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Similarity Solutions of Two Dimensional Turbulent Boundary Layers

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The exact similarity solutions (also called as special exact solutions) of two dimensional laminar boundary layer were obtained by Blasius in 1908, however, no similarity solutions for two dimensional turbulent boundary layers have been reported in the literature. With the help of dimensional analysis and invariance principle, Prandtl mixing length $\ell = \kappa y$ for one dimensional turbulent boundary layer is extended to $\ell(x, y) = \kappa y(1 - \frac{y}{\delta})\sqrt{\frac{\nu}{U\delta}}$ for the two dimensional turbulent boundary layers, furthermore, with a similarity transformation, we successfully transform the two dimensional turbulent boundary layers partial differential equations into a single ordinary differential equation $f''' + ff'' + \beta(1 - f'^2) + \kappa^2[\eta^2(1 - \eta)^2 f''|f''|]' = 0$. As an application, similarity solutions of the two dimensional turbulent boundary layer on a flat plate at zero incidence have been studied in detail. To solve the ordinary differential equation numerically, a complete Maple code is provided.

Keywords: Turbulent boundary layers, laminar boundary layers, similarity transformation, similarity solution, Prandtl mixing length, Reynolds number

INTRODUCTION

Turbulence encountered in different natural and man-made systems is a ubiquitous phenomenon and one of the greatest unsolved mysteries of classical physics [1–19].

Different from bulk homogenous turbulence, when the flow is confined by the presence of solid walls or boundaries, the wall-bounded turbulence is demodulated by the walls and characterized by anisotropic properties. Namely, there is a net mean flow in the streamwise direction along the wall and different flow structures form depending on their distance to the wall [20–25].

The importance influence of solid walls on the flow was studied and a revolutionized concept of boundary layer was introduced by Prandtl in 1904 [20], who theorized that an effect of friction was to cause the fluid immediately adjacent to the surface to stick to the surface - in other words, he assumed that frictional effects were experienced only in a boundary layer, a thin region near the surface. Outside the boundary layer, the flow was essentially the inviscid flow that had been studied for the previous two centuries [14]. The approximation of Navier-Stokes equations based on the boundary layer leads to boundary layer equations, which exhibit a completely different mathematical behavior than the Navier-Stokes equations. The boundary layer equations have parabolic behavior rather than the elliptic one of the Navier-Stokes equations. The parabolic nature of the boundary layer equations affords tremendous analytical and computational simplification. They can be solved step-by-step by marching downstream from where the flow encounters a body, while for the elliptic equations, the complete flow field must be solved simultaneously [2, 3, 14].

Within the boundary layers, the flow can be either laminar or turbulent motion. For the 2D laminar boundary layers, in 1908 Blasius introduced a similarity transformation and found its similarity solutions [21] (which are called as special exact solution by [26]). In 1921 Prandtl discovered that almost all flow motion in the boundary layers are turbulent rather than laminar [22]. However, to the best of the author's knowledge, for the 2D turbulent boundary layers, no similarity solutions have even been obtained [2, 3]. Although Prandtl [24] and Townsend [7] studied a similarity solutions for the free shear turbulent flows, but free turbulence is not the turbulent boundary layers flows, since this kind of flows have no fixed walls. The question is wether or not the similarity transformation that used to solve the 2D laminar boundary layers could be extended to the 2D turbulent boundary layers. If it were possible, how to formulate and what conditions must be hold. Those questions remain open.

In the light of both Prandtl's pioneer work, in this study, the Prandtl mixing length that is for the 1D laminar boundary layer is modified to facilitate the 2D turbulent boundary layer, the partial differential equations of the two dimensional turbulent boundary layers is successfully transformed into a single ordinary differential equation by a similarity transformation. To numerically solve the ordinary differential equation, a Maple code is provided.

FORMULATIONS OF TWO DIMENSIONAL TURBULENT BOUNDARY LAYERS

A thin flat plate is immersed at zero incidence in a uniform stream as shown in Fig.1, which flows with speed $U(x)$ and is assumed not to be affected by the presence of the plate, except in the boundary layer. The fluid is supposed unlimited in extent, and the origin of coordinates is taken at the leading edge, with x measured downstream along the plate and y perpendicular to it. Assuming that the turbulent flow is steady with pressure gradient along the x axis. The pressure gradient drives the against the shear stresses at the wall.

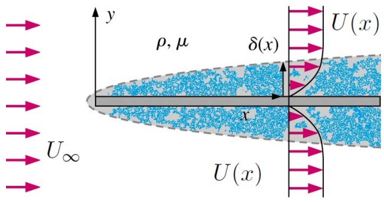

FIG. 1: Turbulent boundary layer. μ is the dynamical viscosity, ρ is the flow density, $\delta(x)$ is "boundary layer thickness". Strictly speaking, $\delta(x)$ is not the boundary layer thickness,but rather a scaled measure of the boundary layer thickness which is equal to the boundary layer thickness up to some numerical factor [26].

The Reynolds-averaged Navier-Stokes equations [1] of the two dimensional turbulent boundary layers flow under gradient, dp/dx, are reduced to

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}=0, \tag{1}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{d\overline{v'^2}}{dy}=0, \tag{2}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=-\frac{1}{\rho}\frac{dp}{dx}+\nu\frac{d^2\bar{u}}{dy^2}-\frac{d\overline{u'v'}}{dy}, \tag{3}$$

and boundary conditions:

$$y=0:\ \bar{u}=\bar{v}=0,\ u'=0,\ v'=0, \tag{4}$$

$$y=\delta(x):\ \bar{u}=U(x),\ u'=0,\ v'=0, \tag{5}$$

where $\bar{u}$ is the mean velocity, ν is the kinematic viscosity, ρ is flow density, p is pressure, u' and v' are velocity fluctuation component, $U(x)$ is outer of boundary layer potential flow velocity. The pressure gradient must be negative, namely $dp/dx<0$, to maintain the flow motion. For a curved boundary layers, the coordinates $(x,\,y)$ should be replaced by $(s,\,n)$, where s is arc length and n is normal to the curve layers.

Integration of Eq.(2) yields

$$\overline{v'^2}+\frac{p}{\rho}=\frac{p_e}{\rho}, \tag{6}$$

where p_e is a function of x only [2]. Because $\overline{v'^2}$ can be neglected comparing with pressure p, then $\partial p/\partial x \approx dp_0/dx$. From Bernoulli equation, we have relation: $p_e+\frac{1}{2}\rho U^2=constant.$, leads to $\frac{dp_e}{dx}=\rho U\frac{dU}{dx}$. The boundary equations are reduced to following:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}=0, \tag{7}$$

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=U\frac{dU}{dx}+\nu\frac{d^2\bar{u}}{dy^2}-\frac{d\overline{u'v'}}{dy}, \tag{8}$$

Introducing a stream function $\Psi(x,y)$ and express the velocity components as follows

$$\bar{u}=\frac{\partial \Psi}{\partial y},\quad \bar{v}=-\frac{\partial \Psi}{\partial x}, \tag{9}$$

with the relation in Eq.(9), the mass conservation Eq.(7) is satisfied, and the momentum conservation Eq.(8) becomes

$$\frac{\partial \Psi}{\partial y}\frac{\partial \Psi}{\partial x\partial y}-\frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial y^2}=U\frac{dU}{dx}+\nu\frac{d^3\Psi}{dy^3}-\frac{d\overline{u'v'}}{dy}, \tag{10}$$

and boundary conditions

$$y=0:\ \frac{\partial \Psi}{\partial y}=0,\frac{\partial \Psi}{\partial x}=0,\ u'=0,\ v'=0, \tag{11}$$

$$y=\delta(x):\ \frac{\partial \Psi}{\partial y}=U(x),\ u'=0,\ v'=0, \tag{12}$$

The Eq.(10) can be solved if $\overline{u'v'}$ can be expressed by the mean velocity.

EXTENDED PRANDTL MIXING LENGTH

Regarding the formulation of $\overline{u'v'}$, Prandtl [3, 24] assumed a greatly simplified model of the fluctuations, according to which the individual fluid elements are displaced in a mean distance (the mixing length) l by the fluctuations, perpendicular to the main flow direction, but still retaining their momentum. The element that was initially at y, and is now at $y+l$, has a higher velocity than its new surroundings. The velocity difference is a measure of the fluctuation velocity in the x direction: $u'=\bar{u}(y+l)-\bar{u}(y)\approx l\frac{d\bar{u}}{dy}$ and $v'\sim u'$. Therefore, the Reynolds stress is proposed to be

$$\tau'_{xy}=-\rho\overline{u'v'}=\rho\overline{l\frac{d\bar{u}}{dy}l\frac{d\bar{u}}{dy}}=\rho\overline{l^2}\frac{d\bar{u}}{dy}\frac{d\bar{u}}{dy}=\rho\ell^2|\frac{d\bar{u}}{dy}|\frac{d\bar{u}}{dy}, \tag{13}$$

where $\ell^2=\overline{l^2}$, $\bar{u}$ is the mean velocity, ρ is the flow density, and u' and v' are the velocity fluctuation components. The Prandtl mixing length design rule is that: a. the mixing length ℓ must have a length scale; b. the mixing length ℓ must satisfies the boundary condition at fixed walls $\ell|_{y=0}=0$, which stems from the fact of $u'=v'=0$ at walls.

After the formulation of the Reynolds stress in Eq.(13), Prandtl [24] applied it to two problems: pipe flow and free turbulence, for the pile flow Prandtl proposed the mixing length to be proportional to $[(a-r)(a+r)]^{6/7}$ where a=pipe radius; and he proposed the mixing length $\ell=cx$ for the free turbulence, the flows without confining walls, x being the distance from the point where the mixing starts. The Prandtl mixing length $\ell=cx$ of the free turbulence was revisited and confirmed by Townsend in 1976 [7].

Although there is no universal mixing length, Prandtl [24] provided a general framework of the mixing length theory. Different situations will have different mixing length. What is the mixing length of a specific question ? it must be analyzed in detail.

For instance, for plane turbulent flows, since the only quantity with length scale is the coordinate y, by dimensional argument, the mixing length is proposed as $\ell = \kappa y$, where κ is a numerical constant, namely, the von Kármán constant.

To include one wall damping effect in the plane turbulent flows, the mixing length is modified to be $\ell = \kappa y[1 - \exp(-y/A)]$ by [27], where A is a constant depending the frequency of oscillation of the plate and kinematic viscosity ν of the fluid. For Poiseuille turbulent flow, to include both walls' boundary conditions and damping effects, Sun [29] obtained a universal high-heels profile of mean velocity by proposing a mixing length as $\ell = \kappa y[1-y/(2h)][1-\exp(-y/A)][1-\exp(-(2h-y)/A)]$.

From publications such as [2, 3, 5–8, 10, 11, 13, 27–29], it is surprised to see that the mixing length $\ell(y) = \kappa y$ that is proposed to the plane turbulent flows has been adopted to deal with the two dimensional turbulent boundary layers flows, without any modification. The question is whether or not we can come out with a specific mixing length that includes the dimensions of all quantities in the two turbulent boundary layers, namely, not only includes the coordinate y but also the potential flow velocity $U(x)$, the kinematic viscosity ν as well as the boundary layer thickness $\delta(x)$. In other words, how to extend the 1D mixing length $\ell(y)$ for the plane turbulence flows to the 2D mixing length $\ell(x, y)$ for the 2D turbulent boundary layers.

In the history of fluid mechanics, Prandtl eventually able to deduce difficulty problems came from the clever and repeated use of dimensional analysis [20, 22]. Following Prandtl's foot step, according to the boundary conditions, $\ell(0) = \ell(\delta) = 0$, the mixing length dedicated to the two dimensional turbulent boundary layers flows can be formulated by the dimensional arguments [18] as follows

$$\ell(x, y) = \kappa y(1 - \frac{y}{\delta}) \left(\frac{U\delta}{\nu}\right)^{\lambda} = \kappa y(1 - \frac{y}{\delta})(Re_\delta)^{\lambda}, \quad (14)$$

where $Re_\delta = \frac{U\delta}{\nu}$, and λ is a constant. In the following, $\lambda = -\frac{1}{2}$ can be determined by invariance principle. The proposed mixing length is called the extended Prandtl mixing length of the two dimensional turbulent boundary layers flows.

Since the wall at $y = 0$ is fixed, according to Driest [27], a damping function, $[1-\exp(-y/\delta)]$ should be introduced to moudify the mixing length, hence we have

$$\ell(x, y) = \kappa y(1 - \frac{y}{\delta})[1 - \exp(-\frac{y}{\delta})] \left(\frac{U\delta}{\nu}\right)^{\lambda}. \quad (15)$$

From our numerical calculations, the results $f(\eta)$, $f'(\eta)$ based on the mixing length with damping in Eq.(15) agrees with the results based on the mixing length in Eq.(14) very well, except minor difference on $f''(\eta)$. In this paper, all calculations are carried out based on the mixing length in Eq.(14). The Maple code can easily modified and to deal with the problem based on the mixing length in Eq.(15).

SIMILARITY SOLUTIONS OF TWO DIMENSIONAL TURBULENT BOUNDARY LAYERS FLOWS

In the analysis of differential equations, it is often useful to examine some special solutions, which allow us to make conclusions about the main properties of the phenomena in question. Such solutions can sometimes be found by dimensional analysis, which can reduced the number of independent variables adopted to characterize the question. Thereby, in some situations, the number of independent variables can be reduced and partial differential equations can even be replaced by ordinary differential equations. The simplest case allowing for the construction of special solutions is that of a problem with the so-called property of self-similarity. This means that there is a system of independent variables and unknown functions which, being introduced into the original equations describing a given phenomenon, reduces the equations to a form that is invariant with respect to affine transformations. A characteristic feature of self-similar problems of the boundary layer theory is that velocity component profiles for the flow form a family of homothetic curves [10, 16, 21, 30].

According to Blasius laminar boundary similarity theory [3, 21, 26], the system has no characteristic length, we can assume that the velocity profiles at different distances from the leading edge are affine or similarity to one another, i.e. that the velocity profile $\bar{u}$ at different distances x can be mapped onto each other by suitable choice of scaling factors for $\bar{u}$ and y. A suitable scaling factor for u could be the free stream velocity $U(x)$, while for y, "boundary-layer thickness" $\delta(x)$, which increases with distance x, could be used. The similarity law of the velocity profile can thus be written as $u/[\delta(x)U] = f(\eta)$ with $\eta = y/\delta(x)$, where the function $f(\eta)$ is independent of x.

Introducing following transformations

$$\Psi = U(x)\delta(x)f(\eta), \quad (16)$$

$$\eta = \frac{y}{\delta(x)}, \quad (17)$$

thus the velocity components become

$$\bar{u} = U\frac{df}{d\eta}, \tag{18}$$

$$\bar{v} = -\left(\delta f\frac{dU}{dx} + Uf\frac{d\delta}{dx} - \eta U\frac{d\delta}{dx}\frac{df}{d\eta}\right). \tag{19}$$

Substituting Eqs.(18) and (19) into Eq.(16), we have

$$\frac{d^3f}{d\eta} + \alpha f\frac{d^2f}{d\eta^2} + \beta[1-(\frac{df}{d\eta})^2] - \frac{\delta}{\nu U}\frac{d}{d\eta}\overline{u'v'} = 0, \tag{20}$$

where the coefficients are

$$\alpha = \frac{\delta}{\nu}\frac{dU\delta}{dx}, \quad \beta = \frac{\delta^2}{\nu}\frac{dU}{dx}. \tag{21}$$

According to the Prandtl mixing length theory [24], and applying the extended Prandtl mixing length in Eq.(14), we have

$$\overline{u'v'} = -\ell^2|\frac{d\bar{u}}{dy}|\frac{d\bar{u}}{dy} = -(\kappa y)^2(1-\frac{y}{\delta})^2(\frac{U\delta}{\nu})^{2\lambda}|\frac{d\bar{u}}{dy}|\frac{d\bar{u}}{dy}, \tag{22}$$

with the Eqs.(18), the above relation can expressed as follows

$$\overline{u'v'} = -\kappa^2U^2(\frac{U\delta}{\nu})^{2\lambda}\eta^2(1-\eta)^2|\frac{d^2f}{d\eta^2}|\frac{d^2f}{d\eta}, \tag{23}$$

hence Eq.20 can be reduced to

$$\begin{aligned}&\frac{d^3f}{d\eta} + \alpha f\frac{d^2f}{d\eta^2} + \beta[1-(\frac{df}{d\eta})^2] \\ &+ \kappa^2\left(\frac{U\delta}{\nu}\right)^{1+2\lambda}\frac{d}{d\eta}[\eta^2(1-\eta)^2\frac{d^2f}{d\eta^2}|\frac{d^2f}{d\eta^2}|] = 0,\end{aligned} \tag{24}$$

and its three boundary conditions:

$$\eta = 0: f = 0, \frac{df}{d\eta} = 0, \tag{25}$$

$$\eta = 1: \frac{df}{d\eta} = 1. \tag{26}$$

General speaking, for a arbitrary λ, the coefficients α, β and $U\delta/\nu$ are the functions of x, then the Eq.(24) changes as the coefficients vary. So that there are no similarity solutions.

From the self-similarity and scaling nature of fully developed turbulence [9, 16, 17, 31–37], it would be a natural attempts to search for similarity solutions in the fully developed turbulent boundary layers.

According to invariance principle of nature [16, 17, 34, 36, 37], to be invariant of Eq.(24), the scaling exponent "$1+2\lambda$" in the 3rd term of Eq.24 must be zero, namely $1+2\lambda = 0$, then leads to $\lambda = -\frac{1}{2}$. Therefore, we have determined the extended Prandtl mixing length as follows:

$$\ell(x,y) = \kappa y(1-\frac{y}{\delta})\sqrt{\frac{\nu}{U\delta}}. \tag{27}$$

If the coefficient α and β are assumed to be constant, then we have the function $U(x)$ as follows

$$U(x) = Cx^m, \quad m = \frac{\beta}{2\alpha-\beta}. \tag{28}$$

With the obtained $U(x)$, we have the boundary layers thickness $\delta(x) = \sqrt{\frac{\nu\beta}{Cm}x^{1-m}}$. Without loss generality, we set $\alpha = 1$, we have

$$\delta(x) = \sqrt{\frac{2}{m+1}\frac{\nu x}{U}}, \tag{29}$$

and similarity variable

$$\eta = \frac{y}{\delta} = y\sqrt{\frac{m+1}{2}\frac{U}{\nu x}}. \tag{30}$$

and the extended mixing length

$$\ell(x,y) = [\frac{1+m}{2C}\nu]^{1/4}\kappa y(1-\frac{y}{\delta})x^{-\frac{1+m}{4}}. \tag{31}$$

Finally, we have successfully transformed the partial differential equations in Eq.(10) and its boundary conditions into a single ordinary differential equation as follows

$$\begin{aligned}&\frac{d^3f}{d\eta} + f\frac{d^2f}{d\eta^2} + \beta[1-(\frac{df}{d\eta})^2] \\ &+ \kappa^2\frac{d}{d\eta}[\eta^2(1-\eta)^2\frac{d^2f}{d\eta^2}|\frac{d^2f}{d\eta^2}|] = 0,\end{aligned} \tag{32}$$

and boundary conditions (25) and (26).

The equation Eq.(32) can be reduced to the Falkner-Skan equation of laminar boundary layers flows when $\kappa = 0$. The parameter $\beta = \frac{2m}{1+m}$ can be determined once the potential flow velocity $U(x) = Cx^m$ is given. If $m \neq 0$, i.e., for boundary layers with outer flow $U(x) \neq 0$, different wedge flows, stagnation point flow, convergent/divergent channel flow and free jets can be studied by Eq.(32) [3, 26].

TWO DIMENSIONAL TURBULENT BOUNDARY LAYER ON A FLAT PLATE AT ZERO INCIDENCE

In the case of two dimensional turbulent plate boundary layers, $m = 0$ then $\beta = 0$, flow velocity $U(x) = U_\infty$, the function $\delta(x) = \sqrt{\frac{2\nu x}{U_\infty}}$, and the extended Prandtl mixing length $\ell(x,y) = \kappa y(1-\frac{y}{\delta})(\frac{\nu}{2xU_\infty})^{1/4}$.

The similarity differential ordinary equation 32 is reduced to

$$\frac{d^3f}{d\eta} + f\frac{d^2f}{d\eta^2} + \kappa^2\frac{d}{d\eta}[\eta^2(1-\eta)^2\frac{d^2f}{d\eta^2}|\frac{d^2f}{d\eta^2}|] = 0, \tag{33}$$

This equation can be numerically solved. The $f(\eta)$, $\frac{df}{d\eta}$ and $\frac{d^2f}{d\eta^2}$ are depicted in Fig.2.

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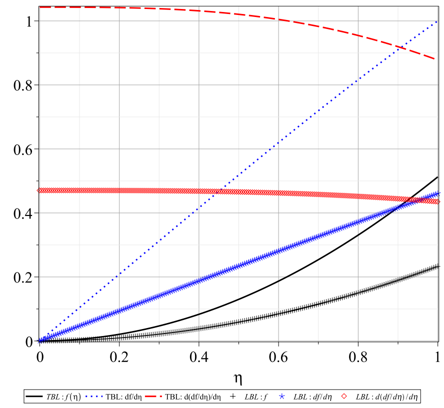

FIG. 2: $f(\eta)$, $\frac{df}{d\eta}$ and $\frac{d^2 f}{d\eta^2}$, in which $f''(0) = \frac{d^2 f}{d\eta^2}|_{\eta=0} = 1.08997$. The Blasius solutions data of the laminar boundary layer (LBL) are from [21], TBL stands for turbulent boundary layer.

Mean velocity profile: With the $U(x)$ and $\delta(x)$, we have the velocity components as follows

$$\bar{u} = U_\infty f'(\eta), \tag{34}$$

$$\bar{v} = \sqrt{\frac{U_\infty}{2\nu x}}[\eta f'(\eta) - f(\eta)]. \tag{35}$$

Shear stress and drag: The shear stress is given by

$$\tau_{xy} = \mu\frac{\partial \bar{u}}{\partial \eta} = \frac{\mu}{\delta(x)}\frac{\partial \bar{u}}{\partial \eta} = \rho\nu\frac{U_\infty}{\delta(x)}f''(\eta)$$
$$= \frac{\rho U_\infty^2}{\sqrt{2Re_x}}f''(\eta). \tag{36}$$

where $Re_x = \frac{xU_\infty}{\nu}$

The wall shear stress $\tau_w(x) = \tau_{xy}(0)$ is given by

$$\tau_w(x) = \frac{\rho U_\infty^2}{\sqrt{2Re_x}}f''(0) = \frac{0.624\rho U_\infty^2}{\sqrt{Re_x}}, \tag{37}$$

where the numerical value for $f''(0) = 0.883$ was computed by the Maple code. The skin-friction coefficient with the reference velocity U_∞ is given by

$$c_f(x) = \frac{\tau_w(x)}{\frac{1}{2}\rho U_\infty^2} = \frac{1.249}{\sqrt{Re_x}}, \tag{38}$$

The drag on one side of a plate of length L and unit breadth is then

$$D = \int_0^L \tau_w dx = 1.249\rho U_\infty\sqrt{\nu L U_\infty}, \tag{39}$$

Then we have the drag coefficient is

$$C_D = \frac{D}{(1/2)\rho L U_\infty^2} = \frac{2.496}{\sqrt{Re}}, \tag{40}$$

where the Reynolds number $Re = \frac{U_0 L}{\nu}$.

Displacement layer thickness: There is no unique boundary layer thickness, since the effect of the viscosity in boundary layer decreases asymptotically as moving outwards from the wall. If we define the boundary layer thickness to be the position where $\bar{u} = 0.99U$, we fund that $\eta = 0.9926$. Therefore, a physical sensible measure for the thickness of the boundary layer id the displacement thickness δ_u. The definition is

$$\delta_u = \int_0^\delta (1 - \frac{\bar{u}}{U})dy = \delta(x)\int_0^1 (1 - f')d]d\eta$$
$$= 0.47704\sqrt{\frac{2\nu x}{U_\infty}} = 0.6746\sqrt{\frac{\nu x}{U_\infty}}, \tag{41}$$

where $f(1) = 0.52296$ is used.

Velocity fluctuation: The velocity fluctuation in x direction is give by

$$u' = \ell\frac{d\bar{u}}{dy} = \frac{\kappa U_\infty}{\sqrt[4]{2Re_x}}\eta(1-\eta)f''(\eta). \tag{42}$$

Reynolds stress: The Reynolds stress is given by

$$\tau'_{xy} = -\rho\ell^2\frac{d\bar{u}}{dy}|\frac{d\bar{u}}{dy}| = \frac{\kappa^2 U_\infty^2}{\sqrt{2Re_x}}\eta^2(1-\eta)^2(f''(\eta))^2. \tag{43}$$

For comparison studies, the results of laminar and turbulent boundary layer on a flat plate at zero incidence are shown in Table I

TABLE I: Laminar vs. turbulent boundary layer.

	Laminar	Turbulent
Mixing length	$\ell(y) = \kappa y$	$\ell(x,y) = \kappa y(1 - \frac{y}{\delta})\sqrt{\frac{\nu}{U\delta}}$
Wall shear stress	$\tau_w = \frac{0.332\rho U_\infty^2}{\sqrt{Re_x}}$	$\tau_w = \frac{0.624\rho U_\infty^2}{\sqrt{Re_x}}$
Skin-friction coeff.	$c_f(x) = \frac{0.664}{\sqrt{Re_x}}$	$c_f(x) = \frac{1.249}{\sqrt{Re_x}}$
Drag coeff.	$C_D = \frac{1.328}{\sqrt{Re}}$	$C_D = \frac{2.496}{\sqrt{Re}}$

With the extended mixing length, Table I show that all of quantities of the laminar and turbulent boundary layer possess non-zero wall values or near-wall peaks - follow a universal $1/\sqrt{Re}$ defect law.

CONCLUSIONS

To the best of the author's knowledge, within the frame of the Prandtl mixing modelling, the similarity solutions of the 2D turbulent boundary layers are obtained for the first time. This study may help facilitate a better understanding of turbulence phenomena.

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A complete Maple code to solve the Eq.32 under the boundary conditions Eqs.25 and 26

restart;with(student);with(plots);beta := 0;kappa := 0.4;b := 1;ode := diff(f(xi), xi, xi, xi) + f(xi)*diff(f(xi),xi,xi)+beta*(1 - diff(f(xi), xi)*(1 - diff(f(xi), xi)) + kappa*kappa*diff(xi*xi*(1-xi)*(1-xi)*abs(diff(f(xi), xi, xi))*diff(f(xi), xi, xi), xi) = 0;sol := dsolve(ode, f(0) = 0, D(f)(0) = 0, D(f)(b) = 1, numeric, output = listprocedure);p0 := plots:-odeplot(sol, [xi, f(xi)], xi = 0 .. b, color = black, linestyle = [1], thickness = 3, legend = [f(xi)], axes = boxed);p1 := plots:-odeplot(sol, [xi, diff(f(xi), xi)], xi = 0 .. b, color = blue, linestyle = [2], thickness = 3, legend = ["df/dxi"], axes = boxed);p2 := plots:-odeplot(sol, [xi, diff(f(xi), xi, xi)], x-

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i = 0 .. b, color = red, linestyle = [3], thickness = 3, legend = ["d(df/dxi;)/dxi"], axes = boxed);u := plots:-odeplot(sol, [xi, diff(f(xi), xi)], xi = 0 .. b, color = red, linestyle = [1], thickness = 3, legend = ["df/dxi"], axes = boxed);v := plots:-odeplot(sol, [xi, xi*diff(f(xi), xi)], xi = 0 .. b, color = blue, linestyle = [3], thickness = 3, legend = ["xi;df/dxi"], axes = boxed);plots:-display([p0, p1, p2], axes = boxed);plots:-display([u, v], axes = boxed);