A Generalization of the Sum of Divisors Function

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Abstract

Superabundant and colossally abundant numbers are generated using the sum of divisors function. If the Riemann hypothesis is false, there will necessarily exist a counterexample to an inequality involving the maximal order of the sum of divisors function which is a colossally abundant number. This paper introduces a generalization of the sum of divisors function having a recursive definition. The multiplicative property of the sum of divisors function is preserved so that infinitely many variants of superabundant and colossally abundant numbers can be generated. Besides the usual method of generating superabundant and colossally abundant numbers, we introduce a new method involving both the sum of divisors function and Euler's totient function. This method is applicable to the variant superabundant and colossally abundant numbers. There is an abundance of material for new research.

1 Introduction

The sum of divisors function is commonly denoted by $\sigma(n)$. An arithmetical function f is called multiplicative if f is not identically zero and if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. The Dirichlet product of arithmetical functions f and g has the form $\sum_{d|n} f(d)g(\frac{n}{d})$. If f and g are multiplicative, so is their Dirichlet product $f \cdot g$. See Theorem 2.14 of Apostol's [1] book. The sum of divisors function is multiplicative. See section 2.13 of Apostol's book. The Dirichlet product of $\sigma(n)$ with the constant function $g(n) = 1$ gives another multiplicative function. Similarly, the Dirichet product of this multiplicative function with the constant function gives yet another multiplicative function. The multiplicative property of these functions is essential for generating variants of superabundant numbers.

The connection between the Riemann hypothesis and the sum of divisors function dates back to Ramanujan's [2] work on highly composite numbers. In 1913, Gronwall [3] determined the maximal order of the sum of divisors function. More recently Alaoglu and Erdős [4] improved Ramanujan's work and proved theorems applicable to superabundant and colossally abundant numbers. In 1984, Robin [5] gave a much more refined version of the asymptotic upper bound of the sum of divisors function than Gronwall and proved a statement equivalent to the Riemann hypothesis. Since then, many statements involving the maximal order of the sum of divisors function have been proved to be equivalent to the Riemann hypothesis.

One purpose of this work is to show that there are infinitely many variants of superabundant numbers and thus that there are likely to be infinitely many statements equivalent to the Riemann hypothesis. No variant superabundant number is proven to lead to a statement equivalent to the Riemann hypothesis - empirical evidence is just given. Another purpose is to investigate the alternate method for generating superabundant and colossally abundant numbers.

2 Results

Let $r_{n,1} = \sigma(n)$, $r_{n,2} = \sum_{d|n} r_{d,1}$, $r_{n,3} = \sum_{d|n} r_{d,2}$, etc.

Theorem 1 *If n is squarefree, $\prod_{p|n}(p+1) = r_{n,1}$. More generally, if p^m divides n and p^{m+1} does not divide n, then $p^m + p^{m-1} + p^{m-2} + \ldots + 1$ divides $r_{n,1}$, that is, $\frac{p^{m+1}-1}{p-1}$ divides $r_{n,1}$.*

Proof 1 *See Lemma 2.1 of Carella's [6] article.*

$\phi(n)$ denotes Euler's totient function. The alternate representation ($\sigma(n) = \prod_{p|n} \frac{p^{m+1}-1}{p-1}$) is of importance due to the similar representation of $\phi(n)$. $\phi(n) = n \prod_{p|n}(1-p^{-1}$. See Theorem 2.4 of Apostol's book.

Theorem 2 *If n is squarefree, $\prod_{p|n}(p+2) = r_{n,2}$. More generally, if p^m divides n and p^{m+1} does not divide n, then $p^m + 2p^{m-1} + 3p^{m-2} + \ldots + (m+1)$ divides $r_{n,2}$, that is, $\frac{\frac{p^{m+2}-1}{p-1}-(m+2)}{p-1}$ divides $r_{n,2}$.*

Proof 2 *If $p^m||n$, then $1+(p+1)+(p^2+p+1)+\ldots+(p^m+p^{m-1}+\ldots+1) = p^m+2p^{m-1}+3p^{m-2}+\ldots+(m+1)$ divides $r_{n,2}$. This can easily be verified to equal $\frac{\frac{p^{m+2}-1}{p-1}-(m+2)}{p-1}$.*

Theorem 3 *If n is squarefree, $\prod_{p|n}(p+3) = r_{n,3}$. More generally, if p^m divides n and p^{m+1} does not divide n, then $p^m + 3p^{m-1} + 6p^{m-2} + \ldots + (m+1)(m+2)/2$ divides $r_{n,3}$. An expression similar to the above can be derived using the binomial theorem.*

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Proof 3 *If $p||n$, then $1+(p+2)+(p^2+2p+3)+(p^3+2p^2+3p+4)+\ldots+(p^m+2p^{m-1}+3p^{m-2}+\ldots+(m+1) = p^m+\frac{2(2+1)}{2}p^{m-1}+\frac{3(3+1)}{2}p^{m-2}+\ldots+\frac{(m+1)(m+2)}{2}$ divides $r_{n,3}$.*

2.1 Superabundant Numbers

A natural number is called a superabundant number if $\sigma(n)/n > \sigma(k)/k$ for $1 \leq k \leq n-1$. The first few superabundant numbers are 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040,.... Gronwall's theorem is

Theorem 4 $\lim_{n\to\infty} \sup \frac{\sigma(n)}{n \log\log n} = e^{\lambda}$

Proof 4 *See Theorem 323 in Hardy and Wright's [7] book.*

λ denotes Euler's constant. A plot of $\frac{r_{n,1}}{n \log\log n}$ for n = 2, 4, 6, 12, 24, 36, 48, 60, 120, 180, 240, 360, 720, 840, 1260, 1680, 2520, 5040, 10080, 15120, 25200, 27720, 55440, 110880, 166320, 277200, 332640, 554400, 665280, 720720, and 1441440 (the superabundant numbers less than 2 million) is

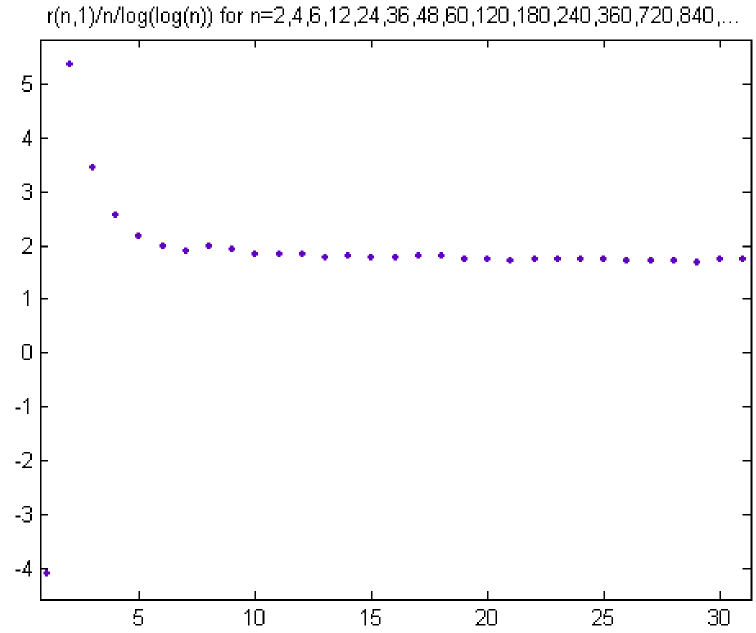

Figure 1: Plot of $\frac{r_{n,1}}{n \log\log n}$ values

e^{λ} is approximately equal to 1.7811. A quadratic least-squares fit of $r_{n,1}/n$ versus $\log\log n$ for these n values is

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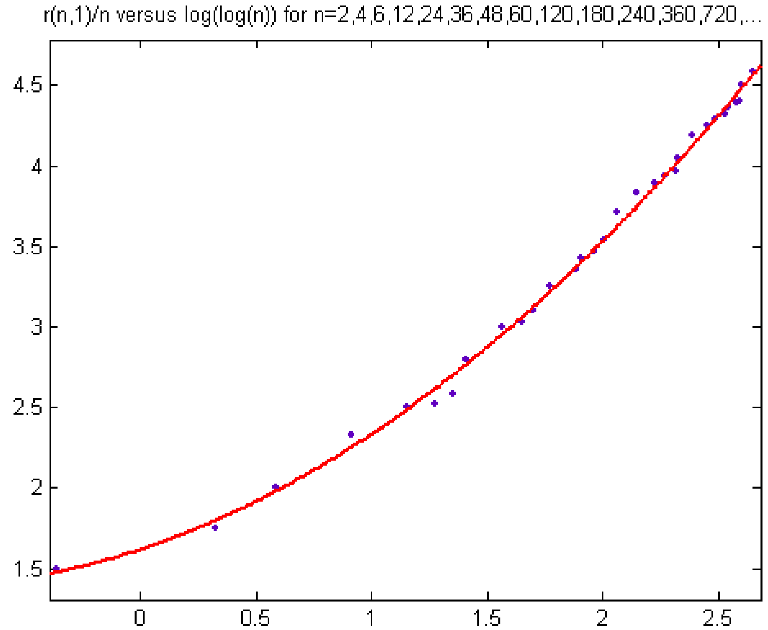

Figure 2: Quadratic least-squares fit of $r_{n,1}/n$ versus $\log\log n$

$p_1 = 0.2386$ with a 95% confidence interval of (0.21, 0.2672), $p_2 = 0.4811$ with a 95% confidence interval of (0.3966, 0.5657), $p_3 = 1.62$ with a 95% confidence interval of (1.556, 1.683), SSE=0.07629, R-squared=0.9966, and RMSE=0.0522. SSE denotes sum-squared error and RMSE denotes root-mean-squared error.

Superabundant numbers consist of a product of all the small primes up to some bound with exponents which are non-increasing as the prime increases. Alaoglu and Erdős' theorems on superabundant numbers (given for comparison purposes) are

Theorem 5 *If $n = 2^{k_2} \cdots p^{k_p}$, then $k_2 \geq k_3 \geq \ldots \geq k_p$.*

Theorem 6 *Let $q < r$, and set $\beta = [k_q \log q/\log r]$. Then k_r has one of the three values: $\beta - 1$, $\beta + 1$, β.*

Theorem 7 *If p is the largest prime factor of n, then $k_p = 1$, except when $n = 4$, 36.*

In the remainder of the theorems, p always denotes the largest prime factor of n.

Theorem 8 *If q is either the greatest prime of exponent k or the least prime of exponent $k-1$, and if $q^{1-\theta} > \log p$, then $q^k = (p\log p/\log q)[1 + O(\log p/q^{1-\theta}\log q)]$.*

If Riemann's hypothesis is true any $\theta > 1/2$ can be used.

Theorem 9 *If $k_q = k$ and $q < (\log p)^\alpha$, where α is a constant, then (i)* $\log \frac{q^{k+1}-1}{q^{k+1}-q} > \frac{\log q}{p \log p}[1 + O((\log\log p)^2/\log p \log q)]$, *(ii)* $\log \frac{q^{k+2}-1}{q^{k+2}-q} < \frac{\log q}{p \log p}[1 + O((\log\log p)^2/\log p \log q)]$.

Let $K_q + 1$ be the integral part and θ_q the fractional part of $\log[(q^{1+\epsilon} - 1)/(q^\epsilon - 1)]/\log q$, where $\epsilon = \log(1 + 1/p)\log p$.

Theorem 10 *(i) If $\epsilon_q < \theta_q \leq 1 - \epsilon_q$, then $k_q = K_q$. (ii) If $\theta_q \leq \epsilon_q$, then $k_q = K_q$ or $k_q = K_q - 1$. (iii) If $1 - \epsilon_q < \theta_q$, then $k_q = K_q$ or $K_q + 1$.*

Theorem 11 $p \sim \log n$.

Theorem 12 *The quotient of two consecutive superabundant numbers tends to 1.*

In the proof of Theorem 12 it was shown that the ratio of two consecutive superabundants n and n' is less than $1 + c(\log\log n)^2/\log n$. The order of $(\log p)^c$ is used.

Theorem 13 *The number of superabundant numbers less than x exceeds $c \log x \log\log x/(\log\log\log x)^2$.*

Similar superabundant numbers can be defined for $r_{n,2}$, $r_{n,3}$, $r_{n,4}$, etc. and usually occur at the same n values. For example, the superabundant numbers for $r_{n,2}$ satisfy $r_{n,2}/n > r_{k,2}/k$ for $1 \leq k \leq n-1$. Like the usual superabundant numbers, these superabundant numbers consist of a product of all the small primes up to some bound with exponents which are non-increasing as the prime increases. For $n_{n,2}$, $r_{n,3}$, $r_{n,4}$,...,$r_{n,13}$ and $n < 2000000$, there are no exceptions to Theorem 5.

For $r_{n,2}$, $r_{n,3}$, and $n < 2000000$ there are no exceptions to Theorem 6. For $r_{n,4}$ and $n < 2000000$ exceptions occur at n=96 and 480. In these cases, k_r has the value $\beta - 2$ for $q = 2$ and $r = 3$. There are more such exceptions for $r_{n,5}$, $r_{n,6}$, $r_7, \ldots, r_{n,13}$ and higher order $r_{n,i}$ values. Typical values are $n = 96$, 480, 960, 1152, 5760, 34560, 40320, 241920, 483840, and 1209600. For $r_{n,9}$ and $r_{n,11}$, there are exceptions at $n = 1088640$. In this case, k_r has the value $\beta - 2$ for $q = 3$ and $r = 5$. For $r_{n,11}$ and $n < 2000000$, exceptions occur at $n = 11520$ and 80640. In these cases, k_r has the value $\beta - 3$ for $q = 2$ and $r = 3$. For higher order $r_{n,i}$ values, typical n values where such exceptions occur are 2304, 11520, 80640, and 967680.

There are also exceptions to Theorem 7 when $q = 2$ and $r = 3$. For $r_{n,1}$, exceptions occur at 4 and 36 (the exceptions for ordinary superabundant numbers). For $r_{n,2}$, exceptions occur at 4, 36, and 72. For $r_{n,3}$, exceptions occur at 4, 8, 36, and 72. For $r_{n,4}$, exceptions occur at 4, 8, 36, 72, and 144. Usually there is one more new exception for the next higher order of variant superabundant numbers. For n_{n13}, the exceptions are 4, 8, 16, 36, 72, 144, 288, 432, 576, 864, 1152, 1728, 2304, 86400, and 172800.

Empirically, it is not practical to determine if Theorems 8, 9, and 10 are applicable to variant superabundant numbers.

A plot of p versus $\log n$ for $n = 2$, 4, 6, 12, 24,...,1441440 (the ordinary superabundant numbers) is

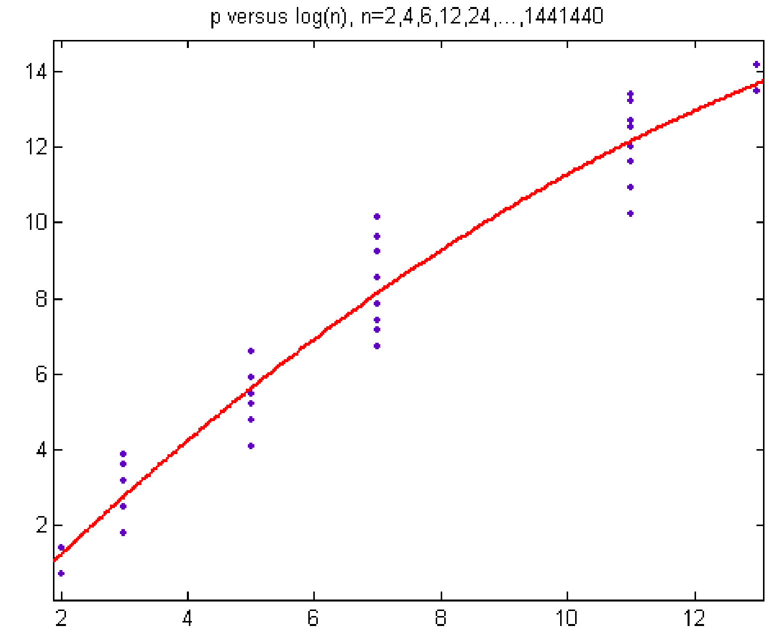

Figure 3: p versus $\log n$ for ordinary superabundant numbers

This is pertinent to Theorem 11. Similar plots occur for the variant superabundant numbers. For $n < 2000000$, the largest p values for $r_{n,1}$, $r_{n,2}$, $r_{n,3}$,...,$r_{n,13}$ are 13, 13, 11, 11, 11, 7, 7, 7, 7, 7, 7, 7, and 7 respectively.

For $r_{n,1}$, $r_{n,2}$, $r_{n,3}$,...,$r_{n,13}$ and $n < 2000000$, a superabundant number (ordinary or variant) is at most twice as large as the previous superabundant number. This is pertinent to Theorem 12.

The number of superabundant numbers less than 2000000 for $r_{n,2}$, $r_{n,3}$, $r_{n,4}$,. . .,$r_{n,13}$ is 38, 39, 44, 43, 41, 41, 42, 48, 48, 49, 51, and 53 respectively. The number of ordinary superabundant numbers less than 2000000 is 31, so the counts for the new superabundant numbers are consistent with Theorem 13.

A plot of $\sqrt{\frac{r_{n,2}}{n \log \log n}}$ for n = 2, 4, 6, 12, 24, 36, 48, 60, 72, 120, 180, 240, 360, 720, 1440, 1680, 2160, 2520, 5040, 10080, 15120, 20160, 25200, 30240, 50400, 55440, 110880, 166320, 221760, 277200, 332640, 554400, 665280, 831600, 1108800, 1330560, 1441440, and 1663200 is

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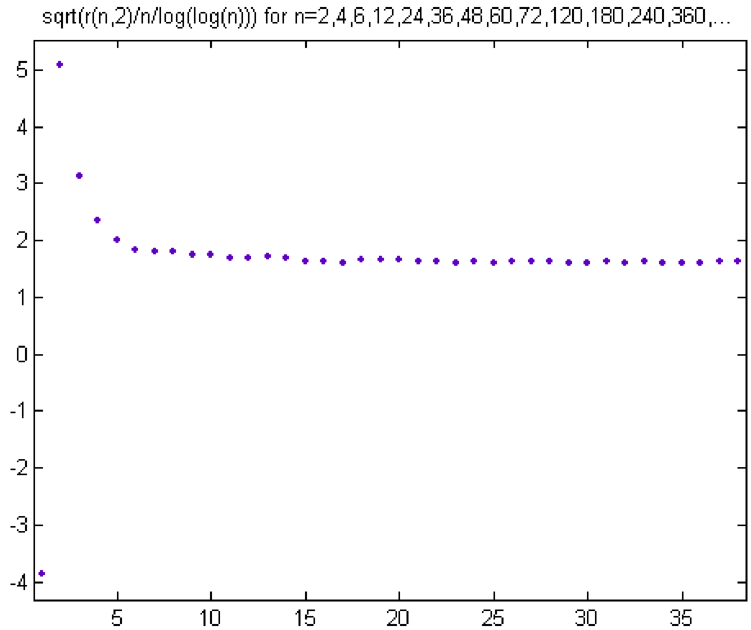

Figure 4: Square root of $\frac{r_{n,2}}{n \log\log n}$

In this variant of Gronwall's formula, the supremum limit is unknown but is slightly less than e^{λ}. For a cubic least-squares fit of $r_{n,2}/n$ versus $\log\log n$ for these n values, SSE=2.172, R-squared=0.9977, and RMSE=0.2527.

A plot of $\sqrt[3]{\frac{r_{n,3}}{n \log\log n}}$ for $n = 2$, 4, 6, 8, 12, 24, 36, 48, 60, 72, 120, 180, 240, 360, 720, 1440, 2160, 2520, 4320, 5040, 10080, 15120, 20160, 30240, 50400, 55440, 60480, 100800, 110880, 151200, 166320, 221760, 302400, 332640, 554400, 665280, 1108800, 1330560, and 1663200 is

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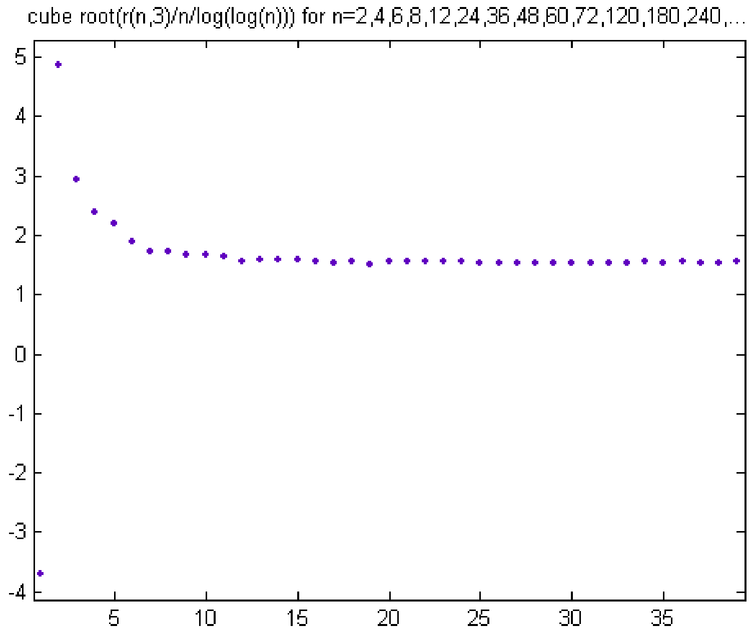

Figure 5: Cube root of $\frac{r_{n,3}}{n \log \log n}$

In this variant of Gronwall's formula, the supremum limit is smaller than above. For a 4th power least-squares fit of $r_{n,3}/n$ versus $\log \log n$ for these n values, SSE=35.02, R-squared=0.9979, and RMSE=1.015.

A cubic least-squares fit of the smallest positive $\sqrt[i]{\frac{r_{n,i}}{n \log \log n}}$ values for $n < 2000000$ and $i = 1, 2, 3, \ldots, 16$ is

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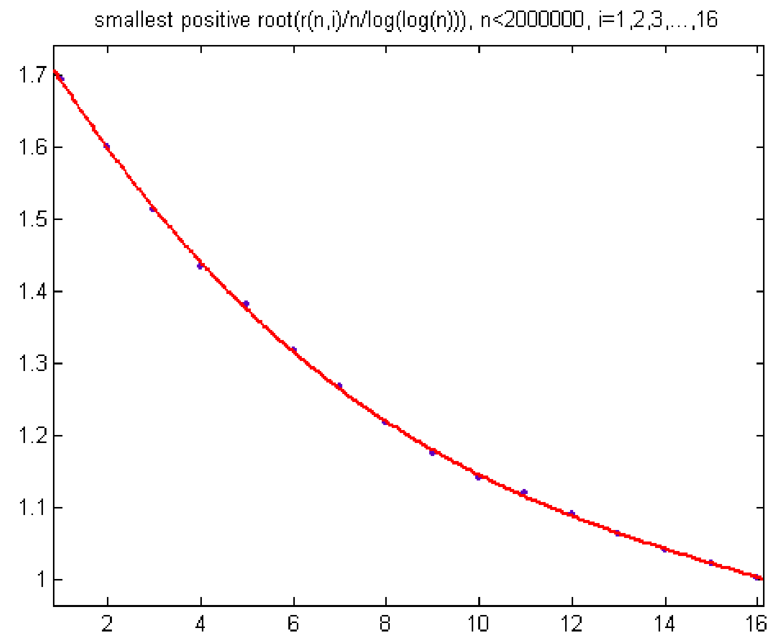

Figure 6: Cubic least-squares fit of smallest positive values of roots of ratios

RMSE=0.0002309, R-squared=0.9997, and RMSE=0.004386. The values are 1.6943, 1.5998, 1.5135, 1.4339, 1.3809, 1.3190, 1.2683, 1.2181, 1.1736, 1.1395, 1.1199, 1.0902, 1.0623, 1.0392, 1.0278, and 1.0018 respectively. For small sample sizes, quadratic and cubic curves give good approximations of some logarithmic functions. Eventually, higher order curves will be required to approximate the values (which are expected to approach zero).

2.2 Euler's Phi Function and the Sum of Divisors Function

Hardy and Wright's Theorem 328 is

Theorem 14 $\lim_{n\to\infty} \inf \frac{\phi(n)\log\log n}{n} = e^{-\lambda}$

Their Theorem 329 is

Theorem 15 $A < \frac{\sigma(n)\phi(n)}{n^2} < 1$ *for a positive constant* A.

The alternate representations of $\phi(n)$ and $\sigma(n)$ are used to prove the above theorem. No counterpart of the alternate representation of $r_{n,2}$ ($\prod_{p|n} \frac{\frac{p^{m+2}-1}{p-1}-(m+2)}{p-1}$) has been found. In general, the relationship between $\phi(n)$ and $\sigma(n)$ is used to prove Gronwall's theorem.

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2.3 New Method for Generating Superabundant Numbers

Superabundant numbers can also be defined for the largest $\frac{r_{n,2}\phi(n)}{n^2}$ values. A plot of $\frac{r_{n,2}\phi(n)}{n^2 \log\log n}$ for n = 3, 4, 8, 16, 24, 32, 48, 72, 144, 288, 432, 576, 720, 864, 1440, 2160, 2880, 3600, 4320, 7200, 8640, 10800, 14400, 15120, 20160, 21600, 30240, 43200, 60480, 75600, 100800, 120960, 151200, 302400, 604800, 907200, 1663200, and 1814400 is

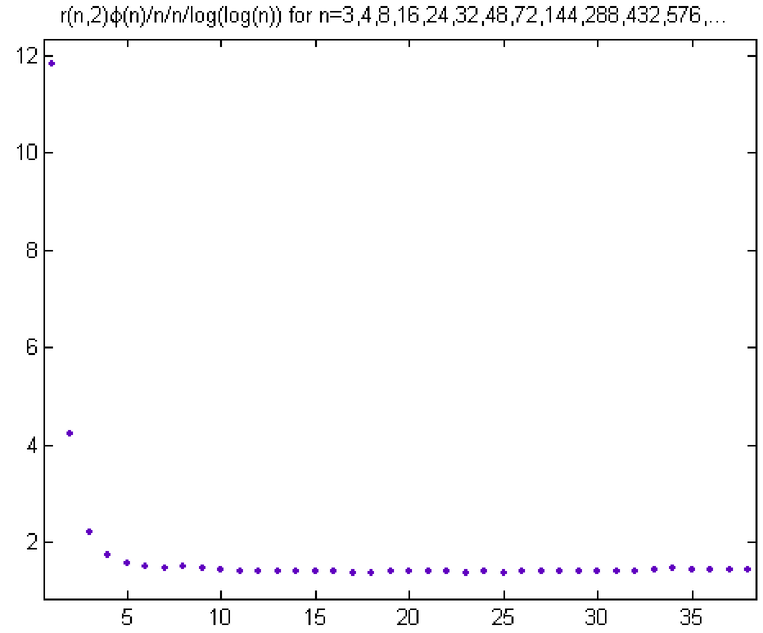

Figure 7: Plot of $\frac{r_{n,2}\phi(n)}{n^2 \log\log n}$

In this variant of Gronwall's formula, the supremum limit is unknown. A quadratic least-squares fit of $\frac{r_{n,2}\phi(n)}{n^2}$ versus $\log\log n$ for these n values is

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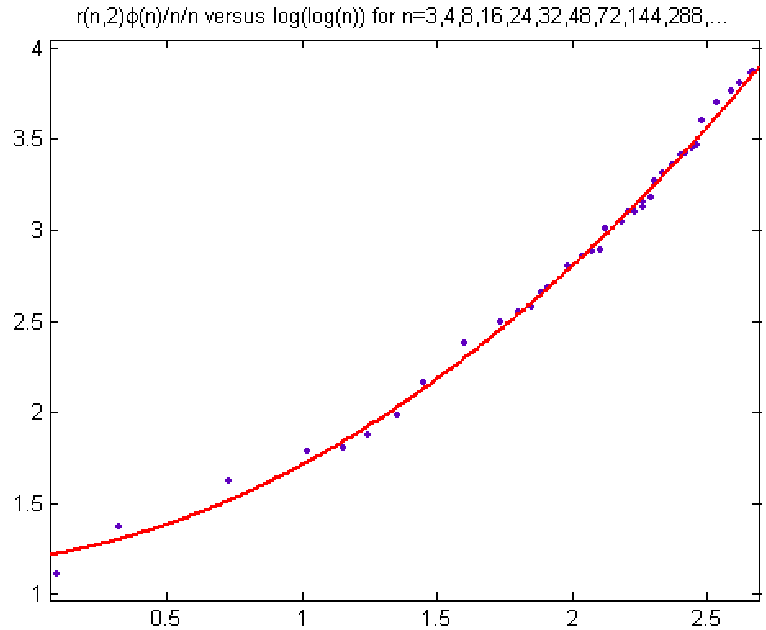

Figure 8: Quadratic least-squares fit of $\frac{r_{n,2}\phi(n)}{n^2}$ versus $\log\log n$

$p_1 = 0.2912$ with a 95% confidence interval of (0.2573, 0.325), $p_2 = 0.2204$ with a 95% confidence interval of (0.1131, 0.3276), $p_3 = 1.201$ with a 95% confidence interval of (1.121, 1.281), SSE=0.08065, R-squared=0.9958, and RMSE=0.048. Note the resemblance to the curve in Figure 2.

Superabundant numbers can also be defined for the largest $\frac{r_{n,3}\phi(n)\phi(n)}{n^3}$ values. A plot of $\sqrt{\frac{r_{n,3}\phi(n)\phi(n)}{n^3 \log\log n}}$ for $n =$ 5, 7, 8, 16, 32, 64, 128, 256, 288, 432, 576, 864, 1728, 3456, 5184, 6912, 10368, 20736, 31104, 41472, 43200, 86400, 129600, 172800, 259200, 518400, 777600, 1036800, 1296000, and 1555200 is

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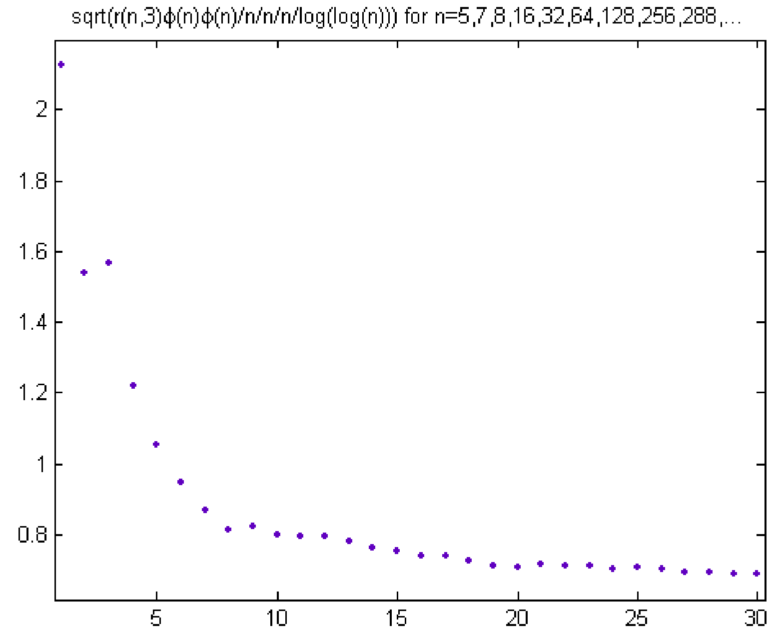

Figure 9: Square root of $\frac{r_{n,3}\phi(n)\phi(n)}{n^3 \log\log n}$

For a cubic least-squares fit of $r_{n,3}/n$ versus $\log\log n$ for these n values, SSE=0.09052, R-squared=0.9935, and RMSE=0.05901.

Superabundant numbers can also be defined for the largest $\frac{r_{n,4}\phi(n)\phi(n)\phi(n)}{n^4}$ values. The smallest value of $\sqrt[3]{\frac{r_{n,4}\phi(n)\phi(n)\phi(n)}{n^4 \log\log n}}$ for $n = 9$, 11, 13, 16, 32, 64, 128, 256, 512, 1024, 1728, 3456, 5184, 6912, 10368, 20736, 31104, 41472, 62208, 124416, 248832, 373248, 497664, 518400, 777600, 1036800, and 1555200 is 0.5427.

Other superabundant numbers are similarly defined for $r_{n,5}$, $r_{n,6}$, $r_{n,7}$, etc. The smallest of the above values for $r_{n,2}$, $r_{n,3}$, $r_{n,4}$, $r_{n,5}$, $r_{n,6}$, $r_{n,7}$, and $r_{n,8}$ are 1.3734, 0.6888, 0.5427, 0.4862, 0.4561, 0.4350, and 0.4187 respectively.

2.4 Variants of Colossally Abundant Numbers

Colossally abundant numbers are those numbers n for which there is a positive exponent ϵ such that $\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(k)}{k^{1+\epsilon}}$ for all $k > 1$. The first few colossally abundant numbers are 2, 6, 12, 60, 120, 360, 2520, 5040, 55440, 720720, 1441440, 4324320, 21621600, 367567200, ... Colossally abundant numbers are a subset of the superabundant numbers. In the above, the n values start with a few odd values followed by powers of 2. Afterwards, the n values are the product of powers of 2 and 3, the product of powers of 2, 3, and 5, the product of powers of 2, 3, 5, and 7, etc. This is similar to colossally abundant numbers. Unlike colossally abundant numbers, the factorization of the n values does not result in a sequence of relatively large

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primes with an exponent of 1. For $r_{n,2}$, the odd numbers and powers of 2 are 3, 4, 8, and 16. For $r_{n,3}$, the odd numbers and powers of 2 are 5, 7, 8, 16, 32, 64, 128, 256. For $r_{n,4}$, the odd numbers and powers of 2 are 9, 11, 13, 16, 32, 64, 128, 256, 512, and 1024. For $r_{n,5}$, the odd numbers and powers of 2 are 13, 17, 19, 23, 25, 27, 32, 64, 128, 256, 512, 1024, 2048, 4096, and 8192. For $r_{n,6}$, the odd numbers and powers of 2 are 19, 23, 29, 31, 37, 49, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, and 32768. For $r_{n,7}$, the odd numbers and powers of 2 are 29, 31, 37, 41, 43, 47, 49, 81, 125, 128, 243, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, and 131072. Note that a power of 3 (243) is interspersed between 128 and 256. For $r_{n,8}$, the odd numbers and powers of 2 are 31, 37, 41, 43, 47, 53, 59, 61, 67, 121, 125, 243, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, and 1048576. For $r_{n,9}$, the odd numbers and powers of 2 (up to 2 million) are 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 121, 169, 243, 289, 343, 512, 729, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... Note that a power of 3 (729) is interspersed between 512 and 1024. For $r_{n,10}$, the odd numbers and powers of 2 (up to 2 million) are 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 169, 289, 343, 625, 729, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... For $r_{n,11}$, the odd numbers and powers of 2 (up to 2 million) are 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 169, 289, 361, 529, 625, 1331, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... For $r_{n,12}$, the odd numbers and powers of 2 (up to 2 million) are 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 289, 361, 529, 841, 1331, 2187, 2401, 3125, 4096, 6561, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... Note that a power of 3 (6561) is interspersed between 4096 and 8192. For $r_{n,13}$, the odd numbers and powers of 2 (up to 2 million) are 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 289, 361, 529, 841, 1331, 2197, 2401, 3125, 6561, 8192, 15625, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... Note that a power of 5 (15625) is interspersed between 8192 and 16384. For $r_{n,14}$, the odd numbers and powers of 2 (up to 2 million) are 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 361, 529, 841, 961, 1331, 2197, 2401, 6561, 14641, 15625, 16384, 32768, 65536, 131072, 262144, 524288, 1048576,.... For $r_{n,15}$, the odd numbers and powers of 2 (up to 2 million) are 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 221. 223, 227, 529, 841, 961, 1331, 2197, 4913, 14641, 15625, 19683, 32768, 59049, 65536, 131072, 262144, 524288, 1048576,.... For $r_{n,16}$, the odd numbers and powers of 2 (up to 2 million) are 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 529, 841, 961, 1369, 1681, 2197, 4913, 6859, 14641, 15625, 16807, 59049, 131072, 262144, 524288, 1048576,.... The odd numbers appear to be prime powers. All the n values appear to satisfy Theorem 2 (with the caveat of the $\beta - 2$ case).

A spectrum similar to the Riemann spectrum can be computed for the odd numbers and the powers of 2 by using the function $F_{\leq C}(t) = -\sum_{p^n \leq C} \frac{\log(p)}{p^{n/2}} \cos(t \log(p^n))$. See Chapter 30 of Mazur and Stein's [8]

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book for graphs of the Riemann spectrum. A plot of the positive values of this function for the odd numbers and powers of 2 corresponding to $r_{n,4}$ and $t \leq 100$ is

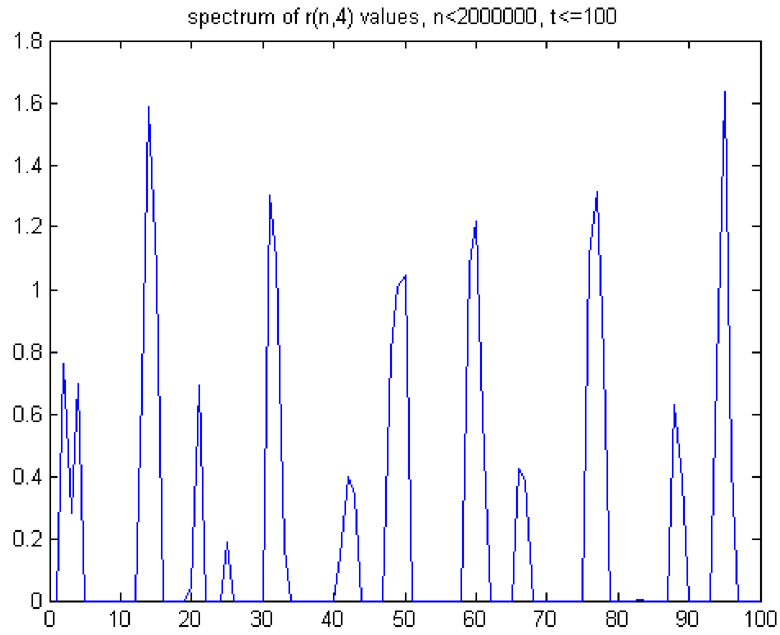

Figure 10: spectrum of odd numbers and powers of 2 corresponding to $r_{n,4}$

All 27 of the variant colossally abundant numbers for $r_{n,4}$ are checked (to determine if they are prime powers), so C is effectively 2 million. The locations of the peaks for $t \leq 300$ are

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2	3	4	
13	14	15	
20	21		
25			
31	32	33	
41	42	43	
48	49	50	
59	60	61	
66	67		
76	77	78	
83			
88	89		
94	95	96	
104	105	106	
111	112	113	
122	123	124	
130			
134			
139	140	141	
150	151	152	
157	158	159	
167	168	169	170
174	175	176	
185	186	187	
192			
196	197	198	
202	203	204	
213	214	215	
220	221	222	
231	232	233	
248	249	250	
259	260		
266	277		
276	277	278	
283			
294	295	296	

The width of the peaks is usually three. The inverse function for the Riemann spectrum is $H_{\leq C}(s) = 1 + \sum_{i \leq C} \cos(\log(s)\theta_i)$. The non-trivial zeta function zeros are denoted by θ_i. In the application of this function here, the maxima of the spectrum are taken to be just the positive values. A plot of the inverse function is

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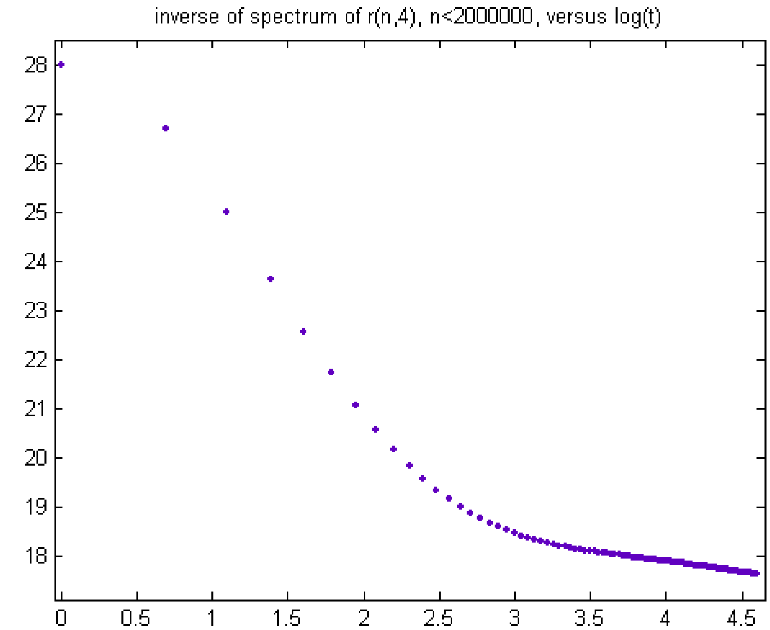

Figure 11: inverse of spectrum of odd numbers and powers of 2 corresponding to r_4

The inverse of the spectrum of the variant colossally abundant numbers for $r_{n,2}$ is a constant 39. The locations of the peaks of the spectrum of the variant colossally abundant numbers for $r_{n,3}$ and $t \leq 300$ are 14, 76, 95, 186, 276, and 295. (Note that these locations are common to the locations for $r_{n,4}$.) The largest value of the inverse is 31.

The locations of the peaks of the spectrum of the variant colossally abundant numbers for $r_{n,5}$ and $t \leq 300$ are 2, 15, 21, 25, 43, 48, (60, 61), 88, 106, 111, 124, 134, 151, 157, 170, 185, 197, 202, (220, 221), 232, 248, (260, 261), 266, 283, and 295. (Except for the location 261, these locations are common to the locations for $r_{n,4}$.) The largest value of the inverse is 28.

The locations of the peaks of the spectrum of the variant colossally abundant numbers for $r_{n,6}$ and $t \leq 300$ are 2, 14, 21, 25, 30, 33, (48, 49), (60, 61), 65, 76, 88, 96, 107, (111,112), (123, 124), 134, 139, 151, 158 (169,170), 174, 185, 197, 202 (220, 221), (232, 233), 248, 260, 279, 283, and 295. (Two of the locations are one less than a location for $r_{n,4}$ and two are one more than a location.) The largest value of the inverse is 27.

The locations of the peaks of the spectrum of the variant colossally abundant numbers for $r_{n,7}$ and $t \leq 300$ are 25, 30, 33, 48, 61, 65, 76, (88, 89), 96, 107, 111, 124, 134, 139, 170, 174, 185, 198, 202, 205, (220, 221), (232, 2233), (247, 248), (260, 261), 279, and 283. (Three of the locations are one less than a location for $r_{n,4}$ and four are one more than a location.) The largest value of the inverse is 28.

The locations of the peaks of the spectrum of the variant colossally abundant numbers for $r_{n,8}$ and $t \leq 300$ are 2, 21, 25, 30, 33, 41, 44, 50, 53, 61, 65, 76, 83, 89, (95, 96), 107, 111, 124 (134, 135), 138, 147, 150, 156, 159, 170, 174, 185, 192, 198 (201, 202), 205, 216, 221, 225, 233, 237, 244, 247, 261, 276, 279, (282, 283), and (295, 296). (Seven of the locations are one less than a location for $r_{n,4}$ and seven are one more than a location. Five more locations do not match a location for $r_{n,4}$.) A plot of the inverse function is

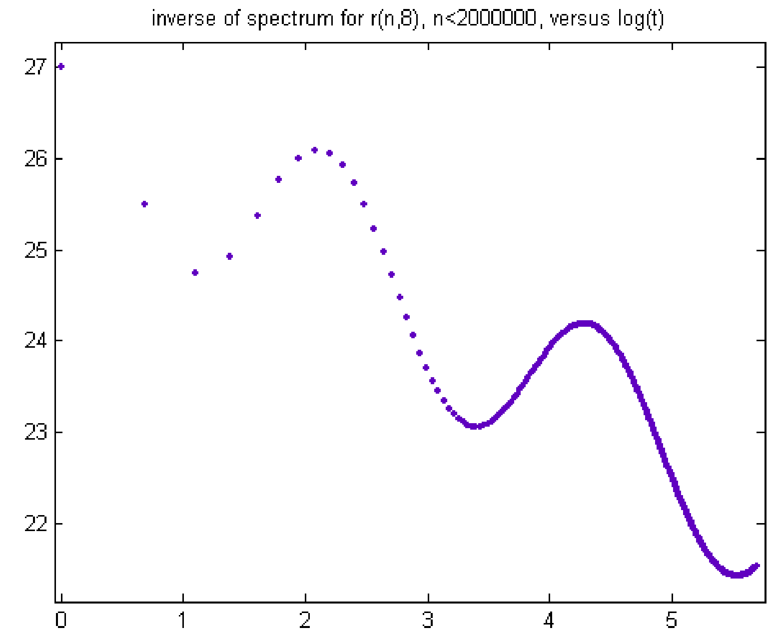

Figure 12: inverse of spectrum variant colossally abundant numbers corresponding to $r_{n,8}$

These are all the variant colossally abundant numbers for $r_{n,i}$ values where the largest power of 2 does not exceed two million. The spectra of the variant colossally abundant numbers appear to have some significance. Interpreting the inverses of the spectra is more difficult.

For the variant colossally abundant numbers corresponding to $r_{n,17}$, the odd numbers are 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 29^2, 31^2, 37^2, 41^2, 43^2, 13^3, 17^3, 19^3, 11^4, 7^5, and 3^{11}. There are 39 of them.

For the variant colossally abundant numbers corresponding to $r_{n,18}$, the odd numbers are 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 29^2, 31^2, 37^2, 41^2, 43^2, 47^2, 17^3, 19^3, 23^3, 11^4, 13^4, 5^7, 3^{11}, and 5^8. There are 44 of them.

For the variant colossally abundant numbers corresponding to $r_{n,19}$, the odd numbers are 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 31^2, 37^2, 41^2, 43^2, 47^2, 53^2, 17^3, 19^3, 23^3, 11^4, 13^4, 5^7, 7^6, 5^8, and 3^{12}. There are 47 of them.

A quadratic least-squares fit of the number of odd numbers is

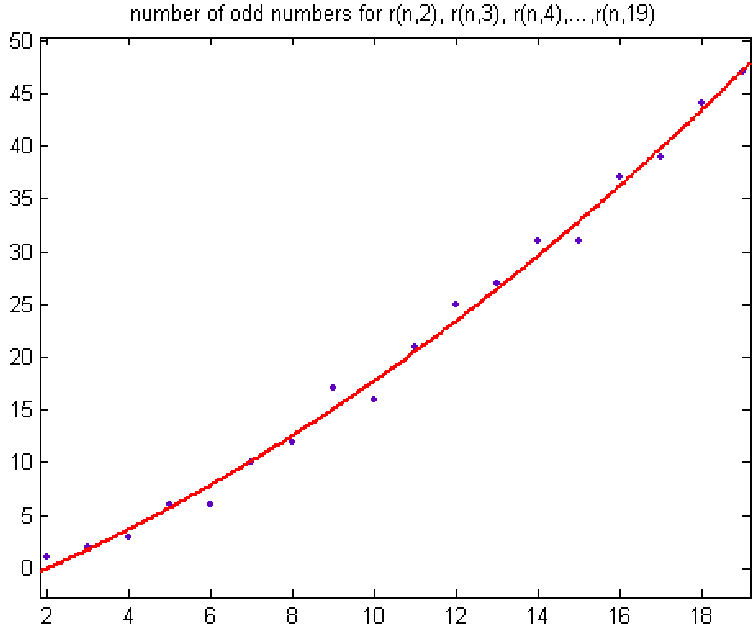

Figure 13: Quadratic least-squares fit of number of odd numbers corresponding to $r_{n,i}$ values

$p_1 = 0.06153$ with a 95% confidence interval of (0.03608, 0.08699), $p_2 = 1.491$ with a 95% confidence interval of (0.9438, 2.038), $p_3 = -3.263$ with a 95% confidence interval of (-5.792, -0.7346), SSE=22.11, R-squared=0.9942, and RMSE=1.214.

Apparently, the number of odd numbers can become arbitrarily large. Perhaps another variant of the Riemann hypothesis can be derived.

2.5 Factorization of the Variants of Colossally Abundant Numbers

Neglecting the odd numbers and powers of 2, the variant colossally abundant numbers less than 2 million corresponding to $r_{n,2}$ are $2^4 \cdot 3$, $2^3 \cdot 3^2$, $2^4 \cdot 3^2$, $2^5 \cdot 3^2$, $2^4 \cdot 3^3$, $2^6 \cdot 3^2$, $2^4 \cdot 3^2 \cdot 5$, $2^5 \cdot 3^3$, $2^5 \cdot 3^2 \cdot 5$, $2^4 \cdot 3^3 \cdot 5$, $2^6 \cdot 3^2 \cdot 5$, $2^4 \cdot 3^2 \cdot 5^2$, $2^5 \cdot 3^3 \cdot 5$, $2^5 \cdot 3^2 \cdot 5^2$, $2^6 \cdot 3^3 \cdot 5$, $2^4 \cdot 3^3 \cdot 5^2$, $2^6 \cdot 3^2 \cdot 5^2$, $2^4 \cdot 3^3 \cdot 5 \cdot 7$, $2^6 \cdot 3^2 \cdot 5 \cdot 7$, $2^5 \cdot 3^3 \cdot 5^2$, $2^5 \cdot 3^3 \cdot 5 \cdot 7$, $2^6 \cdot 3^3 \cdot 5^2$, $2^6 \cdot 3^3 \cdot 5 \cdot 7$, $2^4 \cdot 3^3 \cdot 5 \cdot 7$, $2^6 \cdot 3^2 \cdot 5^2 \cdot 7$, $2^7 \cdot 3^3 \cdot 5 \cdot 7$, $2^5 \cdot 3^3 \cdot 5^2 \cdot 7$, $2^6 \cdot 3^3 \cdot 5^2 \cdot 7$, $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$, $2^6 \cdot 3^4 \cdot 5^2 \cdot 7$,

$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$, and $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$. The numbers $2^5 \cdot 3^3$, $2^5 \cdot 3^3 \cdot 5^2$, $2^6 \cdot 3^3 \cdot 5^2$, and $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ are of interest because they are exceptions to the orderly increase of the last primes of the factorizations. What they have in common is that all the β values of a number equal the exponent of the previous prime. Using $2^5 \cdot 3^3$ as a "seed", the sequence of numbers $2^5 \cdot 3^3$, $2^5 \cdot 3^3 \cdot 5^2$, $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$, $2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11$, $2^{10} \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13$, $2^{13} \cdot 3^8 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17$,... is obtained. When the next-larger prime is added, the exponents of the previous primes are adjusted from right to left. The least exponent of the previous prime that equals β is selected. There is then a well-defined procedure for generating these numbers. Apparently, these numbers can be used to predict when the last primes of the factorizations increase.

These numbers can also be used to generate "building blocks". For example, the "building blocks" for $2^7 \cdot 3^4 \cdot 5^2 \cdot 7$ are $2^7 \cdot 3^5$ ($\beta = 4$), $2^7 \cdot 3^4$ ($\beta = 4$), $2^7 \cdot 3^3$ ($\beta = 4$), $2^7 \cdot 3^2$ ($\beta = 4$), $3^4 \cdot 5^3$ ($\beta = 2$), $3^4 \cdot 5^2$ ($\beta = 2$), $3^4 \cdot 5$ ($\beta = 2$), $5^2 \cdot 7^2$ ($\beta = 1$), $5^2 \cdot 7$ ($\beta = 1$), and $7 \cdot 11$ ($\beta = 0$). The previous exponents can be $\beta + 1$, β, $\beta - 1$, or $\beta - 2$ (permissible values). A cubic least-squares fit of the logarithm of the logarithm of all the building blocks generated from the above numbers is

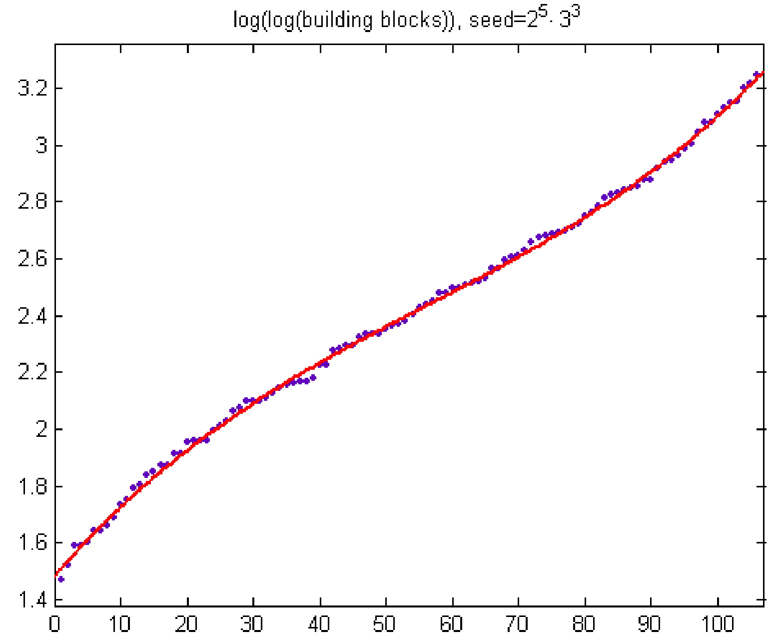

Figure 14: Cubic least-squares fit of the logarithm of the logarithm of all the building blocks

SSE=0.0245, R-squared=0.9989, and RMSE=0.01551. The good fit indicates that the "building blocks" may be useful.

The numbers generated from the "seed" $2^4 \cdot 3^2$ are $2^4 \cdot 3^2 \cdot 5$, $2^5 \cdot 3^3 \cdot 5^2 \cdot 7$,

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$2^8 \cdot 3^5 \cdot 5^3 \cdot 7^2 \cdot 11$, $2^{10} \cdot 3^6 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13$, $2^{13} \cdot 3^8 \cdot 5^5 \cdot 7^4 \cdot 11^3 \cdot 13^2 \cdot 17$, $2^{18} \cdot 3^{11} \cdot 5^7 \cdot 7^5 \cdot 11^4 \cdot 13^3 \cdot 17^2 \cdot 19$, $2^{23} \cdot 3^{14} \cdot 5^9 \cdot 7^7 \cdot 11^5 \cdot 13^4 \cdot 17^3 \cdot 19^2 \cdot 23$,.... These values are not associated with a $r_{n,i}$ value. The cubic least-squares fit of the logarithm of the logarithm of all the building blocks generated from these numbers is also good.

Neglecting the odd numbers and powers of 2, the variant colossally abundant numbers less than 2 million corresponding to $r_{n,3}$ are $2^5 \cdot 3^2$, $2^5 \cdot 3^3$, $2^6 \cdot 3^2$, $2^5 \cdot 3^3$, $2^6 \cdot 3^3$, $2^7 \cdot 3^3$, $2^6 \cdot 3^4$, $2^8 \cdot 3^3$, $2^7 \cdot 3^4$, $2^8 \cdot 3^4$, $2^7 \cdot 3^5$, $2^8 \cdot 3^5$, $2^6 \cdot 3^3 \cdot 5^2$, $2^7 \cdot 3^3 \cdot 5^2$, $2^6 \cdot 3^4 \cdot 5^2$, $2^8 \cdot 3^3 \cdot 5^2$, $2^7 \cdot 3^4 \cdot 5^2$, $2^8 \cdot 3^5 \cdot 5^2$, $2^7 \cdot 3^5 \cdot 5^2$, $2^9 \cdot 3^4 \cdot 5^2$, $2^7 \cdot 3^4 \cdot 5^3$, and $2^8 \cdot 3^5 \cdot 5^2$. Unlike ordinary colossally abundant numbers, the exponents of the primes are strictly decreasing as the primes increase. This appears to be the case for variant colossally abundant numbers corresponding to $r_{n,4}$, $r_{n,5}$, $r_{n,6}$,....

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