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Calculation of the Higgs Mass for Quark and Lepton Electric Charges Swap Lie-Groupoid

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Abstract: Starting from the SU(2) group of weak interactions in the presence of Electric Charge Swap (ECS) symmetry, we show that ordinary and non-regular (ECS) leptons are related by the ECS rotational group SO(3). We find that many Standard Model (SM) algebras depend on the sin of the angle θs of the ECS rotational group SO (3). We call these ECSM algebras. Furthermore, the break of the gauge symmetry of the SM groupoid gives the massive ECS particle. We find that the ECS particle masses are related with the SM particle masses by $\sin\theta_s$. We also investigate the finite subgroups of the ECS Möbius transformations. We find that $\sin\theta_s$ could be derived from the ECS dihedral group D_F, which refers to the symmetry of the fermionic polygon (F-gon). The average value of the anchor of the SM algebroid depends on the fermionic Catalan numbers (C_F). Finally, we find that the ECS physics at loop level differs the SM physics. The ECSM mass is suppressed by the C_F numbers. For 24 fermions, the calculated one-loop radiative correction to the bare Higgs mass μ is 125GeV—a value very close to the experimental one.

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1. Introduction

In the electroweak (EW) theory, $SU(2)_L \times U(1)_Y$[1] is broken down to the electromagnetic gauge group $U(1)_{em}$ by the vacuum expectation value of an elementary scalar field φ. This scalar field should be part of the Grand Unification Theory (GUT)[2]. To produce a vacuum expectation value of a size that imparts to the observed W and Z boson a mass of 82 and 91 GeV, respectively, the Standard Model (SM) Higgs scalar field must obtain a negative mass term, of a size [1]:

$$-\mu_0^2 = -(100)^2 . \quad (1)$$

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Equation (1) is subjected to additive renormalization. Beyond tree level, the radiative corrections to the Higgs mass for the Higgs (H), top quark (t-$\bar{t}$), and boson (W, Z) loops in the SM [4-6], are shown in Figure 1:

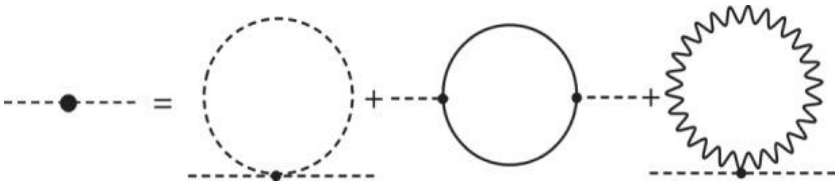

Fig. 1. One-loop corrections to the Higgs mass. The diagrams for the Higgs (H), top quark (t-$\bar{t}$), and boson (W,Z) loops are quadratically divergent, and make the Higgs mass highly UV-sensitive [7] C. Grojean, *Philos Trans A Math Phys Eng Sci*. 2015 Jan 13; 373(2032): 20140042).

Each diagram in Figure 1 shows a contribution to the self-energy $i\Sigma(p^2)$, where p is the four-momentum of the external particle [8]:

$$M_H^2 = \mu_0^2 + \sum(M_H^2)\,. \qquad (2)$$

In Equation (2), μ_0 is the bare Higgs mass. Using a cutoff to regularize the bare amplitudes [8], these radiative corrections lead to the well-known mass correction:

$$M_H^2 = \mu_0^2 + \frac{3\Lambda_C^2}{32\pi^2\upsilon^2}[M_H^2 + 2M_W^2 + M_Z^2 - 4m_t^2] + ... \quad , \qquad (3)$$

where m_t, $M_{W,Z}$, and M_H are the mass of the top quark, W, Z bosons, and Higgs bosons (H), respectively, υ is the vacuum expectation value of the Higgs potential in the SM, and Λ_C is the cutoff energy scale Λ_C [4-6]. The dots indicate logarithmic corrections at the cutoff energy scale Λ_C, as well as contributions independent of the cutoff energy scale, in the large Λ_C limit [8].

Λc can be the energy scale of a new physics coupled to the SM one. In particular, Λc can even be the Planck mass scale, M_p. On the other hand, it is unnatural to have $\Lambda_c >> m_w$, since everything is defined by the EW scale [9]. Equation (3) is the source of the usual fine-tuning problem in the SM: if $\Lambda_c \sim m_W$, then we must suppose that the tree-level μ^2 and the loop contributions cancel each other in order for $|\mu_0^2|$ to be $\sim m^2_W$ [9]. After the discovery of the Higgs boson (of a mass of 125 GeV) in the ATLAS and CMS experiments [10, 11], the SM would be a perfect theory but from the fine-tuning problem. There are two general ways to solve this problem: either by evoking some (super)symmetry to cancel out the huge terms [12], or by introducing some new physics, such as large extra dimensions [13], composite Higgs models[14], etc., at a scale not very far from the electroweak one, thus making the Λ_c to be small [7]. The observation of a light Higgs boson with properties consistent with the SM [10,11] has motivated much reexamination of the notion of the naturalness problem [7].

It is well known that the supersymmetric extension of the SM can solve the fine-tuning problem. Despite many efforts to search for supersymmetry at the Large Hadron Collider (LHC) experiments, the current LHC data do not offer unequivocal proof for the production of supersymmetric particles [15]. For this reason, we investigate the possibility of solving the fine-tuning problem without supersymmetry.

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In the SM, mass is generated due to spontaneous symmetry breaking in the Higgs sector. While the Higgs boson mass was introduced by Peter Higgs et al. [16] from the outset, the tachyon mass term breaks the scale invariance (conformal symmetry) explicitly [16]. However, if the conformal symmetry could be broken sufficiently 'softly' so that symmetry is restored at high energies, the Higgs mass would still be protected from the largest radiative corrections necessitated by the highest energies [17,18,19].

Recently P. Grang´ et.al [8] re-analysed the perturbative radiative corrections to the Higgs mass within the SM in the light of the Taylor-Lagrange renormalisation scheme. This scheme naturally leads to completely finite corrections, depending on an arbitrary scale. Nima Arkani-Hamed et.al [20] proposed the N-naturalness model, which presents a new solution to the electroweak hierarchy problem. The authors introduced N copies of the SM, with varying values of the Higgs mass parameter. This proposition generally yields a sector whose weak scale is parametrically removed from the cutoff by a factor of $1/\sqrt{N}$.

In the framework of fine symmetries, the Monster Group is related to the symmetries of a particular bosonic string theory on the Leech lattice [21]. However, there is no physical reason for the presence of the Monster Group or its subgroups in the Lagrangian: its appearance may merely be a coincidence. Another coincidence is that, in reduced Planck units, the Higgs mass is approximately $48|M|^{-1/3} = 125.5GeV$ (where $|M|$ is the order of the Baby Monster group). This suggests that the small size of the Higgs mass may be due to a redundancy caused by a symmetry of the extra dimensions, which must be divided out [22]. It is also possible that the hierarchy problem is a specific manifestation of Bayesian statistics[23-27] .

In the Fine-Tuning Naturalness (FTN) problem, we suppose that bare SM parameters are physically interpreted as "fundamental parameters," analogous to the microscopic lattice parameters of a Condensed Matter (CM) system [28]. Alternatively, bare SM parameters can be interpreted as unphysical "auxiliary parameters": if the Wilsonian Renormalization Group (RG) transformations are interpreted as invertible re-parametrisations, the bare parameters in High Energy Physics (HEP) and Condensed Matter Physics (CMP) are formally—but not physically—analogous [28]. Furthermore, it is possible that neither the bare mass, μ_0, nor the radiative corrections, δM^2, are directly measurable. Since these are probably not physical, there is no coincidence to be explained (Wetterich 1984[29]), (Bianchi & Rovelli 2010[30]).

For quarks and leptons, an Electric Charge Swap (ECS) symmetry has been proposed by the author [31]. A family of particular transformations may be continuous (e.g., the rotation of a circle) or discrete (e.g., the reflexion of a bilaterally symmetrical figure, or the rotation of a regular polygon) [31-34]. ECS transformation between ordinary families of leptons produces heavy, neutral, non-regular leptons of an O-order mass (TeV). These particles may form cold dark matter [31]. Furthermore, the ECS symmetry could explain certain properties of lepton families within the framework of superstring theories [35-39].

From the mathematical point of view, in R-Category—a category theory with invertible morphisms [40]—the geometric structures under consideration are always associated with local Lie brackets [,] on sections of some vector bundles (Lie algebroids [41,42]). Based on [41-43], in this article, we study the structure of transitive Lie algebroids as a mathematical framework for generalizing the formulation of a gauge theory through an action functional: the integral of a differential form on the algebroid [44]. On Atiyah Lie algebroids [45], the space of ordinary connections 1-forms corresponds with the Ehresmann connections on a

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principal fiber bundle P (see the S. Lazzarini and T. Masson in [46,47]). Cédric Fournel (2013) [44] proved that transitive Lie algebroids equipped with generalized connections contain scalar fields as algebraic parameters. These parameters, absent in differential geometry, have a role similar to that of the scalar field in the Higgs mechanism [16]. In higher-dimensional spacetime, the Double Field Theory (DFT) is a gravity theory with manifest T duality (Hull-Zwiebach, 2009 [48]). The DFT has gauge symmetry (described by the C bracket ($[\ ,\ ]_C$: see[48])), which defines the Vaisman-algebroid (Vaisman, 2013[49]).

This paper is structured as follows: In section 2, taking the SU(2) group of weak interactions in the presence of ECS symmetry as a starting point, we show that ordinary and non-regular (ECS) leptons are related by the ECS rotational SO(3) group. In section 3, we investigate a version of the SM algebroid whose anchor map depends on the sin of the ECS angle θ_s of the ECS rotational SO (3) group. We find many SM algebras that depend on $\sin\theta_s$; we call these algebras ECSM algebras. Furthermore, the broken gauge symmetry of the SM groupoid gives the massive ECS particle. We find that the ECS particle mass is related with the SM particle mass by $\sin\theta_s$. In section 4, we investigate the finite subgroups of the ECS-Möbius transformations. In this case, $\sin\theta_s$ could be originated from the ECS dihedral group D_F, which refers to the symmetry of the Fermionic polygon (F-gon). The average value of the anchor of the SM algebroid then depends on the fermionic Catalan numbers, C_F . Finally, in section 5, we find that the ECS physics at loop-level differs from the SM physics: the ECSM mass is suppressed by the numbers C_F. For 24-fermions, the calculation of one-loop radiative corrections to the bare Higgs mass μ is 125GeV, which is very close to the experimental value.

2. The global ECS symmetry for quarks and leptons

Hypothetical non-regular leptons are, a) a zero-charged version of the electron, $\tilde{e}^0$, and, b) a positively charged version of the electron neutrino, $\tilde{\nu}_e^+$. Non-regular leptons can, therefore, be obtained from the swap of electric charge between electrons and electron neutrinos in the internal space. We call these proposed non-regular leptons, electric charge swap (ECS) leptons [31].

Although ECS leptons have the same mass as ordinary-family leptons, they are distinguished from the latter by their different lepton numbers ($L_s = 1$ for ordinary leptons; $\bar{L}_s = -1$ for ordinary antileptons) and their electric charge (positive or neutral for ordinary leptons; negative or neutral for ordinary antileptons). We hypothesize that ECS leptons are produced from ordinary-family leptons when the latter enter the internal space: in these conditions, the properties of ordinary-family leptons change profoundly as these leptons lose their 'individuality' and swap their electric charge [31]. To formulate the swap of electric charge between ordinary leptons, we have to look for a global symmetry that characterises the swap process in the framework of 2-internal dimensions [31]. We consider the 2-sphere S^2 as a quotient space ($S^2 \equiv SU(2)_L / U(1)_\Upsilon$), and express it in terms of the new symmetry between the original lepton and the new ECS lepton doublets. We do this in the following steps [31]: First, we observe that both the ordinary lepton doublet, $l_0(x^\nu) = (e_L^-, \nu_e)$, and the ECS lepton doublet, $\tilde{l}_0(x^{\nu'}) = (\tilde{e}_L^0, \tilde{\nu}_e^+)$, can form the fundamental representation of $SU(2)_L$ [50]. This fundamental representation is given by:

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$$[I_\alpha, I_\beta] = i\varepsilon_{\alpha\beta\gamma} I_\gamma . \tag{4}$$

The generators are denoted as:

$$I_\alpha = \frac{1}{2}\tau_\alpha , \tag{5}$$

where

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

are the isospin versions of Pauli matrices.

The action of the latter on the new lepton states is represented by:

$$\tilde{e}_L^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tilde{\nu}_e^+ = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{7}$$

To link the two distinct types of lepton, ordinary and ECS leptons, we assume that neither the ordinary (L) nor the ECS (L_s) lepton numbers are conserved, while the overall number of leptons is conserved obligatorily.

$$L_{overall} = L_s + L = 0 . \tag{8}$$

$$L_s = \bar{L} , L_s(\nu_e^+) = \bar{L}(\mathrm{e}^+) = -1. \tag{9}$$

$$\bar{L}_s = L , \bar{L}_s(\tilde{\mathrm{e}}^0) = L(\nu_e) = 1 . \tag{10}$$

The quantum numbers of the new ECS leptons are given in Table 1 [31].

Table 1. Quantum numbers (weak ECS isospin I_s, charge Q, ECS hypercharge Y_s, ECS lepton number L_S) of the ECS leptons $\tilde{e}_L^0$, $\tilde{\nu}_e^+$.

New lepton	I	Is-z	Q	Y_s	L_s
$\tilde{\nu}_e^+$	½	½	1	1	-1
$\tilde{e}_L^0$	½	-½	0	1	-1

The next step is to define the group transformation that can account for the swap of electric charges between the electron and electron neutrino particles. The global ECS transformation must be derived from a transformation from:

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1) $SU(2)_I / U(1)_Y$, in which the fundamental representation of $SU(2)_I$ is $l_0(x^\nu) = (e_L^-, \nu_e)$, and $U(1)_Y$ is the symmetric group generated by hypercharge $Y = -1$
to
2) $SU(2)_{I_s} / U(1)_{Y_s}$, in which the fundamental representation of $SU(2)_{I_s}$ is $\tilde{l}_0(x^{\nu'}) = (\tilde{e}_L^0, \tilde{\nu}_e^+)$, and $U(1)_{Y_S}$ is the symmetric group generated by swap hypercharge $Y_S = 1$.

The quotient space *SU (2)/U (1)* is diffeomorphic to the unit 2-sphere S^2. Consequently, the swap of electric charges between electrons and electron neutrinos must be an automorphism of the 2-sphere to itself [31]. Since the two internal dimensions are endowed with the Fubini-Study[1] metric [51], [52], not all Möbius transformations (e.g., dilations and translations) are isometries. Therefore, the automorphism from $S^2 \equiv SU(2)/U(1)$ to itself, which causes the electric charge swap between the electrons and electron neutrinos, is given by the isometries that form a proper subgroup of the group of projective linear transformations $PGL_2(\hat{C})_{(Charge)}$—namely $PSU_{2(Charge)}$. Subgroup $PSU_{2(Charge)}$ is isomorphic to the global rotation group $SO(3)_{ECS}$ [51], [52],[31], which is the isometric group of the unit sphere in three-dimensional real space R^3. The automophism of the Riemann sphere $\hat{C}$ is given by:

$$Rot_{(ECS)}(\hat{\mathbb{C}}) = PSU_{2(Charge)} = SO(3)_{ECS},$$
$$\hat{\mathbb{C}} = \mathbb{C} \cup \infty = S^2, \tag{11}$$

where $\hat{\mathbb{C}}$ is the extended complex plane, $PSU_{2(Charge)}$ is the proper subgroup of the projective linear transformations, and global swap symmetry, $SO(3)_{ECS}$, is the group of rotations in three-dimensional vector space R^3. This can be consigned in the double fibration on a vector bundle of lines $\Im^2$ in the extended space (*ad infinitum*), that is to say, $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$. The universal cover of $SO(3)_{ECS}$ is the special unitary group $SU(2)_{I(I_S)}$ [31]. This group is also differomorphic to the unit 3 sphere S^3. We regard the ordinary and ECS leptons as different electric charge states of the same particle—analogous, that is, to the proton-neutron isotopic pair. Finally, in terms of global rotational symmetry between the original lepton and the proposed ECS lepton, the ECS 2 sphere S^2_{ECS} is given by:

$$S^2_{ECS} \equiv SU(2)_{I(I_S)} / U(1)_{Y(Y_S)} \text{ [31]}, \tag{12}$$

[1] The round metric of the 2-sphere can be expressed in stereographic coordinates as:

$G = \frac{dy_1^2 + dy_2^2}{(1+\varepsilon^2)^2}$, where $\varepsilon = \sqrt{y_1^2 + y_2^2}$. The metric G is the Fubini-Study metric of the 2-sphere [51], [52]

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where $SU(2)_{I(I_S)}$ is the special unitary group and $U(1)_{Y(Y_s)}$ is the symmetric group generated by hypercharge $Y(Y_s)$. Similar with hypothetical non-regular leptons, hypothetical non-regular quarks are, a) an 1/3-electrically charged version of the up (α) quark types, α̃, and, b) a -2/3-electrically charged version of the down (κ) quark types, κ̃ . Non-regular quarks can, therefore, be obtained from the swap of electric charge between up and down quark types. We call these proposed non-regular quarks electric charge swap (ECS) quarks. We regard ordinary and ECS quarks as different electric charge states of the same particle—analogous, that is, to the proton-neutron isotopic pair. Some quantum numbers of the new ECS quarks are shown in Table 2.

Table 2: Quantum numbers of the proposed ECS quarks.

New (ECS) quarks	Q: electric charge	I_{sz}: ECS isospin component	B_s: Baryonic number
ũ,c̃,t̃	1/3	-½	-1/3
d̃,s̃,b̃	-2/3	½	-1/3

The simplest way to realise the global $SO(3)_{ECS}$ group provided by Equation (11) is by adding the ECS electron $\tilde{e}_L^0$ and grouping it together with the ECS electron neutrino $\tilde{\nu}_e^+$, electron antineutrino $\bar{\nu}_e$, and electron e^- into a triplet, as follows:

$$T_{ECS}^L : \left\{ \left[\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} \tilde{\nu}_e^+ \\ \tilde{e}^0 \end{pmatrix}_L \right] \middle| \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L, \begin{pmatrix} \tilde{\nu}_e^+ \\ \tilde{e}^0 \end{pmatrix}_L \in V^2 \right\} \rightarrow \left\{ \begin{pmatrix} \tilde{\nu}_e^+ \\ \bar{\nu}_e \cos\theta_s + \tilde{e}^0 \sin\theta_s \\ e^- \end{pmatrix}_L \in V^3 \right\}, \tag{13}$$

where θ_s is an arbitrary real parameter, independent of space-time coordinates, which represents the lepton and ECS quark mixing angle. There remains a left-hand singlet:

$$\begin{aligned} &S_{ECS}^L : \left\{ \left[\left(e^-\right)_R, \left(\tilde{\nu}_e^+\right)_R \right] \middle| \left(e^-\right)_R, \left(\tilde{\nu}_e^+\right) \in V_{row}^1 \right\} \\ &\rightarrow \left\{ (\bar{\nu}_e \cos\theta_s - \tilde{e}^0 \sin\theta_s)_L \in \tilde{V}_{row}^1 \right\} \end{aligned} . \tag{14}$$

The muon (μ^-) and its antineutrino ($\bar{\nu}_\mu$), as well the tau (τ^-) and its antineutrino ($\bar{\nu}_\tau$), can be introduced in a manner similar to the above, at the cost of the four ECS leptons ($\tilde{\mu}_L^0$ $\tilde{\nu}_\mu^+$, $\tilde{\tau}_L^0$ $\tilde{\nu}_\tau^+$). For this introduction, we have to add an 1/3-electrically charged version of the up (α) quark types (ECS-α quarks) and group them together with the -2/3-electrically charged version of the down (κ) quark types (ECS-κ quarks), and the anti-up ($\bar{\alpha}$) and down (κ) quark types into a triplet:

$$T_{Q_L} = \begin{pmatrix} \tilde{\alpha} \\ \bar{\alpha} \sin\theta_s + \tilde{\kappa} \cos\theta_s \\ \kappa \end{pmatrix}_L . \tag{15}$$

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There remains a left-hand singlet:

$$S_{Q_L} = \left(\tilde{\kappa}\sin\theta_s + \bar{\alpha}\cos\theta_s\right)_L . \tag{16}$$

Equations (13) and (15) form the fundamental representation space of the $SO(3)_{ECS}$ group:

$$\left\{\begin{pmatrix} \tilde{\nu}_e^+ \\ \bar{\nu}_e \sin\theta_s + \tilde{e}^0 \cos\theta_s \\ e^- \end{pmatrix}, \begin{pmatrix} \tilde{\nu}_\mu^+ \\ \bar{\nu}_\mu \sin\theta_s + \tilde{\mu}^0 \cos\theta_s \\ \mu^- \end{pmatrix}, \begin{pmatrix} \tilde{\nu}_\tau^+ \\ \bar{\nu}_\tau \sin\theta_s + \tilde{\tau}^0 \cos\theta_s \\ \tau^- \end{pmatrix}\right\}_L , \tag{17}$$

$$\left\{\begin{pmatrix} \tilde{u} \\ \bar{u}\sin\theta_s + \tilde{d}\cos\theta_s \\ d \end{pmatrix}, \begin{pmatrix} \tilde{c} \\ \tilde{c}\sin\theta_s + \tilde{s}\cos\theta_s \\ s \end{pmatrix}, \begin{pmatrix} \tilde{t} \\ \tilde{t}\sin\theta_s + \tilde{b}\cos\theta_s \\ b \end{pmatrix}\right\}_L . \tag{18}$$

Equations (17) and (18) are 3-component vectors for the linear combination between fermions and ECS fermions. The representative matrix of a general element of the $SO(3)_{ECS}$ group can be written as:

$$O_{ECS} = \mathrm{e}^{-i\sum_\alpha \theta_s^\alpha X_\alpha}, \ \alpha=(1,2,3), \tag{19}$$

with θ^α = $(\theta^1,\theta^2,\theta^3)$ being arbitrary real group parameters independent of space-time coordinates, and O_{ECS} being an orthogonal 3 × 3 matrix:

$$O_{ECS}^T O_{ECS} = I = O_{ECS} O_{ECS}^T . \tag{20}$$

The three basic ECS rotation matrices that rotate fermions by an angle θ_s^α (α = 1, 2, 3) about the x-, y-, or z-axis in three internal dimensions can be explicitly written as follows:

$$O_{ECS} = \mathrm{e}^{-i\theta_s^\alpha X_\alpha} . \tag{21}$$

In this representative space, the representative matrices of the generators of the $SO(3)_{ECS}$ group are denoted by X_α (α = 1, 2, 3):

$$X_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{22}$$

These matrices satisfy the following condition:

$$[\mathrm{X}_\alpha, \mathrm{X}_\beta] = iC_{\alpha\beta\gamma} X_\gamma , \tag{23}$$

where $C_{\alpha\beta\gamma}$ are structure constants of the $SO(3)_{ECS}$ group. Generator X_α is Hermitian and traceless:

$$X_\alpha^\dagger = X_\alpha \ , \ \mathrm{Tr}\mathrm{X}_\alpha = 0 . \tag{24}$$

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To calculate the trace, we used the sum of the diagonal elements of the rotation matrix given by Equation (21):

$$Tr(O_{ECS}) = 2\cos\| \theta_s \| +1 . \tag{25}$$

From Equation (25), it follows that the arbitrary absolute value of the ECS angle is:

$$\begin{aligned} &\theta_s = \| \theta_s \| = \arccos\left(\frac{Tr(O_{ECS})-1}{2}\right) = \frac{\pi}{2} - \arcsin\left(\frac{Tr(O_{ECS})-1}{2}\right), \\ &F(\theta_s) \equiv \sin\| \theta_s \| = \frac{Tr(O_{ECS})}{2} \end{aligned} . \tag{26}$$

3. The Standard Model of particle physics in the R-Category theory

In general, the U(n) gauge group cannot be identified as a more familiar-looking manifold. It is an 'iterated extension' of the odd-dimensional sphere $S^1, S^3, ..., S^{2n-1}$, and the rational homotopic equivalent to the product $S^1 \times S^3 \times ... \times S^{2n-1}$[53-55]. This means that the U(n) gauge group has the same rational co-homology and rational homotopy groups as this product; however, it is generally not homeomorphic or diffeomorphic to this product[53-55]. 'Iterated extension' means that the unitary groups fit into fiber sequences which are ultimately built from odd spheres, starting with:

$$SU(\mathrm{n}) \to U(\mathrm{n}) \overset{\det}{\to} S^1, \tag{27}$$

and continuing with:

$$SU(\mathrm{n}-1) \to SU(\mathrm{n}) \to S^{2n-1} . \tag{28}$$

The first sequence is a short exact sequence of Lie groups and splits smoothly; therefore, the U(n) is diffeomorphic to $SU(\mathrm{n}) \times S^1$[53-55]. In particular, the SM gauge group U(2) [56-57] is diffeomorphic to the product $S^3 \times S^1$ manifold. However, this is not an isomorphism of groups, since U(n) is a semidirect rather than a direct product. The vector spaces of

$$\sum_{\alpha} \varepsilon^{\alpha}(x) T_{\alpha} = \alpha(x) I + \beta(x)^a \tau_a , \ \alpha=(1,2,3) , \tag{29}$$

together with the commutator relation

$$[I_2, \tau_\alpha] = 0, [\tau_1, \tau_2] = 2i\tau_3, \ \alpha=(1,2,3) \tag{30}$$

$$[\tau_2, \tau_3] = 2i\tau_1, [\tau_3, \tau_1] = 2i\tau_2 , \tag{31}$$

are referred to as the SM algebra $\aleph = u(2) \cong \mathbb{R}^4$. I_2 are the 2×2 unity matrix and the 3-tuple τ (Equation.(31)) of the Hermitian and traceless matrices, respectively; $\varepsilon^a(x)$ are the real arbitrary group parameters of the local U(2) gauge group; and α(x) and β(x) are the real arbitrary group parameters of the local SU(2) and U(1) gauge groups, respectively.

From the mathematical point of view, in R-Category—a category theory with invertible morphisms [40]—the geometric structures we consider here are always associated with local

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Lie brackets [,] on sections of some vector bundles (Lie algebroids). A Lie algebroid over a manifold B is defined to be a vector bundle A over B with a Lie algebra structure on its space of smooth sections [,], together with a bundle map, ρ (called the anchor of the Lie algebroid), from A to the tangent bundle TB. Here, we definite the ECS-Standard Model (ECSM) algebra $\tilde{\aleph} = \tilde{u}(2) \cong \mathbb{R}^4$ from the SM algebroid A_{SM} over the product $M = S^3 \times S^1$ manifold which satisfies the conditions:

$$[\rho(\mathrm{I}_2), \rho(\tau_\alpha)] = \rho([\mathrm{I}_2, \tau_\alpha]) \tag{49}$$

$$[\mathrm{I}_2, \varphi\tau_\alpha] = \varphi[\mathrm{I}_2, \tau_\alpha] + (\rho(\mathrm{I}_2) \cdot \varphi)\tau_\alpha \qquad , \alpha=(1,2,3) \tag{50}$$

$$[\rho(\tau_\alpha), \rho(\tau_\beta)] = \rho([\tau_\alpha, \tau_\beta]) \tag{51}$$

$$[\tau_\alpha, \varphi\tau_\beta] = \varphi[\tau_\alpha, \tau_\beta] + (\rho(\tau_\alpha) \cdot \varphi)\tau_\beta \ , \quad \beta=(1,2,3) \tag{52}$$

(with τ_α being smooth sections of A_{SM}, and ϕ being a smooth function on $M = S^3 \times S^1$), when the anchor's smooth multiplication factor of the SM-algebroid, A_{SM}, is given by Equation (26), and the ECS generators are derived from the SM generators by the anchor map:

$$\tilde{I}_2 = \rho(I_2) = \mathrm{F}(\theta_s) I_2 = I_2 \sin\theta_s \in \Gamma(TM)\,, \tag{53}$$

$$\tilde{\tau}_2 = \rho(\tau_2) = \mathrm{F}(\theta_s)\tau_2 = \tau_2 \sin\theta_s \in \Gamma(TM)\,. \tag{54}$$

In the above equations, $\Gamma(TM)$ indicates the sections of the tangent bundle TM,

$$F = \mathrm{F}(\theta_s) = \sin\theta_s \tag{55}$$

indicates the smooth function on $S^3 \times S^1$, and θ_s are the arbitrary ECS-angles if we parametrise the unit 3-sphere by hyperspherical coordinates (x_0, x_1, x_2, x_3) and use $(\psi, \theta_s, \varphi)$. By restricting the domain of Equation (26), we obtain:

$$\begin{aligned} &\sin : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1,1], \\ &\theta_s \mapsto \sin\theta_s \end{aligned}\,. \tag{56}$$

This function is both one-one and onto; therefore, it has inverse function:

$$\begin{aligned} &\sin^{-1} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ &\theta_s \mapsto \sin^{-1}\theta_s \end{aligned}\,. \tag{57}$$

By Equations (56) and (57), the anchor of the SM-algebroid, A_{SM}, is thus both an one-one and onto map between the SM and ECS generators.

$$[\rho(T_\alpha), \rho(T_\beta)] = [\mathrm{F}(\theta_s)T_\alpha{}', \mathrm{F}(\theta_s)T_\beta] = \mathrm{F}(\theta_s)^2[T_\alpha{}', T_\beta] = \frac{\mathrm{F}(\theta_s)^2}{\mathrm{F}(\theta_s) \pm \varepsilon}[T_\alpha, T_\beta] = i\frac{\mathrm{F}(\theta_s)^2}{\mathrm{F}(\theta_s) \pm \varepsilon}T_\gamma\,, \tag{58}$$

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$T_\alpha{}' \to \frac{T_\alpha}{\mathrm{F}(\theta_s) \pm \varepsilon}$;the changes of scale of the generator T_α, and ε an infinitesimal positive quantity. $\rho\left([T_\alpha, T_\beta]\right) = \rho\left(iT_\gamma\right) = iF(\theta_s)T_\gamma$ (59).For $\varepsilon \to 0$ by Equations (58) and (59) we have: $[\rho(T_\alpha), \rho(T_\beta)] = \rho\left([T_\alpha, T_\beta]\right)$.

Hence Equations (49) and (51) define the ECSM algebra $\tilde{\aleph} = \tilde{u}(2) \cong \mathbb{R}^4$ in terms of the SM algebra $\aleph = u(2) \cong \mathbb{R}^4$ as follows:

$$[\tilde{I}_2, \tilde{\tau}_\alpha] = \mathrm{F}(\theta_s)[I_2, \tau_\alpha] = \sin\theta_s [I_2, \tau_\alpha] = 0 \ , \tag{60}$$

$$[\tilde{\tau}_\alpha, \tilde{\tau}_\beta] = \mathrm{F}(\theta_s)[\tau_\alpha, \tau_\beta] = \sin\theta_s [\tau_\alpha, \tau_\beta] \, . \tag{61}$$

Properties

The ECSM algebra is the $\mathrm{F}(\theta_s)$-valued SM algebra in $\mathrm{A_{SM}}$. Consequently, we have the following properties for the $\tilde{\aleph} = \tilde{u}(2) \cong \mathbb{R}^4$ algebra:

1. *For angle θ_s = 0, we find an ECS trivial algebra {0} and the usual SM algebra.*

2. *For angle $\theta_s = \pm\pi/2$, we find the usual SM-algebra.*

3. *For every other value of angle θs, we find many SM algebras that depend on ECS angle θs. These are called ECSM algebras.*

The terms $\left(\rho(I_2)\cdot\phi\right)$ and $\left(\rho(\tau_\alpha)\cdot\phi\right)$ in Equations (50) and (52) correspond to the Lie derivatives of ϕ with respect to I_2 and τ_α:

$$\mathfrak{I}_{\rho(I_2)}\phi = \sin\theta_s [\mathrm{I}_2, \phi] = 0 \, , \text{where} [\mathrm{I}_2, \phi] = 0 \, , \tag{62}$$

$$\mathfrak{I}_{\rho(\tau_\alpha)}\phi = \sin\theta_s \tau^\alpha \partial_\alpha \phi \, . \tag{63}$$

For a gentle smooth function on $M = S^3 \times S^1$ that satisfies the eigenvalue equation $\partial_\alpha \phi = im\phi$ with eigenvalues ($m = 0, \pm1, \pm2, ...$), Equations (50) and (52) for the SM algebroid $\mathrm{A_{SM}}$ become:

$$[\mathrm{I}_2, \phi\tau_\alpha] = 0 \, , \tag{64}$$

$$[\tau_\alpha, \phi\tau_\beta] = \phi[\tau_\alpha, \tau_\beta](1 + m\mathrm{F}(\theta_s)) \, , \text{where} \ m = 0, \pm1, \pm2, ... \tag{65}$$

For ($\theta_s = 0$) or ($m = 0$), we find the usual SM-algebra. The other possibilities generated by the eigenvalues m are under investigation for a further paper [58]. Following Mark Bugden, Peter Bouwkneg (2018)[59] and Cédric Fournel (2013)[44],when $\mathrm{A_{SM}}$ is an SM algebroid, we can use the anchor map (Equation ()) to lift any vector bundle connection (∇_μ) on $\mathrm{A_{SM}}$, to a so-called A connection (${}^A\nabla_\mu$) on $\mathrm{A_{SM}}$:

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$$ {}^{A}\nabla : \Gamma(A) \otimes \Gamma(A) \to \Gamma(A) . \tag{66} $$

In this paper, we use a specific A connection, given by:

$$ {}^{A}\nabla_{\tilde{T}_\alpha \mu}(\mathrm{T}_\beta) := \nabla_{\rho(\mathrm{T}_\alpha)\mu}(T_\beta) = \rho(\mathrm{T}_\alpha)\nabla_\mu(T_\beta) \ , \tag{67} $$

where $\nabla_\mu(T_\beta)$ is the bundle connection. Once we have an A connection, we can define the A curvature of ${}^{A}\nabla_\mu$ as follows:

$$ \begin{aligned} {}^{A}\tilde{F}_{\gamma\mu\nu} &:= [\nabla_{\rho(T_\alpha)\mu}\nabla_{\rho(T_\beta)\nu}] - \nabla_{[\rho(T_\alpha)\nu,\rho(T_\beta)\nu]} = [\rho(T_\alpha)\nabla_\mu \rho(T_\beta)\nabla_\nu] \\ &-[\rho(T_\alpha),\rho(T_\beta)]\nabla_{[\mathrm{e}_\mu,\mathrm{e}_\nu]} = \rho(T_\gamma)\mathrm{F}_{\mu\nu} + \rho(T_\gamma)\nabla_\mu\nabla_\nu \end{aligned} , \tag{68} $$

where

$$ F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] - \nabla_{[\mathrm{e}_\mu,\mathrm{e}_\nu]} , \tag{69} $$

is the curvature of the bundle connection ∇_μ.

3.1. The ECSM-groupoid $\tilde{U}(2)$

Following Marius Crainic and Rui Loja Fernandes (2003)[42], we now deduce the known inerrability.

1. Lie algebra bundles: For Lie algebroids with zero anchor map (Lie algebra bundles), the orbits are the points of M. Therefore, the conditions of the main theorem [42] are trivially satisfied, and we obtain the results of Douady and Lazard (1966) [60]:

Corollary 1 [42]**:** Any Lie algebra bundle is integrable to a Lie group bundle.

Corollary 2**:** The ECSM algebra is integrable to the ECSM group $\mathrm{U}(2)_{\mathrm{ECS}}$.

By Corollary 1, the SM algebra is integrable to a SM group. By the anchor of the SM algebroid A_{SM} (Equations (53-54)), the SM and ECSM algebras are integrable to the SM group U(2) and ECSM group $\mathrm{U}(2)_{\mathrm{ECS}}$, respectively, up to the anchor homomorphic surjection between the SM and ECSM algebra.

From equations (29) and (53–54), we obtain the vector spaces of

$$ \sum_\alpha \varepsilon^\alpha(x)\tilde{T}_\alpha = \alpha(x)\tilde{I}_2 + \beta(x)^a \tilde{\tau}_a \ , \ \alpha=(1,2,3,4), \tag{70} $$

together with the commutator. Equations (70) and (60), (61) are referred to the ECSM algebra $\tilde{\aleph} = \tilde{u}(2) \cong \mathbb{R}^4$. We observe that:

$$ \begin{aligned} &\alpha(x)\tilde{I}_2(\theta_S) + b(x)\cdot\tilde{\tau}(\theta_S) = \alpha(x)\mathrm{F}(\theta_S)I_2 + b(x)\cdot F(\theta_S)\tau \\ &= F(\theta_S)\begin{pmatrix} (\alpha(x)+b_3(x)) & (b_1(x)-ib_2(x)) \\ (\alpha(x)+b_3(x)) & (\alpha(x)-b_3(x)) \end{pmatrix} = \tilde{\alpha}(x,\theta_S)I_2 + \tilde{b}(x,\theta_S)\cdot\tau \end{aligned} , \tag{71} $$

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with being the real arbitrary ECS parameters. These parameters are as follows:

$$\tilde{a}(x,\theta_s)=a(x)\rho(\theta_s),\tilde{b}_1(x,\theta_s)=b_1(x)\rho(\theta_s),$$
$$\tilde{b}_2(x,\theta_s)=b_2(x)\rho(\theta_s),\tilde{b}_3(x,\theta_s)=b_3(x)\rho(\theta_s), \quad (72)$$

Here, $F(\theta_s)$ is the smooth function on $S^3\times S^1$, given by the Equation(55); θ_s is the ECS mixing angle, which is strictly a global parameter; and $a(x),b_1(x),b_2(x),b_3(x)$ are the real arbitrary group parameters of the local U(2) gauge group [56–57].

A representation of the SM gauge groupoid $\tilde{U}(2)$ on a vector bundle E→M induces a representation ${}^{A}\nabla$ of A_{SM} on E→M, defined by the following system of partial differential equations (PDE):

$$\nabla_{\tilde{\varepsilon}_\alpha}\tilde{U}(\tilde{\varepsilon}_\alpha(x,\theta_s))=\left(\frac{d}{d\tilde{\varepsilon}_\alpha}\tilde{U}(\tilde{\varepsilon}_\alpha(x,\theta_s))|_{\tilde{\varepsilon}_\alpha=0}\right)\tilde{U}(\tilde{\varepsilon}_\alpha(x,\theta_s))=i\mathrm{T}^\alpha\tilde{U}(\tilde{\varepsilon}_\alpha(x,\theta_s)), \quad (73)$$

where

$$\left(\frac{d}{d\tilde{\varepsilon}_\alpha}\tilde{U}(\tilde{\varepsilon}_\alpha(x,\theta_s))|_{\tilde{\varepsilon}_\alpha=0}\right)=i\mathrm{T}^\alpha,\ \alpha=(1,2,3,4). \quad (74)$$

T^α are the generators of the SM algebra, given by Equation (29). Using Equations (71)–(72), the solution of the PDE system is given by:

$$\tilde{U}(\mathrm{x},\theta_S)=\left\{\exp(i\tilde{\alpha}(x,\theta_S)I_2)\exp(i\tilde{b}(x,\theta_S)\cdot\tau)\,|\,\tilde{\alpha}(x,\theta_S)\in\mathbb{R},\theta_S\in\mathbb{R},\tilde{b}(x,\theta_S)\in\mathbb{R}^3\right\}\in\tilde{U}(2) \quad (75)$$

with $\tilde{U}(2)$ being the SM gauge groupoid of two dimensions. Therefore, it follows that

$$\tilde{U}(\mathrm{x},\theta_S)=\left\{\exp(i\tilde{\alpha}(x,\theta_S))\exp(i\tilde{b}(x,\theta_S)\cdot\tau)\,|\,\tilde{\alpha}(x,\theta_S)\in\mathbb{R},\theta_S\in\mathbb{R},\tilde{b}(x,\theta_S)\in\mathbb{R}^3\right\},$$
$$=\tilde{\mathrm{U}}(\mathrm{x})_1\,\tilde{\mathrm{U}}(\mathrm{x})_2\in\tilde{U}(1)\times S\tilde{U}(2) \quad (76)$$

where

$$\tilde{U}(\mathrm{x},\theta_S)_1=\left\{\exp(i\tilde{\alpha}(x,\theta_S))\,|\,\tilde{\alpha}(x,\theta_S),\in\mathbb{R},\theta_S\in\mathbb{R}\right\}\in\tilde{U}(1), \quad (77)$$

$\tilde{U}(1)$ is the unitary gauge groupoid in one dimension, generated by phase I_2 [56–57], and $S\tilde{U}(2)$ is the special unitary gauge groupoid in two dimensions, generated by the 3-tuple $\tau_\alpha=(\tau_1,\tau_2,\tau_3)$ of the Hermitian and traceless matrices, respectively[56–57]. By comparing equations (76) and (77), we have:

$$\tilde{U}(2)=\tilde{U}(1)\times S\tilde{U}(2). \quad (78)$$

For the Equations (71)–(72) and (75), we have:

$$\tilde{U}(\mathrm{x},\theta_S)=\left\{\exp(iF(\theta_S)\alpha(x)I_2)+(iF(\theta_S)b(x)\cdot\tau)\,|\,\alpha(x)\in\mathbb{R},\theta_S\in\mathbb{R},b(x)\in\mathbb{R}^3\right\},$$
$$=U(\mathrm{x})^{\mathrm{F}(\theta_s)}\in\tilde{U}(2) \quad (79)$$

where $U(\mathrm{x})\in U(2)$ [56–57],

$$\tilde{U}(\mathrm{x},\theta_S)=\left\{\exp(iF(\theta_S)\alpha(x))\exp(iF(\theta_S)b(x)\cdot\tau)\,|\,\alpha(x)\in\mathbb{R},\theta_S\in\mathbb{R},b(x)\in\mathbb{R}^3\right\},$$
$$=\mathrm{U}(\mathrm{x})_1^{\mathrm{F}(\theta_s)}\,\mathrm{U}(\mathrm{x})_2^{\mathrm{F}(\theta_s)}\in\tilde{U}(1)\times S\tilde{U}(2) \quad (80)$$

where

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$$\tilde{U}(\mathrm{x},\theta_S)_1 = \{\exp(iF(\theta_S)\alpha(x)) \,|\, \alpha(x), \in \mathbb{R}, \theta_S \in \mathbb{R}\} = U(\mathrm{x})_1^{\mathrm{F}(\theta_s)} \in \tilde{U}(1)\,, \tag{81}$$

$$\tilde{U}(\mathrm{x},\theta_S)_2 = \left\{\exp(iF(\theta_S)b(x)\cdot\tau) \,|\, \theta_S \in \mathbb{R}, b(x) \in \mathbb{R}^3\right\} = U(\mathrm{x})_2^{\mathrm{F}(\theta_s)} \in S\tilde{U}(2)\,, \tag{82}$$

and $U(\mathrm{x})_1 \in U(1)\,, U(\mathrm{x})_2 \in SU(2)$ [56–57].

Therefore from equations (79), (80),we have:

$$U(\mathrm{x})^{\mathrm{F}(\theta_s)} = \mathrm{U}(\mathrm{x})_1^{\mathrm{F}(\theta_s)}\, \mathrm{U}(\mathrm{x})_2^{\mathrm{F}(\theta_s)}\,. \tag{83}$$

We observe that the $\tilde{U}(2), \mathrm{S}\tilde{\mathrm{U}}(2)$, and $\tilde{U}(1)$ gauge groupoids are the $\mathrm{F}(\theta_s)$-valued U(2), SU(2), and U(1) gauge groups [56–57] in θs.

Properties

The gauge groupoid $\tilde{U}(2)$, therefore, has the following properties:

1. *For angle θs = 0, we obtain the trivial group* $\{1\}$.

2. *For angle θs =* $\pm\,\pi/2$*, we obtain the usual SM gauge group* $U(2)$.

3. *For every other value of angle θs, we obtain many SM gauge groups* $U(2)$ *that depend on the ECS angle θs; therefore, these groups are called the ECSM gauge group* $U(2)_{ECS}$.

We regard the ECSM gauge group sectors as mirror SM models [61-63] that are suppressed for small values of the ECS angle θ_s. These groups will be explored in future work [64].

The ECS-angle θ_s in the anchor map (Equation (53-54)) is strictly a global parameter, and may originate form a different group. In this case, the ECS angle θ_s could originate either from the global $\mathrm{SO(3)_{ECS}}$ group or from the finite subgroups of the ECS Möbius transformations (see below). Furthermore, as Weinstein article illustrates [42], there is no assumption that a gauge transformation actually extends to the entire object U(2)*:* it may be that the gauge symmetry does not extend globally but affects only a part of U(2), while the ECS symmetry extend globally.

3.2. The Lagrangian of the gauge groupoid $\tilde{U}(2)$

The A connection coefficient of the *gauge groupoid* $\tilde{U}(2)$ stems from the A connection:

$${}^{A}\tilde{A}_{\mu\alpha\beta}(x) = \tilde{\mathrm{T}}_\alpha({}^{A}\nabla_\mu \mathrm{T}_\beta) = \rho(\mathrm{T}_\alpha)\nabla_\mu(T_\beta) = \sin\theta_s\, \mathrm{T}_\alpha(\nabla_\mu \mathrm{T}_\beta) = F(\theta_s)A_{\mu\alpha\beta}(\mathrm{x})\ , \tag{84}$$

where

$$A_{\alpha\beta\mu}(\mathrm{x}) = \mathrm{T}_\alpha(\nabla_\mu \mathrm{T}_\beta) \tag{85}$$

is the SM connection coefficient, and $F(\theta_s)$ is the smooth function on $S^3 \times S^1$, given by Equation (55). The gauge covariant derivative is as follows:

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$$D_\mu(x,\theta_s)=\partial_\mu-ig\tilde{A}_\mu(x,\theta_s)=\partial_\mu-igF(\theta_s)A_\mu(x)\,. \tag{86}$$

The transformation of $A_\mu(x,\theta_s)$ is derived from the following equation:

$$D'_\mu(x,\theta_s)=\mathrm{e}^{i\varepsilon(x,\theta_s)}\,D_\mu(x,\theta_s)\,. \tag{87}$$

After same matrix manipulations, the solution of Equation (87) for $\tilde{A}'_\mu(x,\theta_s)$ in terms of $\tilde{A}_\mu(x,\theta_s)$ is:

$$\tilde{A}'_\mu(x,\theta_s)=\mathrm{e}^{i\varepsilon(x,\theta_s)}\,\tilde{A}_\mu(x,\theta_s)\mathrm{e}^{-i\varepsilon(x,\theta_s)}-\frac{1}{g}\partial_\mu\varepsilon(x,\theta_s)\,, \tag{88}$$

where $\mathrm{e}^{-i\varepsilon(x,\theta_s)}$ is the inverse of the matrix $\mathrm{e}^{i\varepsilon(x,\theta_s)}$. The strength of the ECSM and ordinary SM gauge fields $\tilde{A}^\alpha_\mu(x,\theta_s)$ and $A^\alpha_\mu(x)$ are defined as follows:

$$\begin{aligned}&\tilde{\mathrm{F}}_{\mu\nu}(x,\theta_s)=\frac{1}{-g}\Big[D_\mu(x,\theta_s),D_\nu(x,\theta_s)\Big]=\partial_\mu\tilde{A}_\nu(x,\theta_s)-\partial_\nu\tilde{A}_\mu(x,\theta_s)\\&-ig\Big[\tilde{A}_\mu(x,\theta_s),\tilde{A}_\nu(x,\theta_s)\Big]=F(\theta_s)\Big(\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)-igF(\theta_s)\Big[A_\mu(\mathrm{x}),A_\nu(x)\Big]\Big)\,.\\&=F(\theta_s)\,\mathrm{F}_{\mu\nu}(x)\end{aligned} \tag{89}$$

Similarly, $\tilde{\mathrm{F}}_{\mu\nu}(x,\theta_s)$ and $\mathrm{F}_{\mu\nu}(x)$ can also be expressed as linear combinations of generators:

$$\tilde{F}_{\mu\nu}(x,\theta_s)=\tilde{F}^\alpha_{\mu\nu}(x,\theta_s)\mathrm{T}_\alpha=F^\alpha_{\mu\nu}(x)\tilde{\mathrm{T}}_\alpha=F(\theta_s)\mathrm{F}^\alpha_{\mu\nu}(x)\mathrm{T}_\alpha=F(\theta_s)\mathrm{F}_{\mu\nu}(x)\,. \tag{90}$$

The transformation law for the matrix $\tilde{F}_{\mu\nu}(x,\theta_s)$ is:

$$\tilde{F}'_{\mu\nu}(x,\theta_s)=\mathrm{e}^{i\varepsilon(x,\theta_s)}\,\tilde{F}_{\mu\nu}(x,\theta_s)\mathrm{e}^{-i\varepsilon(x,\theta_s)}\,. \tag{91}$$

Thus, we can write down a kinetic energy term as follows:

$$\Im_{ECS}=-\frac{1}{4}\tilde{F}^\alpha_{\mu\nu}(x,\theta_s)\tilde{F}^{\mu\nu}_\alpha(x,\theta_s)\,. \tag{92}$$

The Lagrangian density of the ECSM is given by:

$$\begin{aligned}&\Im_{ECSM}(x,\theta_s)=-\bar{\psi}[\gamma^\mu(\partial_\mu-ig\tilde{A}^\alpha_\mu(x,\theta_s)T_\alpha)+m]\psi-\frac{1}{4}\tilde{\mathrm{F}}^{\alpha\mu\nu}(x,\theta_s)\tilde{\mathrm{F}}_{\alpha\mu\nu}(x,\theta_s)\\&=-\bar{\psi}[\gamma^\mu(\partial_\mu-igF(\theta_s)A^\alpha_\mu(x)T_\alpha)+m]\psi-\frac{\mathrm{F}(\theta_s)^2}{4}\mathrm{F}^{\alpha\mu\nu}(\mathrm{x})\,\mathrm{F}_{\alpha\mu\nu}(x)+\mathrm{O}(\rho(\theta_s)^4)\end{aligned}\,. \tag{93}$$

We see that the Lagrangian is given in the initial form before the spontaneous breaking of the groupoid $\tilde{U}(2)$ symmetry. The $\tilde{U}(1)_Y$ ECS gauge groupoid is implicitly assumed in the second term of the summation over $S\tilde{U}(2)_L$. The first term represents the kinetic terms and the gauge interactions of fermions, provided by the covariant derivative(s).

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3.2. ECSM gauge field mass

It turns out that the concept of spontaneous symmetry breaking plays an important role in the proposed theory of ECS electroweak interaction. The broken large groupoid symmetry

$$S\tilde{U}(2)_L \times \tilde{U}(1)_Y \to \tilde{U}(1)_{EM} \tag{94}$$

gives the massive ECS particles $\tilde{W}$ and $\tilde{Z}$ bosons. In equations (80) and (81), we observe that the spontaneous breaking of the large groupoid symmetry can only occur when the symmetry of electroweak interaction breaks spontaneously:

$$\mathrm{SU(2)_L \times U(1)_Y \to U(1)_{EM}} \tag{95}$$

Equation (95) predicts the massive particles W and Z bosons, whose correct mass has already been known since 1983 [4–6]. In Equation (81), $\tilde{U}(1)_{EM}$ is the ECS gauge groupoid of electromagnetism, which is the $\mathrm{F}(\theta_s)$-valued $\mathrm{U(1)_{EM}}$ common electromagnetism in θs. Therefore, we may view the ECS electromagnetism sectors of the $\tilde{U}(1)_{EM}$ gauge groupoid as the mirror sectors [61-63] that are suppressed for small ECS angles θ_s (a proposition to be explored in further work [64]). The most general Lagrangian consistent with the gauge invariance, Lorentz invariance, and renormalizability of the ECS gauge groupoid $\tilde{U}(1) \times S\tilde{U}(2)$:

$$\begin{aligned} \Im_{ECS} &= -\frac{1}{2}|(\partial_\mu - i\tilde{A}_\mu \cdot t^{(\phi)} - i\tilde{B}_\mu y^{(\phi)})\tilde{\phi}|^2 - \frac{\mu^2}{2}\tilde{\phi}^*\tilde{\phi} - \frac{\lambda}{4}(\tilde{\phi}^*\tilde{\phi})^2 \\ &= -\frac{1}{2}|\sin|\theta_s||\{\partial_\mu - i\sin|\theta_s|(A_\mu \cdot t^{(\phi)} - iB_\mu y^{(\phi)})\}\phi|^2 - \frac{\mu^2}{2}\sin^2|\theta_s|\phi^*\phi - \frac{\lambda}{4}\sin^4|\theta_s|(\phi^*\phi)^2 \end{aligned}, \tag{96}$$

where

$$\tilde{A}_\mu(x,\theta_s) = F(\theta_s)A_\mu(x)\,,\ \tilde{B}_\mu(x,\theta_s) = F(\theta_s)\mathrm{B}_\mu(x)\,,\ \tilde{\phi}(x,\theta_s) = F(\theta_s)\phi(x)\,. \tag{97}$$

The scalar Lagrangian (96) then yields an ECSM vector boson mass term:

$$\begin{aligned} -\frac{1}{2}|(\partial_\mu - i\tilde{A}_\mu(\theta_s)\cdot\tau^{(\phi)} - i\tilde{B}_\mu(\theta_s)\cdot\tau^{(\phi)})\tilde{\phi}(\theta_s)|^2 &= \frac{1}{2}\left|\left(\frac{g}{2}\tilde{A}_{\alpha\mu}(\theta_s)T^\alpha - \frac{g'}{2}\tilde{A}_{\alpha\mu}(\theta_s)\right)\begin{pmatrix}0\\ \tilde{v}(\theta_s)\end{pmatrix}\right|^2 \\ &= \frac{\tilde{v}^2(\theta_s)g^2}{4}\tilde{W}^*_\mu(\theta_s)\tilde{W}^\mu(\theta_s) - \frac{\tilde{v}^2(x,\theta_s)g'^2}{4}\tilde{Z}_\mu(\theta_s)\tilde{Z}^\mu(\theta_s) \end{aligned}, \tag{98}$$

where g and g' are the coupling constants. The ECSM masses are given as follows:

$$M_{\tilde{A}}(\theta_s) = 0, M_{\tilde{W}}(\theta_s) = \frac{1}{2}g\tilde{\upsilon}(\theta_s), M_{\tilde{Z}}(\theta_s) = \frac{1}{2}\tilde{\upsilon}(\theta_s)\sqrt{g^2 + g'^2}\ . \tag{99}$$

Here, $\tilde{W}^\pm$ and $\tilde{Z}^0$ are the gauge bosons that mediate the ECS-exchanging electroweak interaction between the families of fermions (for details see [31]). Using Equation (97), the ECSM masses become

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$$M_{\tilde{A}}(\theta_s) = M_A = 0, M_{\tilde{W}}(\theta_s) = M_W \, \mathrm{F}(\theta_s), M_{\tilde{Z}}(\theta_s) = M_Z \, \mathrm{F}(\theta_s), \tag{100}$$

where

$$M_A = 0, M_W = \frac{1}{2} g\upsilon, M_Z = \frac{1}{2}\upsilon\sqrt{g^2 + g'^2} \tag{101}$$

are the gauge boson masses of the SM vector [4–6]. The ECS Higgs potential in Equation (96) is given by

$$V_0(\tilde{\phi}) = \tilde{\mu}^2 \left|\tilde{\phi}\right|^2 + \lambda \left|\tilde{\phi}\right|^4. \tag{102}$$

Decomposing into physical and Nambu-Goldstone (NG) modes, we notice that, when we expand about some general vacuum with $\varphi = \upsilon$ undetermined, the resulting masses are:

$$m_{\tilde{H}}^2(\theta_s) = \tilde{\mu}^2 + 6\lambda\tilde{\upsilon}(\theta_s)^2 \ , \tag{103}$$

$$m_{NGB}^2(\theta_s) = \tilde{\mu}^2 + 2\lambda\tilde{\upsilon}(\theta_s)^2, \tag{104}$$

where m_{GB} is the common mass of the NG bosons. At the minimum of the potential in Equation (102), we obtain the bare ECS Higgs mass:

$$m_{\tilde{H}}^2(\theta_s) = -2\tilde{\mu}^2 = 4\lambda\tilde{\upsilon}(\theta_s)^2 = 4\lambda\upsilon^2 F(\theta_s)^2 = m_H^2 F(\theta_s)^2, \tag{105}$$

where

$$m_H^2 = 4\lambda\upsilon^2. \tag{106}$$

The SM Higgs gauge boson [4–6] and all the NG bosons are massless.

3.3. Masses of the ECSM quarks and leptons

The masses and mixing of the ECSM quarks and leptons have a common origin, as suggested in the SM [4–6]. They arise from the Yukawa interactions with the ECS Higgs condensate:

$$L_Y = -Y_{ij}^d Q_{Li}^I \phi(F(\theta_s))\bar{d}_{Rj}^I - Y_{ij}^u Q_{Li}^I \varepsilon\phi^*(F(\theta_s))\,\bar{\mathrm{u}}_{Rj}^I + h.c\,, \tag{107}$$

where $Y_{ij}^{u,d}$ are 3×3 complex matrices, $\phi(F(\theta_s))$ is the ECS Higgs field, i, j are generation labels, ε is the 2×2 antisymmetric tensor, Q_{Li}^I are left-handed ECSM quark doublets, and $\bar{d}_{Rj}^I$ and $\bar{\mathrm{u}}_{Rj}^I$ are, respectively, right-handed down- and up-type ECS quark singlets in the weak eigenstate.

When $\tilde{\phi}(\theta_s)$ acquires a vacuum expectation value, ($\left\langle\tilde{\phi}(\theta_s)\right\rangle = (0, \tilde{v}(\theta_s)/\sqrt{2})$), Equation (107) yields mass terms for the ECSM quarks, as follows:

$$L_Y = -Q_{Li}^I m_{ij}^{\tilde{d}}(\theta_s)\bar{d}_{Rj}^I - Y_{ij}^u Q_{Li}^I m_{ij}^{\tilde{u}}(\theta_s)\bar{\mathrm{u}}_{Rj}^I + h.c\,, \tag{108}$$

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where

$$m_{ij}^{\tilde{d}}(\theta_s)=\sum_n Y_{ij}^{\tilde{d}}\left\langle\tilde{\phi}^0(\theta_s)\right\rangle=m_{ij}^{d}\,\mathrm{F}(\theta_s)\ ,\ m_{ij}^{\tilde{u}}(\theta_s)=\sum_n Y_{ij}^{\tilde{u}}\left\langle\tilde{\phi}^0(\theta_s)\right\rangle=m_{ij}^{u}\,\mathrm{F}(\theta_s)\ . \tag{109}$$

$$m_{ij}^{d}=\sum_n Y_{ij}^{d}\left\langle\phi^0\right\rangle\ ,\ m_{ij}^{u}=\sum_n Y_{ij}^{u}\left\langle\phi^0\right\rangle \tag{200}$$

are the SM quark masses [4–6]. The ECSM quark masses depend on the arbitrary couplings and cannot be predicted. Furthermore, since ECSM quarks are not observed in isolation, their masses are not precisely defined. Similarly, for ECSM leptons, we have:

$$m_{ij}^{\tilde{l}}(\theta_s)=\sum_n Y_{ij}^{\tilde{l}}\left\langle\tilde{\phi}^0(\theta_s)\right\rangle=m_{ij}^{l}\,\mathrm{F}(\theta_s)\ , \tag{201}$$

where

$$m_{ij}^{l}=\sum_n Y_{ij}^{l}\left\langle\phi^0\right\rangle \tag{203}$$

are the SM quark masses [4–6].

4. Finite subgroups of the ECS Möbius transformations

We observe that the $\tilde{U}(2),\tilde{\mathrm{SU}}(2)$, and the $\tilde{U}(1)$ ECS gauge groupoid are the $\mathrm{F}(\theta_s)$-valued U(2), SU(2), and U(1) gauge groups [4–6] in θs. The ECS angle θ_s in Equation (55) is strictly an arbitrary global parameter, and may originate form a different group—either the $\mathrm{SO}(3)_{\mathrm{ECS}}$ group or from the finite subgroups of the ECS Möbius transformations as we explain in this section.

Let Γ be an ECS subgroup of $\mathrm{PSL}_2(\mathrm{C})_{\mathrm{ECS}}$, consisting of elliptic elements together with the identity. Then Γ_{ECS} is conjugate in $\mathrm{PSL}_2(\mathrm{C})_{\mathrm{ECS}}$ to a subgroup of $\mathrm{PSU}_2(\mathrm{C})_{\mathrm{ECS}}$ [53-54].

Now, by the group isomorphism (12), for every finite subgroup of ECS rotations (i.e. a subgroup of $\mathrm{PSU}_2(\mathrm{C})_{\mathrm{ECS}}$), we have the following:

For a given finite group, Γ_{ECS}, of ECS rotations in C^{∞} (Equation ()),one of the following holds [53-54]:

1 .*Γ is ECS-cyclic;*

2 .*Γ is ECS-dihedral;*

3. *Γ is the ECS symmetry group of a regular ECS tetrahedron (A_4), ECS octahedron (S_4), or ECS icosahedron (A_5).*

One can show that two finite ECS subgroups in $\mathrm{PSL}_2(\mathrm{C})_{\mathrm{ECS}}$ are conjugate if and only if they are isomorphic.

4.1. ECS-dihedral group

Here, D_F (with F being the number of fermions) refers to the symmetries of the Fermionic polygon (F-gon:having *F* fermionicsides)—a group of order 2*F*. In abstract algebra, D_{2F} refers to this same ECS dihedral group (for details of the Dirac equation on the polygon regions see [69-75]). D_F is a subgroup of $O(2)_{ECS}\cong U(1)_{ECS}$, i.e., the group of ECS rotations (about the origin) and ECS reflections (across axes through the origin) of the plane. However, the

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notation 'D_F' is also used for a subgroup of $SO(3)_{ECS} \cong SU(2)_{ECS}$, which is also an abstract group: the proper symmetry group of an F-gon embedded in three-dimensional internal space (if the number of fermions is $F \geq 3$).

Following [69], the Dirac-like equation for the fermionic F-gon, given by decomposition of the equation into two complex conjugate parts through operator calculus, and assuming $\psi_{\text{F-gon}}$ is a two-vector wave function:

$$(\tau_2\partial_\sigma - i\tau_1\partial_\varsigma)\psi_{F-gon}(\sigma,\varsigma) = \lambda(\cos\theta_s \cdot I + i\sin\theta_s \cdot \tau_3)\psi_{F-gon}(\sigma,\varsigma) \ ,$$

where τ_i's are the isospin versions of Pauli matrices, λ is a positive constant, and (σ,ς) internal coordinates . Here, the difference with the original Dirac equation is the inclusion of ECS-angle θs into the equation. By applying finite differences on the partial differentials from [69], we may obtain the following solutions in the continuum limit:

$$\begin{aligned}\psi_{F-gon}(\sigma,\varsigma)_+ &= (i\lambda\cos\theta_s)\psi_+ + (i\lambda\sin\theta_s)\psi_+, \\ \psi_{F-gon}(\sigma,\varsigma)_- &= (i\lambda\cos\theta_s)\psi_- + (i\lambda\sin\theta_s)\psi_-\end{aligned} \ . \quad [69]$$

We can apply representation theory by exchanging the elements into matrices. The fermionic Dihedral group D_F, is the set of the 2F rotation and reflection symmetries of a F-sided fermionic polygon. Assuming we are orienting positive direction as clockwise, then let the group element R_i denote rotation by i units, while S_i denotes rotation of i units, then a reflection. $R_iR_j = R_{i+j}, R_iS_j = S_{i+j}, S_iR_j = S_{i-j}, S_iS_j = R_{i-j}$. Note that using the group representation theory, the elements are isomorphic to

$$R_k = \begin{bmatrix} \cos\frac{2\pi k}{F} & -\sin\frac{2\pi k}{F} \\ \sin\frac{2\pi k}{F} & \cos\frac{2\pi k}{F} \end{bmatrix}, \ S_k = \begin{bmatrix} \cos\frac{2\pi k}{F} & \sin\frac{2\pi k}{F} \\ \sin\frac{2\pi k}{F} & -\cos\frac{2\pi k}{F} \end{bmatrix}, \qquad k \in \mathbb{N}$$

where

$$\theta_s(F) = \frac{2\pi k}{F}$$

ECS-angle and F is the number of fermions. By calculation, one can find the mapping from $\mathbb{C}[D_F]$ to a direct isomorphic sum of $End[V_i]$, where V_1, V_2... are the irreducible representations of D_F group [69].

4.2. The averages *of F(θs) function over the triangulation of a convex fermionic F-gon*

The sum of the interior ECS angles of a simple *F*-gon is $(F-2)\pi$ radians. This is because any simple *F*-gon can be considered to be made up of $(F-2)$ triangles, each of which has an angle sum of π radians:

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$$\theta_s(F)=\left(1-\frac{2}{F}\right)\pi\,, \tag{204}$$

where

$$F=F_{ordinary}+F_{ECS}\,. \tag{205}$$

F is the total number of fermions, given by the sum of ordinary $F_{ordinary}$ and ECS fermions F_{ECS}. Following [76], we find that the number (C_F) of triangulations of a convex fermionic F-gon in the internal space satisfies the recursive formula

$$C_F=C_0C_{F-1}+C_1C_{F-2}+\ldots+C_{F-1}C_0\ , \tag{206}$$

where $C_0 = 1$ [77-80]. The numbers C_F are now called fermionic Catalan numbers. From (206), it follows that $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, and so on. Using generating functions and Segner's formula, an explicit formula for C_F can be developed [80]:

$$C_F=\frac{(2F)!}{(F+1)!F!}, \tag{207}$$

with F being the total number of fermions.

After the triangulation of a convex fermionic F-gon in three-dimensional internal space, Equation (55) becomes:

$$\sin\left(\theta_s(\mathrm{F})\right)_{(\Delta_F)}\equiv\frac{\pi\sin\left(\theta_s(\mathrm{F})\right)_{averge}}{C_F}=\frac{2}{C_F}\ \text{, on } [0,\pi/2], \tag{208}$$

where

$$\sin\left(\theta_s(\mathrm{F})\right)_{averge}=\frac{2}{\pi}\text{, on } [0,\pi/2]. \tag{209}$$

This is the normalised average sin of the ECS angle θ_s over the triangulation of a convex fermionic F-gon by the fermionic Catalan numbers C_F.

5. Results

Now that the averages of the $F(\theta_s)$ function have been determined by the fermionic Catalan numbers C_F, we consider two possible scenarios of ECS contribution to the SM:

Loop level: The ECS Physics at loop-level differs from the SM physics; the ECSM mass is not identical to the SM mass; therefore,

$$\sin\left(\theta_s(\mathrm{F})\right)_{(\Delta_n)}\equiv\frac{2}{C_n}=\frac{M_{\tilde{\mathrm{p}}-(\Delta_n)}}{M_{p-SM}}\neq 1\text{,on } [0,\pi/2] \tag{210}$$

implies

$$M_{\tilde{\mathrm{p}}-(\Delta_n)}=\sin\left(\theta_s(\mathrm{F})\right)_{(\Delta_n)}M_{p-SM}=\frac{2M_{p-SM}}{C_F}\text{, on } [0,\pi/2], \tag{211}$$

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where $M_{\tilde{p}-(\Delta_2)}$ are the triangulation masses of the ECS particles (from equation (208)), M_{p-SM} are the corresponding masses of the SM particles, and C_F are the fermionic Catalan numbers for the triangulations of the fermionic F-gon.

Tree level: The ECS Physics at tree level is the same as the SM physics, and the ECSM masses are identical to the SM masses. Therefore,

$$\sin\left(\theta_s(\mathrm{F})\right)_{(\Delta_2)} \equiv \frac{2}{C_2} = \frac{M_{\tilde{p}-(\Delta_2)}}{M_{p-SM}} = 1, \text{ on } [0,\pi/2] \tag{212}$$

implies

$$M_{p-SM} = M_{\tilde{p}-(\Delta_2)}, \tag{213}$$

where $M_{\tilde{p}-(\Delta_2)}$ are the ECS particle triangulation masses of the fermionic 2-gon, are the corresponding SM particles masses, and $C_2 = 2$ is the fermionic Catalan number for the triangulation of the fermionic 2-gon.

In the proposed ECS model ((211), (212)), the tree-level mass term, $|\mu_0|$, which sets the electroweak scale, is naturally $\sim m_{W,Z}$. We thus only consider the effect of the new ECS physics at loop level. We estimate the ranges of tree-level mass term $|\mu_0|$ as follows.

For $\mu_0^2 < 0$, there is a tree-approximation vacuum expectation value of the stationary point of the Lagrangian. (96).

$$\begin{aligned} \upsilon^2_{Smallest} &= \frac{|\mu_0^2|_{Smallest}}{2\lambda} = \frac{m_W^2}{\lambda} = \frac{m_Z^2}{\lambda}\cos^2\theta_W = \frac{|\mu_0^2|_{Largest}}{2\lambda}\cos^2\theta_W = \upsilon^2_{Largest}\cos^2\theta_W, \\ \upsilon^2_{Largest} &= \frac{|\mu_0^2|_{Largest}}{2\lambda} = \frac{m_Z^2}{\lambda}, \end{aligned} \tag{214}$$

where

$$|\mu_0^2|_{Smallest} = 2\lambda\upsilon^2_{Smallest} = 2m_W^2, |\mu_0^2|_{Largest} = 2\lambda\upsilon^2_{Largest} = 2m_Z^2, m_Z^2 = m_W^2/\cos^2\theta_W \tag{215}$$

and θ_W; the Weinberg angle [1,2]. Implies that the bare parameter $|\mu_0^2|$ must be in the mass ranges of

$$|\mu_0^2|_{Smallest} = 2m_W^2 \leq |\mu_0^2| \leq 2m_Z^2 = |\mu_0^2|_{Largest}, \tag{216}$$

where m_W=82GeV and m_Z=91GeV are the W and Z boson mass respectively.

Considering the one-loop contribution to the effective potential (102), the radiative corrections take the form:

$$V_1(\tilde{\phi}) = \frac{1}{64\pi^2}\int d^4k STr\log(k^2 + M^2(\tilde{\phi}(\Delta_F))) = \frac{\Lambda_c^2}{32\pi^2\upsilon^2} STrM(\tilde{\phi}(\Delta_F)) + \ldots \tag{217}$$

where $STr = Tr\ (-1)^F$ defines the supertrace. This new contribution can be absorbed into V_0 (Equation (102)) by shifting the bare μ_0^2:

$$\begin{aligned} M_{\tilde{H}}^2 &= \mu_0^2 + \frac{3\Lambda_C^2}{32\pi^2\upsilon^2} STrM^2(\tilde{\phi}(\Delta_{F_A})) \\ &= \mu_0^2 + \frac{3\Lambda_C^2}{32\pi^2\upsilon^2}[M_{\tilde{H}}^2(\Delta_{F_A}) + 2M_{\tilde{W}}^2(\Delta_{F_A}) + M_{\tilde{Z}}^2(\Delta_{F_A}) - 4m_{\tilde{t}}^2(\Delta_{F_A})] + \ldots. \end{aligned} \tag{218}$$

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The trace over the φ-dependent triangulations masses is given by

$$\sum_{scalars} M^2(\tilde{\phi}(\Delta_F)) = (\mathrm{m}^2_{\tilde{H}}(\Delta_F)|_{\tilde{\phi}=\upsilon} - \mathrm{m}^2_{\tilde{H}}(\Delta_F)|_{\tilde{\phi}=0)}) \\ +3(\mathrm{m}^2_{NGB}|_{\tilde{\phi}=\upsilon} - \mathrm{m}^2_{NGB}|_{\tilde{\phi}=0)}) = 3\mathrm{m}^2_{\tilde{H}}(\Delta_F)|_{\tilde{\phi}=\upsilon} \quad . \tag{219}$$

Thus Equation (218) is reproducing.

Here, $M_{\tilde{H}}(\Delta_F)$, $M_{\tilde{W},\tilde{Z},}(\Delta_F)$, and $m_{\tilde{t}}(\Delta_F)$ are the triangulation masses of the ECS top quark (from equations (211) and (3)), ECS $\tilde{W},\tilde{Z}$, and ECS Higgs bosons ($\tilde{H}$), respectively, and υ is the vacuum expectation value of the Higgs potential in the SM [4-6]. The dots include logarithmic corrections in Λ_C, as well as contributions independent of Λ_C in the large Λ_C limit. Using equations (211) and (3), we obtain the one-loop radiative corrections to the bare Higgs mass:

$$M^2_{\tilde{H}} = \mu_0^2 - \frac{3m^2_{\tilde{t}}(\Delta_{F_A})\Lambda_C^2}{8\pi^2\upsilon^2} = \mu_0^2 - \frac{3\lambda_t^2\Lambda_C^2}{4\pi^2 C_F^2}. \tag{220}$$

These corrections depend on the fermionic Catalan number C_F (Equation.(207)), the top quark Yukawa coupling λ_t and the cut-off energy scale Λ_C. Using equations (216) and (220), we estimate the ranges of the one-loop radiative corrections to the bare Higgs mass as follows

$$(\mathrm{M}^2_{\tilde{H}})_{Smallest} = -(\mu_0^2)_{Smallest} + \frac{3\lambda_t^2\Lambda_C^2}{4\pi^2 C_F^2} \le M^2_{\tilde{H}} \le -(\mu_0^2)_{Largest} + \frac{3\lambda_t^2\Lambda_C^2}{4\pi^2 C_F^2} = (\mathrm{M}^2_{\tilde{H}})_{Largest} \tag{221}$$

$$(\mathrm{M}^2_{\tilde{H}})_{Smallest} = 2m_W^2 + \frac{3\lambda_t^2\Lambda_C^2}{4\pi^2 C_F^2} \le M^2_{\tilde{H}} \le 2m_Z^2 + \frac{3\lambda_t^2\Lambda_C^2}{4\pi^2 C_F^2} = (\mathrm{M}^2_{\tilde{H}})_{Largest}. \tag{222}$$

Using equations (221) and (222), we obtain the median value of the one-loop radiative corrections to the bare Higgs mass:

$$(M^2_{\tilde{H}})_{Median} = \frac{1}{2}\left((M^2_{\tilde{H}})_{Smallest} + (M^2_{\tilde{H}})_{Largest}\right) = m_W^2 + m_Z^2 + \frac{3\lambda_t^2\Lambda_C^2}{4\pi^2 C_F^2} \quad . \tag{223}$$

Following equations [9], an all-orders result for $M^2_{\tilde{H}}$ is given by the following equation:

$$M^2_{\tilde{H}} = \mu_0^2 + \Lambda_c^2 \sum_{n=0}^{\infty} \tilde{c}_n(\lambda)\log^n\left(\frac{\Lambda_c}{Q}\right), \tag{224}$$

where $c_0 = (32\pi 2)^{-1}\mathrm{STr}\ \mathrm{M}^2/\upsilon^2$. The remaining c_n can be calculated recursively by the following relation:

$$(1+n)c_{n+1} = \frac{dc_n}{d(\log Q)} = \beta_i \frac{\partial c_n}{\partial \lambda_i}, \tag{225}$$

where μ_0^2 should be independent of the renormalization scale Q. For example:

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$$(16\pi^2)c_1 = \frac{\lambda}{C_F^2}(144\lambda - 54g^2 - 18g'^2 + 72\lambda_t^2) + \frac{\lambda_t^2}{C_F^2}(27\,g^2 + 17g'^2 + 96\lambda^2 - 90\lambda_t^2) - \frac{15}{2C_F^2}g^4 + \frac{25}{2C_F^2}g'^4 + \frac{9}{2C_F^2}g^2g'^2 \quad . \quad (226)$$

Since each order in *n* involves more factors of $(16\pi^2C^2_F)^{-1}$, we expect that, for large fermionic Catalan numbers C_F, the higher-loop contributions are unimportant. Our approach to the higher-order contributions yields that each $c_n \neq 0$ is suppressed by the large fermionic Catalan number C_F for all n separately. In the infinite cutoff limit (or in a cutoff-independent solution), this would be the correct procedure for solving the fine-tuning problem [29]. Given that all $c_n \neq 0$ are independent of, and suppressed by, the large C_F over the (λ_i) parameter, a solution exists. Therefore, our approach makes it possible to solve the fine-tuning problem, since an all orders of solutions exist. For an equal number of ordinary and ECS fermions, the calculated one-loop radiative corrections to the bare Higgs mass μ_0^2 is given in Table 3.

Number of Ordinary Fermions ($F_{Ordinary}$)	*12*
Number of ECS Fermions (F_{ECS})	*12*
Total Number of Fermions (F) Eq. (205)	*24*
Fermionic Catalan Number (C_F) Eq (207)	*2.28×10^{12}*
Largest value of the tree-level mass term, $\|\mu_0\|_{(Largest)}$ (GeV) Eq(216)	*128,6*
Smallest value of the tree-level mass term, $\|\mu_0\|_{(Smallest)}$ (GeV) Eq(216)	*115,9*
Largest value of the Higgs mass $M_H^{(Largest)}$ (GeV) Eq(222)	*131*
Smallest value of the Higgs mass $M_H^{(Smallest)}$ (GeV) Eq(222)	*119*
Sum (GeV)	*250*
Count	*2*
Median value of the Higgs mass $M_H^{(Median)}$ (GeV) Eq(223)	*125*
Geometric Mean	*124.85591696031*
Range (GeV)	*12*

Table 3. One-loop radiative corrections to the bare Higgs mass, as calculated from Equation (223). For the reciprocal fermionic Catalan number $1/C_{24}$ = O(10^{13}) (Equation (207)), and top quark Yukawa coupling λ_t=0.99 and the cut-off energy scale Λ_C = O(10^{15}GeV), where m_W=82GeV and m_Z=91GeV are the W and Z boson mass respectively.

This result is very close to the experimental value of the Higgs mass [10,11]. The all-orders result for M_H^2 with a fermionic Catalan number C_{24} becomes:

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$$M_{\tilde{H}}^2 = \mu_0^2 + \left(\frac{\Lambda_c}{C_{24}}\right)^2 \sum_{n=0}^{\infty} c_n(\lambda) \log^n\left(\frac{\Lambda_c}{Q}\right). \quad (227)$$

For instance, the calculated result

$$(16\pi^2)c_1 = \frac{\lambda}{C_{24}^2}(144\lambda - 54g^2 - 18g'^2 + 72\lambda_t^2) + \frac{\lambda_t^2}{C_{24}^2}(27g^2 + 17g'^2 + 96\lambda^2 - 90\lambda_t^2)$$
$$-\frac{15}{2C_{24}^2}g^4 + \frac{25}{2C_{24}^2}g'^4 + \frac{9}{2C_{24}^2}g^2g'^2 \approx O\left(\frac{1}{C_{24}^2}\right) \quad . \quad (228)$$

is of the order of the inverse square of the fermionic Catalan number of the 24F-gon. Our approach, therefore, solves the fine-tuning problem, since it provides an all-orders solution which is suppressed by the fermionic Catalan number C_F for all n separately. For 24 fermions, we calculate a Higgs mass of 125GeV which is very close to the experimental value [10,11].

6. Conclusion

Taking the SU(2) group of weak interactions in the presence of electric charge swap (ECS) symmetry as a starting point, we show that ordinary and non-regular (ECS) leptons are related by the ECS rotational SO(3) group.

We investigate a version of the SM algebroid with the anchor map depending on the ECS angle θs. We find that many SM algebras depend on the ECS angle θs. We call these ECSM algebras. Furthermore, the SM algebroid is integrable to the SM groupoid; so, our results potentially extend well beyond this case. Then, we investigate how the breaking of the SM groupoid symmetry gives the massive ECS particle. We find that the ECS particle mass is related to the SM particle mass through the ECS angle θs.

We investigate the finite subgroups of the ECS Möbius transformations. In this case, the ECS-angle θ_s could originate from the ECS dihedral group that refers to the symmetry of the Fermionic polygon (F-gon). The ECS angle θs can then be determined through the triangulation of a convex fermionic F-gon.

Finally, we find that, at loop-level, the ECS Physics is different from the SM physics, and the ECSM mass is suppressed by the fermionic Catalan numbers C_F. For 24-fermions, the calculated one-loop radiative correction to the bare Higgs mass μ^2 is 125GeV—very close to the experimental value.

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