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On a problem of infinite divisibility

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Abstract

Let $f(t)$ be a characteristic function. The question on infinite divisibility of $g_{2k}(t) = f^{(2k)}(t)/f^{(2k)}(0)$ is considered. There are given the condition for that function not to be infinite divisible. Some examples of infinite divisibility of $g_{2k}(t)$ are given.

Key words: characteristic function; infinite divisibility; singularities

1 Introduction

Suppose that X is a random variable possessing finite moment of even order $2k$ and $f(t)$ is its characteristic function. Then $g_{2k}(t) = f^{(2k)}(t)/f^{(2k)}(0)$ is a characteristic function again. In the case when X is positive (or negative) random variable it is possible consider $g_k(t) = f^{(k)}(t)/f^{(k)}(0)$ when the k^{th} moment exists. An easy question arises. Under what conditions $g_{2k}(t)$ is infinite divisible characteristic function? Clearly, it is possible to consider similar question not for a fixed k only but for a finite or infinite set of such numbers. The aim of this publication is to consider these questions in some details and propose hypothetical answers.

2 Negative results

Let us start with a negative result.

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Theorem 2.1. *Let $f(t)$ be a characteristic function of a random variable X having a symmetric (around origin) distribution. Suppose that $\mathbb{E}X^{2m} < \infty$ for an integer $m \geq 1$. Then the characteristic functions $g_{2k}(t) = f^{(2k)}(t)/f^{(2k)}(0)$ $(k = 1, \dots, m)$ are not infinite divisible.*

Proof. Consider the case $m = 1$. Suppose the opposite. If $g_2(t)$ is infinite divisible it must be positive over the whole real line. Therefore, $f(t)$ must be concave on the whole real line. In this situation its graph should be under a tangent drawn to it at an arbitrary point $(t_o, f(t_0))$. However, it contradicts to the boundness of $f(t)$. Therefore, g_2 has zeros and cannot be infinite divisible.

The similar arguments are applicable to each pair $g_{2k-2}(t)$, $g_{2k}(t)$ for $k = 1, \dots, m$. □

Obviously, the proof essentially uses the reality of symmetric characteristic function on the real line. However, the fact remains true for entire characteristic functions[1]. More precisely, the following result holds.

Theorem 2.2. *Let $f(t)$ be entire characteristic function of non-degenerate distribution. No pair of the functions $g_{2k}(t)$, $g_{2s}(t)$ $(0 \leq k < s \leq m)$ may consists of infinitely divisible characteristic functions.*

Proof. The statement is a simple consequence of the result by Tumura and Clunie (see [2]). Really, if $f(t)$ is entire characteristic function then all g_{2k} are entire functions. It is known (see, for example, [1]) the entire infinite divisible characteristic function has no zeros on the whole complex plain. From Theorem 3.8 [2] it follows that $f(t) = \exp(At + B)$ in contradiction with non-degenerate character of the corresponding distribution. □

From Theorem 2.2 it follows that characteristic functions of the normal and Poisson distributions do not possess the property of infinite divisibility of $g_{2k}(t)$.

3 Positive examples

Here we give some examples of infinite divisible characteristic functions $f(t)$ for which one or more functions $g_{2k}(t)$ $(k > 0)$ are infinite divisible too.

[1]For the definitions and properties see [1]

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Example 3.1. *The main positive example is characteristic function of Gamma-distribution*

$$f(t) = \frac{1}{(1-iat)^{\gamma}}, \quad a>0, \quad \gamma>0, \tag{3.1}$$

which is infinite divisible for all values of the parameters. It is clear that $g_k(t)$ is also characteristic function of the Gamma-distribution with different parameter γ and therefore is infinite divisible. Note that k is not supposed to be even number.

It is obvious that $f(t)$ is an analytic function in the strip $|Imt| < 1/a$ and can be analytically prolonged outside of it, but is not a meromorphic for non-integer γ. *The function $f(-t)$ possesses the same property as well.*

The adding of a location parameter into the distribution with characteristic function $f(t)$ has no impact on its infinite divisibility, but it will change the form of $g_{2k}(t)$.

Example 3.2. *Suppose now that*

$$f(t) = \frac{\exp\{ibt\}}{1-it}, \quad (b>0) \tag{3.2}$$

is a characteristics of shifted exponential distribution (a particular case of shifted gamma-distribution). Then both $f(t)$ and $g_2(t)$ are infinite divisible.

Proof. To (3.2) corresponds characteristic function

$$g_2(t) = \frac{e^{ibt}(2-2ib(i+t)-b^2(i+t)^2)}{(2+2b+b^2)(1-it)^3}. \tag{3.3}$$

We need to show (3.3) represent infinite divisible characteristic function. For the first glance it is not so because $g_2(t)$ has complex zeros. However, these zeros are outside of the strip of analyticity of $g_2(t)$, i.e. outside of the strip $|Im(t)| < 1$. The distributions with characteristic function g_2 is concentrated on semi-axis $x > b$. To verify infinite divisibility of g_2 let us use A.N. Kolmogorov representation of logarithm of the corresponding characteristic function:

$$\log g_2(t) = ict + \int_{-\infty}^{\infty} (e^{itu} - 1 - itu)\frac{dK(u)}{u^2},$$

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where c is a constant and $K(u)$ is a non-decreasing bounded function, $K(-\infty) = 0$. The finding of $K(u)$ is based on the inversion of the Fourier transform

$$-\frac{d^2}{dt^2}\log g_2(t) = \int_{-\infty}^{\infty} e^{itu} dK(u).$$

However,

$$-\frac{d^2}{dt^2}\log g_2(t) = -\frac{3}{(i+t)^2} + b^2\Big(\frac{1}{(-1+i+b(i+t))^2} + \frac{1}{(1+i+b(i+t))^2}\Big).$$

Inverting Fourier transform we find

$$dK(x) = e^{-x}\big(3 - 2e^{-x/b}\cos(x/b)\big) \quad \text{for} \quad x > 0$$

and 0 otherwise. The statement follows now from Kolmogorov representation.

□

Although we considered the case of shifted exponential distribution it is likely the similar hold for Gamma-distribution in general case.

The next example is connected to characteristic function of the negative binomial distribution

$$f(t,r,p) = \Big(\frac{p}{1-(1-p)e^{it}}\Big)^r, \quad 0 < p < 1, \quad r > 0. \tag{3.4}$$

However, not all values of the parameters give suitable condition. Let us give a special example.

Example 3.3. *Consider the function (3.4) for the case $p = 1/2$. It has form*

$$f(t,r) = (2 - e^{it})^{-r}, \quad r > 0. \tag{3.5}$$

Corresponding $g_2(t,r)$ is

$$g_2(t,r) = \frac{e^{it}}{(r+1)(2-e^{it})^{r+1}} + \frac{e^{2it}}{(2-e^{it})^{r+2}}$$

For the use of A.N. Kolmogorov representation we need its second logarithmic derivative:

$$-\frac{d^2}{dt^2}\log g_2(t,r) = \frac{2e^{it}(1+r)(8+4re^{it}+r(1+r)e^{2it})}{(-2+e^{it})^2(2+re^{it})^2} = \tag{3.6}$$

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$$= \sum_{k=1}^{\infty} \frac{k(2 + r - (-r)^k)}{2^k} e^{ikt}$$

If all the coefficients of the last series are non-negative then its sum corresponds to non-decreasing function K. In the opposite case g_2 is not infinite divisible. It is clear that all the coefficients are non-negative if and only if $0 < r \leq 1$.

Probably, the fact remains true for more general case of arbitrary $p \in (0, 1)$ and g_{2k}.

Let us give one more example.

Example 3.4. *Define*

$$f(t) = \exp\{\frac{1}{1 - it} - 1\} = \exp\{\frac{it}{1 - it}\}.$$

Obviously, $f(t)$ is infinite divisible characteristic function. Corresponding $g_2(t)$ has the following form

$$g_2(t) = \frac{e^{-t/(i+t)}(2it - 3)}{(i + t)^4}.$$

$g_2(t)$ is infinite divisible function.

Proof. Apply A.N. Kolmogorov representation again. As it was mentioned above, it is sufficient to define corresponding function $K(x)$ by its Fourier transform:

$$\int_{-\infty}^{\infty} e^{itx} K(x) = -\frac{d^2}{dt^2} \log g_2(t).$$

We have

$$-\frac{d^2}{dt^2} \log g_2(t) = -\frac{2i}{(i + t)^3} - \frac{4}{(i + t)^2} + \frac{4}{(3i + 2t)^2}$$

Applying inverse Fourier transform we obtain

$$\frac{dK}{dx} = xe^{-3x/2}\big((4 + x)e^{x/2} - 1\big) \quad \text{for} \quad x \geq 0$$

and 0 for $x < 0$. □

Given Examples shows that for g_{2k} to be infinite divisible $f(t)$ must have singularities. The types of them may be essentially different. The problem of full description corresponding functions remain open.

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References

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