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## A Proof Of The Riemann Hypothesis Based On A New Expression Of The Completed Zeta Function

Weicun Zhang

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**Abstract** Based on the Hadamard product  $\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho})$ , a new absolute convergent expression of  $\xi(s)$  is obtained by paring  $\rho_i$  and  $\bar{\rho}_i$ , and putting all the  $\rho_i$  related multiple zeros together in one factor, i.e.,

$$\xi(s) = \xi(0) \prod_{i=1}^{\infty} \left( \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i}$$

where  $\xi(0) = \frac{1}{2}$ ,  $\rho_i = \alpha_i + j\beta_i$  and  $\bar{\rho}_i = \alpha_i - j\beta_i$  are the complex conjugate zeros of  $\xi(s)$ ,  $0 < \alpha_i < 1$  and  $\beta_i \neq 0$  are real numbers,  $d_i \geq 1$  are the real (**unique and unchangeable**) multiplicities of  $\rho_i$ ,  $\beta_i$  are in order of increasing  $|\beta_i|$ , i.e.,  $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \dots$ ,  $i = 1, 2, 3, \dots, \infty$ .

Then, by the functional equation  $\xi(s) = \xi(1-s)$ , we have

$$\prod_{i=1}^{\infty} \left( 1 + \frac{(s - \alpha_i)^2}{\beta_i^2} \right)^{d_i} = \prod_{i=1}^{\infty} \left( 1 + \frac{(1-s - \alpha_i)^2}{\beta_i^2} \right)^{d_i}$$

which, owing to the uniqueness and unchangeableness of  $d_i$ , is equivalent to

$$\left\{ \begin{array}{l} \left( 1 + \frac{(s - \alpha_i)^2}{\beta_i^2} \right)^{d_i} = \left( 1 + \frac{(1-s - \alpha_i)^2}{\beta_i^2} \right)^{d_i} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \alpha_i = \frac{1}{2} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{array} \right.$$

Thus, we conclude that the Riemann Hypothesis is true.

**Keywords** Riemann Hypothesis (RH) · Proof · Completed zeta function

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## 1 Introduction

It has been almost 164 years since the Riemann Hypothesis (RH) was proposed in 1859 [1]. Many efforts and achievements have been made towards proving this celebrated hypothesis, but it is still an open problem [2-3].

The Riemann zeta function is the function of the complex variable  $s$ , defined in the half-plane  $\Re(s) > 1$  by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1 \quad (1)$$

The connection between the Riemann zeta function and prime numbers can be established through the well-known Euler product, i.e.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}, \Re(s) > 1 \quad (2)$$

where  $p$  runs over the prime numbers.

Riemann showed how to extend zeta function to the whole complex plane  $\mathbb{C}$  by analytic continuation, i.e.

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{\infty}^{\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} \quad (3a)$$

where " $\int_{\infty}^{\infty}$ " represents contour integral from  $+\infty$  to  $+\infty$  around a domain which includes the value 0 but no other point of discontinuity of the integrand in its interior.

Or equivalently,

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)} \left\{ \frac{1}{s(s-1)} + \int_1^{\infty} (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot \left( \frac{\theta(x)-1}{2} \right) dx \right\} \quad (3b)$$

where  $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}$  being the Jacobi theta function,  $\Gamma$  being the Gamma function in the following Weierstrass expression (Meanwhile, there are also Gauss expression, Euler expression, and integral expression of the Gamma function.)

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \quad (4)$$

where  $\gamma$  is the Euler-Mascheroni constant.

As shown by Riemann,  $\zeta(s)$  extends to  $\mathbb{C}$  as a meromorphic function with only a simple pole at  $s = 1$ , with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \quad (5)$$

The Riemann zeta function  $\zeta(s)$  has zeros at the negative even integers:  $-2, -4, -6, -8, \dots$  and one refers to them as the **trivial zeros**. The other zeros of  $\zeta(s)$  are the complex numbers, i.e., **non-trivial zeros** [2].

In 1896, Hadamard [4] and Poussin [5] independently proved that no zeros could lie on the line  $\Re(s) = 1$ . Together with the functional equation and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the **critical strip**  $0 < \Re(s) < 1$ . This was a key step in their first proofs of the famous **Prime Number Theorem**.

Later on, Hardy (1914) [6], Hardy and Littlewood (1921) [7] showed that there are infinitely many zeros on the **critical line**  $\Re(s) = \frac{1}{2}$ , which was an astonishing result at that time.

As a summary, we have the following results on the properties of the non-trivial zeros of  $\zeta(s)$  [4–9].

**Lemma 1:** Non-trivial zeroes of  $\zeta(s)$ , noted as  $\rho = \alpha + j\beta$ , have the following properties

- 1) The number of non-trivial zeroes is infinity;
- 2)  $\beta \neq 0$ ;
- 3)  $0 < \alpha < 1$ ;
- 4)  $\rho, \bar{\rho}, 1 - \bar{\rho}, 1 - \rho$  are all non-trivial zeroes.

As further study, a completed zeta function  $\xi(s)$  is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) \quad (6)$$

It is well-known that  $\xi(s)$  is an entire function of order 1. This implies  $\xi(s)$  is analytic, and can be expressed as infinite polynomial, in the whole complex plane  $\mathbb{C}$ .

In addition, replacing  $s$  with  $1 - s$  in Eq.(6), and combining Eq.(5), we have the following functional equation

$$\xi(s) = \xi(1 - s) \quad (7)$$

Considering the definition of  $\xi(s)$ , and recalling Eq.(4), the trivial zeros of  $\zeta(s)$  are canceled by the poles of  $\Gamma\left(\frac{s}{2}\right)$ . The zero of  $s - 1$  and the pole of  $\zeta(s)$  cancel; the zero  $s = 0$  and the pole of  $\Gamma\left(\frac{s}{2}\right)$  cancel [9–10]. Thus, all the zeros of  $\xi(s)$  are exactly the nontrivial zeros of  $\zeta(s)$ . Then we have the following Lemma 2.

**Lemma 2:** The zeros of  $\xi(s)$  coincide with the non-trivial zeros of  $\zeta(s)$ .

According to Lemma 2, the following two statements for the RH are equivalent.

**Statement 1 of the RH:** All the non-trivial zeros of  $\zeta(s)$  have real part equal to  $\frac{1}{2}$ .

**Statement 2 of the RH:** All the zeros of  $\xi(s)$  have real part equal to  $\frac{1}{2}$ .

To prove the RH, a natural thinking is to estimate the numbers of non-trivial zeros of  $\zeta(s)$  inside or outside some areas according to Argument Principle. Along this train of thought, there are many research works. Let  $N(T)$

denote the number of zeros of  $\zeta(s)$  inside the rectangle:  $0 < \alpha < 1, 0 < \beta \leq T$ , and let  $N_0(T)$  denote the number of zeros of  $\zeta(s)$  on the line  $\alpha = \frac{1}{2}, 0 < \beta \leq T$ . Selberg proved that there exist positive constants  $c$  and  $T_0$ , such that  $N_0(T) > cN(T), (T > T_0)$  [11], later on, Levinson proved that  $c \geq \frac{1}{3}$  [12], Lou and Yao proved that  $c \geq 0.3484$  [13], Conrey proved that  $c \geq \frac{2}{5}$  [14], Bui, Conrey and Young proved that  $c \geq 0.41$  [15], Feng proved that  $c \geq 0.4128$  [16], Wu proved that  $c \geq 0.4172$  [17].

On the other hand, many zeros have been calculated by hand or by computer programs. Among others, Riemann found the first three non-trivial zeros [18]. Gram found the first 15 zeros based on Euler-Maclaurin summation [19]. Titchmarsh calculated the 138<sup>th</sup> to 195<sup>th</sup> zeros using the Riemann-Siegel formula [20–21]. Here are the first three (pairs of) zeros:  $\frac{1}{2} \pm j14.1347251$ ;  $\frac{1}{2} \pm j21.0220396$ ;  $\frac{1}{2} \pm j25.0108575$ .

The idea of this paper is originated from Euler's work on proving the following famous equality

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{6} \quad (8)$$

This interesting result is deduced by comparing the like terms of two types of infinite expressions, i.e., infinite polynomial and infinite product, as shown in the following

$$\begin{aligned} \frac{\sin x}{x} &= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \\ &= \left(1 - \frac{x^2}{\pi^2}\right) \left(1 - \frac{x^2}{4\pi^2}\right) \left(1 - \frac{x^2}{9\pi^2}\right) \dots \end{aligned} \quad (9)$$

Then it is conjectured that  $\xi(s)$  should be factored into  $\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$  or something like that, which was verified by pairing  $\rho_i$  and  $\bar{\rho}_i$  in the Hadamard product of  $\xi(s)$  to obtain

$$\left(1 - \frac{s}{\rho_i}\right) \left(1 - \frac{s}{\bar{\rho}_i}\right) = \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$$

The Hadamard product of  $\xi(s)$  as shown in Eq.(10) was first proposed by Riemann, however, it was Hadamard [22] who showed the validity of this infinite product expansion.

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \quad (10)$$

where  $\xi(0) = \frac{1}{2}$ ,  $\rho$  runs over all the non-trivial zeros of the Riemann zeta function  $\zeta(s)$ , or in another word,  $\rho$  runs over all the zeros of the completed zeta function  $\xi(s)$ .

To ensure the absolute convergence of the infinite product expansion,  $\rho$  and  $1 - \rho$  are paired. Later in Section 3, we will show that  $\rho$  and  $\bar{\rho}$  can also be paired to ensure the absolute convergence of the infinite product expansion.

## 2 Lemmas

In this section, we first explain the concept of real multiplicity of zeros of  $\xi(s)$ . And then we give three lemmas to support the proof of the RH, among which Lemma 3 is the key lemma.

**Multiple zeros of  $\xi(s)$ :** As shown in Figure 1, the multiple zeros of  $\xi(s)$  are defined in terms of the quadruplet, i.e.,  $\rho, \bar{\rho}, 1 - \rho, 1 - \bar{\rho}$ .

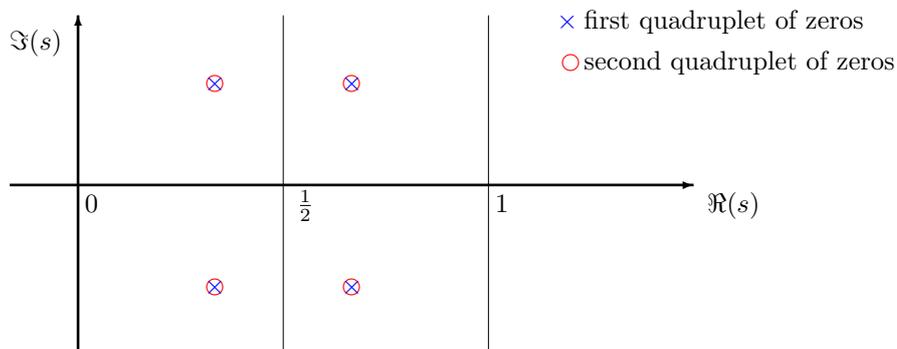


Figure 1 Illustration of the multiple zeros of  $\xi(s)$

It should be noticed that the multiple zeros with their real multiplicities of  $\xi(s)$  are objective existence, but the expression of the corresponding factors of  $\xi(s)$  are optional to some extent. For example, the multiple zeros as shown in Figure 1 have two different expressions as factors of  $\xi(s)$  and  $\xi(1-s)$ , respectively, i.e.,  $\left[ \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^2, \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^2 \right]$ , or  $\left[ \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(s-\alpha_2)^2}{\beta_2^2}\right), \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(1-s-\alpha_2)^2}{\beta_2^2}\right) \right]$ ,  $\alpha_1 + \alpha_2 = 1, \beta_1^2 = \beta_2^2$ .

To exclude the latter expression, we stipulate that the multiple zeros  $\rho_i$  related factors of  $\xi(s)$  take the form of  $\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$ , where  $d_i \geq 1$  is the real multiplicity of  $\rho_i$ .

**Remark:** Although  $d_i$  is unknown, it is an objective existence, unique, and unchangeable. This is the key point in the proofs of Lemma 3 and Lemma 5.

**Lemma 3:** Given two infinite products

$$f(s) = \prod_{i=1}^{\infty} \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \quad (11)$$

and

$$f(1-s) = \prod_{i=1}^{\infty} \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \quad (12)$$

where  $s$  is a complex variable,  $\rho_i = \alpha_i + j\beta_i$  and  $\bar{\rho}_i = \alpha_i - j\beta_i$  are the complex conjugate zeros of  $\xi(s)$ ,  $0 < \alpha_i < 1$  and  $\beta_i \neq 0$  are real numbers,

$d_i \geq 1$  are the real multiplicities of  $\rho_i$ ,  $\beta_i$  are in order of increasing  $|\beta_i|$ , i.e.,  $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \dots$ ,  $i = 1, 2, 3, \dots, \infty$ .

Then we have

$$f(s) = f(1-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{cases} \quad (13)$$

where " $\Leftrightarrow$ " is the equivalent sign.

**Proof:** First of all, we have the following fact:

$$\begin{aligned} \left(1 + \frac{(s-\alpha)^2}{\beta^2}\right)^d &= \left(1 + \frac{(1-s-\alpha)^2}{\beta^2}\right)^d \\ \Leftrightarrow (s-\alpha)^2 &= (1-s-\alpha)^2 \Leftrightarrow \alpha = \frac{1}{2} \end{aligned} \quad (14)$$

where  $d \geq 1$  is a natural number,  $\alpha \neq 0$  and  $\beta \neq 0$  are real numbers.

Next, the proof is based on Transfinite Induction.

Let  $P(n)$  be:

$$\begin{aligned} \prod_{i=1}^n \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} &= \prod_{i=1}^n \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \Leftrightarrow & \\ \begin{cases} \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \dots \\ \left(1 + \frac{(s-\alpha_n)^2}{\beta_n^2}\right)^{d_n} = \left(1 + \frac{(1-s-\alpha_n)^2}{\beta_n^2}\right)^{d_n} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots < |\beta_n| \end{cases} & \quad (15) \\ \Leftrightarrow & \\ \begin{cases} \alpha_i = \frac{1}{2}, i = 1 \dots n \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots < |\beta_n| \end{cases} & \end{aligned}$$

According to Eq.(14),  $P(1)$  is an obvious fact as the **Base Case**, i.e.,

$$\begin{aligned} \prod_{i=1}^1 \left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} &= \prod_{i=1}^1 \left(1 + \frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \Leftrightarrow \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} &= \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \Leftrightarrow \alpha_1 &= \frac{1}{2} \end{aligned} \quad (16)$$

To be more convincing, let's further check  $P(2)$ , i.e.,

$$\begin{aligned}
 \prod_{i=1}^2 \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} &= \prod_{i=1}^2 \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \\
 &\Leftrightarrow \\
 &\begin{cases} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right)^{d_2} = \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right)^{d_2} \\ |\beta_1| < |\beta_2| \end{cases} \quad (17) \\
 &\Leftrightarrow \\
 &\begin{cases} \alpha_1 = \alpha_2 = \frac{1}{2} \\ |\beta_1| < |\beta_2| \end{cases}
 \end{aligned}$$

which is also an obvious fact according to Lemma 5.

As the **Successor Case**, we need to prove  $P(n) \Rightarrow P(n + 1)$ .

Actually, we have

$$\begin{aligned}
 \prod_{i=1}^{n+1} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} &= \prod_{i=1}^{n+1} \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \\
 &\Leftrightarrow \\
 \prod_{i=1}^n \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \left(1 + \frac{(s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} &= \prod_{i=1}^n \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \left(1 + \frac{(1 - s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \\
 &\Leftrightarrow \text{(by Lemma 5)} \\
 &\begin{cases} \prod_{i=1}^n \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^n \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \left(1 + \frac{(s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} = \left(1 + \frac{(1 - s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots < |\beta_n| < |\beta_{n+1}| \end{cases} \\
 &\Leftrightarrow \text{(by Eq.(15))} \\
 &\begin{cases} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \dots \\ \left(1 + \frac{(s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} = \left(1 + \frac{(1 - s - \alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots < |\beta_n| < |\beta_{n+1}| \end{cases} \\
 &\Leftrightarrow \text{(by Eq.(14))} \\
 &\begin{cases} \alpha_i = \frac{1}{2}, i = 1, 2, 3, \dots, n, n + 1 \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots < |\beta_n| < |\beta_{n+1}| \end{cases} \quad (18)
 \end{aligned}$$

Thus the **Successor Case** is true, i.e.,  $P(n) \Rightarrow P(n+1)$ .

Next, we prove that  $P(\infty)$  holds, by considering well-ordered ordinal set  $A$  indexing the family of statements  $P(\gamma : \gamma \in A)$ ,  $A = \mathbb{N} \cup \{\omega\}$  with the ordering that  $n < \omega$  for all natural numbers  $n$ ,  $\omega$  is the first limit ordinal. It is well-known that  $\omega = \bigcup \{\gamma : \gamma < \omega\}$ .

To prove that  $P(\infty)$  holds, it suffices to prove the **Limit Case**, i.e.,  $P(\gamma < \omega) \Rightarrow P(\omega)$ .

Actually, we have

$$\prod_{i=1}^{\omega} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{\omega} \left(1 + \frac{(1-s - \alpha_i)^2}{\beta_i^2}\right)^{d_i}$$

$$\Leftrightarrow$$

$$\prod_{i=1}^{\gamma < \omega} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \left(1 + \frac{(s - \alpha_{\omega})^2}{\beta_{\omega}^2}\right)^{d_{\omega}} = \prod_{i=1}^{\gamma < \omega} \left(1 + \frac{(1-s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \left(1 + \frac{(1-s - \alpha_{\omega})^2}{\beta_{\omega}^2}\right)^{d_{\omega}}$$

$\Leftrightarrow$  (by Lemma 5)

$$\left\{ \begin{array}{l} \prod_{i=1}^{\gamma < \omega} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{\gamma < \omega} \left(1 + \frac{(1-s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \left(1 + \frac{(s - \alpha_{\omega})^2}{\beta_{\omega}^2}\right)^{d_{\omega}} = \left(1 + \frac{(1-s - \alpha_{\omega})^2}{\beta_{\omega}^2}\right)^{d_{\omega}} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots < |\beta_{\omega}| \end{array} \right.$$

$\Leftrightarrow$  (by  $P(\gamma < \omega)$ )

$$\left\{ \begin{array}{l} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1-s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \dots \\ \left(1 + \frac{(s - \alpha_{\omega})^2}{\beta_{\omega}^2}\right)^{d_{\omega}} = \left(1 + \frac{(1-s - \alpha_{\omega})^2}{\beta_{\omega}^2}\right)^{d_{\omega}} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots < |\beta_{\omega}| \end{array} \right.$$

$\Leftrightarrow$  (by Eq.(14))

$$\left\{ \begin{array}{l} \alpha_i = \frac{1}{2}, i = 1, 2, 3, \dots, \omega \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots < |\beta_{\omega}| \end{array} \right.$$

(19)

Thus the **Limit Case** is true, i.e.,  $P(\gamma < \omega) \Rightarrow P(\omega)$ .

Hence, we conclude by Transfinite Induction that  $P(\infty)$  holds, i.e.,

$$\begin{aligned} \prod_{i=1}^{\infty} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} &= \prod_{i=1}^{\infty} \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ \Leftrightarrow & \\ \begin{cases} \left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} = \left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{cases} & \quad (20) \\ \Leftrightarrow & \\ \begin{cases} \alpha_i = \frac{1}{2} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{cases} & \end{aligned}$$

i.e.,

$$f(s) = f(1 - s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{cases} \quad (21)$$

That completes the proof of Lemma 3.

**Lemma 4:** Given

$$\left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right) \quad (22)$$

where  $s$  is a complex variable,  $0 < \alpha_i < 1$  and  $\beta_i \neq 0$  are real numbers,  $|\beta_1| \leq |\beta_2|$ .

Then we have

$$\begin{aligned} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) &= \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right) \\ \Leftrightarrow & \\ \begin{cases} \alpha_1 = \alpha_2 = \frac{1}{2}, & |\beta_1| \leq |\beta_2| \\ \alpha_1 + \alpha_2 = 1, & |\beta_1| = |\beta_2| \end{cases} & \\ \Leftrightarrow & \\ \begin{cases} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right), \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) = \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right), & |\beta_1| \leq |\beta_2| \\ \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) = \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right), \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right), & |\beta_1| = |\beta_2| \end{cases} & \quad (23) \end{aligned}$$

**Proof:** Expanding both sides of Eq.(22), and comparing the coefficients of like terms, we obtain (details are omitted to save space)

$$\begin{aligned} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) &= \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right) \\ \Rightarrow \\ \begin{cases} \alpha_1 = \alpha_2 = \frac{1}{2}, & |\beta_1| \leq |\beta_2| \\ \alpha_1 + \alpha_2 = 1, & |\beta_1| = |\beta_2| \end{cases} \end{aligned} \quad (24)$$

The inverse inference of Eq.(24) is also an obvious fact. i.e.,

$$\begin{aligned} \begin{cases} \alpha_1 = \alpha_2 = \frac{1}{2}, & |\beta_1| \leq |\beta_2| \\ \alpha_1 + \alpha_2 = 1, & |\beta_1| = |\beta_2| \end{cases} \\ \Rightarrow \\ \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) &= \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right) \end{aligned}$$

Then we have

$$\begin{aligned} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) &= \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right) \\ \Leftrightarrow \\ \begin{cases} \alpha_1 = \alpha_2 = \frac{1}{2}, & |\beta_1| \leq |\beta_2| \\ \alpha_1 + \alpha_2 = 1, & |\beta_1| = |\beta_2| \end{cases} \end{aligned} \quad (25)$$

Further, according to Eq.(14), i.e.,

$$\left(1 + \frac{(s - \alpha)^2}{\beta^2}\right)^d = \left(1 + \frac{(1 - s - \alpha)^2}{\beta^2}\right)^d \Leftrightarrow \alpha = \frac{1}{2}$$

and the following similar facts

$$\left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) = \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right) \Leftrightarrow \alpha_1 + \alpha_2 = 1, |\beta_1| = |\beta_2|$$

$$\left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right) \Leftrightarrow \alpha_1 + \alpha_2 = 1, |\beta_1| = |\beta_2|$$

we have

$$\begin{aligned} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) &= \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right) \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right) \\ \Leftrightarrow & \\ \begin{cases} \alpha_1 = \alpha_2 = \frac{1}{2}, & |\beta_1| \leq |\beta_2| \\ \alpha_1 + \alpha_2 = 1, & |\beta_1| = |\beta_2| \end{cases} & \\ \Leftrightarrow & \\ \begin{cases} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right), \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) = \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right), & |\beta_1| \leq |\beta_2| \\ \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) = \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right), \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right), & |\beta_1| = |\beta_2| \end{cases} & \end{aligned} \quad (26)$$

That completes the proof of Lemma 4.

**Remark:** From the viewpoint of the products of (polynomial) factors, Lemma 4 reveals a fact that there are only two possible cases to make Eq. (22) hold, i.e., Case 1:  $\left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right)$ ,  $\left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) = \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right)$ ; and Case 2:  $\left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right) = \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right)$ ,  $\left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right) = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right)$ , respectively. This result is quite different from the similar equation of products of factors  $a_1 \cdot a_2 = b_1 \cdot b_2$ , which has infinite cases of solutions for real numbers  $a_1, a_2, b_1, b_2$ .

**Lemma 5:** Given

$$\left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right)^{d_2} = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right)^{d_2} \quad (27)$$

where  $s$  is a complex variable,  $0 < \alpha_1 < 1, 0 < \alpha_2 < 1$  and  $\beta_1 \neq 0, \beta_2 \neq 0$  are real numbers,  $d_1 \geq 1, d_2 \geq 1$  are natural numbers, denoting the real multiplicities of  $\rho_1 = \alpha_1 + j\beta_1$  and  $\rho_2 = \alpha_2 + j\beta_2$ , respectively,  $|\beta_1| \leq |\beta_2|$ .

Then we have

$$\begin{aligned} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right)^{d_2} &= \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right)^{d_2} \\ \Leftrightarrow & \\ \begin{cases} \alpha_1 = \alpha_2 = \frac{1}{2} \\ |\beta_1| < |\beta_2| \end{cases} & \\ \Leftrightarrow & \\ \begin{cases} \left(1 + \frac{(s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1 - s - \alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \left(1 + \frac{(s - \alpha_2)^2}{\beta_2^2}\right)^{d_2} = \left(1 + \frac{(1 - s - \alpha_2)^2}{\beta_2^2}\right)^{d_2} \\ |\beta_1| < |\beta_2| \end{cases} & \end{aligned} \quad (28)$$

**Proof:** Based on Lemma 4, and considering additional possibilities that  $\left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \mid \left(1 + \frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)^{d_2}$ ,  $\left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \mid \left(1 + \frac{(s-\alpha_2)^2}{\beta_2^2}\right)^{d_2}$  (where "|" is the divisible sign), or vice versa, we have

$$\begin{aligned} & \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \left(1 + \frac{(s-\alpha_2)^2}{\beta_2^2}\right)^{d_2} = \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \left(1 + \frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)^{d_2} \\ & \Leftrightarrow \\ & \begin{cases} \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1}, \left(1 + \frac{(s-\alpha_2)^2}{\beta_2^2}\right)^{d_2} = \left(1 + \frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)^{d_2}, & |\beta_1| \leq |\beta_2| \\ \left\{ \begin{array}{l} \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \mid \left(1 + \frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)^{d_2}, \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \mid \left(1 + \frac{(s-\alpha_2)^2}{\beta_2^2}\right)^{d_2} \\ \left(1 + \frac{(s-\alpha_2)^2}{\beta_2^2}\right)^{d_2-d_1} = \left(1 + \frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)^{d_2-d_1} \end{array} \right. & , \quad |\beta_1| = |\beta_2|, d_1 < d_2 \\ \left\{ \begin{array}{l} \left(1 + \frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)^{d_2} \mid \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1}, \left(1 + \frac{(s-\alpha_2)^2}{\beta_2^2}\right)^{d_2} \mid \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1-d_2} = \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1-d_2} \end{array} \right. & , \quad |\beta_1| = |\beta_2|, d_1 > d_2 \\ \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)^{d_2}, \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(s-\alpha_2)^2}{\beta_2^2}\right)^{d_2}, & |\beta_1| = |\beta_2|, d_1 = d_2 \end{cases} \\ & \Leftrightarrow \\ & \begin{cases} \alpha_1 = \alpha_2 = \frac{1}{2}, & |\beta_1| \leq |\beta_2| \\ \alpha_1 = \alpha_2 = \frac{1}{2}, & |\beta_1| = |\beta_2|, d_1 < d_2 \quad \star \\ \alpha_1 = \alpha_2 = \frac{1}{2}, & |\beta_1| = |\beta_2|, d_1 > d_2 \quad \star \\ \alpha_1 + \alpha_2 = 1, & |\beta_1| = |\beta_2|, d_1 = d_2 \quad \star \end{cases} \end{aligned}$$

(To exclude the duplication situations marked by  $\star$  between the first quadruplet of zeros represented by  $\rho_1 = \alpha_1 + j\beta_1$  and the second quadruplet of zeros represented by  $\rho_2 = \alpha_2 + j\beta_2$ , which contradict the given condition that the real multiplicities of  $\rho_1$  and  $\rho_2$  are  $d_1$  and  $d_2$ , respectively.)

$$\begin{cases} \alpha_1 = \alpha_2 = \frac{1}{2} \\ |\beta_1| \leq |\beta_2| \end{cases}$$

$\Leftrightarrow$

(To further exclude the duplication situation between the first quadruplet and the second quadruplet of zeros.)

$$\begin{cases} \alpha_1 = \alpha_2 = \frac{1}{2} \\ |\beta_1| < |\beta_2| \end{cases}$$

$\Leftrightarrow$  (by Eq.(14))

$$\begin{cases} \left(1 + \frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} = \left(1 + \frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1} \\ \left(1 + \frac{(s-\alpha_2)^2}{\beta_2^2}\right)^{d_2} = \left(1 + \frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)^{d_2} \\ |\beta_1| < |\beta_2| \end{cases}$$

(29)

That completes the proof of Lemma 5.

### 3 A Proof of the RH

This section is planned to present a proof of the Riemann Hypothesis. We first prove that Statement 2 of the RH is true, and then by Lemma 2, Statement 1 of the RH is also true. To be brief, to prove the Riemann Hypothesis, it suffices to show that  $\alpha_i = \frac{1}{2}, i = 1, 2, 3, \dots$  in the new expression of  $\xi(s)$  as shown in Eq. (30).

**Proof of the RH:** The details are delivered in three steps as follows.

**Step 1:** It is well-known that all the zeros of  $\xi(s)$  always come in complex conjugate pairs. Then by pairing  $\rho_i = \alpha_i + j\beta_i$  and  $\bar{\rho}_i = \alpha_i - j\beta_i$  in the Hadamard product as shown in Eq.(10), we have

$$\begin{aligned}\xi(s) &= \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right) \\ &= \xi(0) \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right) \left(1 - \frac{s}{\bar{\rho}_i}\right) \\ &= \xi(0) \prod_{i=1}^{\infty} \left(1 - \frac{s}{\alpha_i + j\beta_i}\right) \left(1 - \frac{s}{\alpha_i - j\beta_i}\right) \\ &= \xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right)\end{aligned}\quad (30)$$

where  $\xi(0) = \frac{1}{2}, 0 < \alpha_i < 1, \beta_i \neq 0$ .

The absolute convergence of the infinite product in Eq.(30) in the form

$$\xi(s) = \xi(0) \prod_{i=1}^{\infty} \left(1 - \frac{s}{\rho_i}\right) \left(1 - \frac{s}{\bar{\rho}_i}\right) = \xi(0) \prod_{i=1}^{\infty} \left(1 - \frac{s(2\alpha_i - s)}{|\rho_i|^2}\right) \quad (31)$$

depends on the convergence of infinite series  $\sum_{i=1}^{\infty} \frac{1}{|\rho_i|^2}$ , or equivalently,  $\sum_{\rho} \frac{1}{|\rho|^2}$ , which is an obvious fact according to Theorem 2 in Section 2, Chapter IV of Ref.[23], as shown in the following.

**Theorem 2.**<sup>[23]</sup> The function  $\xi(s)$  is an entire function of order one that has infinitely many zeros  $\rho_n$  such that  $0 \leq \mathbf{Re} \rho_n \leq 1$ . The series  $\sum |\rho_n|^{-1}$  diverges, but the series  $\sum |\rho_n|^{-1-\varepsilon}$  converges for any  $\varepsilon > 0$ . The zeros of  $\xi(s)$  are the nontrivial zeros of  $\zeta(s)$ .

**Remark:** In the Theorem 2 of Ref.[23],  $\mathbf{Re}(\cdot)$  is identical to  $\Re(\cdot)$  in this paper, both  $\mathbf{Re}(\cdot)$  and  $\Re(\cdot)$  mean the real part of any complex number.

Further, considering the absolute convergence of

$$\xi(s) = \xi(0) \prod_{i=1}^{\infty} \left(1 - \frac{s(2\alpha_i - s)}{|\rho_i|^2}\right) = \xi(0) \prod_{i=1}^{\infty} \left(\frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2}\right) \quad (32)$$

we have the following new expression of  $\xi(s)$  by putting all the  $\rho_i$  related multiple factors (zeros) together in the above Eq.(32)

$$\xi(s) = \xi(0) \prod_{i=1}^{\infty} \left( \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i} \quad (33)$$

where  $d_i \geq 1$  are the real multiplicities of  $\rho_i$ ,  $i$  are natural numbers from 1 to infinity.

**Step 2:** Replacing  $s$  with  $1 - s$  in Eq.(33), we obtain the infinite product expression of  $\xi(1 - s)$ , i.e.,

$$\xi(1 - s) = \xi(0) \prod_{i=1}^{\infty} \left( \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(1 - s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i} \quad (34)$$

**Remark:** According to the new expressions of  $\xi(s)$  and  $\xi(1 - s)$ , i.e., Eq.(33) and Eq.(34), we may conclude that all  $\rho_i$  and  $1 - \rho_i$  related multiple zeros, i.e.,  $(\alpha_i \pm j\beta_i)^{d_i}$ ,  $(1 - \alpha_i \pm j\beta_i)^{d_i}$  are included in the  $i^{th}$  group of factors,  $\left(1 + \frac{(s - \alpha_i)^2}{\beta_i^2}\right)^{d_i}$  and  $\left(1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2}\right)^{d_i}$ , respectively, or in another word, before or after the  $i^{th}$  group of factors of  $\xi(s)$  and  $\xi(1 - s)$ , there are no  $\rho_i$  and  $1 - \rho_i$  related multiple zeros.

**Actually, with such arrangement of  $\rho_i$  and  $1 - \rho_i$  related multiple factors of  $\xi(s)$  and  $\xi(1 - s)$ , we "assigned" a reason for excluding, in the proofs of Lemma 3 and Lemma 5, the "abnormal" situation, i.e., the successor factor and its predecessor factor represent the same quadruplet of zeros.**

**Step 3:** According to the functional equation  $\xi(s) = \xi(1 - s)$ , and considering Eq.(33) and Eq.(34), we have

$$\xi(0) \prod_{i=1}^{\infty} \left( \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i} = \xi(0) \prod_{i=1}^{\infty} \left( \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(1 - s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i} \quad (35)$$

which is equivalent to

$$\prod_{i=1}^{\infty} \left( 1 + \frac{(s - \alpha_i)^2}{\beta_i^2} \right)^{d_i} = \prod_{i=1}^{\infty} \left( 1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2} \right)^{d_i} \quad (36)$$

And that real numbers  $\beta_i$  can be certainly arranged in order of increasing  $|\beta_i|$ , i.e.,  $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \dots$ .

Then, according to Lemma 3, Eq.(36) is equivalent to

$$\begin{cases} \alpha_i = \frac{1}{2} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{cases}$$

Thus, we conclude that all the zeros of the completed zeta function  $\xi(s)$  have real part equal to  $\frac{1}{2}$ , i.e., Statement 2 of the RH is true. According to Lemma 2, Statement 1 of the RH is also true, i.e., all the non-trivial zeros of the Riemann zeta function  $\zeta(s)$  have real part equal to  $\frac{1}{2}$ .

That completes the proof of the RH.

**Remark:** By Lemma 1, there are 2 pairs of complex zeros of  $\zeta(s)$  simultaneously, i.e.,  $\rho = \alpha + j\beta, \bar{\rho} = \alpha - j\beta, 1 - \rho = 1 - \alpha - j\beta, 1 - \bar{\rho} = 1 - \alpha + j\beta$  are all the non-trivial zeroes of  $\zeta(s)$ . With the proof of the RH, i.e.,  $\alpha = \frac{1}{2}$ , these 2 pairs of zeros are actually only one pair, because  $\rho = 1 - \bar{\rho} = \frac{1}{2} + j\beta, \bar{\rho} = 1 - \rho = \frac{1}{2} - j\beta$ . Thus Lemma 1 could be modified more precisely as follows.

**Lemma 1\*:** Non-trivial zeroes of  $\zeta(s)$ , noted as  $\rho = \alpha + j\beta$ , have the following properties

- 1) The number of non-trivial zeroes is infinity;
- 2)  $\beta \neq 0$ ;
- 3)  $0 < \alpha < 1$ ;
- 4)  $\rho = 1 - \bar{\rho}, \bar{\rho} = 1 - \rho$  are all non-trivial zeroes.

#### 4 Conclusion

The celebrated Riemann Hypothesis is proved to be true based on a new expression of the completed zeta function  $\xi(s)$ , i.e.,

$$\xi(s) = \xi(0) \prod_{i=1}^{\infty} \left( \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2} + \frac{(s - \alpha_i)^2}{\alpha_i^2 + \beta_i^2} \right)^{d_i}$$

where  $\xi(0) = \frac{1}{2}$ ,  $\rho_i = \alpha_i + j\beta_i$  and  $\bar{\rho}_i = \alpha_i - j\beta_i$  are the complex conjugate zeros of  $\xi(s)$ ,  $0 < \alpha_i < 1$  and  $\beta_i \neq 0$  are real numbers,  $d_i \geq 1$  are the real multiplicities of  $\rho_i$ ,  $\beta_i$  are in order of increasing  $|\beta_i|$ , i.e.,  $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \dots, i = 1, 2, 3, \dots, \infty$ .

The proof methodology is Transfinite Induction. And the key-point is the use of "real multiplicities" of the zeros of  $\xi(s)$ . Obviously, the real multiplicity of a zero of  $\xi(s)$  is an objective existence, unique, and unchangeable. As a result, the functional equation  $\xi(s) = \xi(1 - s)$  is equivalent to

$$\left\{ \begin{array}{l} \left( 1 + \frac{(s - \alpha_i)^2}{\beta_i^2} \right)^{d_i} = \left( 1 + \frac{(1 - s - \alpha_i)^2}{\beta_i^2} \right)^{d_i} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \alpha_i = \frac{1}{2} \\ |\beta_1| < |\beta_2| < |\beta_3| < \dots \\ i = 1, 2, 3, \dots, \infty \end{array} \right.$$

In the future, the author's research interest will be focused on the GRH (Generalized Riemann Hypothesis).

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