Closed Form Solution of Plane-Parallel Turbulent Flow Along an Unbounded Plane Boundary Surface

Bo-Hua Sun[1]

[1]*School of Civil Engineering & Institute of Mechanics and Technology*
Xi'an University of Architecture and Technology, Xi'an 710055, China
email: sunbohua@xauat.edu.cn
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In this paper, a century-old problem is solved; namely, to find a unified analytic description of the non-uniform distribution of mean velocity across the entire domain of turbulent flow for all Reynolds numbers within the framework of the Prandtl mixing length theory. This study obtains a closed form solution of the mean velocity profile of plane turbulent flow for the Prandtl theory, and as well an approximate analytical solution for the van Driest mixing length theory. The profiles of several useful quantities are given based the closed form solution, such as turbulent viscosity, Reynolds turbulent stress, Kolmogorov's scaling law, and energy dissipation density. The investigation shows that the energy dissipation density at the surface is finite, whereas Landau's energy dissipation density is infinite. Strictly speaking, the closed form solution reveals that the universality of the turbulent velocity logarithmic profile no longer holds, but the von Kármán constant is still universal. Furthermore, a new formulation of the resistance coefficient of turbulent flow in pipes is formulated in implicit form.

Keywords: Turbulent flow, Prandtl mixing length, velocity profile, Reynolds number, plane boundary surface

I. INTRODUCTION

The theory of the turbulence boundary layer dates to the beginning of the 20th century. In 1904, Prandtl [1] proposed that the high Reynolds of plane flow produces a very thin boundary layer at the wall that exhibits basically non-viscous motion; however, the viscous effect within the boundary layer cannot be ignored. In 1908, Blasius [2] obtained a complete description of the laminar boundary layer, which resulted in the classical laminar boundary-layer theory. However, almost all boundary-layer movements in nature are turbulent rather than laminar [3–13, 23], predicting the turbulent mean velocity profile and related quantities are of importance, which have been investigated extensively by [12, 13] from perspectives of the Lie symmetry.
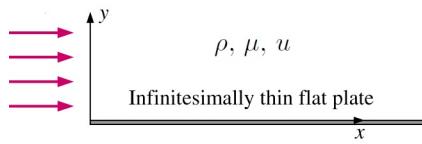
FIG. 1: Plane turbulent flow along an unbounded plane boundary surface, where ρ is flow mass density, μ is dynamical viscosity and u is flow velocity.

We consider here a plane-parallel turbulent flow along an unbounded smooth plane surface (wall) as shown in Fig.1, and take the direction of the flow as the x axis and the plane of the surface as the xz plane, so y is the direction orthogonal to the surface. Assuming that the turbulent flow is steady with constant pressure along the x axis, the y and z components of the mean velocity are zero, and all of the quantities depend only on y. In the following, we only need to study the upper flow due to the symmetry of the problem.

The Reynolds-averaged Navier-Stokes equation of the plane-parallel turbulence is reduced to $\mu\frac{d^2\bar{u}}{dy^2} + \frac{d\tau'_{xy}}{dy} = 0$, where μ is the dynamical viscosity, $\bar{u}$ is the mean velocity, and $\tau'_{xy} = -\rho\overline{u'v'}$ is the Reynolds stress.

Under boundary conditions, i.e., $y = 0$: $\bar{u} = 0$, $u' = 0$, $v' = 0$ and $\mu\frac{d\bar{u}}{dy} = \tau_w$, the above equation can be integrated to

$$\mu\frac{d\bar{u}}{dy} - \rho\overline{u'v'} = \tau_w, \tag{1}$$

where τ_w is the wall friction force on a unit area of the surface. This force is clearly in the x direction. The quantity τ_w is the constant flux of the x component of momentum transmitted by the fluid to the surface per unit time. The first term on the left-hand side of Eq. 1 represents the effect of viscosity on the mean flow, whereas the second term is the Reynolds stress. In turbulent flow located some distance away from a wall, the Reynolds stress is of considerably greater magnitude than the viscous stress; however, the role of viscous stress increases as the distance to a smooth wall decreases until finally, at the wall, viscosity predominates.

According to the Prandtl mixing length theory [5], the Reynolds stress is proposed to be $\tau'_{xy} = \rho\ell^2|\frac{d\bar{u}}{dy}|\frac{d\bar{u}}{dy}$, where the mass density is ρ, the mixing length is constructed by dimensional arguments as $\ell = \kappa y$, and κ is a numerical constant, namely, the von Kármán constant. Hence, Eq. 1 becomes $\mu\frac{d\bar{u}}{dy} + \rho(\kappa y)^2|\frac{d\bar{u}}{dy}|\frac{d\bar{u}}{dy} = \tau_w$, namely,

$$\frac{d\bar{u}}{dy}<0: \quad \mu\frac{d\bar{u}}{dy} - \rho(\kappa y)^2\left(\frac{d\bar{u}}{dy}\right)^2 = \tau_w, \tag{2}$$

$$\frac{d\bar{u}}{dy}>0: \quad \mu\frac{d\bar{u}}{dy} + \rho(\kappa y)^2\left(\frac{d\bar{u}}{dy}\right)^2 = \tau_w, \tag{3}$$

and the boundary condition is

$$y = 0 : \bar{u} = 0. \tag{4}$$

So far, no complete solutions have been obtained for either Eq. 2 or Eq. 3. Instead, asymptotic solutions have been constructed in two different regions (or sub-layers) [8] .

In the inertial sub-layer, the first terms of Eq. 2 and Eq. 3 are neglected, leading to $\rho(\kappa y)^2 \left(\frac{d\bar{u}}{dy}\right)^2 = \tau_w$, the solution of which is a well-known Prandtl logarithmic law,

$$\frac{\bar{u}}{u_\tau} = \frac{1}{\kappa} \log(\frac{y\bar{u}}{\nu}) + \alpha - \frac{1}{\kappa} \log \alpha, \tag{5}$$

where $u_\tau = \sqrt{\tau_w/\rho}$ and $\nu = \mu/\rho$.

According to Nikuradse's famous experiments [7], data fitting gives $\kappa = 0.4$ and $\alpha = 11.5$; hence, the Prandtl log-law is

$$\frac{\bar{u}}{u_\tau} = 5.75 \log(\frac{y\bar{u}}{\nu}) + 5.5 \tag{6}$$

This expression becomes infinite at the boundary $y = 0$ and is inapplicable at very small distances y from the surface, since the effect of viscosity near the surface becomes non-negligible [9] .

To fix the singularity problem, traditionally a viscous sub-layer is introduced in which the viscosity of the fluid begins to be important. The second terms of Eqs. 2 and 3 can be neglected, leading to $\mu \frac{d\bar{u}}{dy} = \tau_w$, the solution of which is the Prandtl linear law: $\bar{u} = \tau_w y/\mu$.

After obtaining these two sub-layer solutions, combining them into one segmental solution as follows:

$$u^+ = \eta + (5.394 - \eta + 2.5 \ln \eta) H(\eta - 5.394), \tag{7}$$

where $H(...)$ is the Heaviside step function, the wall friction velocity $u^+ = \frac{\bar{u}}{u_\tau}$, and the characteristic wall coordinate $\eta = \frac{y u_\tau}{\nu}$ (note: η is denoted as y^+ in some publications). The wall friction velocity is the chracteristic velocity for turbulent flows at a given wall shear stress.

Mathematically speaking, the above two segmental solutions, including linear and log-law solutions, are incomplete, because they are not whole domain solutions. Rather, it is a local solution of the matching area between the inertial sub-layer and viscous sub-layer of turbulence. Between the inertial region and viscous sub-layer, there is an intermediate region whose empirical solution has not been obtained [8–10, 14, 15] .

Finding a unified analytic description of the non-uniform distribution of mean velocity and turbulence intensities across the entire domain of turbulent flow for all Reynolds numbers is an open problem dating back to Prandtl's pioneering work in 1904 [10, 14, 15] .

In this paper, we will focus on the above-mentioned open problems, and find its exact solution and propose a modified approximate analytical solution.

II.   AN EXACT CLOSED FORM SOLUTION BY RIGOROUS MATHEMATICS

To find a singularity-free and consistent solution valid in the whole domain of y, we return to the complete governing equations, Eqs. 2 and 3, and work to find their solutions without any approximations, as in the preceding section. We find the exact solutions for both Eqs. 2 and 3, but the solution of Eq. 2 is not physically possible since two terms of the solution are infinite at $y = 0$. The only physically possible equation is Eq. 3, which can be rewritten as

$$\frac{du^+}{d\eta} + \kappa^2 \eta^2 \left(\frac{du^+}{d\eta}\right)^2 = 1, \tag{8}$$

together with the boundary condition

$$\eta = 0 : u^+ = 0. \tag{9}$$

From Eq.8, we can get $\frac{du^+}{d\eta} = \frac{-1 \mp \sqrt{1 + 4\kappa^2 \eta^2}}{2\kappa^2 \eta^2}$, hence we have two solutions of Eq. 8, namely

$$u^+ = \begin{cases} \int \frac{-1 - \sqrt{1 + 4\kappa^2 \eta^2}}{2\kappa^2 \eta^2} d\eta = \frac{1}{2\kappa^2 \eta} - \int \frac{\sqrt{1 + 4\kappa^2 \eta^2}}{2\kappa^2 \eta^2} d\eta, \\ \\ \int \frac{-1 + \sqrt{1 + 4\kappa^2 \eta^2}}{2\kappa^2 \eta^2} d\eta = \frac{1}{2\kappa^2 \eta} + \int \frac{\sqrt{1 + 4\kappa^2 \eta^2}}{2\kappa^2 \eta^2} d\eta. \end{cases} \tag{10}$$

Introducing a transformation

$$2\kappa\eta = \sinh \xi \tag{11}$$

hence $\sqrt{1 + 4\kappa^2 \eta^2} = \cosh \xi$, and $\coth \xi = \frac{\sqrt{1 + 4\kappa^2 \eta^2}}{2\kappa\eta}$, as well the inversion $\xi = \sinh^{-1}(2\kappa\eta) = \ln(2\kappa\eta + \sqrt{1 + 4\kappa^2 \eta^2})$, and derivative

$$d\eta = \frac{\cosh \xi}{2\kappa} dx. \tag{12}$$

Therefore, the integration can be obtained as follows

$$\int \frac{\sqrt{1 + 4\kappa^2 \eta^2}}{2\kappa^2 \eta^2} d\eta$$
$$= \frac{1}{\kappa} \int (\frac{\cosh \xi}{\sinh \xi})^2 d\xi = \frac{1}{\kappa} (\xi - \coth \xi)$$
$$= \frac{1}{\kappa} \ln(2\kappa\eta + \sqrt{1 + 4\kappa^2 \eta^2}) - \frac{\sqrt{1 + 4\kappa^2 \eta^2}}{2\kappa^2 \eta}. \tag{13}$$

Substitute the integration in Eq. 13 into the first of Eq.10, we have one solution

$$u^+ = \frac{1}{\kappa} \ln(2\kappa\eta + \sqrt{1 + 4\kappa^2 \eta^2})$$
$$- \frac{2\eta}{-1 + \sqrt{1 + 4\kappa^2 \eta^2}} + C, \tag{14}$$

This solution is singular solution because of the singularity of $\frac{2\eta}{-1 + \sqrt{1 + 4\kappa^2 \eta^2}}$ at $\eta \to 0$, we can not use it as a solution.

After carefully checking, the exact closed form and singularity-free solution of Eq.8 is obtained as

$$u^+ = \frac{1}{\kappa} \ln(2\kappa\eta + \sqrt{1 + 4\kappa^2\eta^2})$$
$$- \frac{2\eta}{1 + \sqrt{1 + 4\kappa^2\eta^2}} + C, \tag{15}$$

where C is integration constant that can be determined by the boundary condition in Eq. 9.

The solution in Eq.15 is singularity free because of the cancellation of singularity of the term $\lim_{\eta\to 0}\left(\frac{2\eta}{1+\sqrt{1+4\kappa^2\eta^2}}\right) = 0$, which leads to the limit $\lim_{\eta\to 0} u^+ = C$. Applying the boundary condition in Eq. 9 gives $C = 0$. Finally, the exact closed form solution of Eq. 8 is found as follows

$$u^+ = \frac{1}{\kappa} \ln(2\kappa\eta + \sqrt{1 + 4\kappa^2\eta^2})$$
$$- \frac{2\eta}{1 + \sqrt{1 + 4\kappa^2\eta^2}}. \tag{16}$$

and or in a more compact form

$$u^+ = \frac{1}{\kappa} \ln(2\ell^+ + \sqrt{1 + 4(\ell^+)^2})$$
$$- \frac{1}{\kappa}\frac{2\ell^+}{1 + \sqrt{1 + 4(\ell^+)^2}}. \tag{17}$$

where $\ell^+ = \kappa\eta$.

Strictly speaking, the solution in Eq. 16 reveals that the velocity profile of plane-turbulent flow is not logarithmic as stated in [14], owing to the exist of the 2nd term "$-\frac{2\eta}{1+\sqrt{1+4\kappa^2\eta^2}}$". Roughly speaking, the logarithmic law of the turbulent velocity profile can still be approximately maintained for very large η. Similarly, close to the boundary surface, the velocity profile is not linear of distance to the wall as well, because $\frac{2\eta}{1+\sqrt{1+4\kappa^2\eta^2}} \neq 2\kappa\eta$.

The singularity-free solution in Eq.16 is valid for the whole domain of $\eta \in [0,\infty]$. The only unknown in the solution is the von Kármán constant κ, which could be determined by data fitting from the experiments [7] to be $\kappa \approx 0.4$, confirming that the von Kármán constant is still a universal constant.

Due to the identity of hyperbolic arcsine $\sinh^{-1} x = \ln(x + \sqrt{1 + x^2})$, $x \in (-\infty,\infty)$, the closed form solution can also be expressed as follows

$$u^+ = \frac{1}{\kappa} \sinh^{-1}(2\kappa\eta) - \frac{2\eta}{1 + \sqrt{1 + 4\kappa^2\eta^2}}. \tag{18}$$

III. VALIDATION STUDIES, DISCUSSIONS AND PERSPECTIVES

To visualize the difference between the segmental solution and the closed form solution, Eqs. 7 and 16 are depicted in Fig. 2.
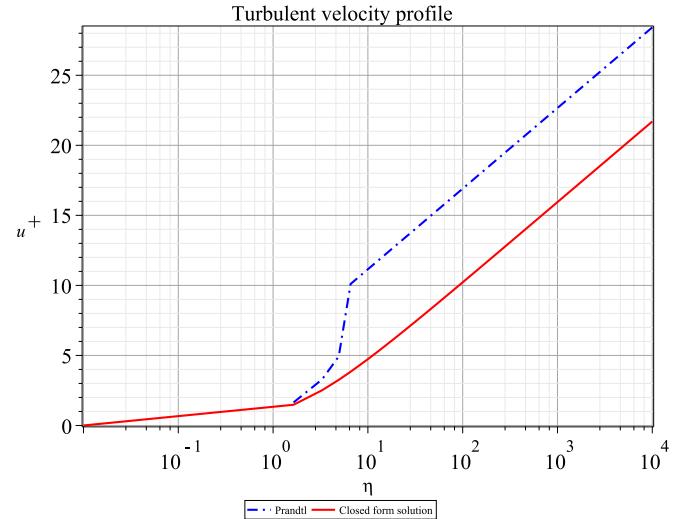
FIG. 2: The $u^+ - \log\eta$ curve. The segmental solution in Eq. 7 is represented by the blue dashline, and the closed form solution in Eq. 16 is represented by the red line. The plot indicates that the mean velocity of the Prandtl solution is larger than the velocity obtained by the closed form solution as a result of neglecting the viscosity in Prandtl's solution.

For further validation, a comparison between the closed form solution in Eq. 16 and direct numerical simulation (DNS) solution [15, 16] is depicted in Fig. 3
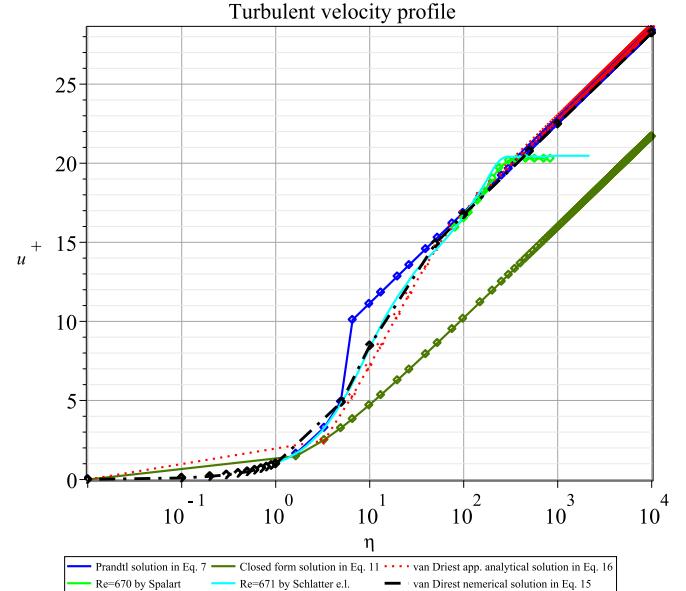
FIG. 3: Comparisons: the closed form solution, the Prandtl solution, van Driest mixing length theory [18], the direct numerical simulation (DNS) solution [15, 16].

Although we successfully obtained the closed form solution of plane turbulent flow based on the Prandtl mixing length model $\ell^+ = \kappa\eta$, where $\ell^+ = \ell u_\tau/\nu$, unfortunately, Fig. 3 shows that the closed form solution does

not perfectly agree with the DNS solution [15, 16] for a smooth wall.

It is worth to mention that the above disagreement is caused by the Prandtl mixing length theory and nothing to do with the mathematical rigorous treatment of the Eq.(8) and its closed form solution. Therefore, if we consider the DNS solution as a high-performance solution, to fix the mess we must modify the Prandtl mixing length model to make the solution as close to the DNS solution as possible.

IV.   THE VAN DRIEST MODIFICATION OF THE PRANDTL MIXING LENGTH THEORY

Regarding the modification, we consider an infinite flat plate undergoing simple harmonic oscillation parallel to the plate in an infinite fluid. According to Stokes [17], the amplitude of the motion diminishes with increasing distance from the surface (wall) as a consequence of the factor $\exp(-y/A)$, where A is a constant depending the frequency of oscillation of the plate and kinematic viscosity ν of the fluid. Hence, van Driest [18] pointed out that when the plate is fixed and the fluid oscillates relative to the plate, the factor $[1 - \exp(-y/A)]$ must be applied to the fluid oscillation to obtain the damping effect on the wall. Furthermore, van Driest believed that fully developed turbulent motion occurs only beyond a distance sufficiently remote from the wall, and eddies are not damped by the nearness of the wall. Indeed, near a wall, the damping factor is $[1-\exp(-y/A)]$ for each mean velocity fluctuation, and the Reynolds stress should be modified to become

$$\tau'_{xy} = \rho\kappa^2 y^2 (\frac{d\bar{u}}{dy})^2 [1 - \exp(-\frac{y}{A})]^2$$
$$= \rho u_\tau^2 \kappa^2 \eta^2 (\frac{du^+}{d\eta})^2 [1 - \exp(-\frac{\eta}{A^+})]^2, \qquad (19)$$

in order to take into account the mean motion all the way to a smooth wall, where $\kappa \simeq 0.4$, $A^+ = \frac{u_\tau A}{\nu} \simeq 26$.

With the van Driest mixing length theory, Eq. 8 should be modified to the following:

$$\frac{du^+}{d\eta} + \kappa^2 \eta^2 [1 - \exp(-\frac{\eta}{A^+}]^2 \left(\frac{du^+}{d\eta}\right)^2 = 1, \qquad (20)$$

whose solution can be expressed in an integration [18]

$$u^+ = \int \frac{2}{1 + \sqrt{1 + 4\kappa^2\eta^2[1 - \exp(-\eta/A^+)]^2}} d\eta, \quad (21)$$

which unfortunately has no analytical solution. However, we can solve it numerically. For easy application, an approximated analytical solution can be proposed as follows

$$u^+ \approx \frac{1}{\kappa} \ln \left[2\kappa\eta\varphi + \sqrt{1 + 4\kappa^2\eta^2\varphi^2}\right]$$
$$+ \frac{3.49\eta\varphi}{1 + \sqrt{1 + 4\kappa^2\eta^2\varphi^2}}, \qquad (22)$$

where $\varphi = 1-\exp(-\eta/B^+)$ is called the van Driest damping function, and $B^+ = 9$. The above expression can also be written in a more compact form

$$u^+ \approx \frac{1}{\kappa} \ln(2\ell^+\varphi + \sqrt{1 + 4(\ell^+\varphi)^2})$$
$$+ \frac{1}{\kappa} \frac{8.725\ell^+\varphi}{1 + \sqrt{1 + 4(\ell^+\varphi)^2}}. \qquad (23)$$

The solution of Eq. 21 and Eq. 22 are depicted in Fig. 3, which shows that both numerical and approximate analytical solution of the van Driest mixing length theory agree well with the DNS solution.

V.   SOME QUALITIES CALCULATED FROM THE CLOSED FORM SOLUTION

With the obtained closed form solution in Eq. 16, we can calculate some useful quantities, as follows:

(1) The turbulent viscosity is $\nu_T = (\kappa y)^2 \frac{d\bar{u}}{dy} = \nu\kappa^2\eta^2 \frac{du^+}{d\eta}$, where $\frac{du^+}{d\eta} = \frac{-1+\sqrt{1+4\kappa^2\eta^2}}{2\kappa^2\eta^2} = \frac{2}{1+\sqrt{1+4\kappa^2\eta^2}}$; hence,

$$\nu_T = \frac{1}{2}\nu(-1 + \sqrt{1 + 4\kappa^2\eta^2}). \qquad (24)$$

This relation reveals that the turbulent viscosity is not constant but rather increases with η. It is depicted in Fig. 4.
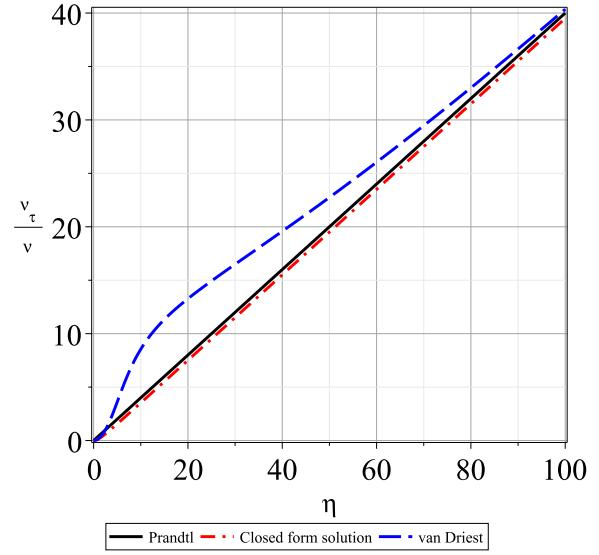
FIG. 4: The turbulent viscosity ratio, the van Driest turbulent viscosity [18] is larger than both the closed form solution's and Prandtl's

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(2) The Reynolds stress is given by

$$\tau'_{xy} = \rho u_\tau^2 \kappa^2 \eta^2 (\frac{du^+}{d\eta})^2 = \frac{4\rho u_\tau^2 \kappa^2 \eta^2}{(1 + \sqrt{1 + 4\kappa^2\eta^2})^2}. \qquad (25)$$

The Reynolds stress ratio $\frac{\tau'_{xy}}{\rho u_\tau^2}$ of the segmental solution and the closed form solution is depicted in Fig. 5.
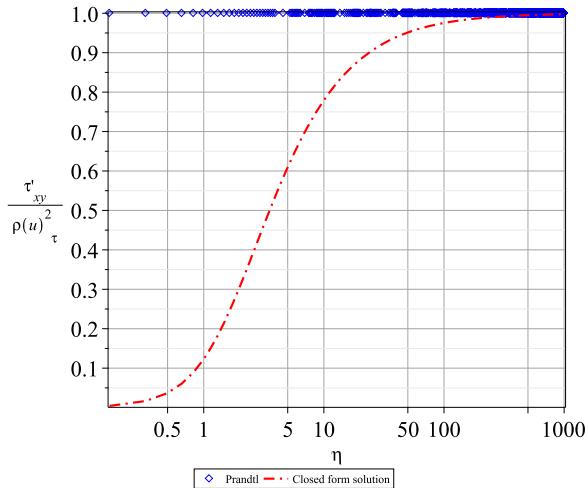
FIG. 5: The Reynolds stress ratio $\frac{\tau'_{xy}}{\rho u_\tau^2}$, generally speaking, the Reynolds stress calculated from the closed form solution increases with the distance from the surface before it becomes a constant, but the segmental solution gives constant Reynolds stress $\frac{\tau'_{xy}}{\rho u_\tau^2} = 1$ for all η.

(3) According to Landau [9], the mean energy flux density is $< q > = \bar{u}\tau_w$, and the energy dissipation density in the turbulent flow is

$$\varepsilon = \frac{1}{\rho}\frac{d < q >}{dy} = \frac{1}{\nu}(\frac{\tau_w}{\rho})^2\frac{du^+}{d\eta} = \frac{u_\tau^4}{\nu}\frac{du^+}{d\eta}$$
$$= \frac{u_\tau^4}{\nu}\frac{2}{1 + \sqrt{1 + 4\kappa^2\eta^2}}, \tag{26}$$

which gives $\lim_{\eta\to 0} \varepsilon = \frac{u_\tau^4}{\nu}$, i.e., a power law u_τ^4; in contrast, Landau's energy dissipation density [9], $\varepsilon_{\text{Landau}} = (\frac{\tau_w}{\rho})^{3/2}\frac{1}{\kappa\eta} = \frac{u_\tau^4}{\nu}\frac{1}{\kappa\eta}$, is infinite at the surface $\eta = 0$, indicating that maintaining turbulent flow requires supplying an infinite energy source, which is physically impossible. Our energy dissipation density and Landau's are depicted in Fig. 6.

(4) According to Kolmogorov [19–22], we have $E(k) = 1.5\varepsilon^{2/3}k^{-5/3}$, namely,

$$E(k) = 1.5\nu^{-2/3}k^{-5/3}u_\tau^{8/3}\left(\frac{2}{1 + \sqrt{1 + 4\kappa^2\eta^2}}\right)^{2/3}, \tag{27}$$

where $E(k)$ is the kinetic energy per unit mass of fluid in eddies with wave number k. Hence, at the surface $\eta = 0$, we have $E(k)_{\eta=0} = 1.5u_\tau^{8/3}k^{-5/3}\nu^{-2/3}$, which is depicted in Fig. 7.

Figure 7 shows that the turbulent energy dissipation is mainly concentrated in a narrow region close to the boundary surface and decays rapidly away from the surface.
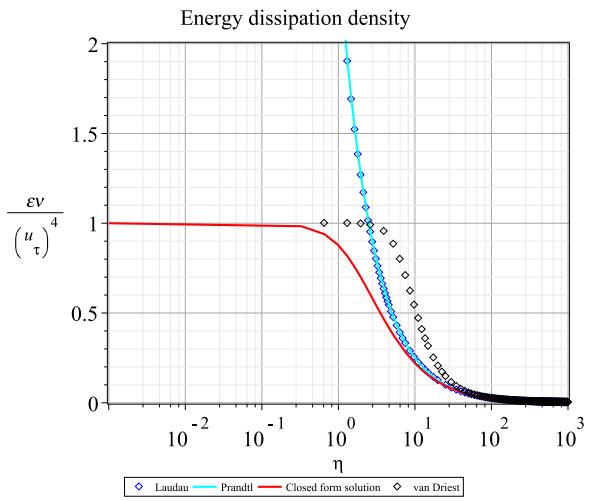
FIG. 6: The energy dissipation density ratio $\frac{\varepsilon\nu}{u_\tau^4}$ decreases to null rapidly as η increases away from the surface.
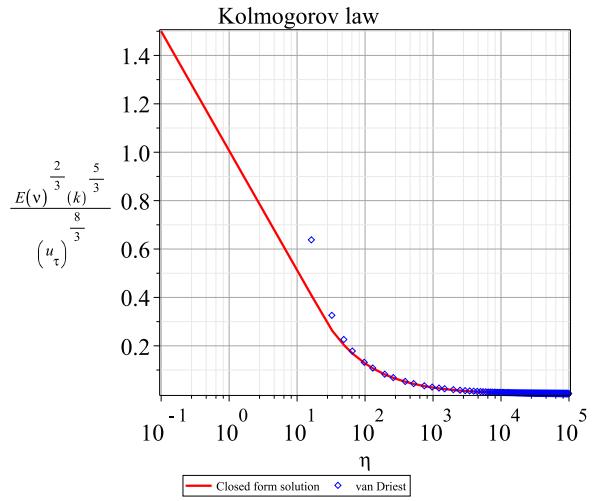
FIG. 7: Kolmogorov's law: Eq. 16 (in red) and van Driest [18] (in blue).

VI.   APPLICATION TO THE FRICTION OF CIRCULAR PIPES

As an application of the solution in Eq. 16, let us now apply Eq. 16 to turbulent flow in a pipe with length l and radius a. The area near the walls of the pipe may be approximately regarded as a plane, and the velocity profile must be given by Eq. 16. The mean velocity $\bar{u}$ of the flow in the pipe is calculated by $\bar{u} = Q/(\rho\pi a^2)$, where Q is the volume of fluid that passes through a cross-section of the pipe per unit time divided by the cross-section area. Since the frictional force per unit area of the wall is $\tau_w = \rho u_\tau^2$, the total frictional force is $2\pi a l \rho u_\tau^2$. Assuming the pressure gradient $\Delta p/l$ maintains the flow, the force on the cross-section is $\pi a^2 \Delta p$. Equating the two forces, namely $\pi a^2 \Delta p = 2\pi a l \rho u_\tau^2$, leads to flow pressure

gradient $\Delta p/l = 2\rho u_\tau^2/a$.

We must point out that the obtained closed form solution of Eq. 16 is valid for turbulent flow with no pressure gradient, namely, $\frac{d}{dy}(p + \rho\overline{v'^2}) = 0$, integrating over the boundary-layer thickness delivers $p + \rho\overline{v'^2} = p_w$, if the outer flow is assumed to be free from turbulence, in which p_w is flow pressure at the wall. In turbulent boundary layers therefore it is not the pressure p which is constant over the boundary layer, but rather the expression $p + \rho\overline{v'^2}$, since the fluctuations vanish at the wall and at the outer edge [8]. Therefore $\Delta p/l = 2\rho u_\tau^2/a \neq 0$, we can still use the exact solution to the pipe because the shear stress near the wall is approximately equal to the wall stress [18].

The dependence of the resistance coefficient is defined as $\lambda = \frac{2a\Delta p/l}{(1/2)\rho\bar{u}^2} = 2a\frac{2\rho u_\tau^2}{a}/(\rho\bar{u}^2/2) = 8(u_\tau/\bar{u})^2 = 8/(u^+)^2$, which gives

$$u^+ = \sqrt{\frac{8}{\lambda}}. \tag{28}$$

The dimensionless Reynolds number is defined as $Re = 2a\bar{u}/\nu = (2au_\tau/\nu)u^+$, hence $\eta|_{y=a} = au_\tau/\nu = a(\bar{u}/u_\tau)/\nu = Re/(2u^+)$ and $2\kappa\eta|_{y=a} = \kappa Re/u^+$. Applying the Eq.28 into the above relation, and leads to

$$2\kappa\eta|_{y=a} = \kappa Re\sqrt{\frac{\lambda}{8}}. \tag{29}$$

The dependence of the resistance coefficient on the dimensionless Reynolds Re number is given in implicit form by the equation

$$\frac{1}{\sqrt{\lambda}} = \frac{1}{\kappa\sqrt{8}}\ln(\kappa Re\sqrt{\frac{\lambda}{8}} + \sqrt{1 + \frac{\kappa^2 Re^2\lambda}{8}})$$
$$- \frac{Re\sqrt{\lambda}}{8}\frac{1}{1 + \sqrt{1 + \kappa^2 Re^2\lambda/8}}. \tag{30}$$

where a is the pipe radius, and $\Delta p/l$ is the pressure gradient. If we set $\kappa = 0.4$, we have

$$\frac{1}{\sqrt{\lambda}} = -\frac{Re\sqrt{\lambda}}{8\left(1 + \sqrt{1 + 0.02\lambda Re^2}\right)}$$
$$+ 0.88\ln\left(0.14Re\sqrt{\lambda} + \sqrt{1 + 0.02\lambda Re^2}\right). \tag{31}$$

Eq. 31 and the Prandtl log-law, $1/\sqrt{\lambda} = 2\log(Re\sqrt{\lambda}) - 0.8$, are depicted in Fig. 8.

VII.   CONCLUSIONS

To the best of the author's knowledge, the solution in Eq. 16 is the first closed form solution of plane turbulent flow ever be obtained in the context of rigorous mathematics. The approximate analytical solution in Eq.22 is also proposed for the first time for the van Driest model. The studies in the paper may help to have a better understanding of turbulence phenomena [23, 24] .
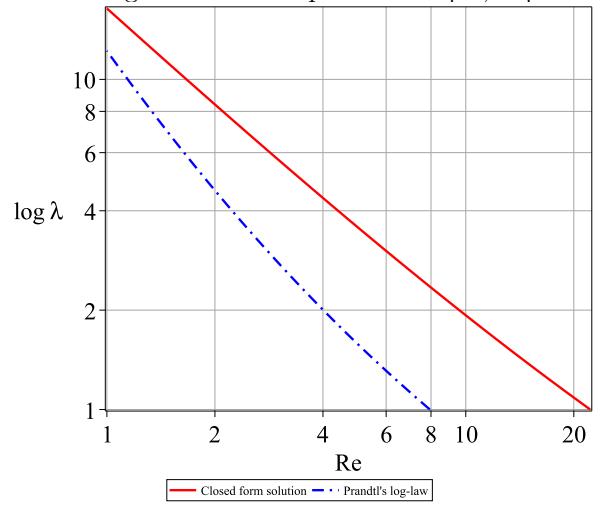
FIG. 8: Pipe resistance coefficient. The Prandtl log-law is represented by a dotted line, and Eq. 31 is represented by a solid line. The resistance coefficient derived from Eq. 31 is higher than that derived from the Prandtl log-law since the closed form solution includes the viscous influence, whereas the Prandtl log-law does not.

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Conflict of interest The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

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