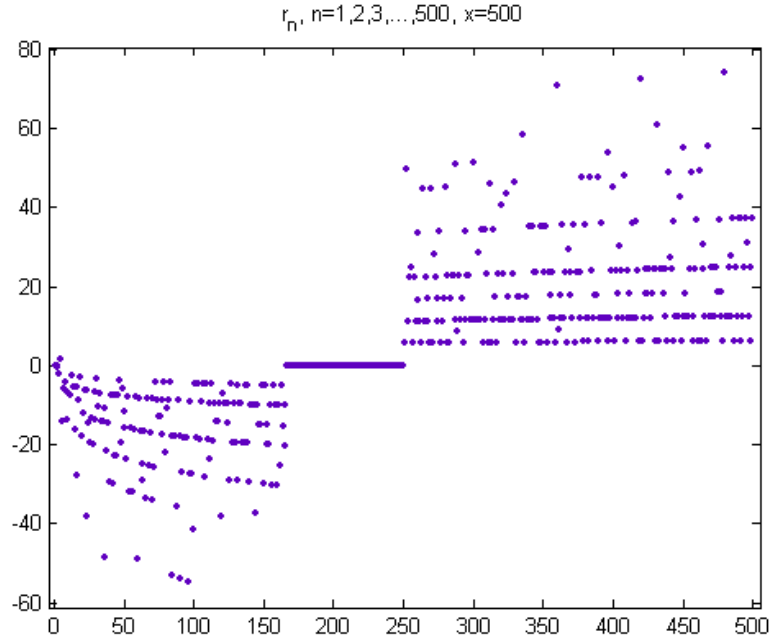
Figure 11: Plot of r_n for $x = 500$

The negative curves are derived from the logarithms of quadratic curves. A plot of the exponential of the negatives of one of the curves (the third one) is given in Figure 12.
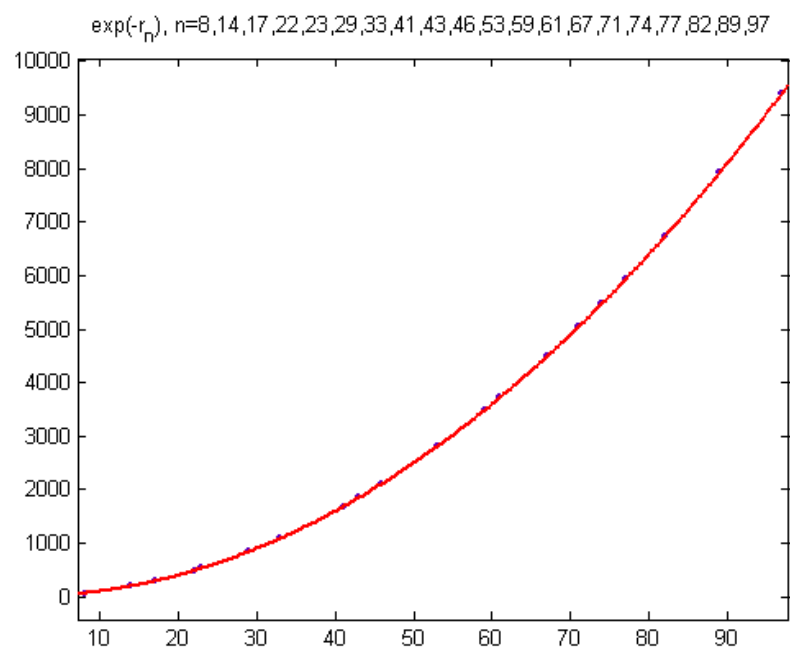
Figure 12: Plot of the exponential of the negatives of one of the curves (the third one)

For a quadratic least-squares fit of the data, $p_1 = 1$ with a 95% confidence interval of $(0.9999, 1)$, $p_2 = 0.002261$ with a 95% confidence interval of $(-0.007718, 0.01224)$, $p_3 = -0.03387$ with a 95% confidence interval of $(-0.2574, 0.1897)$, SSE=0.2631, R-squared=1, and RMSE=0.1244. The positive curves are also quadratic. A plot of one of the curves for $x = 100$ is given in Figure 13.
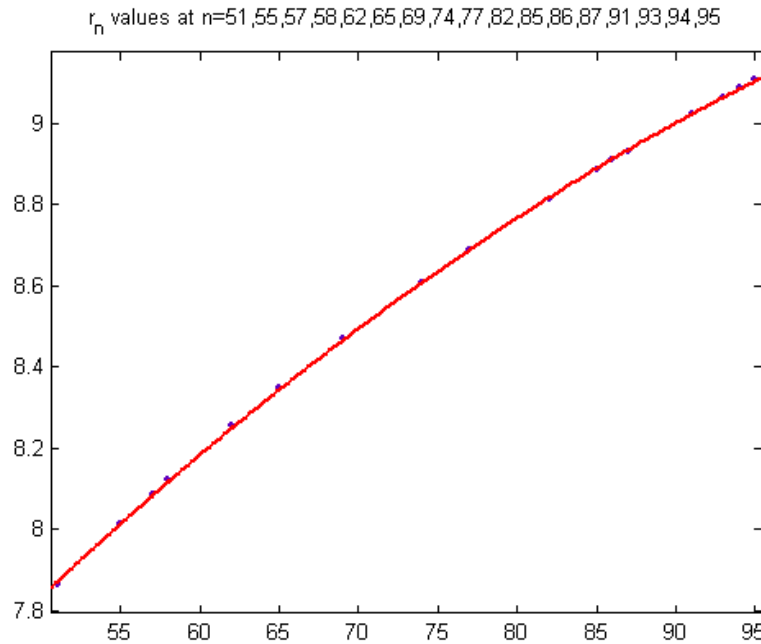
Figure 13: Plot of one of the curves for $x = 100$

For a quadratic least-squares fit of the data, $p_1 = -0.0001926$ with a 95% confidence interval of $(-0.0002054, -0.0001798)$, $p_2 = 0.05611$ with a 95% confidence interval of $(0.05421, 0.05801)$, $p_3 = 5.511$ with a 95% confidence interval of $(5.5443, 5.579)$, SSE=0.0002127, R-squared=0.9999, and RMSE=0.003898. The number of negative or positive curves is equal to the number of distinct non-zero Mertens function values less than or equal to x. Sums involving step functions of certain types can be expressed as integrals by means of the following theorem.

(9) Abel's identity. For any arithmetical function $a(n)$ let $A(x) = \sum_{n \leq x} a(n)$, where $A(x) = 0$ if $x < 1$. Assume f has a continuous derivative on the interval $[y,x]$ where $0 < y < x$. Then we have $\sum_{y < n \leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt$.

See Theorem 4.2 of Apostol's book. In the above case, Riemann-Stieltjes integration of Mertens function values is involved. The integral can be determined empirically - it turns out to be half $A(x)f(x)$. A simplified formula results since $A(1)f(1) = 0$. The following theorem shows that the behavior of $M(x)/x$ is determined by that of $H(x)/(x \log x)$.

(10) We have $\lim_{x \to \infty} \left(\frac{M(x)}{x} - \frac{H(x)}{x \log x} \right) = 0$.

See Theorem 4.13 of Apostol's book. In this case, the trivial estimate $M(x) = O(x)$ of $\int_1^x \frac{M(t)}{t} dt$ is used.

5   The Selberg Identity

Let R_x denote the matrix obtained from T_x by element-by-element multiplication of the columns by $\Lambda(j) \log j + \sum_{d|j} \Lambda(d)\Lambda(\frac{j}{d})$. Let r_n, $n = 1,2,3,\ldots,x$ denote the sums of the rows of R_x. A plot of r_n for $x = 500$ is given in Figure 14.
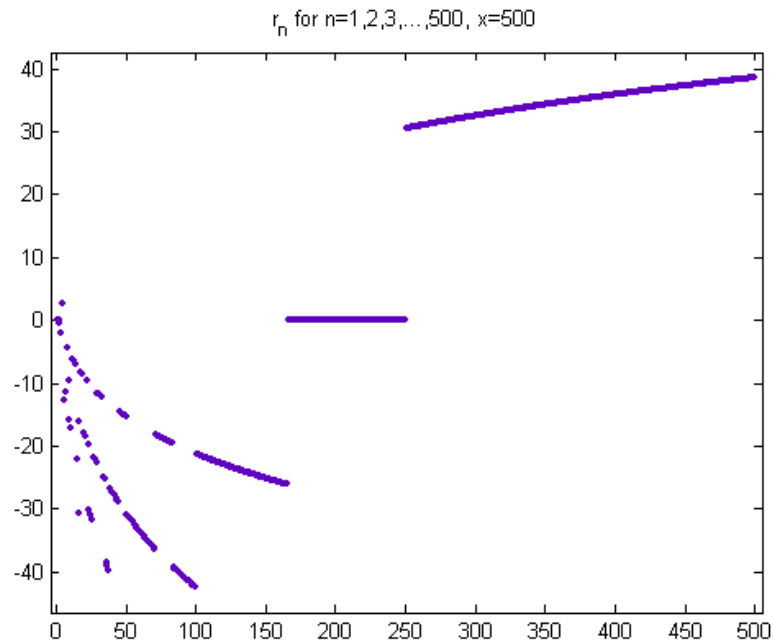
Figure 14: Plot of r_n for $x = 500$

A plot of the r_n for $n = \lfloor x/2 \rfloor + 1$ for $x = 100$ is given in Figure 15.
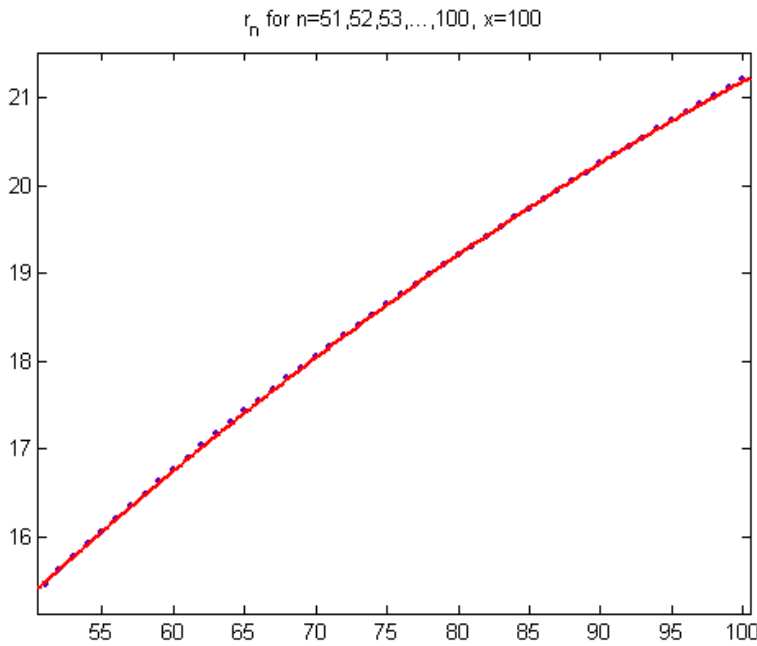
Figure 15: Plot of r_n for $n = 51,52,53,\ldots,100$, $x = 100$

For a quadratic least-squares fit of the data, $p_1 = -0.0006047$ with a 95% confidence interval of $(-0.0006221, -0.0005874)$, $p_2 = 0.2067$ with a 95% confidence interval of $(0.2049, 0.2102)$, $p_3 = 6.475$ with a 95% confidence interval of $(6.379, 6.572)$, SSE=0.006063, R-squared=1, and RMSE=0.01136.

A plot of one of the negative curves is given in Figure 16.
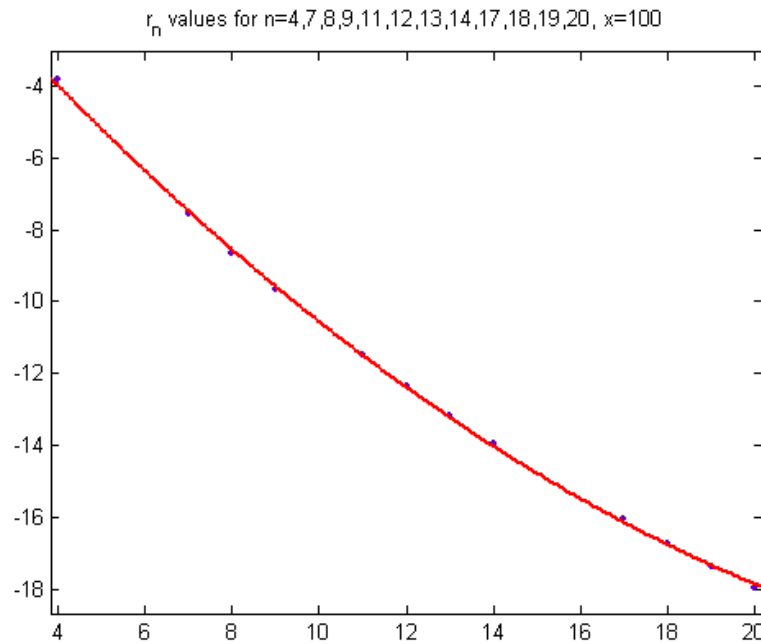
Figure 16: Plot of r_n for $n = 4,7,8,9,11,12,13,14,17,18,19,$and 20, $x = 100$

For a quadratic least-squares fit of the data, $p_1 = 0.02285$ with a 95% confidence interval of $(0.02011, 0.02559)$, $p_2 = -1.415$ with a 95% confidence interval of $(-1.484, -1.345)$, $p_3 = 1.312$ with a 95% confidence interval of $(0.9132, 1.711)$, SSE=0.07941, R-squared=0.9996, and RMSE=0.09393.

6 A Product Formula for $\varphi(n)$

The sum for $\varphi(n)$ in Theorem 2.3 of Apostol's book can also be expressed as a product extended over the distint prime divisors of n.

(11) For $n \geq 1$ we have $\varphi(n) = n \prod_{p|n} (1 - \frac{1}{p})$

See Theorem 2.4 in Apostol's book. Let V_x denote the matrix obtained from T_x by element-by-element multiplication of the columns by j. The sum of the sums of the columns is then $x(x + 1)/2$. Let r_n, $n = 1, 2, 3, \ldots, x$, denote the sums of the rows. A plot of r_n for $x = 200$ is given in Figure 17.
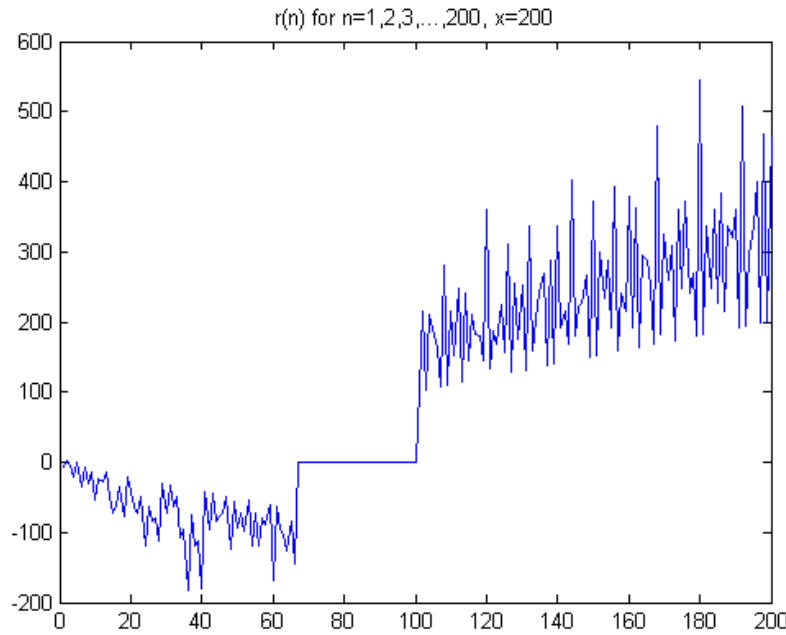
Figure 17: Plot of r_n for $x = 200$

The same intervals corresponding to the squarefree numbers occur and the same multiples of the Mertens function are applicable. In the following, the r_n values are "normalized", that is, divided by the corresponding Mertens function value (except when that value is 0). Let r'_n denote the normalized values. r_n values of zero are neglected. A product formula similar to Euler's is applicable, but $p + 1$ factors occur instead of $p - 1$ factors.

(12) If n is squarefree, $\prod_{p|n}(p+1) = r'_n$.

When n is not squarefree, more complex relationships occur. For example,

(13) If $n = p_1^2 p_2$, $r'_n = (p_2 + 1)(p_1(p_1 + 1) + 1)$.

Let $\sigma(n)$ denote the sum of positive divisors function. Let W_x denote the matrix obtained from T_x by element-by-element multiplication of the columns by $\sigma(j)$. Let r_n, $n = 1, 2, 3, \ldots, x$, denote the sums of the rows. A plot of r_n for $x = 200$ is given in Figure 18.
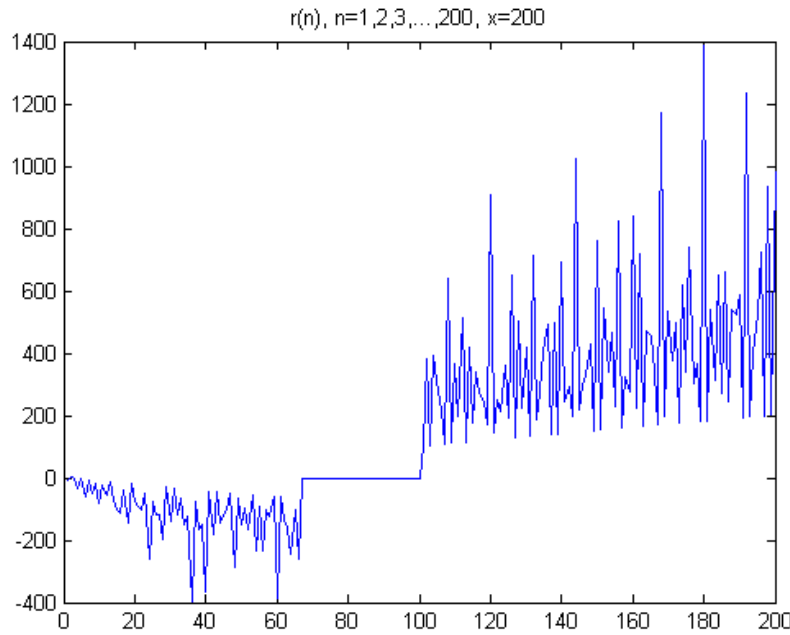
Figure 18: Plot of r_n for $x = 200$

The same intervals corresponding to the squarefree numbers occur and the same multiples of the Mertens function are applicable. In the following, the r_n values are "normalized". A product formula similar to Euler's is applicable, but $p + 2$ factors occur instead of $p - 1$ factors.

(14) If n is squarefree, $\prod_{p|n}(p + 2) = r'_n$.

If n is not squarefree, more complex relationships occur. For example,

(15) If $n = p_1^2 p_2$ and $p_1 > 3$, $r'_n = (p_2 + 2)(p_1(p_1 + 2) + 3)$.

7   An Analogue of $\Phi(x)$

Let Y_x denote the matrix obtained from T_x by element-by-element multiplication of the columns by $\sigma(j) - j$. Let c_n, $n = 1, 2, 3, \ldots, x$ denote the sums of the columns of Y_x. A plot of c_n for $x = 100$ is given in Figure 19.
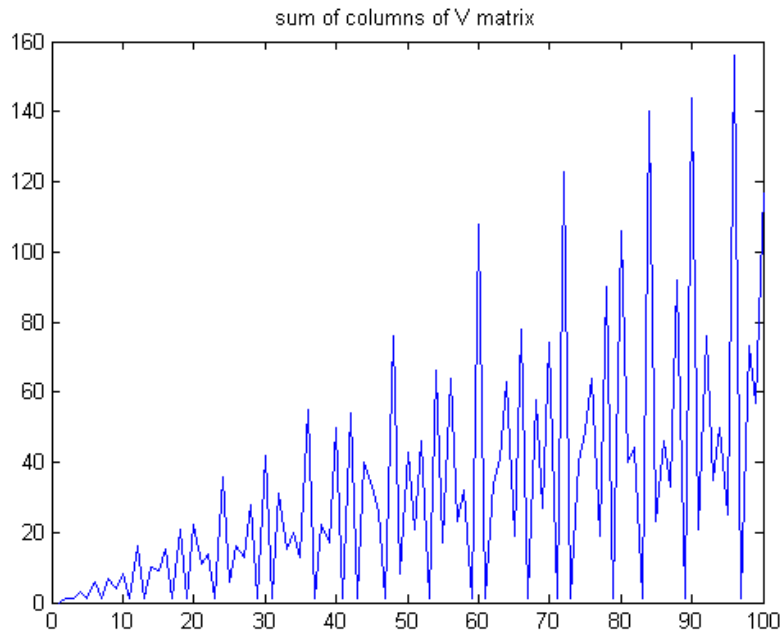
Figure 19: Plot of c_n for $x = 100$

$\sum_{n=1}^{x} c_n$ is approximately equal to $\Phi(x)$. For $x = 100$, the sum of the sums of the columns is 3249 whereas $\Phi(x) = 3044$. An empirical result is

(16) c_n where $n > 1$ is positive. $c_1 = 0$.

Let r_n, $n = 1, 2, 3, \ldots, x$ denote the sums of the rows of Y_x. A plot of r_n for $x = 100$ is given in Figure 20.
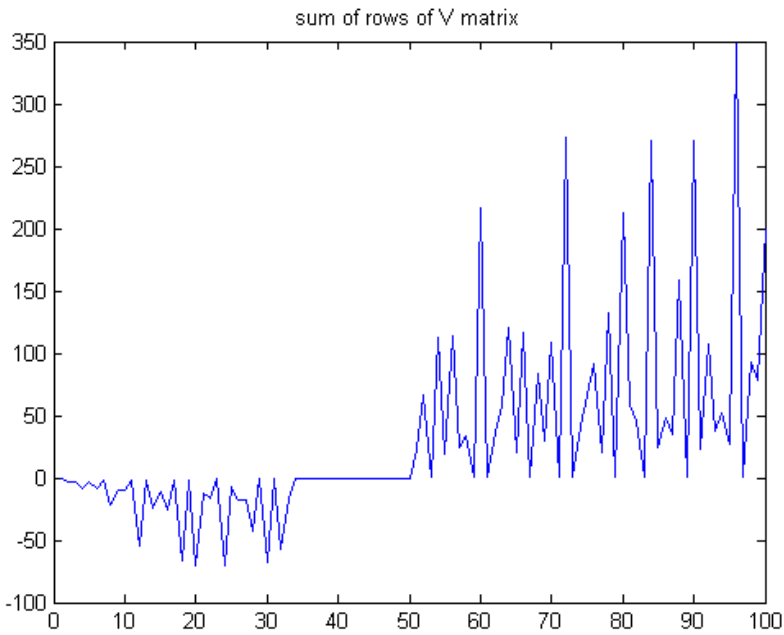
Figure 20: Plot of r_n for $x = 100$

Compared to the sums of the rows of U_x, the sums of the rows of Y_x have been "normalized". An empirical result is

(17) r_n is usually non-positive for $n < \lfloor x/3 \rfloor + 1$, positive for $n > \lfloor x/2 \rfloor$, and 0 in between.

A plot of the upper bounds of the n values, $n < \lfloor x/3 \rfloor + 1$, where r_n becomes greater than 0 for $x = 200, 300, 400, \ldots, 11100$ is given in Figure 21.
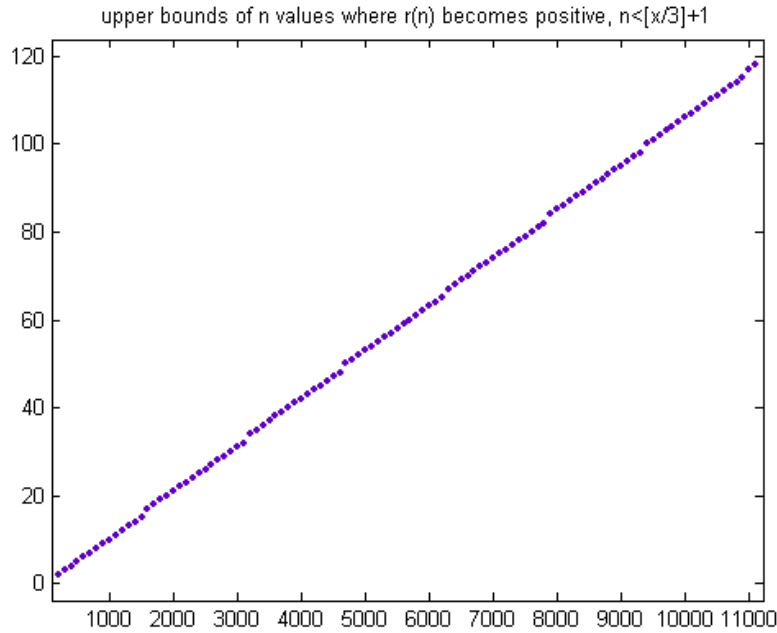
Figure 21: Plot of the upper bounds of the n values, $n < \lfloor x/3 \rfloor + 1$

The plot is not linear - the values jump by 2 at $x = 1600$, 3200, 4800, 6200, 7800, 9400, and 11000. Otherwise, the values increase by 1 at every x multiple of 100. For example, for $x = 1601$, 1602, and 1603, the (n, r_n) values where $r_n > 0$ are (2, 1), (4, 4), (7, 3), (16, 26), and (17, 1). For $x = 1604$, 1605, and 1606, the (n, r_n) values are (2, 2), (7, 2), (16, 26), and (17, 1). For $x = 1607$, 1608, and 1609 the (n, r_n) values are (2, 3), (7, 2), (11, 1), (16, 26), and (17, 1). As x increases, the number of positive r_n values less than the upper bound increases but in an erratic fashion. Note that the r_n values at prime n values are small if not equal to 1.

8   Additive Properties of r_n (for Y_x)

An empirical result is

(18) $r_n = 1$ if and only if n is prime. If $r_n = -1$, then n is prime.

Let p, q, r, and s denote distinct primes. An empirical result is

(19) When $1 < n < \lfloor x/3 \rfloor + 1$, r_n values at $n = p^2$ equal $-(p+2)$, small positive multiples of $-(p+2)$, or 0. For $n > \lfloor x/2 \rfloor$, the r_n values at $n = p^2$ equal $p+2$.

There do not appear to be analogous properties for r_n values at $n = p^3$, $n = p^4$, $n = p^5$, etc.

(20) When $1 < n < \lfloor x/3 \rfloor + 1$, r_n values at $n = pq$ equal $-(p + q + 3)$, small positive multiples of $-(p + q + 3)$, or 0. For $n > \lfloor x/2 \rfloor$, the r_n values at $n = pq$ equal $p + q + 3$.

(21) $|r_n|$ values at $n = pqr$ equal either 0 or rational multiples of $(p + q) + (p + r) + (q + r)$ minus a natural number. For $p = 2$, $q = 3$, and $r = 5$, 7, 11,..., the multiple is 4 and the natural number is 12. For $p = 2$, $q = 5$, and $r = 7$, 11, 13,..., the multiple is 5 and the natural is 32. For $p = 2$, $q = 7$, and $r = 11$, 13, 17,..., the multiple is 6 and the natural number is 60. For $p = 2$, $q = 11$, and $r = 13$, 17, 19,..., the multiple is 8 and the natural number is 140. In general, the multiple is $(q + 5)/2$. For $p = 3$, $q = 5$, and $r = 7$, 11, 13,..., the multiple is $\frac{11}{2}$ and the natural number is 42.

(22) $|r_n|$ values at $n = pqrs$ equal either 0 or rational multiples of $(p + q) + (p + r) + (p + s) + (q + r) + (q + s) + (r + s)$ minus a natural number. For $p = 2$, $q = 3$, $r = 5$, and $s = 7$, 11, 13,..., the multiple is $\frac{68}{3}$ and the natural number is 472.

(23) When $1 < n < \lfloor x/3 \rfloor + 1$, r_n values at $n = p^2 q$ equal $-(m(p + q) + i)$, small positive multiples of $-(m(p + q) + i)$, or 0 where m and i are natural numbers. For $n > \lfloor x/2 \rfloor$, the r_n values at $n = p^2 q$ equal $m(p + q) + i$. For $p = 2$ and $q = 3, 5, 7,...$, $m = 4$ and $i = 7$. For $p = 3$ and $q = 5, 7, 11,...$, $m = 5$ and $i = 8$.

These and similar relationships appear to account for all n values.

The first few superabundant numbers ($\sigma(n)/n > \sigma(k)/k$ for $1 \le k \le n - 1$) are 2, 6, 12, 60, 120, 360, 2520, and 5040. The prime factorizations are 2, $2 \cdot 3$, $2^2 \cdot 3$, $2^2 \cdot 3 \cdot 5$, $2^3 \cdot 3 \cdot 5$, $2^3 \cdot 3^2 \cdot 5$, $2^3 \cdot 3^2 \cdot 5 \cdot 7$, and $2^4 \cdot 3^2 \cdot 5 \cdot 7$ and the $\sigma(n)/n$ values are 1.5, 2.0, 2.333, 2.8, 3.0, 3.25, 3.714, 3.838. Similar superabundant numbers can be found for a fixed x value - the largest $\sigma(|r_n|)/|r_n|$, $n = 1, 2, 3,..., x$ value is found. For $x = 2, 6, 12, 60, 120, 360, 2520$, and 5040, the "superabundant numbers" are 2, 8, 10, 84, 180, -240, 7560, and 32760. The prime factorizations are 2, 2^3, $2 \cdot 5$, $2^2 \cdot 3 \cdot 7$, $2^2 \cdot 3^2 \cdot 5$, $2^4 \cdot 3 \cdot 5$, $2^3 \cdot 3^3 \cdot 5 \cdot 7$, and $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ and the $\sigma(|r_n|)/|r_n|$ values are 1.0, 1.875, 1.8, 2.667, 3.033, 3.1, 3.809, and 4.0.

9   Ramanujan and Robin's Theorems

Let γ denote Euler's constant.

(24) $\lim \sup_{n \to \infty} \frac{\sigma(n)}{n \log \log n} = e^\gamma$

(25) $\liminf_{n\to\infty} \frac{\varphi(n)\log\log n}{n} = e^{-\gamma}$

See 18.3 of Hardy and Wright's [5] book. Ramanujan [6] proved that the Riemann hypothesis implies $\sigma(n) < e^{\gamma} n \log\log n$ for sufficiently large n. Robin [7] proved that this inequality is true for all $n \geq 5041$ if and only if the Riemann hypothesis is true. Robin's theorems are

(26) If the Riemann hypothesis is true, then for each $n \geq 5041$, $\sum_{d|n} d \leq e^{\gamma} n \log\log n$ where γ is Euler's constant.

(27) If the Riemann hypothesis is false, then there exist constant $0 < \beta < \frac{1}{2}$ and $C > 0$ such that $\sum_{d|n} e^{\gamma} n \log\log n + \frac{Cn\log\log n}{(\log n)^{\beta}}$ holds for infinitely many n.

The constant β can be chosen to take any value $1 - b < \beta < \frac{1}{2}$ where $b = \Re(\rho)$ for some zero ρ of $\xi(s)$ with $R(\rho) > \frac{1}{2}$ and $C > 0$ must be chosen sufficiently small, depending on ρ.

A relationship proved by Cox is

(28) $\sum_{n=1}^{x} M(\lfloor x/n \rfloor)\sigma(n) = x(x+1)/2$

An empirical result is

(29) $e^{\gamma}\Phi(x) > x(x+1)/2$

A plot of $e^{\gamma}\Phi(x)$ and $x(x+1)/2$ for $x = 2, 3, 4, \ldots, 20$ is given in Figure 22.
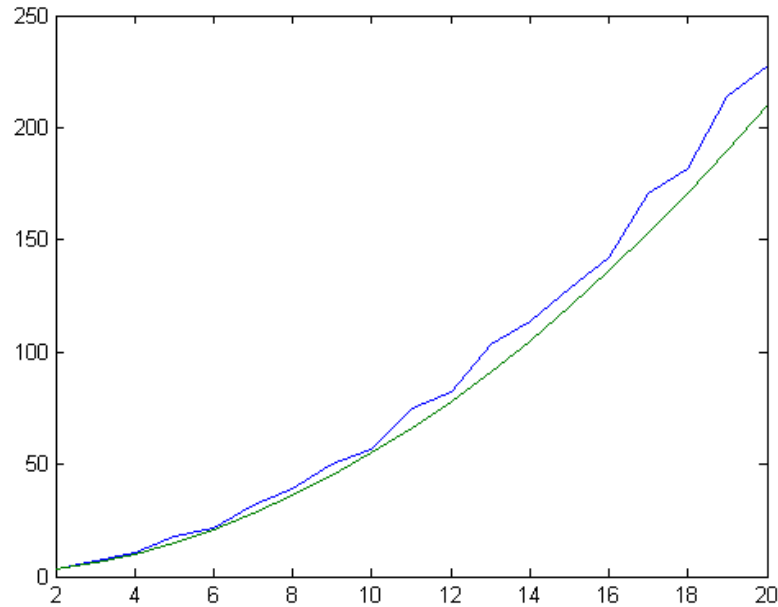
Figure 22: Plot of $e^\gamma \Phi(x)$ and $x(x+1)/2$ for $x = 2, 3, 4, \ldots, 20$

$i > \varphi(i)$ so $\frac{x(x+1)}{2} > \Phi(x)$. Upper and lower bounds of $\Phi(x)$ are then determined. These conditions may be equivalent to the Riemann hypothesis.

10   Conclusion

The modified Redheffer matrix technique is applicable to any formula containing the Möbius function. The Möbius function arises in many different places in number theory. The above is the basis for a more rigorous investigation (currently everything is empirically derived). Abel's identity may result in a proof.

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