A Proof of Riemann Hypothesis Based on MacLaurin Expansion of the Completed Zeta Function

Weicun Zhang

Abstract The basic idea is to expand the completed zeta function $\xi(s)$ in MacLaurin series. Thus, $\xi(s) = 0$ corresponds to an algebraic equation with real coefficients and infinite degree. In addition, by $\xi(s) = \xi(1-s)$, another formally equivalent algebraic equation exists, i.e., $\xi(1-s) = 0$. Then these two simultaneous algebraic equations share the common solution, thus a proof of Riemann Hypothesis (RH) can be obtained.

1 Introduction

It has been 162 years since Riemann Hypothesis (RH) is proposed in 1859 [1]. Many efforts and achievements have been made towards proving the hypothesis, but it is still an open problem [2−3].

The Riemann zeta function is the function of the complex variable s, defined in the half-plane $\Re(s) > 1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (1)$$

Weicun Zhang
School of Automation and Electrical Engineering
University of Science and Technology Beijing
Beijing 100083, China
Tel.: +86-10-62936073
Fax: +86-13601102534
ORCID: 0000-0003-0047-0558
E-mail: weicunzhang@ustb.edu.cn

Riemann showed how to extend zeta function to the whole complex plane $\mathbf{C}$ by analytic continuation

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot (\frac{\theta(x)-1}{2})dx\} \qquad (2)$$

where $\theta(x) = \sum_{-\infty}^\infty e^{-n^2\pi x}$, Γ being the Gamma function in the following equivalent form

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s}\prod_{n=1}^\infty (1+\frac{s}{n})e^{-s/n} \qquad (3)$$

where γ is Euler's constant.

The connection between the Riemann zeta function and prime numbers can be established through the well-known Euler product.

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_p (1-p^{-s})^{-1}, \Re(s) > 1 \qquad (4)$$

the product being over the prime numbers p.

As shown by Riemann, $\zeta(s)$ extends to $\mathbf{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \qquad (5)$$

It is not difficult to see that $\zeta(s)$ has zeros at the negative even integers $-2, -4, -6, -8, \cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

About the non-trivial zeros of $\zeta(s)$, the following results are well established [4].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$ have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho, \bar{\rho}, 1 - \bar{\rho}, 1 - \rho$ are all non-trivial zeroes.

For further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(1-s)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \qquad (6)$$

It is well-known that $\xi(s)$ is an entire function of order 1.

Replacing s with $1 - s$ in Eq.(6), and considering Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, i.e., Eq. (6), and recalling Eq.(3), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s = 0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [5−6]. Thus, all roots of $\xi(s)$ must be the non-trivial roots of the Riemann zeta function, and vice versa. That means the following result, i.e., Lemma 2.

Lemma 2: Zeros of $\xi(s)$ coincide with non-trivial zeros of $\zeta(s)$.

It is well-known that $\xi(s)$ can be expressed by the following infinite product, which was first proposed by Riemann. However, it was Hadamard who showed the validity of this infinite product expansion [7]

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) \tag{8}$$

where ρ are precisely the non-trivial roots of the Riemann zeta function $\zeta(s)$, the factor ρ and $1 - \rho$ are paired.

This paper will use another infinite expansion of $\xi(s)$, i.e., MacLaurin series to open the door to the proof of RH.

The following two statements for Riemann Hypothesis are equivalent according to Lemma 2.

Statement 1 of Riemann Hypothesis: The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

Statement 2 of Riemann Hypothesis: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

2 A proof of RH

This section is planned to give a proof of the Statement 2 of RH. The details are delivered in four steps as follows.

Proof of RH:

Step 1: Since $\xi(s)$ is an entire function, it is analytic in the whole complex plane **C**. Then $\xi(s)$ can be expanded in an infinite MacLaurin series at $s = 0$, i.e.

$$\xi(s) = \xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots, |s| < \infty \tag{9}$$

It is obvious that

$$\frac{\xi^{(n)}(0)}{n!} = \frac{\xi^{(n)}(s)}{n!}\bigg|_{s=0} , n = 0, 1, 2, \cdots$$

are all real numbers.

Step2: In Eq. (9), replace s with $1 - s$, we obtain

$$\xi(1-s) = \xi(0)+\xi^{'}(0)(1-s)+\frac{\xi^{''}(0)}{2!}(1-s)^2+\cdots+\frac{\xi^{(n)}(0)}{n!}(1-s)^n+\cdots , |s| < \infty \tag{10}$$

Step 3: From Eq.(7), Eq.(9), and Eq.(10) we have

$$\xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots$$
$$=\xi(0) + \xi^{'}(0)(1 - s) + \frac{\xi^{''}(0)}{2!}(1 - s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1 - s)^n + \cdots \tag{11}$$

Step 4: To determine the zeros of $\xi(s)$, we only need to set $\xi(s) = \xi(1-s) = 0$, i.e.,

$$\xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots$$
$$=\xi(0) + \xi^{'}(0)(1 - s) + \frac{\xi^{''}(0)}{2!}(1 - s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1 - s)^n + \cdots \tag{12}$$
$$=0$$

Eq.(12) is actually equivalent to the following simultaneous algebraic equations with real number coefficients

$$\xi(0) + \xi^{'}(0)s + \frac{\xi^{''}(0)}{2!}s^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}s^n + \cdots = 0 \tag{13}$$

and

$$\xi(0) + \xi^{'}(0)(1 - s) + \frac{\xi^{''}(0)}{2!}(1 - s)^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}(1 - s)^n + \cdots = 0 \tag{14}$$

Without loss of generality, by Lemma 2, let $s = \alpha \pm j\beta$ be the root of Eq.(13). Next, set $y = 1 - s$, in Eq.(14), we have

$$\xi(0) + \xi^{'}(0)y + \frac{\xi^{''}(0)}{2!}y^2 + \cdots + \frac{\xi^{(n)}(0)}{n!}y^n + \cdots = 0 \tag{15}$$

It is obvious that Eq.(15) formally coincides with Eq.(13), that means these two equations have the same roots, i.e., $y = \alpha \pm j\beta$ being the roots of Eq. (15). Further, recalling $y = 1 - s$, we have

$$1 - s = \alpha \pm j\beta \quad \Rightarrow \quad s = 1 - \alpha \pm j\beta \tag{16}$$

being the roots of Eq.(14)

Since Eq.(13) and Eq.(14) are simultaneous equations, these two equations share the common solution, i.e.,

$$\begin{cases} s = \alpha \pm j\beta \\ s = 1 - \alpha \pm j\beta \end{cases} \tag{17}$$

Thus, we have

$$\alpha \pm j\beta = 1 - \alpha \pm j\beta \quad \Rightarrow \quad \alpha = 1 - \alpha \quad \Rightarrow \quad \alpha = \frac{1}{2} \tag{18}$$

That means all the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.
By Lemma 2, we know that the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of RH.

Remark. All the zeros of $\xi(s)$ should be determined by $\xi(s) = 0$, which is an infinite algebraic equation with real coefficients. According to the theory of algebraic equation, complex roots always come in pairs (complex conjugates), thus, all the n-pair complex roots of $\xi(s) = 0$ can be denoted as $\alpha \pm j\beta$, or to be specific, $\alpha_i \pm j\beta_i, i = 1, 2, \cdots, n, \cdots$. Certainly, $\xi(1-s) = 0$ also has n-pair complex roots $1 - \alpha \pm j\beta$, or to be specific, $1 - \alpha_i \pm j\beta_i, i = 1, 2, \cdots, n, \cdots$. By $\xi(s) = \xi(1-s)$, $1 - \alpha \pm j\beta$ or $1 - \alpha_i \pm j\beta_i, i = 1, 2, \cdots, n, \cdots$ are also zeros of $\xi(s)$. That means the n-pair roots of $\xi(1-s) = 0$ were embezzled to $\xi(s) = 0$. How could it be? But, $\xi(s) = \xi(1-s)$ is also a fact. The only remedy is that $1 - \alpha \pm j\beta = \alpha \pm j\beta$, i.e., $\alpha = 1/2$. Otherwise, the embezzlement of the roots from $\xi(1-s) = 0$ to $\xi(s) = 0$ is unavoidable.

3 Conclusion

This paper presents a proof of Riemann Hypothesis based on a new road map:
1) The completed zeta function $\xi(s)$ can be expressed as MacLaurin series, thus $\xi(s) = 0$ corresponds to an algebraic equation with real coefficients and infinite degree. The roots of $\xi(s) = 0$ can be denoted as $s = \alpha \pm j\beta$;
2) By $\xi(s) = \xi(1-s)$, another formally equivalent algebraic equation exists, i.e., $\xi(1-s) = 0$ with roots $1 - \alpha \pm j\beta$;
3) Solving these two simultaneous algebraic equations, we have $1 - \alpha \pm j\beta = \alpha \pm j\beta \Rightarrow \alpha = 1/2$.
Then we conclude that the celebrated Riemann Hypothesis is true.

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