Article

Mathematical incorrectnes of so called Higuchi's fractal dimension

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Abstract: So called Higuchi's method of fractal dimension estimation is widely used and the term Higuchi's fractal dimension even occurs in many publications. This paper deals with this method from mathematical point of view. Terms distance and dimension and its basic properties are explained and Higuchi's dimension according the original source is defined.   Definition of Higuchi's dimension was compared with mathematical definition of the distance and dimension. It is showed, that the definition of the Higuchi's dimension does not satisfy axioms of distance and dimension. So called Higuchi's method and Higuchi's dimension are mathematically incorrect. Therefore, all results achieved by this method are scientifically unreliable.

Keywords: Higuchi's method, Higuchi's fractal dimension, distance, metric space

1. Introduction

Fractal dimension is any dimension that allows non-integer values. The oldest and the most general among them is the Hausdorff dimension (H-dimension). Nowadays, many other definitions are used as well.

Measurement of the fractal dimension plays an important role in many applications in engineering and science. The fractal dimension is measured either from time series or from digital images. The fractal dimension of various structures has been studied recently. [1] deals with the relation between the fractal dimension of surfaces of metal samples and their wear resistance coefficient as well as with the relation between the volume of pores in ceramics and the Hausdorff dimension of the pore boundary. [2] and [3] perform Hausdorff dimension analysis on fracture surfaces of porous materials such as hydrated cement pastes. The Hausdorff dimension of a fracture surface gives us information on the material and its properties. [4] studies the properties of different fracture surfaces (of metals, ceramics, rocks etc.) and their fractal properties. [5] studied the fracture surfaces of aluminium alloys subjected to four different heat treatments to find that their fractal dimensions were almost identical. [6] showed that the fractal dimension can be a measure of toughness in metals.

Many researchers have suggested using the fractal dimension to quantify rock joint roughness   - see [7 -14] for example. There exist also many papers which deal with fractal dimension for other various purposes: natural phenomena [15], medicine [16], clinical neurophysiology especially [17], seismology [18], computer science [19] or communications technology [20].

Unfortunately, some published papers contain many inaccuracies and even mathematical nonsenses caused by insufficient mathematical foundations   from which their authors deduce their conclusions – see [3, 20 - 28] for example. A special group of incorrect papers are texts which deal with so called Higuchi's method or Higuchi's fractal dimension – see [20, 29 - 35]. Research based on this incorrrect method was published even in physical [36], fractal [37] and chaos [38] journals

2. Materials and Methods

Each definition of a fractal dimension is based on the concept of distance or length of abscissa. In mathematics, they may be abstract terms and they can be defined in many ways. However, each distance or length must satisfy a few simple properties. Formally, metric space is defined in mathematics: A metric space is an ordered pair $(\mathcal{M}; L)$ where

$\mathcal{M}$ is a set and $L: \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is a mapping such that for any $x; y; z \in \mathcal{M}$, the following holds:

a) $L(x; y) \geq 0$
b) $L(x; y) = 0$ if and only if $x = y$
c) $L(x; y) = L(y; x)$
d) $L(x; z) \leq L(x; y) + L(y; z)$

Mapping L is called distance function or metrics on set $\mathcal{M}$.

If we speak of any dimension, we obviously presume its integer value – objects are one-, two- or three dimensional. It is so called topological dimension. Fractal dimension is an arbitrary dimension which allows non-integer values. For its computing, we must measure a series of its approximations. For example, perimeter of a circle can be measured approximately with the series of hypothetical folding rules – polygons with shortenig segments. Each segment shortening in half means approximately twice as many segments. If we denote length of segments as L_k and corresponding numer as N_k then $N_k \cdot L_k^1 \approx constant$. Perimeter of a circle is one-dimensional. For measuring of its interior, square grid can be used by analogy. Each shortening of square side L_k to one half means approximately quadruplicate number of squares N_k, i.e. $N_k \cdot L_k^2 \approx constant$. Interior of a circle is two-dimensional. For interior of a ball is $N_k \cdot L_k^3 \approx constant$ by analogy. It is three-dimensional. Generally, this dimension is defined as the number D, for which is

$$N_k \cdot L_k^D \approx constant$$

However, this dimension is not topological because number D can be non-integer for many geometric shapes. $D = 1.2618 \dots$ for so called Koch curve, $D = 1.5849 \dots$ for so called Sierpinski triangle e.t.c. It is possible to proof that each fractal dimension of each set is greater or equal to its topological dimension. Topological dimension of Koch curve and also Sierpinski triangle is equal to on efor example. See [39 - 41] for more information about these problems.

Higuchi's method [29] is a common method of the estimation of the fractal dimension of a curve The Higuchi's method is defined only for functions $X(t)$, whose values are known only in points $t = 1; 2; \dots; M$. An approximation X_k^m of the graph of function $X(t)$ are polygons going through points

$$X_k^m: X(m); \ X(m+k); \ X(m+2k); \dots; X\left(m + \left\lfloor \frac{N-m}{k} \right\rfloor \cdot k\right); \ \ m = 1; 2; \dots; k \quad (1)$$

and the „length" of the polygon is defined as

$$L_m(k) = \left(\sum_{i=1}^{\left\lfloor \frac{M-m}{n} \right\rfloor} |X(m+ik) - X(m+(i-1)k)| \right) \cdot \frac{M-1}{\left\lfloor \frac{M-m}{n} \right\rfloor \cdot k^2} \quad (2)$$

where $\lfloor \cdot \rfloor$ stands for the nearest lower integer [29].

Principle of this method is illustrated in Fig. 1 where

$$\Delta y_i = |X(m+ik) - X(m+(i-1)k)| \quad (3)$$

for simplicity.

The values of $L_m(k)$ are averaged over m to obtain function $\langle L(k) \rangle$. Then Higuchi claims that if $\langle L(k) \rangle$ is proportional to k^{-D}, then the curve is a fractal with dimension D. However, each fractal dimension must work with concept of length or distance according to points a) – d) in previous text and must be equal or greater than topological dimension. We have verified whether the Higuchi's dimension has these properties or not.
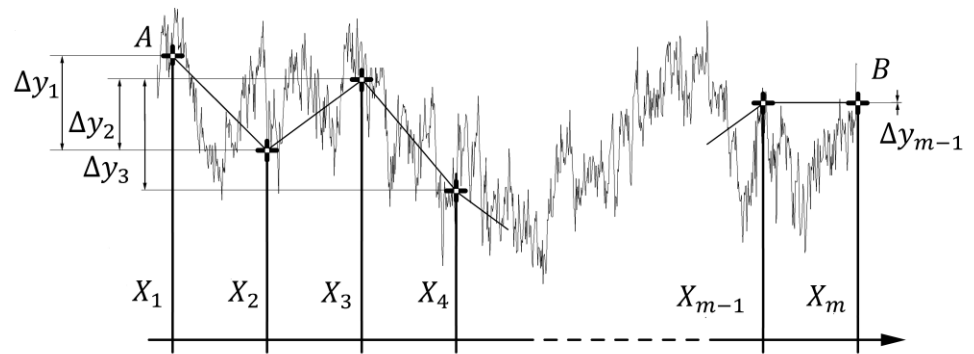
Figure 1. Principle of so called Higuchi's method.

3. Results

Higuchi tested his method on artificially generated noise (see Fig. 1) with good results. However, with the fractal dimension close to one (see Fig. 2), we get meaningless results.
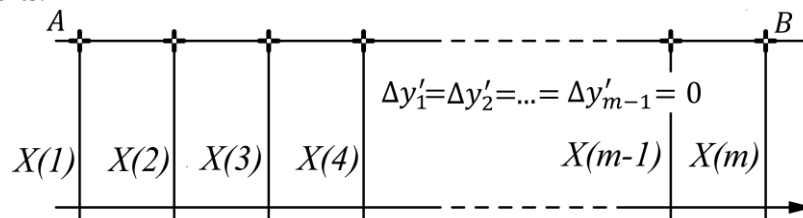

Figure. 2: Higuchi's method applied to a whole straight line gives incorrect result

If the Higuchi's method is applied to a constant time series

$X(1) = X(2) = \cdots = X(M)$,

we obtain $\Delta y_i = 0$ for each i in (3), therefore, sum of Δy_i is equal to zero as well and $L_m(k) = 0$ for each m, k in (2) and $L(AB) = 0$ although points A, B are not identical.

It is contrary to the condition b) in Sec. 2. Moreover, if $\langle L(k) \rangle$ is proportional to k^{-D} (as Higuchi claims) then must be $D = 0$ and fractal dimension would be smaller than topological dimension (it is equal to one). It means that so called Higuchi's dimension is contrary to known properties of a fractal dimension.

4 Conclusion

As was shown in previous section, so called Higuchi's method for fractal dimension measurement and also the term Higuchi's dimension is mathematically incorrect. Therefore, all results achieved by this method must be considered scientifically unreliable.

Conflicts of Interest: The authors declare no conflict of interest.

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