Solutions and Drift Homogenization for a Class of Viscous Lake Equations in $L_2(\Omega)$

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Abstract

In this paper we study solutions and drift homogenization for a class of viscous lake equations by using the method of semigroups of bounded operators. Suppose that the initial value $(t_0, u_0) \in U$, i.e., $u_0 = u(t_0)$ for some Hölder continuous function u on $[0, T]$ with smooth function value $u(t) \in DL_2(\Omega)_{1/2}$ satisfing $\partial_j u_i = 0 (i \neq j)$ and $b(x) \in C^\infty(\Omega)$. Then the initial value problem (2) for viscous lake equations has a unique smooth local strong solution. Using this result we study the drift homogenization for three-dimensional stationary Stokes equation in the usual sense $bDL_2(\Omega)$.

（1）Introduction

The viscous lake equations considered in this paper have the following equations (see the formula (1.3）in [17])

$$\begin{cases} \partial_t(bu^\mu) + div(bu^\mu \otimes u^\mu) + A_b(u^\mu) + b\nabla p^\mu = 0, x \in \Omega, t \in (0, T] \\ div(bu^\mu) = 0, x \in \Omega \end{cases} \tag{1}$$

for $(t, x) \in (0, T] \times \Omega$ with $\Omega \subset R^3$, a bounded domain with smooth boundary $\partial\Omega$ of class C^3, $\mu > 0$ represents the eddy viscosity coefficient, $u^\mu = u^\mu(u_1^\mu(t, x), u_2^\mu(t, x), u_3^\mu(t, x))$ stands for the three-dimensional fluid velocity, $p = p(t, x)$ is the pressure. Moreover, the depth $b(x)$ is a given function. We assume that the boundary is the only place where the depth can vanish,

namely: there is a positive constant M such that $0 < b(x) \le M$ in Ω. A_b is a viscous second order operator depending on b which can satisfy the two expressions

$$(i) A_b(\bullet) = -2\mu div(bD(\bullet) + 2bdiv(\bullet)I), \quad (ii) A_b(\bullet) = -\mu b\Delta(\bullet)$$

where I is the 3×3 identity matrix and $D(\bullet) = (\nabla \bullet + (\nabla \bullet)^t)/2$ is the deformation tensor. The equation (1) shows that the system does not describe incompressible flow, it is a constraint that plays a role similar to that played by the incompressibility condition for the incompressible Navier–Stokes system. We do not have an existence result concerning the solution of viscous shallow water equations with viscosity term given by (i) until now.

We consider in this paper the well posedness of system (1)-(ii) with initial and boundary conditions which is given as follows

$$\begin{cases} \partial_t u^\mu + (u^\mu \bullet \nabla)u^\mu - \mu\Delta u^\mu + \nabla p^\mu = 0, x \in \Omega, t \in (t_0, T] \\ div(bu^\mu) = 0, x \in \Omega \\ u^\mu\big|_{t=t_0} = u_0^\mu, x \in \Omega \\ u^\mu(t, x) = 0, x \in \partial\Omega \end{cases} \tag{2}$$

The existence，uniqueness and regularity properties of solutions for the viscous lake equations are extensively studied. There is an extensive literature on the solvability of the initial value problem for viscous lake equations. The terms and symbols in this paper are the same as [18]. For some narratives and background, please refer to [18].

In case when $b(x)$ is a constant, system (2) becomes similar to the classical 3D incompressible Navier-Stokes equations with the external force $f = 0$ as follows

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla p - (u \bullet \nabla)u, x \in \Omega, t \in (t_0, T] \\ \nabla \bullet u \equiv divu = 0 \\ u\big|_{\partial\Omega} = 0, t \in (t_0, T] \\ u\big|_{t=t_0} = u_0, x \in \Omega \end{cases} \tag{3}$$

Let $L_2(\Omega)$ be the Hilbert space of real vector functions in $L^2(\Omega)$. That is

$$L_2(\Omega) = \left\{ u : \Omega \to R^3, u = (u_1, u_2, u_3), u_i \in L^2(\Omega)(i = 1, 2, 3) \right\}.$$

For $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in L_2(\Omega)$, we define the norm and the inner

$$\|u\|_{L_2(\Omega)} = (\sum_{i=1}^{3} \|u_i\|_{L^2(\Omega)}^2)^{\frac{1}{2}}, \ (u, v) = \sum_{i=1}^{3}(u_i, v_i)$$

then $L_2(\Omega)$ is also a Hilbert space. The set of all real vector functions u such that $div \ bu = 0$

and $u \in C_0^\infty(\Omega)$ is denoted by $C_{0,b}^\infty(\Omega)$. Let $bDL_2(\Omega)$ be the closure of $C_{0,b}^\infty(\Omega)$ in

$L_2(\Omega)$. The set of all real vector functions u such that $div\ u = 0$ and $u \in C_0^\infty(\Omega)$ is denoted

by $C_{0,\sigma}^\infty(\Omega)$. Let $DL_2(\Omega)$ be the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L_2(\Omega)$. In the case when b is a

constant $bDL_2(\Omega) = DL_2(\Omega)$. Similarly to p.270 in [4] we can prove that if $u \in C^\infty(\Omega)$ then

$$u \in bDL_2(\Omega) \text{ if and only if } div\ bu = 0 \text{ in } \Omega \text{ and } u_n = 0 \text{ on } \partial\Omega. \tag{4}$$

(see lemma 1) We will see in lemma 7 that if $b \in C^\infty(\Omega)$ then

$$C_{0,\sigma}^\infty(\Omega) \subseteq C_{0,b}^\infty(\Omega) \subseteq L_2(\Omega), \qquad DL_2(\Omega) \subseteq bDL_2(\Omega) \subseteq L_2(\Omega) = W^{0,2}(\Omega),$$

and

$$\|\bullet\|_{DL_2(\Omega)} = \|\bullet\|_{bDL_2(\Omega)} = \|\bullet\|_{L_2(\Omega)},$$

$$L_2(\Omega) = DL_2(\Omega) \oplus (DL_2(\Omega))^\perp = bDL_2(\Omega) \oplus (bDL_2(\Omega))^\perp.$$

From [4] and [9] we have $(DL_2(\Omega))^\perp = \{\nabla h; h \in W^{1,2}(\Omega)\}$ and $(bDL_2(\Omega))^\perp \supseteq$

$\{b\nabla h; h \in W^{1,2}(\Omega)\}$ (see lemma 2) Let P be the orthogonal projection from $L_2(\Omega)$ onto

$bDL_2(\Omega)$. $A = P\Delta$ is called the Stokes operator. Since $\nabla p = b\nabla h$ has a solution $h \in W^{1,2}(\Omega)$

for $b \in C^\infty(\Omega)$. By applying P to the first equation of (3) and taking account of the other

equations , we are let the following abstract initial value problem, $\mathrm{Pr. II}$

$$\begin{cases} \dfrac{du}{dt} = P\Delta u + Fu, t \in (t_0, T] \\ u\big|_{t=t_0} = u_0, x \in \Omega \end{cases} \tag{5}$$

where $Fu = -P(u \bullet \nabla)u$ and $P\nabla p = 0$. We consider equation（5）in integral form Pr.III

$$u(t) = e^{tP\Delta}u_0 + \int_{t_0}^t e^{(t-s)P\Delta} Fu(s)ds. \tag{6}$$

For $u = (u_1, u_2, u_3) \in L_2(\Omega)$ we define $\Delta u = (\Delta u_1, \Delta u_2, \Delta u_3)$. We take a rectangular

coordinate system. If $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ is a velocity field, define $\nabla u = (\dfrac{\partial u_1}{\partial x_1},$

$\dfrac{\partial u_2}{\partial x_2}, \dfrac{\partial u_3}{\partial x_3})$. Since the operator $-\nabla = -\sum_{i=1}^3 \dfrac{\partial^2}{\partial x_i^2}$ is strongly elliptic of order 2. Theorem7.3.6

in [11] implies that Δ is the infinitesimal generator of an analytic semigroup of contractions on $L^2(\Omega)$ with $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. Hence Δ is also the infinitesimal generator of an analytic semigroup of contraction on $L_2(\Omega)$ with $D(-\Delta) = H_2(\Omega) \cap H_{1,0}(\Omega)$, where $H_2(\Omega)$ and $H_{1,0}(\Omega)$ are the Sobolev spaces of vector value in $H^2(\Omega)$ and $H_0^1(\Omega)$ respectively. We will prove that Δ is also the infinitesimal generator of an analytic semigroup of contraction on $bDL_2(\Omega)$.

(2) Some lemmas

For $u \in L_2(\Omega)$, if $div\ bu = 0$ then u is called b divergence free.

Lemma 1. If $u \in C^\infty(\Omega)$ then $u \in bDL_2(\Omega)$ implies $div\ bu = 0$.

Proof. Suppose that $u = (u_1, u_2, u_3) \in C^\infty(\Omega) \cap bDL_2(\Omega)$. Then there exists a sequence $\{u^n \in C_{0,b}^\infty(\Omega) : n = 1, 2, \ldots\}$ such that $u = \lim_{n\to\infty} u^n$, that is $\lim_{n\to\infty} \|u - u^n\|_{bDL_2(\Omega)} = 0$ uniformly on Ω and $\lim_{n\to\infty} \|u_i - u_i^n\|_{L^2(\Omega)} = 0$ uniformly on Ω for $i = 1, 2, 3$ where $div\ bu^n = 0$. It follows from Theorem 7.16 in [12] that

$$\lim_{n\to\infty} \left[\int_\Omega |u_i(x) - u_i^n(x)|^2\, dx \right]^{\frac{1}{2}} = \left[\int_\Omega \lim_{n\to\infty} |u_i(x) - u_i^n(x)|\, dx \right]^{\frac{1}{2}} = 0$$

And so $\lim_{n\to\infty} |u_i(x) - u_i^n(x)| = 0$ and $\lim_{n\to\infty} u_i^n(x) = u_i(x)$ uniformly on Ω for $i = 1, 2, 3$. (see Theorem 1.39 in [14]) From the proof of Theorem 7.11 in [13] it follows that

$$divbu = \sum_{i=1}^3 \frac{\partial bu_i}{\partial x_i} = \sum_{i=1}^3 \frac{\partial (b \lim_{n\to\infty} u_i^n)}{\partial x_i} = \sum_{i=1}^3 \left(\frac{\partial b}{\partial x_i} \lim_{n\to\infty} u_i^n + b \frac{\partial}{\partial x_i} \lim_{n\to\infty} u_i^n \right)$$

$$= \sum_{i=1}^3 \left[\lim_{n\to\infty} \frac{\partial b}{\partial x_i} u_i^n + b \lim_{x_i \to x_{i0}} \frac{\lim_{n\to\infty} u_i^n(x_i) - \lim_{n\to\infty} u_i^n(x_{i0})}{x_i - x_{i0}} \right]$$

$$= \sum_{i=1}^3 \left[\lim_{n\to\infty} \frac{\partial b}{\partial x_i} u_i^n + b \lim_{n\to\infty} \lim_{x_i \to x_{io}} \frac{u_i^n(x_i) - u_i^n(x_{i0})}{x_i - x_{i0}} \right]$$

$$= \sum_{i=1}^3 \left[\lim_{n\to\infty} \frac{\partial b}{\partial x_i} u_i^n + b \lim_{n\to\infty} \frac{\partial u_i^n}{\partial x_i} \right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^3 \left[\frac{\partial bu_i^n}{\partial x_i} \right]$$

$$= 0$$

This is to say $div\ bu = 0$. $\square$

Lemma 2. $(bDL_2(\Omega))^{\perp} \supseteq \{b\nabla h; h \in W^{1,2}(\Omega)\}.$

Proof. Suppose $u \in C_{0,b}^{\infty}(\Omega)$. Then from integration by parts we have for $h \in W^{1,2}(\Omega)$

$$\left(u, b\nabla h\right) = \sum_{i=1}^{3}(u_i, \frac{b\partial h}{\partial x_i}) = \sum_{i=1}^{3}\int_{\Omega} bu_i \frac{\partial h}{\partial x_i} dx = \int_{\partial\Omega}(\sum_{i=1}^{3} bu_i h)dx - \int_{\Omega}(\sum_{i=1}^{3}\frac{\partial(bu_i)}{\partial x_i}h)dx$$

$$= -\int_{\Omega}(\sum_{i=1}^{3}\frac{\partial(bu_i)}{\partial x_i})hdx = -\int_{\Omega} div(bu)hdx = 0.$$

If $u \in bDL_2(\Omega)$, then there exists a sequence $\{u^n \in C_{0,b}^{\infty}(\Omega): n = 1,2,...\}$ such that $u = \lim_{n\to\infty} u^n$ uniformly on Ω. Then

$$(u, b\nabla h) = (\lim_{n\to\infty} u^n, b\nabla h) = \lim_{n\to\infty}(u^n, b\nabla h) = 0. \quad \square$$

Lemma 3. For every $u \in L_2(\Omega)$, $div\ u = 0$ if and only if $div\ (\lambda I - \Delta)u = 0$ for

$$\lambda \in \Sigma_{\vartheta} = \{\lambda : \vartheta - \pi\ < \arg \lambda < \pi - \vartheta, |\lambda| \geq r\}$$

where $0 < \vartheta < \pi/2$.

Proof. Let $u = (u_1, u_2, u_3) \in L_2(\Omega)$. Then

$$\Delta u = (\frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}, \frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} + \frac{\partial^2 u_2}{\partial x_3^2}, \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_3^2}),$$

$$div(\Delta u) = \frac{\partial^3 u_1}{\partial x_1^3} + \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_1}{\partial x_1 \partial x_3^2} + \frac{\partial^3 u_2}{\partial x_2 \partial x_1^2} + \frac{\partial^3 u_2}{\partial x_2^3} + \frac{\partial^3 u_2}{\partial x_2 \partial x_3^2} + \frac{\partial^3 u_3}{\partial x_3 \partial x_1^2} + \frac{\partial^3 u_3}{\partial x_3 \partial x_2^2} + \frac{\partial^3 u_3}{\partial x_3^3}$$

$$= \frac{\partial^2}{\partial x_1^2}(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}) + \frac{\partial^2}{\partial x_2^2}(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}) + \frac{\partial^2}{\partial x_3^2}(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3})$$

$$= (\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2})(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}) = \Delta(divu).$$

So we have

$$div[(\lambda I - \Delta)u] = (\lambda I - \Delta)(divu). \tag{7}$$

From (7) it is clear that $div\ u = 0$ implies that $div\ (\lambda I - \Delta)u = 0$.

Since $-\Delta$ is a strongly elliptic operator of order 2 on Ω. From Theorem 7.3.2 in [11] it follows that there exist constant $C > 0$, $r \geq 0$ and $0 < \vartheta < \pi/2$ such that

5

$$\|u\|_{L_2(\Omega)} \le \frac{C}{|\lambda|} \|(\lambda I - \Delta)u\|_{L_2(\Omega)} \tag{8}$$

for $u \in D(\Delta) = H_2(\Omega) \cap H_{1,0}(\Omega) \subset L_2(\Omega)$ and

$$\lambda \in \Sigma_\vartheta = \{\lambda : \vartheta - \pi < \arg \lambda < \pi - \vartheta, |\lambda| \ge r\}.$$

From (8) it follows that for every $\lambda \in \Sigma_\vartheta$ the operator $\lambda I - \Delta$ is injective from $D(\Delta)$ into $L_2(\Omega)$. From (7) it follows that $div(\lambda I - \Delta)u = 0$ implies that $div\ u = 0$. $\square$

Lemma 4. (1.5.12 in [6]) Let $\{T(t) : t \ge 0\}$ be a C_0-semigroup on a Banach space X. If Y is a closed subspace of X such that $T(t)Y \subset Y$ for all $t \ge 0$, i.e., if Y is $T(t)_{t \ge 0}$-invariant, then the restrictions

$$T(t)_| := T(t)_{|Y}$$

form a C_0-semigroup $\{T(t)_| : t \ge 0\}$, called the subspace semigroup, on the Banach space Y.

Lemma 5. (Proposition 2.2.3 in [6] Let $(A, D(A))$ be the generator of a C_0-semigroup $\{T(t) : t \ge 0\}$ on a Banach space X and assume that the restricted semigroup (subspace semigroup) $\{T(t)_| : t \ge 0\}$ is a C_0-semigroup on some $(T(t))_{t \ge 0}$-invariant Banach space $Y \to X$. Then the generator of $\{T(t)_| : t \ge 0\}$ is the part $(A_|, D(A_|))$ of A in Y.

Lemma 6. The operator $\Delta_{|bDL_2(\Omega)}$ with $D(\Delta_{|bDL_2(\Omega)}) = \{u \in D(\Delta) \cap bDL_2(\Omega), \Delta u \in bDL_2(\Omega)\}$ is the infinitesimal generator of an analytic semigroup of contractions on $bDL_2(\Omega)$.

Proof. From Theorem 7.3.6 in [11] Δ is the infinitesimal generator of an analytic semigroup of contractions on $L^2(\Omega)$. Then Δ is also the infinitesimal generator of an analytic semigoup of contractions on $L_2(\Omega)$. Let $\{T(t) | t \ge 0\}$ be the restriction of the analytic semigroup generated by Δ on $L_2(\Omega)$ to the real axis. $\{T(t) | t \ge 0\}$ is a C_0 semigroup of contractions by Theorem 7.2.5 and Theorem 3.1.1 in [11]. We have already noted that $bDL_2(\Omega)$ is a closed subspace of $L_2(\Omega)$ and is also a Hilbert space. We want to show that $bDL_2(\Omega)$ is $T(t)_{t \ge 0}$-invariant.

For every $u \in C_{0,b}^\infty(\Omega)$, $div\ bu = 0$ and $\lambda \in \rho(\Delta) \cap \Sigma_\vartheta = \{\lambda : \vartheta - \pi < \arg \lambda < \pi - \vartheta, |\lambda| \ge r\}$ we have $(\lambda I - \Delta)[R(\lambda : \Delta)bu] = bu$ where Σ_ϑ is the same as in the proof of

lemma 3. Since $bR(\lambda:\Delta)u = b\int_0^\infty e^{-\lambda t}T(t)u\,dt = \int_0^\infty e^{-\lambda t}T(t)b(x)u\,dt = R(\lambda:\Delta)bu.$ From

Lemma 3 it follows that $div\ bR(\lambda:\Delta)u = div(R(\lambda:\Delta)bu) = 0$. Since $R(\lambda:\Delta)$ is bounded

and so is continuous . Hence $R(\lambda,\Delta)u \in C_0^\infty$. That is to say that $C_{0,b}^\infty$ is $R(\lambda:\Delta)$-invariant for

$\lambda \in \rho(\Delta) \cap \Sigma_g$. From Theorem 2.5.2 (c) in [11] it follows that $\rho(\Delta) \supset R^+$, and so

$\rho(\Delta) \cap \Sigma_g \supset \{\lambda : \lambda \geq r\}$. Hence $C_{0,b}^\infty$ is $R(\lambda:\Delta)$ -invariant for every $\lambda \geq r$. Let

$u \in bDL_2(\Omega)$ then there exists a sequence u_n such that $\lim_{n\to\infty} u_n = u$ and $u_n \in C_{0,b}^\infty$ for

$n = 1,2,....$ Hence $R(\lambda,\Delta)u_n \in C_{0,b}^\infty$ and

$$\lim_{n\to\infty} R(\lambda:\Delta)u_n = R(\lambda:\Delta)u.$$

Therefore $R(\lambda:\Delta)u \in bDL_2(\Omega)$ for every $\lambda \geq r$. It follows that $bDL_2(\Omega)$ is $R(\lambda:\Delta)$

-invariant for every $\lambda \geq r$. Now the Theorem 4.5.1 in [11] implies that $bDL_2(\Omega)$ is $T(t)_{t\geq0}-$

invariant. From Lemma 4 and Lemma 5 it follows that $\Delta_{|bDL_2(\Omega)}$ is the infinitesimal generator of

the C_0 semigroup $\{T(t)_{|bDL_2(\Omega)} : t \geq 0\}$ of contractions on $bDL_2(\Omega)$.

We will prove that $\{T(t)_{|bDL_2(\Omega)} : t \geq 0\}$ can also be extended to an analytic semigroup on

$bDL_2(\Omega)$. Suppose that $\lambda \in \rho(\Delta)$, i.e. there exists $R(\lambda:\Delta)$ from $L_2(\Omega)$ into $D(\Delta)$.

Then for any $u \in bDL_2(\Omega) \subset L_2(\Omega)$, we have

$$(\lambda I - \Delta)R(\lambda:\Delta)u = u \text{ and } R(\lambda:\Delta)(\lambda I - \Delta)u = u. \tag{9}$$

Thus the formula (9) becomes

$$(\lambda I - \Delta_{|bDL_2(\Omega)})R(\lambda:\Delta)_{|bDL_2(\Omega)}u = u \text{ and } R(\lambda:\Delta)_{|bDL_2(\Omega)}(\lambda I - \Delta_{|bDL_2(\Omega)})u = u.$$

Hence $(\lambda I - \Delta_{|bDL_2(\Omega)})R(\lambda:\Delta)_{|bDL_2(\Omega)} = I$ and $R(\lambda:\Delta)_{|bDL_2(\Omega)}(\lambda I - \Delta_{|bDL_2(\Omega)}) = I.$ We get

$$(\lambda I - \Delta_{|bDL_2(\Omega)})^{-1} = R(\lambda:\Delta)_{|bDL_2(\Omega)}. \tag{10}$$

From the formula (10) and Theorem 2.5.2(c) in [11] we have

$$\rho(\Delta_{|bDL_2(\Omega)}) \supset \rho(\Delta) \supset \Sigma = \{\lambda : |\arg \lambda| \langle \pi/2 + \delta\} \cup \{0\}$$

where $0 < \delta < \pi/2$. Thus, for $\lambda \in \Sigma$, $\lambda I - \Delta_{|bDL_2(\Omega)}$ is invertible. From Theorem 2.5.2(c) in

[11] we have for $\lambda \in \Sigma, \lambda \neq 0$

$$\left\| R(\lambda : \Delta_{|bDL_2(\Omega)} \right\| = \operatorname*{Sup}_{\substack{u \in bDL_2(\Omega) \\ \|u\|_{bDL_2(\Omega)} = 1}} \left\| R(\lambda : \Delta_{|bDL_2(\Omega)} |u \right\|$$

$$\leq \operatorname*{Sup}_{\substack{u \in L_2(\Omega) \\ \|u\|_{L_2(\Omega)} = 1}} \left\| R(\lambda : \Delta)u \right\| = \left\| R(\lambda, \Delta) \right\|_{B(L_2(\Omega), L_2(\Omega))} \leq \frac{M}{|\lambda|}.$$

Now Theorem 2.5.2(c) in [11] implies that $\left\{ T(t)_{|bDL_2(\Omega)} : \lambda \geq 0 \right\}$ can also be extended to an

analytic semigroup on $bDL_2(\Omega)$. Therefor $\Delta_{|bDL_2(\Omega)}$ is a infinitesimal generator of an analytic

semigroup of contraction on $bDL_2(\Omega)$. We always denote $\Delta_{|bDL_2(\Omega)}$ by Δ in the follows. □

Since $bDL_2(\Omega) = DL_2(\Omega)$ in the case b is a constant. So we have

Corollary 1. The operator Δ is the infinitesimal generator of an analytic semigroup of

contractions on $DL_2(\Omega)$.

Theorem 2 in [9] is similar to the above Lemma 6. Sobolevskii proved the fact that $A_p =$

$P\Delta$ generates an analytic semigroup on $L_\sigma^2(\Omega) = DL_2(\Omega)$ in [17]. Giga gave a different proof

in [9]. Our proof using the theory of semigroups of bounded operators is more simple.

Lemma 7. Suppose that $b \in C^\infty(\Omega)$ then $DL_2(\Omega) \subseteq bDL_2(\Omega)$.

Proof. If $b \in C^\infty(\Omega)$ then $b \in W^{1,2}(\Omega), \nabla b \in (DL_2(\Omega))^\perp$, so $(u, \nabla b) = 0$ for every

$u \in C_{0,\sigma}^\infty(\Omega) \subset DL_2(\Omega)$, that is

$$\int_\Omega \left(\sum_{i=1}^3 u_i \frac{\partial b}{\partial x_i} \right) dx = 0.$$

So $\sum_{i=1}^3 u_i \frac{\partial b}{\partial x_i} = 0$ for every $u \in C_{0,\sigma}^\infty$. Hence

$$div[bu] = \sum_{i=1}^3 \frac{\partial [bu_i]}{\partial x_i} = \sum_{i=1}^3 \frac{\partial b}{\partial x_i} u_i + b \sum_{i=1}^3 \frac{\partial u_i}{\partial x_i} = 0.$$

So $u \in C_{0,b}^\infty(\Omega)$ and $C_{0,\sigma}^\infty(\Omega) \subset C_{0,b}^\infty(\Omega)$. Therefore

$$DL_2(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)} \subseteq \overline{C_{0,b}^\infty(\Omega)} = bDL_2(\Omega). \ \square$$

8

Suppose that $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ on a Banach space X. From the results of section 2.6 in [11] we can define the fraction powers A^{α} for $0 \leq \alpha \leq 1$ and A^{α} is a closed linear invertible operator with domain $D(A^{\alpha})$ dense in X. $D(A^{\alpha})$ equipped with the norm $\|x\|_{\alpha} = \|A^{\alpha}x\|$ is a Banach space denoted by X_{α}. It is clear that $0 < \alpha < \beta$ implies $X_{\alpha} \supset X_{\beta}$ and that the embedding of X_{β} into X_{α} is continuous.

If $-A = \Delta$ and $\gamma > \frac{1}{2}$ then $X_{\gamma} \subset X_{\frac{1}{2}}$ and $D\left((-\Delta)^{\gamma}\right) \subset D((-\Delta)^{\frac{1}{2}}) \subset DL_2(\Omega)$, the embedding of $D((-\Delta)^{\gamma})$ into $D((-\Delta)^{\frac{1}{2}})$ is continuous. If a sequence $u_n (n = 1, 2, ...) \in D((-\Delta)^{\gamma})$, $\lim_{n\to\infty}\|u_n\|_{DL_2(\Omega)_{\gamma}} = 0$, then $\lim_{n\to\infty}\|u_n - 0\|_{DL_2(\Omega)_{\gamma}} = 0$, that is $\lim_{n\to\infty} u_n = 0$ in $D((-\Delta)^{\gamma})$, and so $\lim_{n\to\infty} u_n = 0$ in $D((-\Delta)^{\frac{1}{2}})$. It follows from Proposition 1.17 in [3] that $\lim_{n\to\infty}\|u_n\|_{DL_2(\Omega)_{\frac{1}{2}}} = 0$. Hence $\|u_n\|_{DL_2(\Omega)_{\gamma}} \to 0$ always implies $\|u_n\|_{DL_2(\Omega)_{\frac{1}{2}}} \to 0$. We can prove that convergence in $(D((-\Delta)^{\gamma}), \|\bullet\|_{DL_2(\Omega)_{\gamma}})$ implies convergence in $(D((-\Delta)^{\frac{1}{2}}), \|\bullet\|_{DL_2(\Omega)_{1/2}})$ and conversely, that is the norms $\|\bullet\|_{DL_2(\Omega)_{\gamma}}$ and $\|\bullet\|_{DL_2(\Omega)_{\frac{1}{2}}}$ in $D((-\Delta)^{\gamma})$ are equivalent (see p291 Problem 8 in [7]）, therefore there exists $L_0 > 0$ such that for any $u \in D((-\Delta)^{\gamma})$

$$\left\|(-\Delta)^{\gamma}u\right\|_{DL_2(\Omega)} = \|u\|_{DL_2(\Omega)_{\gamma}} \leq L_0 \|u\|_{DL_2(\Omega)_{1/2}} \quad . \tag{11}$$

For $u \in D(\Delta)$ we have

$$\|\nabla u\|_{DL_2(\Omega)} = \|-\nabla u\|_{DL_2(\Omega)} = \left\|(-\Delta)^{1/2}u\right\|_{DL_2(\Omega)} = \|u\|_{DL_2(\Omega)_{1/2}}. \tag{12}$$

In [9] Giga proved the following result. Suppose that $1 < r < \infty, n \geq 2$.

Lemma 8. Let $0 \leq \delta < \frac{1}{2}\left|-n(1-r^{-1})/2\right|$. Then

$$\left\|A^{-\delta}P(u, \nabla)v\right\|_{0,r} \leq M\left\|A^{\vartheta}u\right\|_{0,r}\left\|A^{\omega}v\right\|_{0,r}$$

with some constant $M = M(\delta, \vartheta, \omega, r)$, provided $\delta + \vartheta + \omega \geq n/2r + 1/2$, $\vartheta > 0, \omega > 0$, $\omega + \delta > 1/2$.

From the Lemma 8 and the formulas (11) (12) we see that if take $n = 3, r = 2$,

$\delta = 0$, $\vartheta = 3/4$ and $\omega = 3/4$, then

$$\left\| (u \bullet \nabla) v \right\|_{DL_2(\Omega)} = \left\| (u \bullet \nabla) \; v \right\|_{L_2(\Omega)} = \left\| (u \bullet \nabla) v \right\|_{W^{0,2}(\Omega)}$$

$$\leq M \left\| (-\Delta)^{\vartheta} u \right\|_{DL_2(\Omega)} \left\| (-\Delta)^{\omega} v \right\|_{DL_2(\Omega)}$$

$$\leq M L_0^2 \left\| u \right\|_{DL_2(\Omega)_{\frac{1}{2}}} \left\| v \right\|_{DL_2(\Omega)_{\frac{1}{2}}}$$

with some constant M for any $u, v \in DL_2(\Omega)$. Hence we have

Lemma 9. Suppose that u, $v \in DL_2(\Omega)$ are velocity fields and $(u \bullet \nabla) v \in DL_2(\Omega)$, then

$$\left\| (u \bullet \nabla) v \right\|_{DL_2(\Omega)} \leq M L_0^2 \left\| u \right\|_{DL_2(\Omega)_{\frac{1}{2}}} \left\| v \right\|_{DL_2(\Omega)_{\frac{1}{2}}}.$$

Assumption (F). Let $X = bDL_2(\Omega)$ and U be an open subset in $R^+ \times X_\alpha$ ($0 < \alpha < 1$). The function $f : U \to X$ satisfies the assumption (F) if for every $(t, u) \in U$ there is a neighborhood $V \subset U$ and constants $L \geq 0$, $0 < \vartheta \leq 1$ such that for all $(t_i, u_i) \in V (i = 1,2)$

$$\left\| f(t_1, u_1) - f(t_2, u_2) \right\| \leq L(\left| t_1 - t_2 \right|^{\vartheta} + \left\| u_1 - u_2 \right\|_\alpha). \tag{13}$$

Lemma 10. (Theorem 6.3.1 in [11]) Let $-A$ be the generator of an analytic semigroup $T(t)$ on the Banach space $X = bDL_2(R^3)$ satisfying $\left\| T(t) \right\| \leq M$ and assume that $0 \in \rho(-A)$. If, $0 < \alpha < 1$ and f satisfies the assumption (F) then for every initial date $(t_0, u_0) \in U$ the initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) = f(t, u(t)), t \in (t_0, T] \\ \qquad u(t_0) = u_0 \end{cases} \tag{14}$$

has a unique local solution $u \in C([t_0, t_1] : X) \cap C^1((t_0, t_1) : X)$ where $t_1 = t_1(t_0, u_0) > t_0$.

In what follows we will need Banach lattice (see [1]). A real vector space G which is ordered by some order relation $\leq$ is called a vector lattice (or Riesz space) if any two elements $f, g \in G$ have a least upper bound, denoted by $f \vee g$, and a greatest lower bound, denoted by $f \wedge g$, and the following properties are satisfied:

(i) If $f \leq g,$ then $f + h \leq g + h$ for all $f, g, h \in G,$

(ii) If $0 \leq f$, then $0 \leq tf$ for all $f \in G$ and $0 \leq t \in R.$

A Banach lattice is a real Banach space G endowed with an ordering $\leq$ such that $(G, \leq)$ is a vector lattice and the norm is a lattice norm, that is $|f| \leq |g|$ implies $\|f\| \leq \|g\|$ for $f, g \in G,$ where $|f| = f \vee (-f)$ is the absolute value of f and $\|\bullet\|$ is the norm in $G.$ In a Banach lattice G we define for $f \in G$

$$f^{+} := f \vee 0, \qquad f^{-} := (-f) \vee 0.$$

The absolute value of f is $|f| = f^{+} + f^{-}$ and $f = f^{+} - f^{-}.$

$$0 \leq f \leq g \Rightarrow |f| \leq |g| \Rightarrow \|f\| \leq \|g\|.$$

In what follows we will need the above formula .

In $L^{2}(\Omega)$ we define the order for $f, g \in L^{2}(\Omega)$

$$f \leq g \Leftrightarrow f(x) \leq g(x) \text{ for } a.e. \ x \in \Omega$$

and $(f \vee g)(x) := \max\{f(x), g(x)\}, \quad (f \wedge g)(x) := \min\{f(x), g(x)\}$

for $a.e.$ $x \in \Omega.$ Then $W^{1,2}(\Omega), L^{2}(\Omega), \ L_{2}(\Omega)$ and $DL_{2}(\Omega)$ are all Banach lattices. (see [1] p.148)

Lemma11. Suppose that $u, v \in DL_{2}(\Omega)$ are divergence free satisfying $\dfrac{\partial u_{i}}{\partial x_{j}} = 0, \dfrac{\partial v_{i}}{\partial x_{j}} = 0$

$(i \neq j).$ Then $(u \bullet \nabla)v, \ (v \bullet \nabla)u \in DL_{2}(\Omega).$

Proof. If $u, v \in DL_{2}(\Omega)$ are divergence free satisfying $\dfrac{\partial u_{i}}{\partial x_{j}} = 0, \dfrac{\partial v_{i}}{\partial x_{j}} = 0 (i \neq j).$ From [4] and [8]we have

$$\int_{\Omega} u \bullet \nabla h dx = 0, \int_{\Omega} v \bullet \nabla h dx = 0 \text{ for all } h \in W^{1,2}(\Omega).$$

That is $\int_{\Omega} \sum_{i=1}^{3} u_{i} \dfrac{\partial h}{\partial x_{i}} dx = 0.$ Since $L_{2}(\Omega)$ is a Banach lattice and $\nabla h \in DL_{2}(\Omega)^{\perp}$ for

$h \in W^{1,2}(\Omega),$ then $u = u^{+} - u^{-}, u^{+}, u^{-} \in DL_{2}(\Omega), \nabla h = (\nabla h)^{+} - (\nabla h)^{-}.$ From proposition 10.8

in [1] the lattice operations $\wedge$ and $\vee$ are continuous, we have

$$
(\nabla h)^+ = \begin{pmatrix} \dfrac{\partial h}{\partial x_1} \\[2mm] \dfrac{\partial h}{\partial x_2} \\[2mm] \dfrac{\partial h}{\partial x_3} \end{pmatrix}^+ = \begin{pmatrix} (\dfrac{\partial h}{\partial x_1})^+ \\[2mm] (\dfrac{\partial h}{\partial x_2})^+ \\[2mm] (\dfrac{\partial h}{\partial x_3})^+ \end{pmatrix} = \begin{pmatrix} (\lim_{x_1 \to x_{10}} \dfrac{h(x_1,x_2,x_3)-h(x_{10},x_2,x_3)}{x_1-x_{10}})^+ \\[2mm] (\lim_{x_2 \to x_{20}} \dfrac{h(x_1,x_2,x_3)-h(x_1,x_{20},x_3)}{x_2-x_{20}})^+ \\[2mm] (\lim_{x_3 \to x_{30}} \dfrac{h(x_1,x_2,x_3)-h(x_1,x_2,x_{30})}{x_3-x_{30}})^+ \end{pmatrix}
$$

$$
= \begin{pmatrix} \lim_{x_1 \to x_{10}} \dfrac{h^+(x_1,x_2,x_3)-h^+(x_{10},x_2,x_3)}{x_1-x_{10}} \\[2mm] \lim_{x_2 \to x_{20}} \dfrac{h^+(x_1,x_2,x_3)-h^+(x_1,x_{20},x_3)}{x_2-x_{20}} \\[2mm] \lim_{x_3 \to x_{30}} \dfrac{h^+(x_1,x_2,x_3)-h^+(x_1,x_2,x_{30})}{x_3-x_{30}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial h^+}{\partial x_1} \\[2mm] \dfrac{\partial h^+}{\partial x_2} \\[2mm] \dfrac{\partial h^+}{\partial x_3} \end{pmatrix} = \nabla(h^+),
$$

Similarly, $(\nabla h)^- = \nabla(h^-)$. $W^{1,2}(\Omega) \subset W^{0,2}(\Omega) = L_2(\Omega)$ are all Banach spaces. $h \in W^{1,2}(\Omega)$

implies $h^+, h^- \in W^{1,2}(\Omega)$, $\int_\Omega u^+ \bullet \nabla(h^+)dx = 0$. Since v is divergence free, $\sum_{i=1}^3 \dfrac{\partial v_i}{\partial x_i} = 0$,

so $\dfrac{\partial v_i}{\partial x_i}$ are all bounded, $\left| \dfrac{\partial v_i}{\partial x_i} \right| \leq L$ (i = 1,2,3) for some constant $L > 0$. We have

$$
0 = -L\int_\Omega \sum_{i=1}^3 u_i^+ (\dfrac{\partial h}{\partial x_i})^+ dx \leq \int_\Omega (u^+ \bullet \nabla)v \bullet (\nabla h)^+ dx
$$

$$
= \int_\Omega \sum_{i=1}^3 u_i^+ \dfrac{\partial v_i}{\partial x_i}(\dfrac{\partial h}{\partial x_i})^+ dx \leq L\int_\Omega \sum_{i=1}^3 u_i^+ \dfrac{\partial(h^+)}{\partial x_i} dx = 0
$$

Hence $\int_\Omega (u^+ \bullet \nabla)v \bullet (\nabla h)^+ dx = 0$. Similarly, $\int_\Omega (u^+ \bullet \nabla)v \bullet (\nabla h)^- dx = 0$. So we have

$$
\int_\Omega (u^+ \bullet \nabla)v \bullet \nabla h dx = \int_\Omega (u \bullet \nabla)v \bullet [(\nabla h)^+ - (\nabla h)^-]dx = 0.
$$

Similarly, $\int_\Omega (u^- \bullet \nabla)v \bullet \nabla h dx = 0$. Therefore

$$
\int_\Omega (u \bullet \nabla)v \bullet \nabla h dx = \int_\Omega ((u^+ - u^-) \bullet \nabla)v \bullet \nabla h dx = \int_\Omega (u^+ \bullet \nabla)v \bullet \nabla h dx - \int_\Omega (u^- \bullet \nabla)v \bullet \nabla h dx = 0
$$

and so $(u \bullet \nabla)v \in DL_2(\Omega)$. Similarly $(v \bullet \nabla)u \in DL_2(\Omega)$. $\square$

(3)The solutions of lake equations

Now we study the viscous lake equations (2). In the following proof of Theorem 1 we will need valuation for nonlinear term $(u \bullet \nabla)u$. Lemma 9 gives a good valuation for $(u \bullet \nabla)u$ in $DL_2(\Omega)$. But we do not know that whether this valuation is established in $bDL_2(\Omega)$. From lemma 7 we see that if $b \in C^\infty(\Omega)$ then $DL_2(\Omega) \subseteq bDL_2(\Omega)$. Therefore we can consider the solution of (2) in the state space $DL_2(\Omega)$ and use lemma 9.

A function u which is differentiable almost everywhere on $\left[0, \ T\right]$ such that $u^{\cdot} \in L^1\left[0, T : DL_2(\Omega)\right]$ is called a strong solution of the initial value problem (2) if $u(0) = u_0$ and u satisfies (2).

Let $H\left(\left[0, T\right]; DL_2(\Omega)_{1/2}\right)$ denote the space of all Hölder continuous functions $u(t)$ on $\left[0, T\right]$ with different exponents in $(0,1]$ and with smooth functions values $u(t)$ satisfying $\dfrac{\partial u_i}{\partial x_j} = 0$ $(i \neq j)$ in the Banach space $DL_2(\Omega)_{1/2}$. From the formula (4) we see that these functions values are all divergence free. Then from lemma 11 for any $u \in H\left(\left[0, T\right]; DL_2(\Omega)_{1/2}\right)$ and any $t_1, t_2 \in \left[0, T\right]$, $(u(t_1) \bullet \nabla)u(t_2) \in DL_2(\Omega)$; and for any $u_1, u_2 \in H\left(\left[0, T\right]; DL_2(\Omega)_{1/2}\right)$ and any $t \in \left[0, T\right]$, $(u_1(t) \bullet \nabla)u_2(t) \in DL_2(\Omega)$. In the following we will use these facts. The bilinear form $(v(t) \bullet \nabla)u(t)$ on $DL_2(\Omega)_{1/2}$ takes value in $DL_2(\Omega)$.

Let $u_k(t, x) = (k_1, k_2, k_3)$ $(t \in \left[0, T\right], \ x \in \Omega, \ k_i \in R, i = 1,2,3))$. Then $u_k \in H\left(\left[0, T\right]\right.$ $\left.; DL_2(\Omega)_{1/2}\right)$ for all $k_1, k_2, k_3 \in R$ and all $t \in \left[0, T\right]$. Suppose that $u(x) = (u_1(x_1), u_2(x_2), u_3(x_3)) \in DL_2(\Omega)_{1/2}$ with smooth $u_i(x_i) (i = 1,2,3)$. Let $u(t, x) \equiv u(x)$ for $t \in \left[0, T\right]$. Then $u(t, x) \in H\left(\left[0, T\right]; DL_2(\Omega)_{1/2}\right)$. Hence $H\left(\left[0, T\right]; DL_2(\Omega)_{1/2}\right)$ is not empty. Consider the graph

$$G := \left\{ (t, u(t)) : t \in (0, T), \forall u \in H\left(\left[0, T\right]; DL_2(\Omega)_{1/2}\right) \right\}$$

1

Take the open kernel $U = G^0$ of G in $(0,T) \times DL_2(\Omega)_{1/2}$. Then U is an open subset of

$(0,T) \times DL_2(\Omega)_{1/2}$. It is clear that G is not empty, and so U is also not empty. $F(t, u(t))$

$= -(u \bullet \nabla)u$ is a function : $U \to DL_2(\Omega)$.

Theorem 1. The initial value problem (2) for viscous lake equations has a unique local strong

solution if the initial value $(t_0, u_0) \in U$, i.e., $u_0 = u(t_0)$ for some Hölder continuous function u

on $[0,T]$ with smooth function value $u(t) \in DL_2(\Omega)_{1/2}$ satisfying $\partial_j u_i = 0 (i \neq j)$ and $b(x) \in$

$C^\infty(\Omega)$.

Proof. We will find that by incorporating the divergence-free condition, we can remove the

pressure term from our equation. (see p. 271^3 in [4], p. 234_6 and p. 239_9 in [12]) In fact, from

$DL_2(\Omega)^\perp = \left\{ \nabla h; h \in W^{1,2}(\Omega) \right\}$ we see that $\nabla p \in DL_2(\Omega)^\perp$ and so $P\nabla p = 0$. For $u \in$

$DL_2(\Omega)$ we have $\Delta u \in DL_2(\Omega)$ because to Lemma 6. Hence by applying P to the equation (2)

we have $P\Delta u = \Delta u$. It follows from $(u \bullet \nabla)u \in DL_2(\Omega)$ that $P(u \bullet \nabla)u = (u \bullet \nabla)u$.

Therefore we can first rewrite (2) into an abstract initial value problem on $DL_2(\Omega)$

$$
\begin{cases}
\dfrac{du}{dt} = \Delta u + F(t, u(t)), t \in (t_0, T] \\
\quad u\big|_{t=t_0} = u_0, x \in \Omega
\end{cases}
\tag{15}
$$

where $F(t, u(t)) = -(u \bullet \nabla)u$ is an abstract function. From corollary 1 Δ is the infinitesimal

generator of an analytic semigroup $T(t)$ of contraction on $DL_2(\Omega)$ and $\|T(t)\| \leq 1, 0 \in \rho(\Delta)$.

If $u(t)$ is Hölder continuous about t on $[t_0, T]$ in $DL_2(\Omega)_{1/2}$, then there is a constant C

and $0 < \beta \leq 1$ such that

$$
\left\| u(t_1, x) - u(t_2, x) \right\|_{DL_2(R^3)_{1/2}} \leq C \left| t_1 - t_2 \right|^\beta \quad \text{for} \quad t_1, t_2 \in [t_0, T]
\tag{16}
$$

For any $(t_1, u_1(t_1)), (t_2, u_2(t_2)) \in U$ we have $(u_1(t_1) \bullet \nabla)u_1(t_1), (u_2(t_1) \bullet \nabla)u_2(t_1) \in DL_2(\Omega)$

and

$$\left\|(u_1(t_1)\bullet\nabla)u_1(t_1)-(u_2(t_1)\bullet\nabla)u_2(t_1)\right\|_{DL_2(\Omega)}$$

$$\leq\left\|(u_1(t_1)\bullet\nabla)u_1(t_1)-(u_1(t_1)\bullet\nabla)u_2(t_1)\right\|_{DL_2(\Omega)}$$

$$+\left\|(u_1(t_1)\bullet\nabla)u_2(t_1)-(u_2(t_1)\bullet\nabla)u_2(t_1)\right\|_{DL_2(\Omega)}$$

$$=\left\|(u_1(t_1)\bullet\nabla)(u_1(t_1)-u_2(t_1))\right\|_{DL_2(\Omega)}$$

$$+\left\|\left[(u_1(t_1)-u_2(t_1))\bullet\nabla\right]u_2(t_1)\right\|_{DL_2(\Omega)}$$

$$\leq ML_0^2\left\|u_1(t_1)\right\|_{DL_2(\Omega)_{1/2}}\left\|(u_1(t_1)-u_2(t_1))\right\|_{DL_2(\Omega)_{1/2}}$$

$$+\left\|u_1(t_1)-u_2(t_1)\right\|_{DL_2(\Omega)_{1/2}}\left\|u_2(t_2)\right\|_{DL_2(\Omega)_{1/2}})$$

$$=ML_0^2\left(\left\|u_1(t_1)\right\|_{DL_2(\Omega)_{1/2}}+\left\|u_2(t_2)\right\|_{DL_2(\Omega)_{1/2}}\right)\left\|u_1(t_1)-u_2(t_1)\right\|_{DL_2(\Omega)_{1/2}}$$

$$\leq ML_0^2\left(\left\|u_1(t_1)\right\|_{DL_2(\Omega)_{1/2}}+\left\|u_2(t_2)\right\|_{DL_2(\Omega)_{1/2}}\right)$$

$$\left(\left\|u_1(t_1)-u_2(t_2)\right\|_{DL_2(\Omega)_{1/2}}+\left\|u_2(t_2)-u_2(t_1)\right\|_{DL_2(\Omega)_{1/2}}\right)$$

$$\leq ML_0^2\left(\left\|u_1(t_1)\right\|_{DL_2(\Omega)_{1/2}}+\left\|u_2(t_2)\right\|_{DL_2(\Omega)_{1/2}}\right) \qquad (17)$$

$$\left(\left\|u_1(t_1)-u_2(t_2)\right\|_{DL_2(\Omega)_{1/2}}+C_1\left|t_1-t_2\right|^{\beta_1}\right)$$

$$=2ML_0^2L\left\|u_1(t_1)-u_2(t_2)\right\|_{DL_2(\Omega)_{1/2}}+2ML_0^2LC_1\left|t_1-t_2\right|^{\beta}.$$

We used lemma 9 in the above third step . For any $u(t_1),u(t_2)\in DL_2(\Omega)_{1/2}$ we have

$$\left\|(u(t_1)\bullet\nabla)u(t_1)-(u(t_2)\bullet\nabla)u(t_2)\right\|_{DL_2(\Omega)}$$

$$\leq\left\|(u(t_1)\bullet\nabla)u(t_1)-(u(t_1)\bullet\nabla)u(t_2)\right\|_{DL_2(\Omega)}+\left\|(u(t_1)\bullet\nabla)u(t_2)-(u(t_2)\bullet\nabla)u(t_2)\right\|_{DL_2(\Omega)}$$

$$=\left\|(u(t_1)\bullet\nabla)(u(t_1)-u(t_2))\right\|_{DL_2(\Omega)}+\left\|\left[(u(t_1)-u(t_2))\bullet\nabla\right]u(t_2)\right\|_{DL_2(\Omega)}$$

$$\leq ML_0^2(\left\|u(t_1)\right\|_{DL_2(\Omega)_{1/2}}\left\|(u(t_1)-u(t_2))\right\|_{DL_2(\Omega)_{1/2}}+\left\|u(t_1)-u(t_2)\right\|_{DL_2(\Omega)_{1/2}}\left\|u(t_2)\right\|_{DL_2(\Omega)_{1/2}})$$

$$=ML_0^2(\left\|u(t_1)\right\|_{DL_2(\Omega)_{1/2}}+\left\|u(t_2)\right\|_{DL_2(\Omega)_{1/2}})\left\|u(t_1)-u(t_2)\right\|_{DL_2(\Omega)_{1/2}}$$

$$\leq ML_0^2C(\left\|u(t_1)\right\|_{DL_2(\Omega)_{1/2}}+\left\|u(t_2)\right\|_{DL_2(\Omega)_{1/2}})\left|t_1-t_2\right|^{\beta}. \qquad (18)$$

We used the Lemma 9 in the above third step and the formula (16) in fifth step.

Suppose that $(t_0,u_0)\in U$. Set

$$V=B_\varepsilon(t_0,u_0)=\left\{(t,u(t))\in U:\left|t-t_0\right|\langle\varepsilon\leq1,\left\|u-u_0\right\|_{DL_2(\Omega)_{1/2}}\langle\varepsilon\right\}.$$

Then for $(t,u(t))\in V$,

$$\|u\|_{DL_2(\Omega)_{1/2}} = \|u - u_0 + u_0\|_{DL_2(\Omega)_{1/2}} \le \|u - u_0\|_{DL_2(\Omega)_{1/2}} + \|u_0\|_{DL_2(\Omega)_{1/2}} \le \varepsilon + \|u_0\|_{DL_2(\Omega)_{1/2}}.$$

Let $\quad L = \varepsilon + \|u_0\|_{DL_2(\Omega)_{1/2}}, \quad L_1 = 2ML_0^2 L, \quad L_2 = 2ML_0^2 L(C_1 + C), \quad L_3 = Max(L_2, L_1) \quad$ and

$\beta_2 = Min(\beta, \beta_1),$ then from (16),(17) and (18) for all $(t_i, u_i) \in V (i=1,2)$ we have

$$\left\| F(t_1, u_1(t_1)) - F(t_2, u_2(t_2)) \right\|_{DL_2(\Omega)}$$

$$\le \left\| F(t_1, u_1(t_1)) - F(t_1, u_2(t_1)) \right\|_{DL_2(\Omega)} + \left\| F(t_1, u_2(t_1)) - F(t_2, u_2(t_2)) \right\|_{DL_2(\Omega)}$$

$$= \left\| (u_1(t_1) \bullet \nabla) u_1(t_1) - (u_2(t_1) \bullet \nabla) u_2(t_1) \right\|_{DL_2(\Omega)} + \left\| (u_2(t_1) \bullet \nabla) u_2(t_1) - (u_2(t_2) \bullet \nabla) u_2(t_2) \right\|_{DL_2(\Omega)}$$

$$\le 2ML_0^2 L \left\| u_1(t_1) - u_2(t_2) \right\|_{DL_2(\Omega)_{1/2}} + 2ML_0^2 LC_1 |t_1 - t_2|^{\beta_1} + 2ML_0^2 LC |t_1 - t_2|^{\beta}$$

$$\le L_1 \left\| u_1(t_1) - u_2(t_2) \right\|_{DL_2(\Omega)_{1/2}} + L_2 |t_1 - t_2|^{\beta_2}$$

$$\le L_3 \left(|t_1 - t_2|^{\beta_2} + \left\| u_1(t_1) - u_2(t_2) \right\|_{DL_2(\Omega)_{1/2}} \right).$$

Hence $F(t, u(t))$ satisfies the assumption (F), then by lemma 10 for every initial data

$(t_0, u_0) \in U$ the initial value problem （15） has a unique local solution

$$u \in C\left([t_0, t^{'}) : DL_2(\Omega) \right) \cap C^1\left((t_0, t^{'}) : DL_2(\Omega) \right) \tag{19}$$

where $t^{'} = t^{'}(u_0)$. Since $b \in C^\infty(\Omega)$ (19) and lemma 7 imply

$$u \in C\left([t_0, t^{'}) : bDL_2(\Omega) \right) \cap C^1\left((t_0, t^{'}) : bDL_2(\Omega) \right).$$

Changing the value of u on $\partial\Omega$ to zero we get a unique local strong solution for (2).

Using a similar induction way as Theorem 3.9 in [9] or as Theorem 5.1 in [15] we can prove

that the solution $u(t, x) \in \left(C^\infty([t_0, t'] \times \Omega) \right)^3$. We can also prove directly that $u(t, x)$ is smooth.

In fact, the solution (19) of (15) is also the solution of (6). The Theorem 3.4 in [9] mean that as

long as the solution of (6) exists, this solution is smooth. From Theorem 3.4 in [9] we have the

solution $u(t, x) \in \left(C^\infty([t_0, t'] \times \Omega) \right)^3$. Substituting $u(t, x)$ into (2) we get the solution $p(t, x)$.

We also have $p(t, x) \in C^\infty([t_0, t'] \times \Omega)$. Since $u(t, x) \in (C^\infty([t_0, t'] \times \Omega))^3$. It follows from

$u \in bDL_2(\Omega)$ and lemma 1 that $div\, bu = 0$. So the solution $u(t, x)$ is divergence-free. Hence

$u = u(t, x), \ p(t, x)$ is the unique local strong solution of the initial value problem (2) for viscous

lake equations.$\square$

The Theorem 1 has the following corollary.

Corollary 2. The Navier-Stokes initial value problem (3) has a unique local strong solution if the initial value $(t_0, u_0) \in U$.

(3) **Drift homogenization**

A composite is a material containing two or more finely mixed components.   Composite materials are widely used nowadays in any kind of industries. How to determine the properties of a composite material, for example thermal, electrical or linear elastic properties of materials ? It can be solved by the homogenization of a set of partial differential equations. The homogenization theory allows to describe the asymptotic behaviour as $\varepsilon \to 0$ of partial differential equations of many types. A classical problem is the elliptic Dirichlet problem

$$\begin{cases} -div(A^\varepsilon \nabla u^\varepsilon) = f, x \in \Omega \\ \quad u^\varepsilon = 0, x \in \partial\Omega \end{cases}$$

where f is given in $H^{-1}(\Omega)$ and the matrix A^ε is the $Y-$periodic matrix.

The prescription of traditional Chinese medicine consists of a variety of Chinese herbal medicines. Its efficacy can be obtained from the homogenization of a group of partial differential equations controlled by each drug. The actual speed of a ship can be obtained from the homogenization of a set of Lake equations disturbed by the ship's dynamic speed (drift) v_ε . The homogenization of PDEs is completed by the following three steps:

1. Fist Construct a set of partial differential equations satisfied by objective functions $u_\varepsilon (\varepsilon \to 0)$ and prove the existence of solutions u_ε ; One of the ways to solve these equations is to change the equations into a variational formulation, that is multiplying the equation by a arbitrary(test function) $v \in H_0^1(\Omega)$, then by integrating by parts. And then use lax-Milgram theorem to prove the existence of solutions. But in this paper we will use the method of semigroups .

2. Prove that u_ε strongly or weakly converge to u in some state space when $\varepsilon \to 0$ and u is the homogenization value;

3. Find the equation satisfied by u . Solve this equation and get u .

Now we discuss the lake equation in a bounded domain Ω of R^3, perturbed by oscillating term $(v_\varepsilon \bullet \nabla)u_\varepsilon$, i.e.

$$\begin{cases} \partial_t u_\varepsilon + (u_\varepsilon \bullet \nabla)u_\varepsilon + (v_\varepsilon \bullet \nabla)u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon, x \in \Omega, t \in (t_0, T] \\ \qquad\qquad div(bu_\varepsilon) = 0, x \in \Omega \\ \qquad\qquad u_\varepsilon \big|_{t=t_0} = u_{\varepsilon 0}, x \in \Omega \\ \qquad\qquad u_\varepsilon(t, x) = 0, x \in \partial\Omega \end{cases}$$

$$(20)$$

where the oscillations are produced by the sequence of vector-value functions v_ε which strongly

converges to some v in $bDL_2(\Omega)_{1/2}$ when $\varepsilon \to 0$. Here v is the scheduled ship speed and

v_ε is the actual approximate speed, $\lim\limits_{\varepsilon \to 0} v_\varepsilon(t) = v(t)$ in $bDL_2(\Omega)_{1/2}$ for each $t \in [t_0, T]$. In

what follows all $u_\varepsilon, v_\varepsilon$ are velocity fields.

Theorem 2. Suppose that the smooth initial value $(t_0, u_{\varepsilon 0}) \in U$ for each $\varepsilon > 0$, $b(x) \in$

$C^\infty(\Omega)$, v_ε and $f_\varepsilon(t, x)$ are Hölder continuous in $[t_0, T]$ with exponent β and the values in

$DL_2(\Omega)$, $\|v_\varepsilon\|_{DL_2(\Omega)_{1/2}} \le L^0$ and v_ε are smooth and satisfy $\dfrac{\partial v_{\varepsilon i}}{\partial x_j} = 0 (i \ne j)$. Then the initial

value problem (20) for viscous lake equations have local smooth strong solutions .

Proof. The proof is similar to Theorem 1. First we rewrite (20) into an abstract initial value

problem on the state space $DL_2(\Omega)_{\frac{1}{2}}$.

$$\begin{cases} \dfrac{du_\varepsilon}{dt} = \Delta u_\varepsilon + F(t, u_\varepsilon(t)), t \in (t_0, T] \\ \left. u_\varepsilon \right|_{t=t_0} = u_{\varepsilon 0}, x \in \Omega \end{cases}.$$

(21)

where $F(t, u_\varepsilon(t)) = -(u_\varepsilon(t) \bullet \nabla)u_\varepsilon(t) - (v_\varepsilon(t) \bullet \nabla)u_\varepsilon(t) + f_\varepsilon$ where v_ε is fixed for $\varepsilon > 0$. Let

$$G := \left\{ (t, u(t)) : t \in (0, T), \forall u \in H\left([0, T]; DL_2(\Omega)_{1/2}\right) \right\}$$

Take the open kernel $U = G^0$ of G in $(0, T) \times DL_2(\Omega)_{1/2}$. Then U is an open subset of

$[0, T] \times DL_2(\Omega)_{1/2}$. The bilinear form $(v \bullet \nabla)u$ on H^0 takes value in $DL_2(\Omega)$. That is,

$F(t, u_\varepsilon(t)) = -(u_\varepsilon \bullet \nabla)u_\varepsilon + (v_\varepsilon \bullet \nabla)u_\varepsilon$ is a function : $U \to DL_2(\Omega)$.

Suppose that $(t_0, u_{\varepsilon 0}) \in U$. Set

$$V = B_\varepsilon(t_0, u_{\varepsilon 0}) = \left\{ (t, u) \in U : |t| < \varepsilon \le 1, \|u - u_{\varepsilon 0}\|_{DL_2(\Omega)_{1/2}} < \varepsilon \right\}.$$

Let $L = \varepsilon + \|u_0\|_{DL_2(\Omega)_{\frac{1}{2}}}$, $L_1 = 2ML_0^2 L$. Similarly to Theorem 1. from (16),(17) and (18) for any

$(t_i, u_{\varepsilon i}) \in V (i = 1, 2)$ we have

$$\left\| F(t_1, u_{\varepsilon 1}(t_1)) - F(t_2, u_{\varepsilon 2}(t_2)) \right\|_{DL_2(\Omega)}$$

$$\leq \left\| (u_{\varepsilon 1}(t_1) \bullet \nabla) u_{\varepsilon 1}(t_1) - (u_{\varepsilon 2}(t_1) \bullet \nabla) u_{\varepsilon 2}(t_1) \right\|_{DL_2(\Omega)} + \left\| (u_{\varepsilon 2}(t_1) \bullet \nabla) u_{\varepsilon 2}(t_1) - (u_{\varepsilon 2}(t_2) \bullet \nabla) u_{\varepsilon 2}(t_2) \right\|_{DL_2(\Omega)}$$

$$+ \left\| (v_{\varepsilon}(t_1) \bullet \nabla) u_{\varepsilon 1}(t_1) - (v_{\varepsilon}(t_1) \bullet \nabla) u_{\varepsilon 1}(t_2) \right\|_{DL_2(\Omega)} + \left\| (v_{\varepsilon}(t_1) \bullet \nabla) u_{\varepsilon 1}(t_2) - (v_{\varepsilon}(t_2) \bullet \nabla) u_{\varepsilon 1}(t_2) \right\|_{DL_2(\Omega)}$$

$$+ \left\| (v_{\varepsilon}(t_2) \bullet \nabla) u_{\varepsilon 1}(t_2) - (v_{\varepsilon}(t_2) \bullet \nabla) u_{\varepsilon 2}(t_2) \right\|_{DL_2(\Omega)} + \left\| f_{\varepsilon}(t_1) - f_{\varepsilon}(t_2) \right\|_{DL_2(\Omega)}$$

$$\leq 2 M L_0^2 L \left\| u_{\varepsilon 1}(t_1) - u_{\varepsilon 2}(t_2) \right\|_{DL_2(\Omega)_{1/2}} + 4 M L_0^2 L C_0 |t_1 - t_2|^{\beta}$$

$$+ \left\| (v_{\varepsilon}(t_1) \bullet \nabla)((u_{\varepsilon 1}(t_1) - u_{\varepsilon 1}(t_2)) \right\|_{DL_2(\Omega)} + \left\| [(v_{\varepsilon}(t_1) - (v_{\varepsilon}(t_2)) \bullet] u_{\varepsilon 1}(t_2) \right\|_{DL_2(\Omega)}$$

$$+ \left\| (v_{\varepsilon}(t_2) \bullet \nabla)(u_{\varepsilon 1}(t_2) - u_{\varepsilon 2}(t_2)) \right\|_{DL_2(\Omega)} + \left\| f_{\varepsilon}(t_1) - f_{\varepsilon}(t_2) \right\|_{DL_2(\Omega)}$$

$$\leq 2 M L_0^2 L \left\| u_{\varepsilon 1}(t_1) - u_{\varepsilon 2}(t_2) \right\|_{DL_2(\Omega)_{1/2}} + 4 M L_0^2 L C_0 |t_1 - t_2|^{\beta}$$

$$+ M L_0^2 \left\| v_{\varepsilon}(t_1) \right\|_{DL_2(\Omega)_{1/2}} \left\| u_{\varepsilon 1}(t_1) - u_{\varepsilon 1}(t_2) \right\|_{DL_2(\Omega)_{1/2}} + M L_0^2 \left\| v_{\varepsilon}(t_1) - v_{\varepsilon}(t_2) \right\|_{DL_2(\Omega)_{1/2}} \left\| u_{\varepsilon 1}(t_2) \right\|_{DL_2(\Omega)_{1/2}}$$

$$+ M L_0^2 \left\| v_{\varepsilon}(t_2) \right\|_{DL_2(\Omega)_{1/2}} (\left\| u_{\varepsilon 1}(t_2) - u_{\varepsilon 1}(t_1) \right\|_{DL_2(\Omega)_{1/2}} + \left\| u_{\varepsilon 1}(t_1) - u_{\varepsilon 2}(t_2) \right\|_{DL_2(\Omega)_{1/2}}) + \left\| f_{\varepsilon}(t_1) - f_{\varepsilon}(t_2) \right\|_{DL_2(\Omega)}$$

$$\leq 2 M L_0^2 L \left\| u_{\varepsilon 1}(t_1) - u_{\varepsilon 2}(t_2) \right\|_{DL_2(\Omega)_{1/2}} + 4 M L_0^2 L C_0 |t_1 - t_1|^{\beta}$$

$$+ M L_0^2 L^0 C_1 |t_1 - t_2|^{\beta} + M L_0^2 L C_2 |t_1 - t_2|^{\beta} + M L_0^2 L^0 (C_1 |t_1 - t_2|^{\beta} + \left\| u_{\varepsilon 1}(t_1) - u_{\varepsilon 2}(t_2) \right\|_{DL_2(\Omega)_{1/2}})$$

$$+ C_3 |t_1 - t_2|^{\beta}$$

$$\leq (2 M L_0^2 L + M L_0^2 L^0) \left\| u_{\varepsilon 1}(t_1) - u_{\varepsilon 2}(t_2) \right\|_{DL_2(\Omega)_{1/2}} + C_4 |t_1 - t_2|^{\beta}$$

$$\leq L_1 (|t_1 - t_2|^{\beta} + \left\| u_1(t_1) - u_2(t_2) \right\|_{DL_2(\Omega)_{1/2}}),$$

where $C_4 = Max(4 M L_0^2 L C_0, M L_0^2 L^0 C_1, M L_0^2 L C_2, M L_0^2 L^0 C_1, C_3)$, $L_1 = Max(M L_0^2 L + C_4)$.

Hence $F(t, u(t))$ satisfies the assumption (F), then by lemma 10 for the initial data

$(t_0, u_0) \in U$ the initial value problem （21）has a unique smooth local strong solution

$$u_{\varepsilon} \in C([t_0, t_1] : DL_2(\Omega)) \cap C^1((t_0, t_1) : DL_2(\Omega)). \tag{22}$$

Since $b \in C^{\infty}(\Omega)$ (22) and lemma 7 imply

$$u_{\varepsilon} \in C([t_0, t_1] : b DL_2(\Omega)) \cap C^1((t_0, t_1) : b DL_2(\Omega))$$

Similarly to Theorem 1 we get smooth strong solutions u_{ε} and p_{ε} for $\varepsilon > 0.\square$

Theorem 3. Suppose that $\lim_{\varepsilon \to 0} v_{\varepsilon}(t) = v(t)$ in $b DL_2(\Omega)_{\frac{1}{2}}$ for each $t \in [t_0, T]$. Then there

exists $u(t) \in b DL_2(\Omega)_{\frac{1}{2}}$ such that $\lim_{\varepsilon \to 0} u_{\varepsilon}(t) = u(t)$ in $b DL_2(\Omega)_{1/2}$ for each $t \in [t_0, t_1]$.

1

Proof. For each fixed $\varepsilon > 0$ let $y_\varepsilon(t)$ be the fixed point in the proof of Theorem 6.3.1 in [11]. Then

$$\lim_{\varepsilon \to 0}\left[((-\Delta)^{-\frac{1}{2}} y_\varepsilon \bullet \nabla)(-\Delta)^{-\frac{1}{2}} y_\varepsilon\right] = \lim_{\varepsilon \to 0}\begin{pmatrix} \sum_{i=1}^3 (-\Delta)^{-\frac{1}{2}} y_{\varepsilon i} \frac{\partial (-\Delta)^{-\frac{1}{2}} y_{\varepsilon 1}}{\partial x_i} \\ \sum_{i=1}^3 (-\Delta)^{-\frac{1}{2}} y_{\varepsilon i} \frac{\partial (-\Delta)^{-\frac{1}{2}} y_{\varepsilon 2}}{\partial x_i} \\ \sum_{i=1}^3 (-\Delta)^{-\frac{1}{2}} y_{\varepsilon i} \frac{\partial (-\Delta)^{-\frac{1}{2}} y_{\varepsilon 3}}{\partial x_i} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^3 (-\Delta)^{-\frac{1}{2}} \lim_{\varepsilon \to 0} y_{\varepsilon i} (-\Delta)^{-\frac{1}{2}} \frac{\partial (\lim_{\varepsilon \to 0} y_{\varepsilon 1})}{\partial x_i} \\ \sum_{i=1}^2 (-\Delta)^{-\frac{1}{2}} \lim_{\varepsilon \to 0} y_{\varepsilon i} (-\Delta)^{-\frac{1}{2}} \frac{\partial (\lim_{\varepsilon \to 0} y_{\varepsilon 2})}{\partial x_i} \\ \sum_{i=1}^3 (-\Delta)^{-\frac{1}{2}} \lim_{i \to 0} y_{\varepsilon i} (-\Delta)^{-\frac{1}{2}} \frac{\partial (\lim_{\varepsilon \to 0} y_{\varepsilon 3})}{\partial x_i} \end{pmatrix},$$

So
$$\lim_{\varepsilon \to 0}((-\Delta)^{-\frac{1}{2}} y_\varepsilon \bullet \nabla)(-\Delta)^{-\frac{1}{2}} y_\varepsilon = ((-\Delta)^{-\frac{1}{2}} \lim_{\varepsilon \to 0} y_\varepsilon \bullet \nabla)(-\Delta)^{-\frac{1}{2}} \lim_{\varepsilon \to 0} y_\varepsilon.$$

Similarly,
$$\lim_{\varepsilon \to 0}(v_\varepsilon \bullet \nabla)v_\varepsilon = (\lim_{\varepsilon \to 0} v_\varepsilon \bullet \nabla)\lim_{\varepsilon \to 0} v_\varepsilon.$$

From (3.10)in [11] we have

$$y_\varepsilon(t) = T(t)(-\Delta)^{\frac{1}{2}} u_0 + \int_{t_0}^t (-\Delta)^{\frac{1}{2}} T(t-s)\left[((-\Delta)^{-\frac{1}{2}} y_\varepsilon \bullet \nabla)(-\Delta)^{-\frac{1}{2}} y_\varepsilon + (v_\varepsilon \bullet \nabla)v_\varepsilon\right]ds,$$

$$y_\varepsilon(t) - T(t)(-\Delta)^{\frac{1}{2}} u_0 - \int_{t_0}^t (-\Delta)^{\frac{1}{2}} T(t-s)((-\Delta)^{-\frac{1}{2}} y_\varepsilon(s) \bullet \nabla)(-\Delta)^{-\frac{1}{2}} y_\varepsilon(s)ds$$

$$= \int_{t_0}^t (-\Delta)^{\frac{1}{2}} T(t-s)(v_\varepsilon(s) \bullet \nabla)v_\varepsilon(s)ds \qquad (23)$$

Since $(-\Delta)^{\frac{1}{2}} T(t-s)$ is bounded by Theorem 2.16.13(c) in [11], hence the below of (23) has a

limit when $\varepsilon \to 0$ by
Theorem 7.16 in [13].

$$-\lim_{\varepsilon \to 0} \int_{t_0}^t (-\Delta)^{\frac{1}{2}} T(t-s)(v_\varepsilon(s) \bullet \nabla)v_\varepsilon(s)ds$$

$$= -\int_{t_0}^t (-\Delta)^{\frac{1}{2}} T(t-s)(\lim_{\varepsilon \to 0} v_\varepsilon(s) \bullet \nabla)\lim_{\varepsilon \to 0} v_\varepsilon(s)ds$$

$$= -\int_{t_0}^t (-\Delta)^{\frac{1}{2}} T(t-s)(v(s) \bullet \nabla)v(s)ds$$

Therefor the above of (23) also has a limit and so there exists $\lim_{\varepsilon \to 0} y_\varepsilon(t) = y(t)$. Otherwise,

$y_\varepsilon(t), (-\Delta)^{-\frac{1}{2}} y_\varepsilon(t)$ and $\int_{t_0}^t (-\Delta)^{\frac{1}{2}} T(t-s)((-\Delta)^{-\frac{1}{2}} y_\varepsilon(s) \bullet \nabla)(-\Delta)^{-\frac{1}{2}} y_\varepsilon(s)ds$ are all

unbounded or oscillatory, and they cannot be offset, inducing the above of (23) unbounded or oscillating, a contradiction.

Since $u_\varepsilon(t) = (-\Delta)^{-\frac{1}{2}} y_\varepsilon(t),$ so there exists limit

$$\lim_{\varepsilon \to 0} u_\varepsilon(t) = \lim_{\varepsilon \to 0}(-\Delta)^{-\frac{1}{2}} y_\varepsilon(t) = (-\Delta)^{-\frac{1}{2}} \lim_{\varepsilon \to 0} y_\varepsilon(t) = (-\Delta)^{-\frac{1}{2}} y(t) = u(t)$$

for $t \in [t_0, t_1]$. $\square$

Theorem 4. Assume that $\lim_{\varepsilon \to 0} v_\varepsilon = v, \lim_{\varepsilon \to 0} f_\varepsilon = f$ in $bDL_2(\Omega)$ and there exists $p \in$

$C^\infty([t_0, T] \times \Omega)$ such that $\lim_{\varepsilon \to 0} \nabla p_\varepsilon = \nabla p$, then u, p satisfies

$$\begin{cases} \partial_i u + (u \bullet \nabla)u + (v \bullet \nabla)u - \Delta u + \nabla p = f \\ div(bu) = 0, x \in \Omega \\ u\big|_{t=t_0} = u_0, x \in \Omega \\ u(t,x) = 0, x \in \partial\Omega \end{cases} \qquad (24)$$

Proof. From theorem 3 $\lim_{\varepsilon \to 0} v_\varepsilon(t) = v(t)$ implies $\lim_{\varepsilon \to 0} u_\varepsilon(t) = u(t)$, i.e. $\lim_{\varepsilon \to 0} \|u_\varepsilon(t) - u(t)\|_{bDL_2(\Omega)} =$

0 for each $t \in [t_0, T]$. This strong convergence is uniform for $x \in \Omega$. Now by using similar

Theorem 7.11 in [13] we have

(1) $\lim_{\varepsilon \to 0} \dfrac{\partial u_\varepsilon}{\partial t} = \lim_{\varepsilon \to 0} \lim_{t \to t'} \dfrac{u_\varepsilon(t) - u_\varepsilon(t')}{t - t'} = \lim_{t \to t'} \lim_{\varepsilon \to 0} \dfrac{u_\varepsilon(t) - u_\varepsilon(t')}{t - t'} = \lim_{t \to t'} \dfrac{u(t) - u(t')}{t - t'} = \dfrac{\partial u}{\partial t}$.

(2) Let $u_\varepsilon = (u_{\varepsilon 1}, u_{\varepsilon 2}, u_{\varepsilon 3}), u = (u_1, u_2, u_3)$.

$$\lim_{\varepsilon \to 0} \dfrac{\partial u_{\varepsilon i}}{\partial x_i} = \lim_{\varepsilon \to 0} \lim_{x_i \to x_{i0}} \dfrac{u_{\varepsilon i}(x_i) - u_{\varepsilon i}(x_{i0})}{x_i - x_{i0}} = \lim_{x_i \to x_{i0}} \lim_{\varepsilon \to 0} \dfrac{u_{\varepsilon i}(x_i) - u_{\varepsilon i}(x_{i0})}{x_i - x_{i0}} = \lim_{x_i \to x_{i0}} \dfrac{u_i(x_i) - u_i(x_{i0})}{x_i - x_{i0}} = \dfrac{\partial u_i}{\partial x_i}$$

for $i = 1,2,3$. This is to say that $\lim_{\varepsilon \to 0} u_\varepsilon = u$ implies $\lim_{\varepsilon \to 0} \dfrac{\partial u_{\varepsilon i}}{\partial x_i} = \dfrac{\partial u_i}{\partial x_i}$ for $i = 1,2,3$.

$$\lim_{\varepsilon \to 0} \dfrac{\partial^2 u_{\varepsilon i}}{\partial x_i^2} = \lim_{\varepsilon \to 0} \dfrac{\partial}{\partial x_i} \dfrac{\partial u_{\varepsilon i}}{\partial x_i} = \lim_{\varepsilon \to 0} \lim_{x_i \to x_{i0}} \dfrac{\frac{\partial u_{\varepsilon i}}{\partial x_i}(x_i) - \frac{\partial u_{\varepsilon i}}{\partial x_i}(x_{i0})}{x_i - x_{i0}}$$

$$= \lim_{x_i \to x_{i0}} \lim_{\varepsilon \to 0} \dfrac{\frac{\partial u_{\varepsilon i}}{\partial x_i}(x_i) - \frac{\partial u_{\varepsilon i}}{\partial x_i}(x_{i0})}{x_i - x_{i0}} = \lim_{x_i - x_{i0}} \dfrac{\frac{\partial u_i}{\partial x_i}(x_i) - \frac{\partial u_i}{\partial x_i}(x_{i0})}{x_i - x_{i0}} = \dfrac{\partial^2 u_i}{\partial x_i^2}$$

for $i = 1,2,3$. This to say that $\lim_{\varepsilon \to 0} u_\varepsilon = u$ implies $\lim_{\varepsilon \to 0} \dfrac{\partial^2 u_{\varepsilon i}}{\partial x_i^2} = \dfrac{\partial^2 u_i}{\partial x_i^2}$ for $i = 1,2,3$. $\lim_{\varepsilon \to 0} \Delta u_{\varepsilon i} =$

$$\lim_{\varepsilon \to 0} \sum_{j=1}^3 \dfrac{\partial^2 u_{\varepsilon i}}{\partial x_j^2} = \sum_{j=1}^3 \dfrac{\partial^2 u_i}{\partial x_j^2} = \Delta u_i \quad \text{for } i = 1,2,3. \quad \text{So } \lim_{\varepsilon \to 0} \Delta u_\varepsilon = \Delta u.$$

(3) For $i = 1,2,3$

2

$$\lim_{\varepsilon \to 0} u_{\varepsilon i} \frac{\partial u_{\varepsilon i}}{\partial x_i} = \lim_{\varepsilon \to 0}\Big[u_{\varepsilon i} \lim_{x_i \to x_{i0}} \frac{u_{\varepsilon i}(x_i) - u_{\varepsilon i}(x_{i0})}{x_i - x_{i0}} \Big] = \lim_{\varepsilon \to 0} u_{\varepsilon i} \lim_{\varepsilon \to 0} \lim_{x_i \to x_{i0}} \frac{u_{\varepsilon i}(x_i) - u_{\varepsilon i}(x_{i0})}{x_i - x_{i0}}$$

$$= u_i \lim_{x_i \to x_{i0}} \lim_{\varepsilon \to 0} \frac{u_{\varepsilon i}(x_i) - u_{\varepsilon i}(x_{i0})}{x_i - x_{i0}} = u_i \lim_{x_i \to x_{i0}} \frac{u_i(x_i) - u_i(x_{i0})}{x_i - x_{i0}} = u_i \frac{\partial u_i}{\partial x_i}.$$

And so

$$\lim_{\varepsilon \to 0}(u_\varepsilon \bullet \nabla)u_\varepsilon = \lim_{\varepsilon \to 0}\left(\sum_{i=1}^{3} u_{\varepsilon i} \frac{\partial}{\partial x_i} \right)\begin{pmatrix} u_{\varepsilon 1} \\ u_{\varepsilon 2} \\ u_{\varepsilon 3} \end{pmatrix} = \lim_{\varepsilon \to 0}\begin{pmatrix} \sum_{i=1}^{3} u_{\varepsilon i} \frac{\partial u_{\varepsilon 1}}{\partial x_i} \\ \sum_{i=1}^{3} u_{\varepsilon i} \frac{\partial u_{\varepsilon 2}}{\partial x_i} \\ \sum_{i=1}^{3} u_{\varepsilon i} \frac{\partial u_{\varepsilon 3}}{\partial x_i} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{2} u_i \frac{\partial u_1}{\partial x_i} \\ \sum_{i=1}^{3} u_i \frac{\partial u_2}{\partial x_i} \\ \sum_{i=1}^{3} u_i \frac{\partial u_3}{\partial x_i} \end{pmatrix} = (u \bullet \nabla)u.$$

Similarly, $\lim_{\varepsilon \to 0}(v_\varepsilon \bullet \nabla)u_\varepsilon = (v \bullet \nabla)u.$

(4) Now from (1),(2),(3) and the equations (20) we take limit and get

$$\partial_t u + (u \bullet \nabla)u + (v \bullet \nabla)u - \Delta u + \nabla p = \lim_{\varepsilon \to 0}\big(\partial_t u_\varepsilon + (u_\varepsilon \bullet \nabla)u_\varepsilon + (v_\varepsilon \bullet \nabla)u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon \big) = 0.$$

It is clear that $div\ bu_\varepsilon = 0$ for $\varepsilon > 0$ implies

$$div\ bu = div \lim_{\varepsilon \to 0} bu_\varepsilon = \lim_{\varepsilon \to 0} div\ bu_\varepsilon = 0,$$

$$u\Big|_{t=t_0} = \lim_{\varepsilon \to 0} u_\varepsilon \Big|_{t=t_0} = \lim_{\varepsilon \to 0} u_{\varepsilon 0} = u_0,$$

$$u(t,x) = \lim_{\varepsilon \to 0} u_\varepsilon(t,x) = 0 \quad \text{when } x \in \partial\Omega.$$

Therefore u, p satisfies (24). $\square$

Now we consider a special case (stationary Stokes equation). In [1] M.Braine and P.Gerard studied a scalar and a two-dimensional stationary Stokes equations perturbed by a drift in the sense of distributions. Under higher integrability conditions the Tartar approach based on the oscillations test functions method applies and leads to a limit equation with an extra zero-order term. However, the lack of integrability makes difficult the direct use of the Tartar method for the three-dimensional Stokes equations. But we can use the semigroup method to study the three-dimensional stationary Stokes equation in the usual sense $bDL_2(\Omega)$.

When a ship is sailing in the lake, the set dynamic speed is v. Due to the operation error, the actual dynamic speed is $v_\varepsilon(t,x)$, v_ε infinitely close to v, that is $\lim_{\varepsilon \to 0}\|v_\varepsilon - v\|_{bDL_2(\Omega)} = 0$. How to calculate the relative speed of the ship, the speed of the combination of dynamic speed and lake speed? In order to keep the ship's speed stable, it is assumed that the ship's dynamic force $f_\varepsilon(t,x)$ makes the time-varying acceleration $\dfrac{\partial u_\varepsilon}{\partial t}$ and the spatial position dependent migration acceleration $(u_\varepsilon \bullet \nabla)u_\varepsilon$ to be zero. The transfer acceleration with the change of space

position produced by ship dynamic force $f_\varepsilon(t,x)$ is $(v_\varepsilon \bullet \nabla)u_\varepsilon$. This acceleration $(v_\varepsilon \bullet \nabla)$ u_ε counteracts the accelerations produced by the viscous force Δu_ε and the pressure gradient force ∇p_ε such that $\dfrac{\partial u_\varepsilon}{\partial t} = 0$ and $(u_\varepsilon \bullet \nabla)u_\varepsilon = 0$. This is the result of zero resultant force of $f_\varepsilon, \Delta u_\varepsilon$ and ∇p_ε. The relative speed of the ship is the homogenization of u_ε, $u = \lim\limits_{\varepsilon \to 0} u_\varepsilon$. Our mathematical problem is as following

$$\begin{cases} (v_\varepsilon \bullet \nabla)u_\varepsilon - \Delta u_\varepsilon + \nabla p_\varepsilon = f_\varepsilon, x \in \Omega, t \in (t_0, T] \\ \qquad div(bu_\varepsilon) = 0, x \in \Omega \\ \qquad u_\varepsilon(t,x) = 0, x \in \partial\Omega \end{cases} \qquad (25)$$

From Theorem 2, Theorem 3 and Theorem 4 we have the following corollaries

    Corollary 3. Suppose that for each $\varepsilon > 0, (t_0, u_{\varepsilon 0}) \in U, b(x) \in C^\infty(\Omega),$ $v_\varepsilon, f_\varepsilon$ are Hölder continuous in $DL_2(\Omega)_{1/2}$ and in $DL_2(\Omega)$ respectively with exponent β, and $\left\| v_\varepsilon(t) \right\|_{DL_2(\Omega)_{\frac{1}{2}}} \leq L^0$ $(t \in [t_0, T])$ for some constant L^0. Then the viscous lake equations (25) has a unique smooth local strong solutions .

    Corollary 4. Suppose that $\lim\limits_{\varepsilon \to 0} v_\varepsilon = v$ in $bDL_2(\Omega)$. Then there exists $u \in bDL_2(\Omega)_{\frac{1}{2}}$ such that $\lim\limits_{\varepsilon \to 0} u_\varepsilon = u$ in $bDL_2(\Omega)$.

    Corollary 5. Assume that $\lim\limits_{\varepsilon \to 0} v_\varepsilon = v, \lim\limits_{\varepsilon \to 0} f_\varepsilon = f$ and there exists $p \in bDL_2(\Omega)$ such that $\lim\limits_{\varepsilon \to 0} \nabla p_\varepsilon = \nabla p,$ then u, p satisfies

$$\begin{cases} -\Delta u + (v \bullet \nabla)u + \nabla p = f, x \in \Omega \\ \qquad div(bu) = 0, x \in \Omega \\ \qquad u = 0, x \in \partial\Omega \end{cases} .$$

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