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Article

Factoring continuous characters defined on subgroups of products of topological groups

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Abstract: We study factorization properties of continuous homomorphisms defined on subgroups (or submonoids) of products of (para)topological groups (or monoids). A typical result is the following one: Let $D = \prod_{i\in I} D_i$ be a product of paratopological groups, S be a dense subgroup of D, and χ a continuous character of S. Then one can find a finite set $E \subset I$ and continuous characters χ_i of D_i, for $i \in E$, such that $\chi = \left(\prod_{i\in E} \chi_i \circ p_i\right) \restriction S$, where $p_i \colon D \to D_i$ is the projection.

Keywords: Monoid; Group; Character; Homomorphism; Factorization; Roelcke uniformity

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To my colleague and friend M.J. Chasco, with great respect

1. Introduction

The present article is a natural continuation of [18,19], where we study continuous homomorphisms of subgroups (submonoids) of products of topological groups (monoids). We establish there that in many cases, a continuous homomorphism $f\colon S \to H$ of a submonoid (subgroup) S of a product $D = \prod_{i\in I} D_i$ of topological monoids (groups) to a topological monoid (group) H admits a factorization in the form

$$f = g \circ p_J \restriction S, \tag{1}$$

where J is a "small" subset of the index set I, $p_J\colon D \to D_J = \prod_{i\in J} D_i$ is the projection, and $g\colon p_J(S) \to H$ is a continuous homomorphism. If one can find a finite (countable) set J for which (1) holds true, we say that f has a *finite* (*countable*) *type*. Most of the results in [18,19] present different conditions on S and/or H under which f has a countable or even finite type. Purely algebraic aspects of this study can be found in [4].

In this article we go further and try to decompose a given continuous homomorphism $f\colon S \to H$ into a product of 'coordinate' homomorphisms as explained below.

It follows from the Pontryagin–van Kampen duality theory that every continuous homomorphism of a product $D = \prod_{i\in I} D_i$ of compact abelian groups to the circle group $\mathbb{T}$ (called *character*) has a finite type. Hence, every continuous character of D is a linear combination of finitely many continuous characters, each of which depends on exactly one coordinate. This fact remains valid in a considerably more general situation presented by S. Kaplan in [15]:

Proposition 1. *Let χ be a continuous character of a product $\Pi = \prod_{i\in I} G_i$ of (reflexive) topological abelian groups. Then one can find pairwise distinct indices $i_1, \ldots, i_n \in I$ and continuous characters $\chi_1, \ldots, \chi_n$ of the respective groups $G_{i_1}, \ldots, G_{i_n}$ such that the equality*

$$\chi(x) = \prod_{k=1}^{n} \chi_k(x_{i_k}) \tag{2}$$

holds for each $x \in \Pi$.

An analysis of the argument presented in [15] shows that one can drop 'reflexive' in the assumptions of Proposition 1. Hence we can reformulate the conclusion of Proposition 1 by saying that the dual group $\Pi^\wedge$ is algebraically isomorphic with $\bigoplus_{i\in I} D_i^\wedge$, the direct sum of the duals of the factors. Our aim is to extend the conclusion of Proposition 1 to

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considerably wider classes of objects like subgroups or submonoids of Cartesian products of monoids or paratopological groups (see Theorem 2, Corollary 1 and Theorems 4, 5 and 6).

An important property of the torus $\mathbb{T}$ is that it is an *NSS group*, which means that there exists an open neighborhood of the identity in $\mathbb{T}$ containing no nontrivial subgroups. Every Lie group is an NSS group. According to [19, Theorem 3.11], every continuous homomorphism of an arbitrary subgroup of a product of topological monoids to a Lie group has a finite type. This is an essential ingredient in several arguments presented in Section 2.

In Section 3 we complement several results from [19, Section 2] about continuous characters of a dense submonoid S of the *P-modification* of a product $D = \prod_{i\in I} D_i$ of *topologized* monoids. We show in Proposition 3 and Example 3 that if $\varphi\colon S \to H$ is a nontrivial continuous homomorphism of S to a topologized monoid of countable pseudocharacter, then the family $\mathcal{J}(\chi)$ of the subsets J of the index set I such that φ *depend on* J is often a filter on I, and this filter can have empty intersection, even if $S = D$ and the product $D = \mathbb{Z}(2)^\omega$ is a compact metrizable topological group (hence the P-modification of D is a discrete group).

Notation and Auxiliary Results

Let $\mathbb{C}$ be the field of complex numbers with the usual Euclidean topology. The torus $\mathbb{T}$ is identified with the multiplicative subgroup $\{z \in \mathbb{C} : |z| = 1\}$ of $\mathbb{C}$.

A *semigroup* is a nonempty set S with a binary associative operation (called *multiplication*). A semigroup with identity is called a *monoid*. Clearly a monoid has a unique identity.

A semigroup S with topology is said to be a *semitopological semigroup* if multiplication in S separately continuous. This is equivalent to saying that the left and right shifts in S are continuous. If multiplication in S is jointly continuous, we say that S is a *topological semigroup*. The concept of *topological monoid* is defined similarly.

Assume that G is a semigroup (monoid, group) with a topology. If the left shifts in G are continuous, then G is called *left topological* semigroup (monoid, group). If both left and right shifts in G are continuous, then G is said to be a *semitopological* semigroup (monoid, group). Further, if G is a group and multiplication in G is jointly continuous, we say that G is a *paratopological* group. A paratopological group with continuous inversion is a *topological* group.

A *topologized* monoid (group) is a monoid (group) with an arbitrary topology that may have no relation to multiplication in the monoid (group). We say that a left topological monoid G has *open left shifts* if for every $x \in G$, the left shift λ_x of G defined by $y \mapsto x \cdot y$ for each $y \in G$, is an open mapping of G to itself.

A *character* of an arbitrary monoid G is a (not necessarily continuous) homomorphism of G to the torus $\mathbb{T}$. The continuity of a character, if applies, will always be specified explicitly.

In the sequel we follow notation of Proposition 1. For every $i \in I$, let p_i be the projection of Π onto the factor G_i. Then the conclusion of the proposition is equivalent to saying that $\chi = \prod_{k=1}^{n} \chi_k \circ p_{i_k}$. It is worth noting that the projections p_i are continuous open homomorphisms, so the characters $\chi_1, \ldots, \chi_n$ are 'automatically' continuous. This assertion follows from the next simple result which shows that for finitely many factors, the conclusion of Proposition 1 remains valid, even if the factors are topologized monoids.

Lemma 1. *Let $G = G_1 \times \cdots \times G_n$ be a product of topologized monoids and χ be a continuous homomorphism of G to a topologized semigroup K. Then there exist homomorphisms $\chi_1, \ldots, \chi_n$ of the respective monoids $G_1, \ldots, G_n$ to K such that $\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n)$, for each $x = (x_1, \ldots, x_n) \in G$. This representation of χ is unique and the homomorphisms $\chi_1, \ldots, \chi_n$ are continuous.*

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Proof. For every $k = 1, \ldots, n$, let e_k be the identity of G_k and p_k be the projection of G onto the factor G_k. We define a homomorphism χ_k of G_k to K by $\chi_k(y) = \chi(e_1, \ldots, y, \ldots, e_n)$ for every $y \in G_k$, where y stands at the kth position in $(e_1, \ldots, y, \ldots, e_n)$. A direct verification shows that $\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n)$, for each $x = (x_1, \ldots, x_n) \in G$.

Let $\psi_1, \ldots, \psi_n$ be homomorphisms of $G_1, \ldots, G_n$, respectively, to K satisfying $\chi(x) = \psi_1(x_1) \cdots \psi_n(x_n)$, for each $x \in G$. We fix an integer k with $1 \leq k \leq n$ and for every $y \in G_k$, consider the element $\hat{y} = (e_1, \ldots, y, \ldots, e_n) \in G$, where y stands at the kth position in $\hat{y}$. Then $\chi_k(y) = \chi(\hat{y}) = \psi_k(y)$, so $\psi_k = \chi_k$ for each $k \leq n$ and, hence, the representation $\chi(x) = \chi_1(x_1) \cdots \chi_n(x_n)$ is unique.

It follows from the continuity of the homomorphism χ and the equalities $\chi_k(y) = \chi(e_1, \ldots, y, \ldots, e_n)$, where $1 \leq k \leq n$ and $y \in G_k$, that $\chi_1, \ldots, \chi_n$ are continuous. □

Let $X = \prod_{i \in I} X_i$ be the Tychonoff product of a family $\{X_i : i \in I\}$ of spaces and $a \in X$ be an arbitrary point. For every $i \in I$, the projection of X to the factor X_i is denoted by p_i. Also, for every $x \in X$, we put

$$\mathrm{diff}(x, a) = \{i \in I : p_i(x) \neq p_i(a)\}.$$

Then

$$\Sigma X(a) = \{x \in X : |\mathrm{diff}(x, a)| \leq \omega\}$$

and

$$\sigma X(a) = \{x \in X : |\mathrm{diff}(x, a)| < \omega\}$$

are dense subspaces of X which are called respectively the Σ-*product* and σ-*product* of the family $\{X_i : i \in I\}$ with center at a. If every X_i is a monoid (group), we will always choose a to be the identity e of X. In the latter case, $\Sigma X(e)$ and $\sigma X(e)$ are dense submonoids (subgroups) of the product monoid (group) X and we shorten $\Sigma X(e)$ and $\sigma X(e)$ to ΣX and σX, respectively.

Assume that Z is a nonempty subset of the product $X = \prod_{i \in I} X_i$ of a family $\{X_i : i \in I\}$ of sets and $f \colon Z \to Y$ is an arbitrary mapping. We say that f *depends on* J, for some $J \subset I$, if the equality $f(x) = f(y)$ holds for all $x, y \in Z$ with $p_J(x) = p_J(y)$, where $p_J \colon X \to \prod_{i \in J} X_i$ is the projection. It is clear that if f depends on J, then there exists a mapping g of $p_J(Z)$ to Y satisfying $f = g \circ p_J \restriction Z$. Conversely, if there exists such a mapping g of $p_J(Z)$ to Y, then f depends on J.

Definition 1. *Assume that D_i is a monoid with identity e_i, where $i \in I$. For a nonempty subset J of I, we define a retraction r_J of $D = \prod_{i \in I} D_i$ by letting*

$$r_J(x)_i = \begin{cases} x_i & \text{if } i \in J; \\ e_i & \text{if } i \in I \setminus J, \end{cases}$$

for each element $x \in D$. A subset S of D is said to be retractable *if $r_J(S) \subset S$, for each $J \subset I$. If the inclusion $r_J(S) \subset S$ holds for each finite set $J \subset I$, we call S* finitely retractable.

The concept of finite retractability will be used in Theorem 5.

Given a space X, we denote by PX the underlying set X with the topology whose base consists of all nonempty G_δ-sets in X. The space PX is usually referred to as the *P-modification* of X. If X is a (left) topological group (monoid), then PX with the same multiplication is also a (left) topological group (monoid).

The family of countable subsets of a given set I is denoted by $[I]^{\leq \omega}$.

2. Factoring continuous characters

In this section, we deal with not necessarily Hausdorff objects of Topological Algebra. Since a major part of research articles and books on this subject treat exclusively the Hausdorff case, we need to extend several well-known facts to non-Hausdorff monoids

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and groups. We start with the following result that, informally, goes back to Graev's article [12, pp. 52–53].

Lemma 2. *Let G be a topological group with identity e, N be the closure of the singleton $\{e\}$ in G and $\pi\colon G \to G/N$ the quotient homomorphism. For every continuous homomorphism $f\colon G \to H$ to a Hausdorff topological group H, there exists a unique homomorphism $g\colon G/N \to H$ satisfying $f = g \circ \pi$ and g is automatically continuous.*

Proof. Notice that N is a closed invariant subgroup of G, so the quotient topological group G/N is a T_1-space. Hence G/N is Hausdorff. Denote by K the kernel of f. Since H is Hausdorff, K is a closed subgroup of G. Hence $\ker \pi = N \subset K = \ker f$. It now follows from [2, Proposition 1.5.10] that there exists a homomorphism $g\colon G/N \to H$ satisfying $f = g \circ \pi$. Assume that a homomorphism $\tilde{g}\colon G/N \to H$ also satisfies $f = \tilde{g} \circ \pi$. If $y \in G/N$, we take an element $x \in G$ with $\pi(x) = y$. Then $g(y) = g(\pi(x)) = f(x)$ and, similarly, $\tilde{g}(y) = \tilde{g}(\pi(x)) = f(x)$. Hence $\tilde{g}(y) = g(y)$ for each $y \in G/N$, so $\tilde{g} = g$. As π is open and continuous, we conclude that g is continuous. □

The pair $(G/N, \pi)$ in Lemma 2 is called the *Hausdorff reflection* of G. Abusing terminology, we usually refer to G/N as the Hausdorff reflection of G, thus omitting the quotient homomorphism π. We also denote G/N by $T_2(G)$.

Informally speaking, the following lemma states that the functor of Hausdorff reflection in the category of topological groups and continuous homomorphisms *respects* arbitrary subgroups.

Lemma 3. *Let G be a topological group with identity e, N be the closure of the singleton $\{e\}$ in G and $\pi\colon G \to G/N$ the quotient homomorphism. Let also S be an arbitrary subgroup of G and $N_S = S \cap N$. Then the quotient group $T_2(S) = S/N_S$ is topologically isomorphic to the subgroup $\pi(S)$ of $T_2(G) = G/N$ and the restriction of π to S is an open continuous homomorphism of S onto $\pi(S)$.*

Proof. It follows from the definition of π that every closed subset C of G satisfies $C = \pi^{-1}\pi(C)$. Therefore, if the subgroup S is closed in G then $N \subset S$, $S = \pi^{-1}\pi(S)$, and the restriction of π to S is an open continuous homomorphism of S onto the subgroup $\pi(S)$ of G/N. By the first isomorphism theorem, the groups $\pi(S)$ and S/N are topologically isomorphic.

In the general case, let K be the closure of S in G. Then K is a closed subgroup of G, $N \subset K$ and, by the above argument, the groups $T_2(K) = K/N$ and $\pi(K) \subset T_2(G)$ are topologically isomorphic. Hence it suffices to verify that the group $T_2(S)$ is topologically isomorphic to the subgroup $\pi(S)$ of K/N. To this end we show that the restriction of π to S is an open homomorphism onto the subgroup $\pi(S)$ of K/N. Let U be a nonempty open set in K and $V = U \cap S$. Since $K = \pi^{-1}\pi(K)$ and $N \subset K$, the set U satisfies the equality $U = \pi^{-1}\pi(U)$. Hence the set $\pi(U) \cap \pi(S) = \pi(U \cap S) = \pi(V)$ is open in $\pi(S)$. Thus $\pi{\restriction}S$ is an open homomorphism of S onto $\pi(S)$ whose kernel is $S \cap N$, so the groups $T_2(S)$ and $\pi(S)$ are topologically isomorphic. □

Let us recall that the *precompact Hausdorff reflection* of a given topological group G is a pair (H, φ_G), where H is a precompact Hausdorff topological group and $\varphi_G\colon G \to H$ is a continuous onto homomorphism, such that for every continuous homomorphism $g\colon G \to K$ to a Hausdorff precompact topological group K, there exists a continuous homomorphism $h\colon H \to K$ satisfying $g = h \circ \varphi_G$. Every topological group G has a precompact Hausdorff reflection and this reflection is unique up to topological isomorphism [13]. The homomorphism φ_G is referred to as *universal* for G.

Lemma 4. *Let S be a dense subgroup of a topological group G and (H, φ_G) be the precompact Hausdorff reflection of G. Let also $T = \varphi_G(S)$ and $\psi = \varphi_G{\restriction}S$. Then (T, ψ) is the precompact Hausdorff reflection of the group S.*

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Proof. Since H is a precompact Hausdorff topological group, so is its dense subgroup T. Therefore it suffices to verify that the continuous onto homomorphism $\psi\colon S \to T$ is universal for S. Let $g\colon S \to K$ be a continuous homomorphism to a precompact Hausdorff group K. The completion of K, say, ϱK is a compact Hausdorff topological group. Hence the group ϱK is complete. Since S is dense in G, g extends to a continuous homomorphism $g^*\colon G \to \varrho K$. By the universality of φ_G, there exists a continuous homomorphism $h^*\colon H \to \varrho K$ such that $g^* = h^* \circ \varphi_G$. Let h be the restriction of h^* to T. Then $g = g^* \restriction S = h^* \circ \varphi_G \restriction S = h^* \circ \psi = h \circ \psi$. This proves the universality of ψ for S. □

A subgroup S of a topological abelian group G is said to be *dually embedded* in G if every continuous character of S extends to a continuous character of G. The next lemma is well known in the special case of Hausdorff topological groups [11, Lemma 2.2].

Lemma 5. *Every subgroup S of a precompact topological abelian group G is dually embedded in G.*

Proof. Let e be the identity of G and N be the closure of the singleton $\{e\}$ in G. Let also $p\colon G \to G/N$ be the quotient homomorphism. Since G is precompact, the pair $(G/N, p)$ is the precompact Hausdorff reflection of G. Let S be a subgroup of G. Denote by K the closure of S in G. It follows from the definition of N that $N \subset K$ and $K = p^{-1}p(K)$, so $K/N \cong p(K)$ and $(p(K), q)$ is the precompact Hausdorff reflection of K, where $q = p\restriction K$. Since S is dense in K, Lemma 4 implies that $(q(S), q\restriction S) = (p(S), p\restriction S)$ is the precompact Hausdorff reflection of S.

Let χ be a continuous character of S. There exists a continuous character λ of the subgroup $T = p(S)$ of the precompact Hausdorff group G/N such that $\chi = \lambda \circ p\restriction S$. By [11, Lemma 2.2], T is dually embedded in the Hausdorff precompact abelian group G/N, so λ extends to a continuous character λ^* of G/N. Hence $\chi^* = \lambda^* \circ p$ is an extension of χ to a continuous character of G and S is dually embedded in the group G. □

The following fact complements Lemma 5 in the non-abelian case.

Lemma 6. *Every dense subgroup S of an arbitrary topological group G is dually embedded in G.*

Proof. Let (H, φ_G) be the precompact Hausdorff reflection of the group G. We put $T = \varphi_G(S)$ and $\psi = \varphi_G\restriction S$. By Lemma 4, the pair (T, ψ) is the precompact Hausdorff reflection of S.

Let χ be a continuous character of S. Then there exists a continuous character χ_T of T such that $\chi = \chi_T \circ \psi$. Since the group H is precompact and Hausdorff, it follows from [11, Lemma 2.2] that T is dually embedded in H. Hence χ_T extends to a continuous character λ of H. Then $\chi^* = \lambda \circ \varphi_G$ is a continuous character of G which extends χ. □

Lemma 6 is not valid for *closed* subgroups of Hausdorff topological groups. In fact, even a compact subgroup of a separable metrizable topological abelian group can fail to be dually embedded [2, Example 9.9.61].

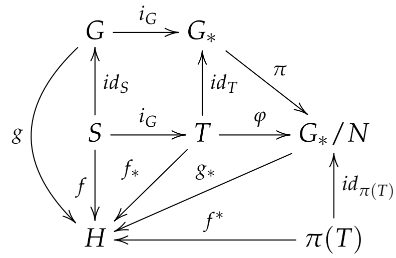
According to Proposition 3.6.12 of [2], a continuous homomorphism of a dense subgroup S of a Hausdorff topological group G to a complete Hausdorff topological group H extends to a continuous homomorphism of G to H. Below we generalize this fact by showing that it remains valid for dense subgroups of arbitrary paratopological groups. Our argument makes use of the *topological group reflection* of a paratopological group (see [16]).

Theorem 1. *Let S be a dense subgroup of a paratopological group G and $f\colon S \to H$ be a continuous homomorphism of S to a complete Hausdorff topological group H. Then f extends to a continuous homomorphism $g\colon G \to H$.*

Proof. Let $i_G\colon G \to G_*$ be the identity mapping of G onto the topological group reflection G_* of G. It follows from [16, Theorem 12] that the subgroup $T = i_G(S)$ of G_* is topologically isomorphic to the topological group reflection S_* of S, so we can identify the groups T and S_*, algebraically and topologically.

Since H is a topological group, there exists a continuous homomorphism $f_*\colon T \to H$ satisfying $f = f_* \circ i_G \restriction S$. It follows from the continuity of i_G that T is a dense subgroup of G_*. However, the groups G_* and T may fail to be Hausdorff.

To reduce our further argument to the case of Hausdorff groups, we denote by N the closure of the singleton $\{e_G\}$ in G_* and consider the quotient homomorphism $\pi\colon G_* \to G_*/N$. Then the quotient group G_*/N is the Hausdorff reflection of G_*. By Lemma 3, the subgroup $\pi(T)$ of G_*/N is the Hausdorff reflection of T and the homomorphism $\varphi = \pi \restriction T$ of T onto $\pi(T)$ is open and continuous. Since the group H is Hausdorff, Lemma 2 implies the existence of a continuous homomorphism $f^*\colon \pi(T) \to H$ satisfying the equality $f_* = f^* \circ \varphi$. Notice that T is dense in G_* and $\pi(T)$ is dense in G_*/N. Therefore, by [2, Corollary 3.6.17], f_* extends to a continuous homomorphism $g_*\colon G_*/N \to H$ (we use the completeness of H here).



Then $g = g_* \circ \pi \circ i_G$ is a continuous homomorphism of G to H which extends f. This proves the theorem. □

We complement Theorem 1 in Proposition 2 by considering continuous homomorphisms defined on dense submonoids of topological monoids.

Example 1. Closed subgroups of completely regular paratopological groups need not be dually embedded. Hence Theorem 1 does not extend to closed subgroups of paratopological groups.

Proof. Let $\mathbb{S}$ be the Sorgenfrey line endowed with the usual topology and addition. Clearly $\mathbb{S}$ is a regular (even hereditarily normal) paratopological group. Let also $\Delta = \{(x, -x) : x \in \mathbb{S}\}$ be the *second diagonal* of $\mathbb{S} \times \mathbb{S}$. It is well known and easy to verify that the subgroup Δ is discrete and closed. Hence every character of Δ is continuous and Δ can be identified with the real line $\mathbb{R}_d$ endowed with the discrete topology. On the one hand, an easy calculation shows that the family of characters of Δ has the cardinality $\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}$, where $\mathfrak{c} = 2^{\omega}$. On the other hand, the groups $\mathbb{S}$ and $\mathbb{S} \times \mathbb{S}$ are separable, so there are at most $\mathfrak{c}^{\omega} = \mathfrak{c}$ continuous characters of $\mathbb{S} \times \mathbb{S}$. Therefore, not every character of Δ extends to a continuous character of $\mathbb{S} \times \mathbb{S}$. In other words, Δ fails to be dually embedded in $\mathbb{S} \times \mathbb{S}$. It is also clear that not every character of Δ admits the representation described in Lemma 1 (or in Theorem 2 that follows). □

The next result is a considerable generalization of Proposition 1.

Theorem 2. *Let $D = \prod_{i \in I} D_i$ be a product of paratopological groups and S be a subgroup of D. Assume that for every finite set $F \subset I$, the subgroup $p_F(S)$ of $D_F = \prod_{i \in F} D_i$ is dually embedded in D_F, where $p_F\colon D \to D_F$ is the projection. Then for every continuous character χ of S, one can find a finite set $E \subset I$ and continuous characters χ_i of $p_i(S)$, for $i \in E$, such that $\chi = \left(\prod_{i \in E} \chi_i \circ p_i\right) \restriction S$.*

Proof. By Corollary 3.12 in [19], one can find a finite set $E \subset I$ and a continuous character χ_E of $p_E(S)$ such that $\chi = \chi_E \circ p_E \restriction S$, where $p_E\colon D \to \prod_{i \in E} D_i$ is the projection. By assumptions of the theorem, $T = p_E(S)$ is a dually embedded subgroup of $D_E = \prod_{i \in E} D_i$.

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Hence χ_E extends to a continuous character ψ of D_E. According to Lemma 1, for every $i \in E$, there exists a continuous character ψ_i of G_i such that $\psi = \prod_{i\in E} \psi_i \circ q_i$, where $q_i \colon D_E \to D_i$ is the projection. Let $p_i \colon D \to D_i$ be the projection, for each $i \in E$. Since $p_i = q_i \circ p_E$ and $\chi = \psi \circ p_E \restriction S$, we conclude that the required equality $\chi = \left(\prod_{i\in E} \chi_i \circ p_i\right) \restriction S$ is valid. □

Example 1 explains why in Theorem 2, we require the projections of a subgroup $S \subset D$ to finite subproducts to be dually embedded, though this does not exclude the possibility that the theorem be valid for *arbitrary* subgroups of products of (para)topological groups. Later, in Example 2, we will show that such a generalization of Theorem 2 is impossible, even if the factors of the product $D = \prod_{i\in I} D_i$ are topological groups.

By Theorem 1, a dense subgroup of a paratopological group is dually embedded. Hence the next corollary is immediate from Theorem 2.

Corollary 1. *Let $D = \prod_{i\in I} D_i$ be a product of paratopological groups, S be a dense subgroup of D, and χ a continuous character of S. Then one can find a finite set $E \subset I$ and continuous characters χ_i of D_i, for $i \in E$, such that $\chi = \left(\prod_{i\in E} \chi_i \circ p_i\right) \restriction S$, where $p_i \colon D \to D_i$ is the projection.*

The next example shows that the conditions on S to be 'dually embedded' in Theorem 2 and 'dense' in Corollary 1 are essential.

Example 2. *There exist countably infinite, metrizable topological abelian groups G_1, G_2 and a closed discrete subgroup Δ of the product $\Pi = G_1 \times G_2$ such that $p_1(\Delta) = G_1$, $p_2(\Delta) = G_2$, and the only continuous character of the group Π is the trivial one. Here p_1 and p_2 are projections of Π onto G_1 and G_2, respectively. In particular, the trivial character of Δ is the only one representable in the form described in Corollary 1.*

Proof. Let G be a countable, infinite Boolean group. Then G is the direct sum of countably many copies of the group $\mathbb{Z}(2) = \{0, 1\}$, so G is as in item 2) of Lemma 0 in [1]. Therefore, Theorem′ on page 22 of [1] implies that G admits a metrizable topological group topology τ_1 such that the only continuous character of $G_1 = (G, \tau_1)$ is the trivial one.

Our first observation is that the group G_1 is not precompact—otherwise continuous characters of G_1 would separate elements of G_1. Since every non-zero element of the countable group G_1 has order 2, one can apply [5, Theorem 5.28] to find an open neighborhood U of zero e_1 in G_1 and a (necessarily discontinuous) automorphism f of the group G_1 such that $f(U) \cap U = \{e_1\}$. In other words, the group G_1 is *self-transversal.*

Let $\tau_2 = \{f(V) : V \in \tau_1\}$ be the image of the topology τ_1 under the automorphism f and $G_2 = (G, \tau_2)$. Then f is a topological isomorphism of G_1 onto G_2 and the only continuous character of G_2 is the trivial one. By Lemma 1 the product group $\Pi = G_1 \times G_2$ has the same property. Denote by Δ the subgroup $\{(x, x) : x \in G\}$ of the group Π. It is clear that $p_1(\Delta) = G_1$ and $p_2(\Delta) = G_2$. The set $O = U \times f(U)$ is open in Π and it follows from our choice of the set U that the intersection $O \cap \Delta$ contains only the identity element of $G_1 \times G_2$. Hence the subgroup Δ of Π is discrete and closed. It is clear that every character of Δ is continuous, and that the only character of Δ that can be expressed in the form presented in Corollary 1 is the trivial one. □

Since the subgroup Δ of the group $G_1 \times G_2$ in Example 2 is discrete, we see that Corollary 1 is not valid for locally compact subgroups of products of topological groups. However, it is valid for *precompact* abelian subgroups of product groups.

First, we present a well-known result from [10] often called the *Comfort-Ross duality* for precompact topological abelian groups. We denote the family of all characters of an abstract group G to the torus $\mathbb{T}$ by $Hom(G, \mathbb{T})$. Clearly, the pointwise multiplication of characters in $Hom(G, \mathbb{T})$, $(\chi_1 \cdot \chi_2)(x) = \chi_1(x) \cdot \chi_2(x)$, makes it an abelian group.

Theorem 3. *For every abelian group G, there exists a natural (i.e., functorial) monotone bijection between the family of precompact topological group topologies on G and the subgroups of the group $Hom(G, \mathbb{T})$.*

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'Monotone' in Theorem 3 means that a finer precompact topological group topology on G corresponds to a bigger subgroup of $Hom(G, \mathbb{T})$. For more details on this correspondence, see [10].

In the following theorem we do not impose any separation restrictions on the factors D_i:

Theorem 4. *Let C be a precompact abelian subgroup of a product $D = \prod_{i\in I} D_i$ of topological groups and χ be a continuous character of C. Then one can find a finite set $E \subset I$ and continuous characters χ_i of $p_i(C)$, for $i \in E$, such that $\chi = \left(\prod_{i\in E} \chi_i \circ p_i\right)\restriction C$, where $p_i\colon D \to D_i$ is the projection.*

Proof. The projection $p_i(C)$ is a precompact abelian subgroup of the group D_i, for each $i \in I$. We can assume, therefore, that each factor $D_i = p_i(C)$ is a precompact abelian group. Then D is also a precompact topological abelian group. For every $i \in I$, let $D_i^\wedge$ be group of continuous characters of D_i. By [10, Theorem 1.2], the topology of D_i is initial with respect to $D_i^\wedge$. Consider the family

$$\mathcal{A} = \{\chi \circ p_i : i \in I,\ \chi \in D_i^\wedge\}.$$

Then each element of $\mathcal{A}$ is a continuous character of D, so $\mathcal{A} \subset D^\wedge$. Let H be the subgroup of $D^\wedge$ generated by $\mathcal{A}$. Every element χ of H has the form

$$\chi = \prod_{k=1}^{n} \chi_k \circ p_{i_k}, \tag{3}$$

where $i_1, \ldots, i_n$ are pairwise distinct elements of I and $\chi_k \in D_{i_k}^\wedge$ for each $k = 1, \ldots, n$. It is clear that the topology of D is initial with respect to H. Since C is a topological subgroup of D, the family of restrictions $H_C = \{\chi\restriction C : \chi \in H\}$ generates the topology of C. Notice that H_C is a subgroup of $C^\wedge \cap Hom(C, \mathbb{T})$, so Theorem 3 implies that $H_C = C^\wedge$. The latter equality together with (3) imply the required conclusion. □

Problem 1. *Does Theorem 4 extend to precompact subgroups of products of paratopological abelian groups?*

The main difficulty in solving Problem 1 is the fact that the topological group reflection of a subgroup C of a paratopological abelian group D can have a strictly finer topology than the topology of C inherited from D_*. In other words, Lemma 4 cannot be extended to paratopological groups. Even the very special case of Problem 1, where C is a precompact subgroup of the product of two (precompact) paratopological groups, is not clear.

The following result extends a well-known property of continuous homomorphisms of topological groups to a more general case when the domain of a homomorphism is a dense *submonoid* of a topological monoid with open shifts. First we recall the notions of Roelcke uniformity and Roelcke-completeness in topological groups.

Let G be a topological group and $\mathcal{N}(e)$ be the family of open neighborhoods of the identity e in G. For every $U \in \mathcal{N}(e)$, the set

$$O_U = \{UxU : x \in G\}$$

is an open entourage of the diagonal in $G \times G$ and the family $\{O_U : U \in \mathcal{N}(e)\}$ constitutes a base for a compatible uniformity on G, say, $\mathcal{V}_G$ which is called the *Roelcke uniformity* of G (see [2, Section 1.8]). If the uniform space $(G, \mathcal{V}_G)$ is complete, we say that the group G is *Roelcke-complete*.

Proposition 2. *Let S be a dense submonoid of a topological monoid D with open shifts. Then every continuous homomorphism $f\colon S \to K$ to a Roelcke-complete Hausdorff topological group K extends to a continuous homomorphism $f^*\colon D \to K$.*

Proof. Let $\mathcal{N}(e)$ be the family of open neighborhoods of the identity e in D. We denote by $\mathcal{Q}$ the *quasi-Roelcke* uniformity of D whose base consists of the sets

$$Q_V = \{(x,y) \in D \times D : Vx \cap yV \neq \emptyset \neq Vy \cap xV\},$$

where $V \in \mathcal{N}(e)$ (see [3]). It is easy to see that the topology of D generated by $\mathcal{Q}$ is weaker than the original topology of D. Let also $\mathcal{V}_K$ be the Roelcke uniformity of the group K.

Consider a continuous homomorphism $f\colon S \to K$ to a Roelcke-complete Hausdorff topological group K with identity e_K. We claim that f is uniformly continuous considered as a mapping of $(S, \mathcal{Q}{\restriction}S)$ to $(K, \mathcal{V}_K)$. To this end, take an arbitrary symmetric element $U \in \mathcal{N}(e_K)$ and choose an element $W \in \mathcal{N}(e_K)$ such that $W^2 \subset U$. Then $\overline{W} \subset U$. By the continuity of f, we can find an element $V \in \mathcal{N}(e)$ satisfying $f(V \cap S) \subset W$. It remains to verify that $(f(x), f(y)) \in O_U$ whenever $(x,y) \in Q_V \cap S^2$ or, equivalently, $(f \times f)(Q_V \cap S^2) \subset O_U$.

Let $(x,y) \in Q_V \cap S^2$. Then $Vx \cap yV \neq \emptyset$ and $Vy \cap xV \neq \emptyset$. Since S is dense in D and the sets Vx and yV are open in D, we can choose a point $z \in S \cap Vx \cap yV$. It follows from the continuity of shifts in D and the density of $S \cap V$ in V that $z \in Vx \subset \overline{(S \cap V) \cdot x}$, the closure is taken in D. As $z \in S$, we see that z is in the closure of $(S \cap V) \cdot x$ in S. Hence $f(z) \in \overline{f(V \cap S) \cdot f(x)} = \overline{f(V \cap S)} \cdot f(x)$, by the continuity of f; the closure is taken in K. Since $\overline{f(V \cap S)} \subset \overline{W} \subset U$, the latter implies that $f(z) \in Uf(x)$. A similar argument, starting with $z \in yV$, shows that $f(z) \in f(y)U$. Thus $f(z) \in Uf(x) \cap f(y)U \neq \emptyset$, whence $f(y) \in Uf(x)U^{-1} = Uf(x)U$. This implies that $(f(x), f(y)) \in O_U$ and proves the uniform continuity of f as a mapping of $(S, \mathcal{Q}{\restriction}S)$ to $(K, \mathcal{V}_K)$.

Since the space $(K, \mathcal{V}_K)$ is complete, f extends to a uniformly continuous mapping $f^*\colon (D, \mathcal{Q}) \to (K, \mathcal{V}_K)$. It follows from the density of S in D and the Hausdorffness of K that f^* is a homomorphism. □

Corollary 2. *Let S be a dense submonoid of a topological monoid D with open shifts. Then every continuous homomorphism $f\colon S \to K$ to a locally compact topological group K extends to a continuous homomorphism $f^*\colon D \to K$.*

Proof. According to Proposition 2 it suffices to verify that every locally compact topological group K is Roelcke-complete. The latter fact is immediate since for every compact neighborhood U of the identity in K, every Cauchy filter ξ in the uniform space $(K, \mathcal{V}_K)$ has an element contained in the compact set UxU, for some $x \in K$. Hence ξ converges to an element of K and $(K, \mathcal{V}_K)$ is complete, where $\mathcal{V}_K$ is the Roelcke uniformity of K. □

Now we apply Proposition 2 in a less obvious way.

Theorem 5. *Let S be a dense submonoid of a product $D = \prod_{i \in I} D_i$ of topological monoids with open shifts and $f\colon S \to K$ be a continuous homomorphism to a Lie group K. If S is either finitely retractable or open in D, then f extends to a continuous homomorphism $f^*\colon D \to K$. Hence, one can find a finite set $E \subset I$ and continuous homomorphisms $\chi_i\colon D_i \to K$ for $i \in E$, such that $f^*(x) = \prod_{i \in E} \chi_i(x_i)$ for each $x = (x_i)_{i \in I} \in D$.*

Proof. Depending on whether S is finitely retractable or open, we apply respectively Theorem 2.12 or Theorem 3.8(b) of [19] to conclude that f depends on a finite set $E \subset I$. In either case, there exists a continuous homomorphism $g\colon p_E(S) \to K$ satisfying $f = g \circ p_E{\restriction}S$, where p_E is the projection of D to $D_E = \prod_{i \in E} D_i$. Then $p_E(S)$ is a dense submonoid of D_E and D_E is a topological monoid with open shifts, by [19, Lemma 3.5]. So we entitled to apply Proposition 2 to the homomorphism g. Hence, there exists a continuous homomorphism

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$g^*\colon D_E \to K$ extending g. According to Lemma 1 we can find continuous homomorphisms $\chi_i\colon D_i \to K$ for $i \in E$ such that $g(y) = \prod_{i\in E}\chi_i(y_i)$, for each $y = (y_i)_{i\in E}$. Then $f^* = g^* \circ p_E$ is a continuous homomorphism of D to K extending f and satisfying $f^*(x) = \prod_{i\in E}\chi_i(x_i)$, for each $x \in D$. This implies the required equality for the homomorphism f. □

According to [19, Theorem 5], every continuous homomorphism $f\colon S \to K$ of an arbitrary subgroup S of a product D of topological monoids to a Lie group K has a *finite type*, i.e., can be represented as the composition of the projection p_E of S to a finite subproduct D_E of D and a continuous homomorphism of $p_E(S)$ to K. Therefore, arguing as in the proof of Theorem 5 and applying Proposition 2 we deduce the following:

Theorem 6. *Let $D = \prod_{i\in I} D_i$ be a product of topological monoids with open shifts, S be a dense subgroup of D and $f\colon S \to K$ a continuous homomorphism to a Lie group K. Then f extends to a continuous homomorphism $f^*\colon D \to K$, so one can find a finite set $E \subset I$ and continuous homomorphisms $\chi_i\colon D_i \to K$, for $i \in E$, such that $f^*(x) = \prod_{i\in E}\chi_i(x_i)$ for each $x = (x_i)_{i\in I} \in D$.*

3. More on continuous homomorphisms of P-modifications of products and their dense submonoids

First we introduce notation which is used in this section and clarifies our aim.

Let $X = \prod_{i\in I} X_i$ be the product of a family $\{X_i : i \in I\}$ of sets, Z be a subset of X, and $f\colon Z \to Y$ be a mapping. Denote by $\mathcal{J}(f)$ the family of all sets $J \subset I$ such that f depends on J. Our main concern is to determine the properties of the family $\mathcal{J}(f)$. For example, one can ask whether $\mathcal{J}(f)$ is a filter or whether it has minimal, by inclusion, elements, or even the smallest element. It has been shown by W. Comfort and I. Gotchev in [7–9] that the family $\mathcal{J}(f)$ can have quite a complicated set-theoretic structure, even if X is a Cartesian product of topological spaces and f is a continuous mapping to a space Y. It is worth mentioning that the thorough study of the family $\mathcal{J}(f)$ was motivated by a somewhat simpler question on whether $\mathcal{J}(f)$ had a countable element $J \subset I$. The reader can find an extensive bibliography related to this question in the aforementioned articles and in the earlier survey article [14] by M. Hušek.

It turns out that the intersection of the family $\mathcal{J}(f)$, denoted by J_f, admits a clear description in terms of f. We say that an index $i \in I$ is *f-essential* if there exist points $x, y \in Z$ such that $\mathrm{diff}(x, y) = \{i\}$ and $f(x) \neq f(y)$. Let E_f be the set of all f-essential indices in I. By Proposition 2.2 in [17], $J_f = E_f = \bigcap \mathcal{J}(f)$. In particular, the set J_f is empty if and only if no index $i \in I$ is f-essential.

Below we present a useful fact which is not valid for arbitrary dense subgroups of the topological group PD, the P-modification of the product $D = \prod_{i\in I} D_i$ of topologized monoids D_i, not even if the factors D_i are finite discrete groups (see [18, Example 1]).

Proposition 3. *Let $D = \prod_{i\in I} D_i$ be a Cartesian product of topologized monoids, S be a submonoid of D with $\Sigma D \subset S$, and $\varphi\colon PS \to H$ a nontrivial continuous homomorphism of the P-modification of S to a topologized monoid H of countable pseudocharacter. Then the family*

$$\mathcal{J}(\varphi) = \{J \subset I : \varphi \text{ depends on } J\}$$

is a filter on the index set I.

Proof. Since the subspace PS of PD is a P-space, the homomorphism $\varphi\colon PS \to PH$ remains continuous (see e.g. [18, Lemma 6]). Notice that PH is a discrete space. Therefore, we can assume that H carries the discrete topology. Applying [18, Proposition 2] we find a countable subset E of I and a continuous homomorphism φ_E of $p_E(S) \subset PD_E$ to H such that $\varphi = \varphi_E \circ p_E{\restriction}S$, where $p_E\colon D \to D_E = \prod_{i\in E} D_i$ is the projection. It follows from $\Sigma D \subset S$ that $p_E(S) = D_E$. Hence $\bar\varphi = \varphi_E \circ p_E$ is a continuous homomorphism of PD to H. It follows from the definition of $\bar\varphi$ that this homomorphism depends on E. Furthermore, if $\bar\varphi$ depends on F, for some $F \subset I$, then so does φ. It is now clear that $\mathcal{J}(\bar\varphi) = \mathcal{J}(\varphi)$.

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Therefore, we can assume without loss of generality that φ is a continuous character of $PD = S$. Assume that $J_1 \subset J_2 \subset I$ and $J_1 \in \mathcal{J}(\varphi)$. Then there exists a mapping $g\colon D_{J_1} = \prod_{i\in J_1} D_i \to H$ satisfying $\varphi = g \circ p_{J_1}$, where $p_{J_1}\colon PD \to PD_{J_1}$ is the projection. Clearly g is a homomorphism. Since the projection p_{J_1} is open, the homomorphism g is continuous. Therefore, g is a continuous homomorphism of PD_{J_1} to H. Let $p^{J_2}_{J_1}$ be the projection of D_{J_2} to D_{J_1}. Then $\varphi = g \circ p_{J_1} = g \circ p^{J_2}_{J_1} \circ p_{J_2} = f \circ p_{J_2}$, where $f = g \circ p^{J_2}_{J_1}$ is a continuous homomorphism of PD_{J_2}. Hence, φ depends on J_2 and $J_2 \in \mathcal{J}(\varphi)$.

Let J_1 and J_2 be arbitrary elements of $\mathcal{J}(\varphi)$. It is easy to see that $\ker p_{J_1} \subset \ker \varphi$ and $\ker p_{J_2} \subset \ker \varphi$. Put $J = J_1 \cap J_2$. Then

$$\ker p_J = \ker p_{J_1} \cdot \ker p_{J_2} \subset \ker \varphi \neq D.$$

In particular, $J \neq \emptyset$ (we identify $p_\emptyset$ with the constant mapping of D to the identity e_D of D). It follows from the inclusion $\ker p_J \subset \ker \varphi$ that there exists a homomorphism $h\colon D_J \to H$ satisfying $\varphi = h \circ p_J$ (see [6, Theorem 1.48] or [18, Lemma 2]). We conclude that $J \in \mathcal{J}(\varphi)$.

Summing up, the family $\mathcal{J}(\varphi)$ is a filter. □

The reader can find several results about continuous homomorphisms or characters defined on dense submonoids and subgroups of Cartesian (equivalently, *Tychonoff*) products in [18] and [19]. On many occasions, the conclusions there are stronger than the one in Proposition 3.

It is natural to ask whether the filter $\mathcal{J}(\varphi)$ in Proposition 3 contains a minimal by inclusion element. The next example answers this question in the negative, even if S is the P-modification of the compact metrizable group $\mathbb{Z}(2)^\omega$ (so S is discrete). Notice that the continuous characters of the compact group $\mathbb{Z}(2)^\omega$ are described in Proposition 1.

Example 3. Let the group $G = \mathbb{Z}(2)^\omega$ carry the discrete topology. There exist a non-trivial character χ of G and a decreasing sequence $\{J_n : n \in \omega\}$ of infinite subsets of ω with empty intersection such that χ depends on J_n, for each $n \in \omega$. Hence the filter $\mathcal{J}(\chi)$ does not have minimal elements.

Proof. Let $J_n = \omega \setminus \{0, 1, \ldots, n\}$, for each $n \in \omega$. Denote by $\mathbf{1}$ the point of $\mathbb{Z}(2)^\omega$ all coordinates of which are equal to 1. Let also

$$H_n = \{x \in \mathbb{Z}(2)^\omega : x(i) = 0 \text{ for each } i \in J_n\}.$$

Clearly, H_n is a subgroup of G and $H_n \subset H_{n+1}$, for each $n \in \omega$. Hence $H = \bigcup_{n=0}^\infty H_n$ is also a subgroup of G. Since $\mathbf{1} \notin H$, there exists a character χ of G such that $\chi(H) = \{1\}$ and $\chi(\mathbf{1}) = -1$. It is immediate from the definition that χ depends on J_n, for each $n \in \omega$. Since $\bigcap_{n=0}^\infty J_n = \emptyset$, the family $\mathcal{J}(\chi)$ has no smallest element. Taking into account that $\mathcal{J}(\chi)$ is a filter (see Proposition 3), we infer that it does not contain minimal elements either. □

Since the subgroup H of G in the proof of Example 3 is dense in $G = \mathbb{Z}(2)^\omega$ provided the latter group is endowed with the usual Tychonoff product topology, the above character χ is discontinuous on the compact group $\mathbb{Z}(2)^\omega$. It turns out that considering the Tychonoff product topology improves the situation greatly—the family $\mathcal{J}(\chi)$ always has a finite *minimal* (by inclusion) element, for each continuous character χ of an arbitrary subgroup G of a product of left topological groups. This conclusion can be recovered using techniques from [15] in the special case where G itself is a product of *topological* groups, but the reader can find a direct argument in the more general [19, Proposition 2.1].

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