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A Proof of the Riemann Hypothesis Based on a New Expression of the Completed Zeta Function

Weicun Zhang *

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Keywords: Riemann Hypothesis (RH); proof; completed zeta function

Article

A Proof of the Riemann Hypothesis Based on a New Expression of the Completed Zeta Function

Weicun Zhang

University of Science and Technology Beijing, Beijing 100083, China

Abstract: The Riemann Hypothesis (RH) is proved based on a new absolutely convergent expression of $\xi(s)$, which was obtained from the Hadamard product, through paring ρ_i and $\bar{\rho}_i$, and taking the multiple zeros into consideration in advance, i.e. $\xi(s) = \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) = \xi(0)\prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2}+\frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{d_i}$ where $\xi(0)=\frac{1}{2}$, $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\xi(s)$, $0<\alpha_i<1$ and $\beta_i \neq 0$ are real numbers, $d_i \geq 1$ is the real (**unique and unchangeable**) multiplicity of ρ_i, β_i are arranged in order of increasing $|\beta_i|$, i.e., $0<|\beta_1|\leq|\beta_2|\leq|\beta_3|\leq\cdots$, $i=1,2,3,\cdots,\infty$. Then, according to the functional equation $\xi(s)=\xi(1-s)$, we have $\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$ which, owing to the uniqueness and unchangeableness of d_i (see Lemma 3 for the proof details), is finally equivalent to $\alpha_i = \frac{1}{2}; 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots; i=1,2,3,\cdots,\infty$ Thus, we conclude that the RH is true.

Keywords: proof of the Riemann Hypothesis (RH); completed zeta function; new expression; hadamard product

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Significance Statement: The Riemann Hypothesis is one of the great unsolved problems in mathematics. It is crucial for understanding the nature of prime numbers. In addition, it is also related to the field of physics: there are striking similarities between the non-trivial zeros of Riemann zeta function and the quantum energy levels of chaotic systems.

1. Introduction

The Riemann Hypothesis [1] is one of the most important unsolved problems in mathematics. Although many efforts and achievements have been made towards proving this celebrated hypothesis, it still remains an open problem [2–3]. The Riemann zeta function is originally defined in the half-plane $\Re(s)>1$ by the absolutely convergent series [2]

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}, \Re(s)>1 \tag{1}$$

The connection between the above-defined Riemann zeta function and prime numbers was discovered by Euler, i.e., the famous Euler product

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p}(1-p^{-s})^{-1}, \Re(s)>1 \tag{2}$$

where p runs over the prime numbers.

Riemann showed in his paper in 1859 how to extend the zeta function to the whole complex plane $\mathbb{C}$ by analytic continuation, i.e.

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{\infty}\frac{(-x)^s}{e^x-1}\cdot\frac{dx}{x} \tag{3a}$$

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where " $\int_{\infty}^{\infty}$ " is the symbol adopted by Riemann to represent the contour integral from $+\infty$ to $+\infty$ around a domain which includes the value 0 but no other point of discontinuity of the integrand in its interior.

Or equivalently,

$$\zeta(s) = \frac{\pi^{s/2}}{\Gamma(s/2)}\{\frac{1}{s(s-1)} + \int_1^{\infty}(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}) \cdot (\frac{\theta(x)-1}{2})dx\} \tag{3b}$$

where $\theta(x) = \sum_{-\infty}^{\infty} e^{-n^2\pi x}$ is the Jaccobi theta function, Γ is the Gamma function in the following Weierstrass expression

$$\frac{1}{\Gamma(s)} = s \cdot e^{\gamma s}\prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-s/n} \tag{4}$$

where γ is the Euler-Mascheroni constant.

As shown by Riemann, $\zeta(s)$ extends to $\mathbb{C}$ as a meromorphic function with only a simple pole at $s = 1$, with residue 1, and satisfies the following functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{5}$$

The Riemann zeta function $\zeta(s)$ has zeros at the negative even integers: $-2, -4, -6, -8, \cdots$ and one refers to them as the **trivial zeros**. The other zeros of $\zeta(s)$ are the complex numbers, i.e., **non-trivial zeros** [2].

In 1896, Hadamard [4] and Poussin [5] independently proved that no zeros could lie on the line $\Re(s) = 1$, together with the functional equation $\xi(s) = \xi(1-s)$ and the fact that there are no zeros with real part greater than 1, this showed that all non-trivial zeros must lie in the interior of the **critical strip** $0 < \Re(s) < 1$. Later on, Hardy (1914) [6], Hardy and Littlewood (1921) [7] showed that there are infinitely many zeros on the **critical line** $\Re(s) = \frac{1}{2}$.

To give a summary of the related research on the RH, we have the following results on the properties of the non-trivial zeros of $\zeta(s)$ [4−9].

Lemma 1: Non-trivial zeroes of $\zeta(s)$, noted as $\rho = \alpha + j\beta$, have the following properties
1) The number of non-trivial zeroes is infinity;
2) $\beta \neq 0$;
3) $0 < \alpha < 1$;
4) $\rho, \bar{\rho}, 1-\bar{\rho}, 1-\rho$ are all non-trivial zeroes.

As further study, a completed zeta function $\xi(s)$ is defined as

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) \tag{6}$$

It is well-known that $\xi(s)$ is an entire function of order 1. This implies $\xi(s)$ is analytic, and can be expressed as infinite polynomial, in the whole complex plane $\mathbb{C}$. In addition, replacing s with $1-s$ in Eq.(6), and combining Eq.(5), we obtain the following functional equation

$$\xi(s) = \xi(1-s) \tag{7}$$

Considering the definition of $\xi(s)$, and recalling Eq.(4), the trivial zeros of $\zeta(s)$ are canceled by the poles of $\Gamma(\frac{s}{2})$. The zero of $s-1$ and the pole of $\zeta(s)$ cancel; the zero $s = 0$ and the pole of $\Gamma(\frac{s}{2})$ cancel [9−10]. Thus, all the zeros of $\xi(s)$ are exactly the nontrivial zeros of $\zeta(s)$. Then we have the following Lemma 2.

Lemma 2: The zeros of $\xi(s)$ coincide with the non-trivial zeros of $\zeta(s)$.
Accordingly, the following two statements of the RH are equivalent.

Statement 1: All the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

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Statement 2: All the zeros of $\xi(s)$ have real part equal to $\frac{1}{2}$.

To prove the RH, a natural thinking is to estimate the numbers of non-trivial zeros of $\zeta(s)$ inside or outside some certain areas according to Argument Principle. Along this train of thought, there are many research works. Let $N(T)$ denote the number of non-trivial zeros of $\zeta(s)$ inside the rectangle: $0 < \alpha < 1, 0 < \beta \le T$, and let $N_0(T)$ denote the number of non-trivial zeros of $\zeta(s)$ on the line $\alpha = \frac{1}{2}, 0 < \beta \le T$. Selberg proved that there exist positive constants c and T_0, such that $N_0(T) > cN(T), (T > T_0)$ [11], later on, Levinson proved that $c \ge \frac{1}{3}$ [12], Lou and Yao proved that $c \ge 0.3484$ [13], Conrey proved that $c \ge \frac{2}{5}$ [14], Bui, Conrey and Young proved that $c \ge 0.41$ [15], Feng proved that $c \ge 0.4128$ [16], Wu proved that $c \ge 0.4172$ [17].

On the other hand, many non-trivial zeros have been calculated by hand or by computer programs. Among others, Riemann found the first three non-trivial zeros [18]. Gram found the first 15 zeros based on Euler-Maclaurin summation [19]. Titchmarsh calculated the 138^{th} to 195^{th} zeros using the Riemann-Siegel formula [20−21]. Here are the first three (pairs of) non-trivial zeros: $\frac{1}{2} \pm j14.1347251$; $\frac{1}{2} \pm j21.0220396$; $\frac{1}{2} \pm j25.0108575$.

The idea of this paper is originated from Euler's work on proving the following famous equality

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6} \tag{8}$$

This interesting result is deduced by comparing the like terms of two types of infinite expressions, i.e., infinite polynomial and infinite product, as shown in the following

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = (1 - \frac{x^2}{\pi^2})(1 - \frac{x^2}{4\pi^2})(1 - \frac{x^2}{9\pi^2}) \cdots \tag{9}$$

Then the author of this paper conjectured that $\xi(s)$ should be factored into $\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$ or something like that, which was verified by paring ρ_i and $\bar{\rho}_i$ in the Hadamard product of $\xi(s)$, i.e. $(1 - \frac{s}{\rho_i})(1 - \frac{s}{\bar{\rho}_i}) = \frac{\beta_i^2}{\alpha_i^2 + \beta_i^2}\left(1 + \frac{(s-\alpha_i)^2}{\beta_i^2}\right)$

The Hadamard product of $\xi(s)$ as shown in Eq.(10) was first proposed by Riemann, however, it was Hadamard who showed the validity of this infinite product expansion [22].

$$\xi(s) = \xi(0) \prod_{\rho} (1 - \frac{s}{\rho}) \tag{10}$$

where $\xi(0) = \frac{1}{2}$, ρ runs over all the zeros of the completed zeta function $\xi(s)$.

Hadamard pointed out that to ensure the absolute convergence of the infinite product expansion, ρ and $1 - \rho$ are paired. Later in Section 3, we will show that ρ and $\bar{\rho}$ can also be paired to ensure the absolute convergence of the infinite product expansion.

2. Lemmas

In this section, we first explain the concept of the real multiplicity of a zero of $\xi(s)$. And then we give four lemmas (Lemma 3 to Lemma 6) to support the proof of the RH, among which Lemma 3 is the key lemma.

Multiple zeros of $\xi(s)$ and their real multiplicities: As shown in Figure 1, the multiple zeros of $\xi(s)$ are defined in terms of the quadruplet, i.e., $\rho, \bar{\rho}, 1 - \rho, 1 - \bar{\rho}$.

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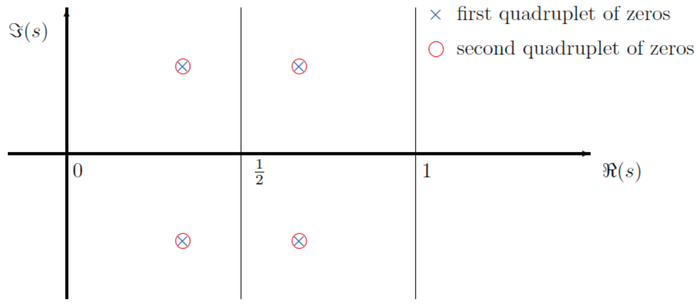

Figure 1. Illustration of the multiple zeros of $\xi(s)$.

There are two different expressions of factors of $\xi(s)/\xi(1-s)$ for the multiple zeros in Figure 1, respectively, i.e., $\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)^2/\left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^2$, or $\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)\left(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\right)/\left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)\left(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)$ with $\alpha_1+\alpha_2=1, \beta_1^2=\beta_2^2$.

To exclude the latter expression, we stipulate that zero ρ_i related factors of $\xi(s)/\xi(1-s)$ take the unique form of $\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}/\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$, where $d_i \geq 1$ is the real multiplicity of ρ_i, here "real" means unique and unchangeable. In Figure 1, the multiplicity of ρ_1 is 2, i.e., $d_1 = 2$.

Remark: Although the real multiplicity d_i of zero ρ_i is unknown, it is an objective existence, unique, and unchangeable. This is the key point in the proofs of Lemma 3, Lemma 4, and Lemma 5.

Lemma 3: Given two absolutely convergent infinite products

$$f(s)=\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{11}$$

and

$$f(1-s)=\prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{12}$$

where s is a complex variable, $\rho_i=\alpha_i+j\beta_i$ and $\bar{\rho}_i=\alpha_i-j\beta_i$ are the complex conjugate zeros of $\xi(s)$, $0<\alpha_i<1$ and $\beta_i \neq 0$ are real numbers, $d_i \geq 1$ are the real multiplicities of ρ_i, β_i are in order of increasing $|\beta_i|$, i.e., $0<|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots, i=1,2,3,\cdots,\infty$.

Then we have

$$f(s)=f(1-s) \Leftrightarrow \begin{cases} \alpha_i=\frac{1}{2} \\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots \\ i=1,2,3,\cdots,\infty \end{cases} \tag{13}$$

where " $\Leftrightarrow$ " is the equivalent sign.

Proof: First of all, we have the following fact:

$$(1+\frac{(s-\alpha)^2}{\beta^2})^d=(1+\frac{(1-s-\alpha)^2}{\beta^2})^d \Leftrightarrow (s-\alpha)^2=(1-s-\alpha)^2 \Leftrightarrow \alpha=\frac{1}{2} \tag{14}$$

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where $d \geq 1$ is a natural number, $\alpha \neq 0$ and $\beta \neq 0$ are real numbers.
Next, the proof is based on Transfinite Induction.
Let $P(n)$ be:

$$\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$$
$$\Leftrightarrow$$
$$\begin{cases} (1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1} \\ \cdots \\ (1+\frac{(s-\alpha_n)^2}{\beta_n^2})^{d_n}=(1+\frac{(1-s-\alpha_n)^2}{\beta_n^2})^{d_n} \\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots<|\beta_n| \end{cases} \tag{15}$$
$$\Leftrightarrow$$
$$\begin{cases} \alpha_i=\frac{1}{2}, i=1\cdots n \\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots<|\beta_n| \end{cases}$$

According to Eq.(14), $P(1)$ is an obvious fact as the **Base Case**, i.e.,

$$\begin{aligned} &\prod_{i=1}^{1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ &\Leftrightarrow\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1}=\left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1}\Leftrightarrow\alpha_1=\frac{1}{2} \end{aligned} \tag{16}$$

To be more convincing, let's further check $P(2)$, i.e.,

$$\prod_{i=1}^{2}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{2}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$$
$$\Leftrightarrow$$
$$\begin{cases} (1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1} \\ (1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2} \\ 0<|\beta_1|<|\beta_2| \end{cases} \tag{17}$$
$$\Leftrightarrow$$
$$\begin{cases} \alpha_1=\alpha_2=\frac{1}{2} \\ 0<|\beta_1|<|\beta_2| \end{cases}$$

which is also an obvious fact according to Lemma 4.

As the **Successor Case**, we need to prove $P(n) \Rightarrow P(n+1)$.
Actually, we have

$$
\begin{aligned}
&\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\
&\Leftrightarrow \Big(\text{by Lemma 5}\Big)\\
&\begin{cases}
\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\\
(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}=(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}\\
0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots<|\beta_n|<|\beta_{n+1}|
\end{cases}\\
&\Leftrightarrow \Big(\text{by Eq.(15)}\Big)\\
&\begin{cases}
(1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1}\\
\cdots\\
(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}=(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}\\
0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots<|\beta_n|<|\beta_{n+1}|
\end{cases}\\
&\Leftrightarrow \Big(\text{by Eq.(14)}\Big)\\
&\begin{cases}
\alpha_i=\frac{1}{2}, i=1,2,3,\cdots,n,n+1\\
0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots<|\beta_n|<|\beta_{n+1}|
\end{cases}
\end{aligned}
\tag{18}
$$

Thus the **Successor Case** is true, i.e., $P(n) \Rightarrow P(n+1)$.

Next, we prove that $P(\infty)$ holds, by considering well-ordered ordinal set A indexing the family of statements $P(\gamma : \gamma \in A)$, $A = \mathbb{N} \bigcup \{\omega\}$ with the ordering that $n < \omega$ for all natural numbers n, ω is the first limit ordinal.
It is well-known that $\omega = \bigcup\{\gamma : \gamma < \omega\}$.

To prove that $P(\infty)$ holds, it suffices to prove the **Limit Case**, i.e., $P(\gamma < \omega) \Rightarrow P(\omega)$.
Assume, for the sake of contradiction, that $P(\gamma < \omega)$ holds, but $P(\omega)$ does not hold, then by

$$\prod_{i=1}^{\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \not\Rightarrow \alpha_i=\frac{1}{2} \tag{19a}$$

and

$$P(\gamma<\omega):\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \Leftrightarrow \alpha_i=\frac{1}{2}, \tag{19b}$$

we have

$$\prod_{i=1}^{\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \Rightarrow \alpha_i\neq\frac{1}{2} \tag{19c}$$

$\Big[$Otherwise, since $f(s)/f(1-s)$ are absolutely convergent, then through rearrangement of the order of factors in the left-hand part of (19c), we have $\alpha_i \neq \frac{1}{2}, i = 1, ..., \gamma, \gamma < \omega$, and $\alpha_i = \frac{1}{2}, i = \gamma+1, ..., \omega$, then by canceling the corresponding factors of thess $\alpha_i = \frac{1}{2}$ in the left-hand part of (19c), we obtain

$$\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{19d}$$

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Taking (19b) into consideration, (19d) leads to $\alpha_i = \frac{1}{2}, i = 1, ..., \gamma, \gamma < \omega$, this together with $\alpha_i = \frac{1}{2}, i = \gamma + 1, ..., \omega$, is a contradiction to the assumption (19a).]

Next, let's check the "divisibility" in $\prod_{i=1}^{\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$ based on (19c). We have

$$\begin{aligned}
&\prod_{i=1}^{\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\
&\Leftrightarrow \\
&\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega} = \prod_{i=1}^{\gamma<\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(1-s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega}
\end{aligned} \tag{19e}$$

It follows from (19e) that

$$\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega} / \left(1+\frac{(1-s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega} = \prod_{i=1}^{\gamma<\omega}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \tag{19f}$$

Eq.(19f) means that $\left(1+\frac{(1-s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega}$ divides $\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega}$. Meanwhile we know that $\left(1+\frac{(1-s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega}$ can not divide $\prod_{i=1}^{\gamma<\omega}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$, because, as shown in the proof of Lemma 4, that leads to new multiple zeros, which contradics the definition of the real multiplicities of zeros. Then, by Lemma 6, $\left(1+\frac{(1-s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega}$ divides $\left(1+\frac{(s-\alpha_\omega)^2}{\beta_\omega^2}\right)^{d_\omega}$, that means $\alpha_\omega = \frac{1}{2}$, which contradicts (19c).
Thus, the assumption that $P(\gamma < \omega)$ holds, but $P(\omega)$ does not hold, is false.

Then the **Limit Case** is true, i.e., $P(\gamma < \omega) \Rightarrow P(\omega)$.
Hence, we conclude by Transfinite Induction that $P(\infty)$ holds, i.e.,

$$\begin{aligned}
&\prod_{i=1}^{\infty}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} = \prod_{i=1}^{\infty}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\
&\Leftrightarrow \\
&\begin{cases} (1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{d_i} = (1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{d_i} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \\ i = 1, 2, 3, \cdots, \infty \end{cases} \\
&\Leftrightarrow \\
&\begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \\ i = 1, 2, 3, \cdots, \infty \end{cases}
\end{aligned} \tag{20}$$

i.e.,

$$f(s) = f(1-s) \Leftrightarrow \begin{cases} \alpha_i = \frac{1}{2} \\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots \\ i = 1, 2, 3, \cdots, \infty \end{cases} \tag{21}$$

That completes the proof of Lemma 3.

Lemma 4: Given

$$\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2}=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2} \tag{22}$$

where s is a complex variable, $0<\alpha_1<1, 0<\alpha_2<1$ and $\beta_1\neq 0, \beta_2\neq 0$ are real numbers, $d_1\geq 1, d_2\geq 1$ are natural numbers, denoting the real multiplicities of $\rho_1=\alpha_1+j\beta_1$ and $\rho_2=\alpha_2+j\beta_2$, respectively, and $0<|\beta_1|\leq|\beta_2|$.
Then we have

$$\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2}=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2}$$

$$\Leftrightarrow$$

$$\begin{cases} (1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1} \\ (1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2} \\ 0<|\beta_1|<|\beta_2| \end{cases} \tag{23}$$

$$\Leftrightarrow$$

$$\begin{cases} \alpha_1=\alpha_2=\frac{1}{2} \\ 0<|\beta_1|<|\beta_2| \end{cases}$$

Proof: Eq.(22) has an obvious solution, i.e.,

$$\begin{cases} (1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1} \\ (1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2} \\ 0<|\beta_1|\leq|\beta_2| \end{cases}$$
$$\Leftrightarrow \tag{24}$$
$$\begin{cases} \alpha_1=\alpha_2=\frac{1}{2} \\ 0<|\beta_1|\leq|\beta_2| \end{cases}$$

Other solutions can be obtained by Lemma 6, and excluding the above obvious solution, i.e., we can find other solutions of Eq.(22) by considering $\alpha_1\neq\frac{1}{2}, \alpha_2\neq\frac{1}{2} \Leftrightarrow gcd(1+(s-\alpha_1)^2/\beta_1^2, 1+(1-s-\alpha_1)^2/\beta_1^2)=gcd(1+(s-\alpha_2)^2/\beta_2^2, 1+(1-s-\alpha_2)^2/\beta_2^2)=1$, where "gcd" stands for: greatest common divisor.

Then from Eq.(22), and suppose $d_2\geq d_1$ without loss of generality, we have

$$\Big(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2}=\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2}/\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1} \tag{25}$$

i.e.,

$$\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big|\Big(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2} \tag{26}$$

where "$|$" is the "divisible" sign.

By Lemma 6, and the fact $gcd(1+(s-\alpha_1)^2/\beta_1^2, 1+(1-s-\alpha_1)^2/\beta_1^2)=1$, it follows that

$$\Big(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\Big)^{d_1}\Big|\Big(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\Big)^{d_2} \tag{27}$$

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Then we have

$$\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)=\left(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\right) \tag{28}$$

By comparing like terms of both sides of Eq.(28), it follows that $\alpha_1+\alpha_2=1, |\beta_1|=|\beta_2|$. Obviously there is no need to further solve Eq.(25) to get the final solutions, because the solution of Eq.(28), $\alpha_1+\alpha_2=1, |\beta_1|=|\beta_2|$ already mean that the quadruplet zeros $\alpha_2 \pm j\beta_2/1-\alpha_2 \pm j\beta_2$ is a duplicate of $\alpha_1 \pm j\beta_1/1-\alpha_1 \pm j\beta_1$, which is a similar situation as shown in Figure 1. That contradicts the definition of real multiplicities in Eq.(22), i.e., in Eq.(22) the real multiplicity of $\alpha_1 \pm j\beta_1/1-\alpha_1 \pm j\beta_1$ is d_1, if $\alpha_1+\alpha_2=1, |\beta_1|=|\beta_2|$, then the real multiplicity of $\alpha_1 \pm j\beta_1/1-\alpha_1 \pm j\beta_1$ would be d_1+d_2.

Therefore, the obvious solution Eq.(24) is the only solution of Eq.(22), i.e.

$$\left(1+\frac{(s-\alpha_1)^2}{\beta_1^2}\right)^{d_1}\left(1+\frac{(s-\alpha_2)^2}{\beta_2^2}\right)^{d_2}=\left(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2}\right)^{d_1}\left(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2}\right)^{d_2}$$

$$\Leftrightarrow$$

$$\begin{cases} (1+\frac{(s-\alpha_1)^2}{\beta_1^2})^{d_1}=(1+\frac{(1-s-\alpha_1)^2}{\beta_1^2})^{d_1} \\ (1+\frac{(s-\alpha_2)^2}{\beta_2^2})^{d_2}=(1+\frac{(1-s-\alpha_2)^2}{\beta_2^2})^{d_2} \\ 0<|\beta_1|\le|\beta_2| \end{cases} \tag{29}$$

$$\Leftrightarrow$$

$$\begin{cases} \alpha_1=\alpha_2=\frac{1}{2} \\ 0<|\beta_1|\le|\beta_2| \end{cases}$$

Further excluding the contradictory situation (violating the definition of the real multiplicities of zeros) in Eq.(29), i.e., $\alpha_1=\alpha_2=\frac{1}{2}, |\beta_1|=|\beta_2|$, then we know that Eq.(23) holds.

That completes the proof of Lemma 4.

Similarly, we have the following extended result of Lemma 4 without proof details.

Lemma 5: Given

$$\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$$

where s is a complex variable, $0<\alpha_i<1$, and $\beta_i\neq 0$ are real numbers, $d_i\geq 1$ are natural numbers, denoting the real multiplicities of $\rho_i=\alpha_i+j\beta_i$, $0<|\beta_1|\le|\beta_2|\le|\beta_3|\cdots\le|\beta_{n+1}|$.
Then we have

$$\prod_{i=1}^{n+1}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n+1}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}$$

$$\Leftrightarrow$$

$$\prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}\left(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2}\right)^{d_{n+1}}$$

$$\Leftrightarrow$$

$$\begin{cases} \prod_{i=1}^{n}\left(1+\frac{(s-\alpha_i)^2}{\beta_i^2}\right)^{d_i}=\prod_{i=1}^{n}\left(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2}\right)^{d_i} \\ (1+\frac{(s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}}=(1+\frac{(1-s-\alpha_{n+1})^2}{\beta_{n+1}^2})^{d_{n+1}} \\ 0<|\beta_1|<|\beta_2|<|\beta_3|<\cdots<|\beta_n|<|\beta_{n+1}| \end{cases}$$

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Lemma 6: Let F be a field, $p(x), q(x), m(x) \in F[x]$. If $m(x)$ divides the product $p(x)q(x)$, but $m(x)$ and $p(x)$ are relatively prime, i.e., $gcd(m(x), p(x)) = 1$, then $m(x)$ divides $q(x)$.

Remark: The content of Lemma 6 can be found in many textbooks of Linear Algebra or Advanced Algebra. It is the foundation to find other possible solutions in addition to the obvious solution in Lemma 4 and Lemma 5.

3. A Proof of the RH

This section is planned to present a proof of the Riemann Hypothesis. We first prove that Statement 2 of the RH is true, and then by Lemma 2, Statement 1 of the RH is also true. To be brief, to prove the Riemann Hypothesis, it suffices to show that $\alpha_i = \frac{1}{2}, i = 1, 2, 3, \cdots$ in the new expression of $\xi(s)$ as shown in Eq. (30).

Proof of the RH: The details are delivered in three steps as follows.

Step 1:
It is well-known that all the zeros of $\xi(s)$ always come in complex conjugate pairs. Then by pairing $\rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ in the Hadamard product as shown in Eq.(10), we have

$$\begin{aligned} \xi(s) &= \xi(0)\prod_{\rho}(1-\frac{s}{\rho}) = \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) \\ &= \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\alpha_i + j\beta_i})(1-\frac{s}{\alpha_i - j\beta_i}) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}) \end{aligned} \tag{30}$$

where $\xi(0) = \frac{1}{2}, 0 < \alpha_i < 1, \beta_i \neq 0$.
The absolute convergence of the infinite product in Eq.(30) in the form

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}(1-\frac{s}{\rho_i})(1-\frac{s}{\bar{\rho}_i}) = \xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i - s)}{|\rho_i|^2}\right) \tag{31}$$

depends on the convergence of infinite series $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$, which is an obvious fact according to Theorem 2 in Section 2, Chapter IV of Ref.[23].

Further, considering the absolute convergence of

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i - s)}{|\rho_i|^2}\right) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}) \tag{32}$$

we have the following new expression of $\xi(s)$ by putting all the ρ_i related multiple factors (zeros) together in the above Eq.(32)

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{33}$$

where $d_i \geq 1$ are the real multiplicities of ρ_i, i are natural numbers from 1 to infinity.

Step 2: Replacing s with $1-s$ in Eq.(33), we obtain the infinite product expression of $\xi(1-s)$, i.e.,

$$\xi(1-s) = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \tag{34}$$

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Step 3: According to the functional equation $\xi(s) = \xi(1-s)$, and considering Eq.(33) and Eq.(34), we have

$$\xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} = \xi(0)\prod_{i=1}^{\infty}(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(1-s-\alpha_i)^2}{\alpha_i^2+\beta_i^2})^{d_i} \quad (35)$$

which is equivalent to

$$\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})^{d_i} = \prod_{i=1}^{\infty}(1+\frac{(1-s-\alpha_i)^2}{\beta_i^2})^{d_i} \quad (36)$$

where β_i are in order of increasing $|\beta_i|$, i.e., $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots$.

To check the absolute convergence of both sides of Eq.(36), it suffices to make a comparison with Eq.(31) without considering multiple zeros in Eq. (36), i.e., to make a comparison between $\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})$ and $\xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i-s)}{|\rho_i|^2}\right)$. It is well-known that the absolute convergence of $\xi(0)\prod_{i=1}^{\infty}\left(1-\frac{s(2\alpha_i-s)}{|\rho_i|^2}\right)$ depends on the convergence of infinite series $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$ (already proved in Step 1); the absolute convergence of $\prod_{i=1}^{\infty}(1+\frac{(s-\alpha_i)^2}{\beta_i^2})$ depends on the convergence of infinite series $\sum_{i=1}^{\infty}\frac{1}{\beta_i^2}$, which is also an obvious fact because $0 < \alpha_i < 1, |\rho_i| \to \infty, |\beta_i| \to \infty$, as $i \to \infty$, $\lim_{i\to\infty}\frac{\beta_i^2}{|\rho_i|^2} = \lim_{i\to\infty}\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} = 1$, that means $\sum_{i=1}^{\infty}\frac{1}{\beta_i^2}$ and $\sum_{i=1}^{\infty}\frac{1}{|\rho_i|^2}$ have the same convergence.

Then, according to Lemma 3, Eq.(36) is equivalent to

$$\alpha_i = \frac{1}{2};\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots;\ i = 1,2,3,\cdots,\infty \quad (37)$$

Thus, we conclude that all the zeros of the completed zeta function $\xi(s)$ have real part equal to $\frac{1}{2}$, i.e., Statement 2 of the RH is true. According to Lemma 2, Statement 1 of the RH is also true, i.e., all the non-trivial zeros of the Riemann zeta function $\zeta(s)$ have real part equal to $\frac{1}{2}$.

That completes the proof of the RH.

4. Conclusion

This paper presents a proof of the RH based on a new expression of $\xi(s)$, i.e.,

$$\xi(s) = \xi(0)\prod_{i=1}^{\infty}\left(\frac{\beta_i^2}{\alpha_i^2+\beta_i^2} + \frac{(s-\alpha_i)^2}{\alpha_i^2+\beta_i^2}\right)^{d_i}$$

where $\xi(0) = \frac{1}{2}, \rho_i = \alpha_i + j\beta_i$ and $\bar{\rho}_i = \alpha_i - j\beta_i$ are the complex conjugate zeros of $\xi(s)$, $0 < \alpha_i < 1$ and $\beta_i \neq 0$ are real numbers, $d_i \geq 1$ are the real multiplicities of ρ_i, β_i are in order of increasing $|\beta_i|$, i.e., $0 < |\beta_1| \leq |\beta_2| \leq |\beta_3| \leq \cdots, i = 1,2,3,\cdots,\infty$.

The proof is conducted according to Transfinite Induction, and the key-point is the use of "real multiplicities" of the zeros of $\xi(s)$. Obviously, the real multiplicity of a zero of $\xi(s)$ is an objective existence, unique, and unchangeable. As a result, the functional equation $\xi(s) = \xi(1-s)$ finally leads to $\alpha_i = \frac{1}{2};\ 0 < |\beta_1| < |\beta_2| < |\beta_3| < \cdots;\ i = 1,2,3,\cdots,\infty$.

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