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Entropy and Its Application to Number Theory

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ABSTRACT. In this paper, we propose the expansion of the Planck distribution functions which is derived from the Boltzmann principle. Furthermore, we examine to expand Planck's law using new distribution functions. Moreover, using the ideas applied to the expansion of the Planck distribution function, we show that the derivation of Von Koch's inequality without using the Riemann Hypothesis and the negative consequence of the abc conjecture. Besides, we describe some issues for the future. Namely, we discuss that the Entropy is associated with the dynamical system, and the classical gravity theory of Newton's law and the electromagnetism of Coulomb's law by the law of inverse squares.

Key words: Entropy; Boltzmann principle; Planck's law; Dynamical system; Von Koch's inequality; Riemann Hypothesis; abc conjecture

1. INTRODUCTION.

In this paper, we will explain in the following order.

1.1. First, we explain the Boltzmann principle of entropy S and the Planck distribution function for ease of understanding. The Planck distribution function divides particles P into resonators N and applies this division method to entropy S. Furthermore, this entropy S is made to correspond to the average energy of resonators U and an energy element ε. In addition, The Planck distribution function is derived by differentiating with the average energy of resonators U.

1.2. Second, we describe that the expansion of the Planck distribution function which is main contents of this article. We consider the entropy $S_{\pi_f}(x)$ which is the Boltzmann principle divided by function $x/f(x)$, where set the function $f(x)$ to $\log(x)$ and let x be a positive real number. The function $x/\log(x)$ is an approximation of the number of prime numbers $\pi(x)$. The function $R_\alpha^\pm(x)$ is defined and describe the relation between $S_{\pi_f}(x)$ and $R_\alpha^\pm(x)$. Furthermore, we attempt to compare the possibility of expanding the Planck distribution function by using the function $R_\alpha^\pm(x)$. Besides, the relation between the constant α of the function $R_\alpha^\pm(x)$ and fine-structure constant will be considered.

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Key words and phrases. Entropy, Boltzmann principle, Planck's law, Dynamical system, Von Koch's inequality, Riemann Hypothesis, abc conjecture.

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1.3. Third, we consider to apply the constant α of the function $R^{\pm}_{\alpha}(x)$ to number theory. Namely, we prove the Riemann Hypothesis. Moreover, we verify that the ABC conjecture is the negative consequence.

1.4. Finally, we will describe some considerations and issues for the future. Namely, the entropy is associated with the dynamical system, and the classical gravity theory of Newton's law and the electromagnetism of Coulomb's law by the law of inverse squares.

2. The Boltzmann principle and the Planck distribution function.

2.1. Introduction for Entropy S and the Planck distribution function. We examine to be apply statistical mechanics concept to natural numbers. To make it easier the understanding, we would first let us introduce the Boltzmann principle and the Planck distribution function as follows.

Definition 2.1. *We define symbols using on this article as follows :*

$$\begin{aligned} P &: The\ number\ of\ particles, \\ N &: The\ number\ of\ resonators, \\ U &: The\ average\ energy\ per\ a\ resonator, \\ U_N &: Total\ energy, \\ \varepsilon &: An\ element\ of\ energy, \\ \nu &: Frequency, \\ T &: Temperature, \\ k_B &: The\ Boltzmann\ constant, \\ h &: The\ Planck\ constant, \\ \beta &: Inverse\ temperature. \end{aligned} \tag{2.1}$$

□

Using the definitions above, the following equations are satisfied :

$$U_N = NU = P\varepsilon, \tag{2.2}$$

$$\frac{P}{N} = \frac{U}{\varepsilon}, \tag{2.3}$$

$$\beta = \frac{1}{k_B T} \tag{2.4}$$

where the inequality $P > N$ is satisfied.

The concept of the Planck distribution is that the number of particles P is divided by the number of resonators N. Namely, the number of particles P is divided by the number of partitions $N-1$. The number of particles P and resonators N can be regarded positive integer numbers. Therefore, we define the number of states $W_{N,P}$ and the entropy (the Boltzmann Principle) S as follows :

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Definition 2.2. *Let the number of particles P and the number of resonators N be positive integer numbers* $(P, N \in \mathbb{N})$.

$$W_{N,P} := \frac{(N+P-1)!}{(N-1)!P!}, \quad (\textit{the number of states}) \tag{2.5}$$

$$S_{N,P} := k_B \log W_{N,P}, \quad (\textit{the Boltzmann Principle}) \tag{2.6}$$

$$S := \frac{S_{N,P}}{N}. \quad (\textit{the average of } S_N) \tag{2.7}$$

□

Using the Stirling's formula, for sufficiently large natural number P and N, the following conditions are satisfied :

$$W_{N,P} = \frac{(N+P-1)!}{(N-1)!P!} \approx \frac{(N+P)^{N+P}}{N^N P^P}. \tag{2.8}$$

Using the Boltzmann principle above, for sufficiently large the number of particles P and resonators N, we can obtain the following equations :

$$\begin{aligned} S_{N,P} &= k_B \log W_{N,P} \\ &= k_B \big\{ (N+P)\log(N+P) - \log N^N - \log P^P \big\} \\ &= k_B \big\{ (N+P)\log(N+P) - N\log N - P\log P \big\} \\ &= k_B N \big\{ (\frac{P}{N})\log N + (1+\frac{P}{N})\log(1+\frac{P}{N}) - \frac{P}{N}\log P \big\} \\ &= k_B N \big\{ (1+\frac{P}{N})\log(1+\frac{P}{N}) - \frac{P}{N}\log\frac{P}{N} \big\}. \end{aligned} \tag{2.9}$$

Using the definition above, the equality(2.3) $P/N = U/\varepsilon$ and (2.7) $S = S_{N,P}/N$, the equality(2.9) is satisfied as follows :

$$S = k_B \Big\{ (1+\frac{U}{\varepsilon})\log(1+\frac{U}{\varepsilon}). - \frac{U}{\varepsilon}\log\frac{U}{\varepsilon} \Big\}. \tag{2.10}$$

Differentiate both sides of the equation(2.10) with respect to average energy per resonator U. Hence, the following equation is satisfied :

$$\frac{dS}{dU} = \frac{k_B}{\varepsilon} \Big\{ \log(1+\frac{U}{\varepsilon}) - \log\frac{U}{\varepsilon} \Big\}. \tag{2.11}$$

Furthermore, differentiate both sides of the equation(2.11) with respect to U, the following an equation is satisfied :

$$\frac{d^2S}{dU^2} = \frac{-k_B}{U(\varepsilon+U)}. \tag{2.12}$$

The rate of change of entropy dS is the multiplication of the rate of change of energy U and the reciprocal of temperature T. Namely, the following relation between entropy S, total energy U and temperature T are satisfied :

$$\frac{dS}{dU} = \frac{1}{T}. \tag{2.13}$$

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Thus, using the equation(2.12) and (2.13), the following relation is satisfied :

$$\frac{d}{dU}\left(\frac{1}{T}\right) = \frac{-k_B}{U(\varepsilon + U)}. \tag{2.14}$$

Integrating both sides of the equation(2.14) with respect to U, the following relation is satisfied :

$$U = \frac{\varepsilon}{\exp(\frac{\varepsilon}{k_B T}) - 1}. \tag{2.15}$$

Here, put ε as follows :

$$\varepsilon = h\nu. \tag{2.16}$$

Therefore, the following equations is satisfied :

$$U = \frac{h\nu}{\exp(\frac{h\nu}{k_B T}) - 1} = \frac{h\nu}{\exp(h\nu\beta) - 1}, \quad (Planck's\ law). \tag{2.17}$$

The equation above(2.17) is determined Planck's the average number of particles in a single mode of frequency ν in thermal equilibrium, that is, called Planck's law.

Besides, we define the distribution function $\bar{n}(\nu, \beta)$ as follows:

$$\bar{n}(\nu, \beta) = \frac{1}{\exp(h\nu\beta) - 1}, \quad (the\ Planck\ distribution\ function). \tag{2.18}$$

This is expressed the mean particle occupation number in thermal equilibrium. This is called the Planck distribution funcion on this paper. Moreover, the equation(2.18) is transformed as follows :

$$\frac{\bar{n}(\nu, \beta)}{\bar{n}(\nu, \beta) + 1} = \exp(-h\nu\beta). \tag{2.19}$$

The function $\exp(-h\nu\beta)$ is called the Boltzmann factor.

Besides, let N_g and N_e be the mean number of atoms in the ground state and in the excited state. the following equation also satisfied :

$$\frac{N_e}{N_g} = \exp(-h\nu\beta). \tag{2.20}$$

3. Expansion of the Planck distribution function.

3.1. The Entropy S_{π_f}.

We will continue the discussion with reference to ideas in subsection 2.1. The number of particles P is replaced to the positive real number x. The number of resonator N is replaced to the number of primes number $\pi(x)$. We consider to divide the positive real number x by approximately of the number of prime numbers $\pi(x)$, that is, the function $x/\log(x)$. We show the function $R_\alpha^{\pm}(x)$ is derived as follows. First, we start with some definitions.

Definition 3.1. *Let $x > 1$ be a positive real number ($x \in \mathbb{R}$) and $f(x)$ be a positive real valued function on x.*

$$\pi(x) := \sum_{\substack{p \le x \\ p:prime}} 1 \ , \tag{3.1}$$

The number of prime numbers less than or equal to x.

$$\pi_f(x) := \frac{x}{f(x)} \ , \tag{3.2}$$

$$Q_f(x) := \frac{x}{\pi_f(x)} \ . \tag{3.3}$$

By the definition above, it is satisfied that $Q_f(x) = f(x)$.

□

We define the number of states $W_{\pi_f,x}$. Therefore, the entropy $S_{\pi_f,x}$ under $W_{\pi_f,x}$ is defined by $W_{\pi_f,x}$. Moreover, the entropy $S_{\pi_f}(x)$ under π_f is defined divided by the entropy $S_{\pi_f,x}$ by $\pi_f(x)$ as follows :

Definition 3.2. *The entropy $S_{\pi_f}(x)$ divided by $\pi_f(x)$.*
Let $x > 1$ be a positive real number ($x \in \mathbb{R}$).

$$W_{\pi_f,x} := \frac{(\pi_f(x) + x - 1)!}{(\pi_f(x) - 1)! x!}, \tag{3.4}$$

$$S_{\pi_f,x} := \log W_{\pi_f,x}, \tag{3.5}$$

$$S_{\pi_f}(x) := \frac{S_{\pi_f,x}}{\pi_f(x)}. \tag{3.6}$$

□

In discussion below, unless otherwise specified, let the function $f(x)$ set to $\log(x)$. Namely, the following is satisfied :

$$f(x) = \log(x). \tag{3.7}$$

Therefore, using definitions above and the Primes theorem, the following conditions are satisfied :

$$Q_f(x) = Q_{\log}(x) = \frac{x}{\pi_{\log}(x)} = \log(x) \sim \frac{x}{\pi(x)}, \quad (x \in \mathbb{R}). \tag{3.8}$$

Using Stirling's formula, for sufficiently large $x > 0$, the following condition are satisfied :

$$W_{\pi_f,x} = \frac{(\pi_f(x) + x - 1)!}{(\pi_f(x) - 1)! x!} \approx \frac{(\pi_f(x) + x)^{\pi_f(x)+x}}{\pi_f(x)^{\pi_f(x)} x^x}. \tag{3.9}$$

Therefore, for sufficiently large $x > 0$, the following equations are satisfied :

$$\begin{aligned} S_{\pi_f,x} &= (\pi_f(x) + x) \log(\pi_f(x) + x) - \pi_f(x) \log(\pi_f(x)) - x \log(x) \\ &= \pi_f(x) \Big(\big(1 + \frac{x}{\pi_f(x)}\big) \log\big(1 + \frac{x}{\pi_f(x)}\big) - \frac{x}{\pi_f(x)} \log\big(\frac{x}{\pi_f(x)}\big) \Big), \end{aligned} \tag{3.10}$$

$$S_{\pi_f}(x) = (1 + \frac{x}{\pi_f(x)}) \log(1 + \frac{x}{\pi_f(x)}) - \frac{x}{\pi_f(x)} \log(\frac{x}{\pi_f(x)}). \tag{3.11}$$

Using the function $Q_f(x)$ above, the function $S_{\pi_f}(x)$ under π_f is express as follows :

$$S_{\pi_f}(x) = (1 + Q_f(x)) \log(1 + Q_f(x)) - Q_f(x) \log Q_f(x). \tag{3.12}$$

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Differentiating the entropy $S_{\pi_f}(x)$ under π_f as follows :

$$\begin{aligned} S'_{\pi_f}(x) &= (\frac{x}{\pi_f(x)})' \log(1+\frac{x}{\pi_f(x)}) + (\frac{x}{\pi_f(x)})' \\ &\quad - \Bigl((\frac{x}{\pi_f(x)})' \log(\frac{x}{\pi_f(x)}) + (\frac{x}{\pi_f(x)})' \Bigr) \\ &= (\frac{x}{\pi_f(x)})' \Bigl(\log(1+\frac{x}{\pi_f(x)}) - \log(\frac{x}{\pi_f(x)}) \Bigr). \end{aligned} \tag{3.13}$$

Furthermore, differentiating $S'_{\pi_f}(x)$ as follows :

$$\begin{aligned} S''_{\pi_f}(x) &= (\frac{x}{\pi_f(x)})'' \Bigl(\log(1+\frac{x}{\pi_f(x)}) - \log(\frac{x}{\pi_f(x)}) \Bigr) \\ &\quad + (\frac{x}{\pi_f(x)})' \Bigl(\log(1+\frac{x}{\pi_f(x)}) - \log(\frac{x}{\pi_f(x)}) \Bigr)'. \end{aligned} \tag{3.14}$$

Therefore, the equations above is expressed by using $Q_f(x)$ as follows :

$$S'_{\pi_f}(x) = Q'_f(x) \Bigl(\log(1+Q_f(x)) - \log Q_f(x) \Bigr), \tag{3.15}$$

$$\begin{aligned} S''_{\pi_f}(x) &= Q''_f(x) \Bigl(\log(1+Q_f(x)) - \log Q_f(x) \Bigr) \\ &\quad + Q'_f(x) Q'_f(x) \Bigl(\frac{-1}{Q_f(x)(1+Q_f(x))} \Bigr). \end{aligned} \tag{3.16}$$

Repeating differential of the part of $Q''_f(x)\Bigl(\log(1+Q_f(x)) - \log Q_f(x)\Bigr)$ on (3.16), the following conditions are satisfied :

$$\begin{aligned} &\Bigl(Q''_f(x) \Bigl(\log(1+Q_f(x)) - \log Q_f(x) \Bigr) \Bigr)' \\ &= Q^{(3)}_f(x) \Bigl(\log(1+Q_f(x)) - \log Q_f(x) \Bigr) \\ &\quad + Q^{(2)}_f(x) Q'_f(x) \Bigl(\frac{-1}{Q_f(x)(1+Q_f(x))} \Bigr), \end{aligned} \tag{3.17}$$

$$\begin{aligned} &\Bigl(Q^{(n)}_f(x) \Bigl(\log(1+Q_f(x)) - \log Q_f(x) \Bigr) \Bigr)' \\ &= Q^{(n+1)}_f(x) \Bigl(\log(1+Q_f(x)) - \log Q_f(x) \Bigr) \\ &\quad + Q^{(n)}_f(x) Q'(x)_f \Bigl(\frac{-1}{Q_f(x)(1+Q_f(x))} \Bigr). \end{aligned} \tag{3.18}$$

Therefore, for all sufficiently large $x > 1$, the following conditions are satisfied :

$$\begin{aligned} &Case1)\ n > 1\ is\ even\ number : \\ &Q^{(n+1)}_f(x) = \frac{(-1)^n (n)!}{x^{n+1}} > \frac{(-1)^{n-1}(n-1)!}{x^n} = Q^{(n)}_f(x), \end{aligned} \tag{3.19}$$

$$\begin{aligned} &Case2)\ n > 1\ is\ odd\ number : \\ &Q^{(n+1)}_f(x) = \frac{(-1)^n (n)!}{x^{n+1}} < \frac{(-1)^{n-1}(n-1)!}{x^n} = Q^{(n)}_f(x). \end{aligned} \tag{3.20}$$

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Furthermore, for all sufficiently large $x > 1$, the following are satisfied :

$$Q_f^{(n)}(x) > Q_f^{(2)}(x), \tag{3.21}$$

$$\left|Q_f^{(n+1)}(x)\right| > \left|Q_f^{(n)}(x)\right|. \tag{3.22}$$

Next, we define some functions $k_f(x)$, $R_m^+(x)$ and $R_m^-(x)$ as follows :

Definition 3.3. *The definition of the function $k_f(x)$.*

Let $x > 1$ and $f(x)$ be a positive real number and a real-valued function ($x \in \mathbb{R}$). The function $k_f(x)$ is defined as follows :

$$k_f(x) = S''_{\pi_f}(x)\left(\frac{-Q_f(x)(1+Q_f(x))}{Q'_f(x)}\right). \tag{3.23}$$

Namely, the following equation is satisfied :

$$S''_{\pi_f}(x) = k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right). \tag{3.24}$$

□

let us call this function $k_f(x)$ the Boltzmann variable function in the function $f(x)$.

Definition 3.4. *The function $R_m^+(x)$ and $R_m^-(x)$ are defined as follows :*

$$R_m^+(x) := \sum_{n=1}^{m}\left|\frac{(-1)^{n-1}(n-1)!}{x^n}\right|, \quad (x \in \mathbb{R}). \tag{3.25}$$

Therefore, the following equations are satisfied :

$$R_m^+(x) = \sum_{n=1}^{m}\left|\frac{(-1)^{n-1}(n-1)!}{x^n}\right| = \sum_{n=1}^{m}\left|\big(\log(x)\big)^{(n)}\right|, \quad (x \in \mathbb{R}). \tag{3.26}$$

Same as discussion, the following inequality are satisfied :

$$R_m^-(x) := \sum_{n=1}^{m}\frac{(-1)^{n-1}(n-1)!}{x^n}, \quad (x \in \mathbb{R}). \tag{3.27}$$

Therefore, the following equations are satisfied :

$$R_m^-(x) = \sum_{n=1}^{m}\frac{(-1)^{n-1}(n-1)!}{x^n} = \sum_{n=1}^{m}\big(\log(x)\big)^{(n)}, \quad (x \in \mathbb{R}). \tag{3.28}$$

□

The function $R_m^+(x)$ is called an m-th absolute lower bound approximation of the Boltzmann variable function $k_f(x)$ in the function $f(x)$. Similarly, the function $R_m^-(x)$ is called an m-th lower bound approximation of the Boltzmann variable function $k_f(x)$ in the function $f(x)$. Using the defintion above, the following inequality is satisdied :

$$R_m^+(x) \geq R_m^-(x). \tag{3.29}$$

where the function $(\log(x))^{(n)}$ represents the nth derivation of $\log(x)$. Moreover, the function $(\log(x))^n$ and $\log^n(x)$ to the n-th power represents $\log(x)$.

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Using equivalent(3.24), for all sufficiently large $x > 1$, the following conditions are satisfied :

$$k_f(x) = -S''_{\pi_f}(x)\frac{Q_f(x)(1+Q_f(x))}{Q'_f(x)} \leq \frac{1}{x}(2+\log(x)). \tag{3.30}$$

where the function $Q_f(x)$ is $\log(x)$. Because, by the equation(3.16),

$$\begin{aligned} S''_{\pi_f}(x) &= Q''_f(x)\Big(\log(1+Q_f(x)) - \log Q_f(x)\Big) \\ &\quad + Q'_f(x)Q'_f(x)\left(\frac{-1}{Q_f(x)(1+Q_f(x))}\right). \end{aligned} \tag{3.31}$$

Therefore, for sufficiently large $x > 1$, the following are satisfied :

$$\begin{aligned} k_f(x) &= \frac{1}{x^2}\log\left(1+\frac{1}{\log(x)}\right)x\log(x)\big(1+\log(x)\big) + \frac{1}{x} \\ &= \frac{1}{x}\log\left(1+\frac{1}{\log(x)}\right)^{\log(x)}\big(1+\log(x)\big) + \frac{1}{x} \\ &\leq \frac{1}{x}\log(e)\Big(1+\log(x)\Big) + \frac{1}{x} \qquad \because) \lim_{x\to\infty}\left(1+\frac{1}{\log(x)}\right)^{\log(x)} \to e \\ &= \frac{1}{x}\Big(1+\log(x)\Big) + \frac{1}{x} \\ &= \frac{1}{x}(2+\log(x)). \end{aligned} \tag{3.32}$$

Furthermore, there is a positive integer $m \geq 1$ such that the following conditions are satisfied:

$$\begin{aligned} S''_{\pi_f}(x) &= k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right) \\ &\geq \Big(|Q'_f(x)| + |Q''_f(x)| + \cdots + |Q^{(m)}_f(x)|\Big)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \\ &\geq R^+_m(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right). \end{aligned} \tag{3.33}$$

Using the same discussion above, there is a positive integer $m \geq 1$ such that the following conditions are satisfied:

$$\begin{aligned} S''_{\pi_f}(x) &= k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right) \\ &\geq \Big(Q'_f(x) + Q''_f(x) + \cdots + Q^{(m)}_f(x)\Big)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \\ &\geq R^-_m(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right). \end{aligned} \tag{3.34}$$

First order differentiation of the entropy S_{π_f} is always positive, so that $S'_{\pi_f} > 0$. Moreover, second order differentiation of the entropy S_{π_f} has always negative values, so that $S''_{\pi_f} < 0$. Therefore, the entropy S_{π_f} has no inflection points.

3.2. Derivation of the functions $R^{\pm}_{\alpha}(x)$.

Next, the function $R^{+}_{\alpha}(x)$ and $R^{-}_{\alpha}(x)$ are derived as follows :

Definition 3.5. *$R^{+}_{\alpha}(x)$, $R^{-}_{\alpha}(x)$ and $R^{\pm}_{\alpha}(x)$*
Let the constant $\alpha > 0$ be a positive real number $(\in \mathbb{R})$. For all $x > 0$ $(\in \mathbb{R})$, the function $R^{+}_{\alpha}(x)$ and $R^{-}_{\alpha}(x)$ are defined as follows :

$$R^{+}_{\alpha}(x) = \frac{\sqrt{2\pi}\alpha}{ex+1}, \tag{3.35}$$

$$R^{-}_{\alpha}(x) = \frac{\sqrt{2\pi}\alpha}{ex-1}. \tag{3.36}$$

The function $R^{\pm}_{\alpha}(x)$ are combined $R^{+}_{\alpha}(x)$ and $R^{-}_{\alpha}(x)$ as follows :

$$R^{\pm}_{\alpha}(x) := \frac{\sqrt{2\pi}\alpha}{ex \pm 1}. \tag{3.37}$$

Therefore, the following conditions are satisfied :

$$xR^{\pm}_{\alpha}(x) = \frac{\sqrt{2\pi}\alpha x}{ex \pm 1}, \tag{3.38}$$

$$\frac{1}{xR^{\pm}_{\alpha}(x)} = \frac{e}{\sqrt{2\pi}\alpha}\left(1 \pm \frac{1}{ex}\right). \tag{3.39}$$

□

This function $R^{\pm}_{\alpha}(x)$ is called an $\pm$lower bound approximation of the Boltzmann variable function $k_f(x)$ in the function $f(x)$ and the constant $\alpha \in \mathbb{R}$.

The relations of functions $R^{\pm}_{\alpha}(x)$, $R^{+}_{m}(x)$ and $R^{-}_{m}(x)$ are satisfied as follows :

Lemma 3.6. *The relation $R^{+}_{m}(x) \geq R^{+}_{\alpha}(x)$.*
Let $\alpha > 0$ be a positve real number. There is an integer $m \geq 1$ such that for all sufficiently large $x > 1$, the following inequality is satisfied :

$$R^{+}_{m}(x) \geq R^{+}_{\alpha}(x) = \frac{\sqrt{2\pi}\alpha}{ex+1} \tag{3.40}$$

where the positive real number α is satisfied as follows :

$$x \geq \frac{-1}{e - \sqrt{2\pi}\alpha}, \tag{3.41}$$

that is, satisfied as follows:

$$\frac{e}{\sqrt{2\pi}}\big(1 + \frac{1}{x}\big) \geq \alpha. \tag{3.42}$$

Using same as discussion, the following conditions are satisfied :

Lemma 3.7. *The relation $R^{-}_{m}(x) \geq R^{-}_{\alpha}(x)$.*
Let $\alpha > 0$ be a positve real number. There is an integer $m \geq 1$ such that for all sufficiently large $x > 1$, the following inequality is satisfied :

$$R^{-}_{m}(x) \geq R^{-}_{\alpha}(x) = \frac{\sqrt{2\pi}\alpha}{ex-1} \tag{3.43}$$

where the positive real number α is satisfied as follows :

$$x \geq \frac{1}{e-\sqrt{2\pi}\alpha}, \tag{3.44}$$

that is, satisfied as follows:

$$\frac{e}{\sqrt{2\pi}}\left(1-\frac{1}{x}\right) \geq \alpha. \tag{3.45}$$

Proof. The proof of Lemma 3.6 and Lemma 3.7 are described the following the section 6.1.

□

Consequently, for sufficiently large real number $x > 1$, real-valued function $f(x)$ and a positive integer $m > 1$, the following inequalities are satisfied :

$$S''_{\pi_f}(x) \geq R^{+}_{m}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \geq R^{+}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}, \tag{3.46}$$

$$S''_{\pi_f}(x) \geq R^{-}_{m}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))} \geq R^{-}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}. \tag{3.47}$$

Namely, the following inequality is satisfied :

$$S''_{\pi_f}(x) \geq R^{\pm}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}. \tag{3.48}$$

The second derivative of the entropy $S''_{\pi_f}(x)$ is suppressed from the bottom side by formula. Besides, the second derivative of the entropy $S''_{\pi_f}(x)$ is suppressed from the upper side by formula as follows.

Lemma 3.8. $R^{+}_{m}(x) \geq R^{+}_{\alpha}(x)$, *and* $R^{-}_{m}(x) \geq R^{-}_{\alpha}(x)$.
For all sufficiently large $x > 1$ and a positive integer $m > 1$, the following inequalities are satisfied :

$$\frac{1}{(x-1)^{1/2}}\frac{e}{e+1} \geq R^{+}_{m}(x) \geq R^{+}_{\alpha}(x) = \frac{\sqrt{2\pi}\alpha}{ex+1}, \tag{3.49}$$

$$\frac{1}{(x-1)^{1/2}}\frac{e}{e+1} \geq R^{-}_{m}(x) \geq R^{-}_{\alpha}(x) = \frac{\sqrt{2\pi}\alpha}{ex-1}. \tag{3.50}$$

Proof. The proof of Lemma 3.8 are described the following the section 6.2.

□

On the next subsection, we discuss the meaning of inequalities above Lemma 3.6 and Lemma 3.7.

3.3. The Expansion of the Planck distribution $n^{\pm}_{\alpha}(x)$ by using $R^{\pm}_{\alpha}(x)$.

Next, we examine to define the distribution functions $n^{\pm}_{\alpha}(x)$ by using $R^{\pm}_{\alpha}(x)$. The function $R^{\pm}_{\alpha}(x)$ and $Q_f(x)$ regards as the constant and the variable. Beside, integrate the inequality(3.46) and (3.47) as the variable x, Therefore, the following inequality is satisfied :

$$S''_{\pi_f}(x)dx \geq R^{\pm}_{\alpha}(x)\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}dx. \tag{3.51}$$

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Namely, the following inequality is satisfied :

$$\int S''_{\pi_f}(x)dx \geq R^{\pm}_{\alpha}(x) \int \frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}dx. \tag{3.52}$$

Therefore, the following equation is satisfied :

$$S'_{\pi_f}(x) = \int S''_{\pi_f}(x)dx. \tag{3.53}$$

Integrate the right side of inequality(3.52), the function $Q_f(x)$ regards as a variable. Hence, the following formulas are satisfied :

$$\begin{aligned} S'_{\pi_f}(x) &\geq R^{\pm}_{\alpha}(x)(\log(1+Q_f(x)) - \log Q_f(x)) + C \\ &= R^{\pm}_{\alpha}(x)\log\big(1+\frac{1}{Q_f(x)}\big) + C. \end{aligned} \tag{3.54}$$

where the constant C is a positive real number.

Here, for all sufficiently large $Q_f(x) > 0$, the following equation is satisfied :

$$\log(1+\frac{1}{Q_f(x)}) = 0. \tag{3.55}$$

Hence, the first order differentiation $S'_{\pi_f}(x)$ is satisfied as follows :

$$S'_{\pi_f}(x) = Q'_f(x)\log(1+\frac{1}{Q_f(x)}) = 0. \tag{3.56}$$

Thus, using inequality(3.54), the constant C is satisfied as follows :

$$C = 0. \tag{3.57}$$

Therefore, the inequality(3.54) is satisfied as follows :

$$S'_{\pi_f}(x) \geq R^{\pm}_{\alpha}(x)\log\big(1+\frac{1}{Q_f(x)}\big). \tag{3.58}$$

For sufficiently large positive real number $x > 0$, the function $S'_{\pi_f}(x)$ is satisfied as follows :

$$\frac{1}{x} \geq S'_{\pi_f}(x) = \frac{1}{x}\log\big(1+\frac{1}{\log(x)}\big). \tag{3.59}$$

According to inequalities(3.58) and (3.59),

$$\frac{1}{xR^{\pm}_{\alpha}(x)} \geq \log(1+\frac{1}{Q_f(x)}). \tag{3.60}$$

Therefore, by (3.60) the following inequality is derived :

$$Q_f(x) \geq \frac{1}{\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1}. \tag{3.61}$$

Focusing on equality of the inequality (3.61), we define the distribution function $n^{\pm}_{\alpha}(x)$ as follows:

Definition 3.9. *The distribution functions $n^{\pm}_{\alpha}(x)$ are defined as follows:*

$$n^{\pm}(x,\alpha) = \frac{1}{\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1} \tag{3.62}$$

where $\alpha > 0 \in \mathbb{R}$. □

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The definition above is transformed as follows :

$$\frac{n^{\pm}(x,\alpha)}{n^{\pm}(x,\alpha)+1} = \exp\left(\frac{-1}{xR_{\alpha}^{\pm}(x)}\right), \quad (\alpha > 0). \tag{3.63}$$

Thus, this distribution function $n^{\pm}(x,\alpha)$ is regards as the approximately of the density of prime numbers $x/\pi(x)$ until the number x. Besides, this function $n^{\pm}(x,\alpha)$ is regarded as one of the distribution functions. Furthermore, this function $n^{\pm}(x,\alpha)$ is seems to expand the Planck distribution function $\bar{n}(\nu,\beta)$. According to imitate Boltzmann factor, the following function

$$\exp\left(\frac{-1}{xR_{\alpha}^{\pm}(x)}\right) \tag{3.64}$$

is called the expansion of Boltzmann factor or $R_{\alpha}^{\pm}$ factor . We will consider the further relationship in the next subsection.

3.4. Corresponding the Planck distribution function and the distribution function $n^{\pm}(x,\alpha)$.

We examine to correspond the Planck distribution function $\bar{n}(\nu,\beta)$ and the distribution function $n^{\pm}(x,\alpha)$ as follows :

$$\bar{n}(\nu,\beta) = \frac{1}{\exp(h\nu\beta)-1} \tag{3.65}$$

where

$$\begin{aligned} h &: \textit{ the Planck constant}, \\ \nu &: \textit{Frequency}, \\ \beta &: \textit{Inverse temperature}. \end{aligned} \tag{3.66}$$

Here, we consider to correspond the internal parameter of the Boltzmann factor $\exp(-h\nu\beta)$

$$h\nu\beta \tag{3.67}$$

and the internal function of $R_{\alpha}^{\pm}$ factor $\exp(\frac{-1}{xR_{\alpha}^{\pm}(x)})$

$$\frac{1}{xR_{\alpha}^{\pm}(x)} \quad \left(= \frac{e}{\sqrt{2\pi\alpha}}(1 \pm \frac{1}{ex})\right). \tag{3.68}$$

Namely, we suppose the correspondence as follows:

$$h\nu\beta = \frac{e}{\sqrt{2\pi\alpha}}(1 \pm \frac{1}{ex}). \tag{3.69}$$

Furthermore, we can consider by separating the correspondence between the constant part and the variable part as follows :

$$\nu\beta = (1 \pm \frac{1}{ex}), \tag{3.70}$$

$$h = \frac{e}{\sqrt{2\pi\alpha}}. \tag{3.71}$$

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$$
\begin{array}{ccc}
W_{N,P} = \frac{(N+P-1)!}{(N-1)!P!}, & & S_{N,P} = k_B \log \frac{(N+P-1)!}{(N-1)!P!}, \\
\exp(-h\nu\beta) & \xrightarrow{\log} & -h\nu\beta \\
{\scriptstyle N:=\pi_f(x),\ P:=x}\downarrow \uparrow {\scriptstyle \alpha:=\frac{e}{\sqrt{2\pi h}},\ x:=\frac{\mp 1}{e(1-\nu\beta)}} & & {\scriptstyle N:=\pi_f(x),\ P:=x}\downarrow \uparrow {\scriptstyle \alpha:=\frac{e}{\sqrt{2\pi h}},\ x:=\frac{\mp 1}{e(1-\nu\beta)}} \\
W_{\pi_f,x} = \frac{(\pi_f(x)+x-1)!}{(\pi_f(x)-1)!x!}, & & S_{\pi_f,x} = \log \frac{(\pi_f(x)+x-1)!}{(\pi_f(x)-1)!x!}, \\
\exp(\frac{-1}{xR^{\pm}_{\alpha}(x)}) & \xrightarrow{\log} & \frac{-1}{xR^{\pm}_{\alpha}(x)}
\end{array}
\tag{3.72}
$$

Corresponding the above, the distribution function $n^{\pm}(x, \alpha)$ becomes to expand the Planck distribution function. Namely, the following conditions are satisfied :

Suggestion 3.10. *The expansion of the Planck distribution $\bar{n}(\nu, \beta)$.*
Let $\alpha > 0$ be a real number constant. For all real number $x > 0$ the following equation is satisfied :

$$\bar{n}(\nu, \beta) = n^{\pm}(x, \alpha) \tag{3.73}$$

where

$$
\begin{aligned}
x &= \frac{\mp 1}{e(1-\nu\beta)}, \\
\alpha &= \frac{e}{\sqrt{2\pi h}}, \quad \nu, \beta, h \in \mathbb{R}.
\end{aligned}
\tag{3.74}
$$

Namely, the distribution function $n^{\pm}(x, \alpha)$ can be regarded as representing an expansion of the Planck distribution function $\bar{n}(\nu, \beta)$. □

For sufficiently large $x > 1$, the correspondence of equation(3.71) is satisfied as follows:

$$\nu\beta = \lim_{x\to\infty} (1 \pm \frac{1}{ex}) = 1. \tag{3.75}$$

Moreover, according to the method to divide each S and $S_{\pi_f}(x)$, we remember that the following corresponds :

(1) The number of particles P is replaced to the positive real number x.
(2) The number of resonators N is replaced to the number of primes number $\pi(x) \sim \frac{x}{\log(x)}$.

Therefore, we consider the correspondence the between

$$\frac{U}{\varepsilon} = \frac{P}{N} \tag{3.76}$$

and

$$Q_f(x) = \frac{x}{\frac{x}{\log(x)}} \sim \frac{x}{\pi(x)}. \tag{3.77}$$

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Namely, we suppose the following correspondence is considered :

$$\begin{aligned} U &\longleftrightarrow x, \\ \varepsilon &\longleftrightarrow \frac{x}{\log(x)} \sim \pi(x). \end{aligned} \tag{3.78}$$

Thereby, We consider the correspondence the between Planck's law U and the following function $U^{\pm}_{x,\alpha}$.

Definition 3.11. *The real valued function* $U^{\pm}_{x,\alpha}$.
Let $\alpha > 0$ be a real number constant. For sufficiently large real number $x > 1$, the real valued function $U^{\pm}_{x,\alpha}$ is define as follows :

$$\begin{aligned} U^{\pm}_{x,\alpha} &= n^{\pm}(x,\alpha)\frac{x}{\log(x)} \\ &= \frac{x}{\log(x)\big(\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1\big)} \\ &\sim \frac{\pi(x)}{\exp(\frac{1}{xR^{\pm}_{\alpha}(x)}) - 1} \quad (\alpha > 0). \end{aligned} \tag{3.79}$$

□

Using the suggestion3.10 and the definition3.11, the following suggestion is given :

Suggestion 3.12. *The expansion of Planck's law U.*
Let $h > 0$, $\nu > 0$ and $\beta > 0$ be real numbers. Each values h, ν and β means the planck constant, frequency and inverse temperature.
There exists real numbers $x > 0$ and $\alpha > 0$ such that the following equality is satisfied :

$$U = U^{\pm}_{x,\alpha} \tag{3.80}$$

where the following conditions are satisfied :

$$\begin{aligned} h\nu &= \frac{x}{\log(x)} \sim \pi(x), \\ \nu\beta &= (1 \pm \frac{1}{ex}), \\ \alpha &= \frac{e}{\sqrt{2\pi h}}, \quad \nu, \beta, h \in \mathbb{R}. \end{aligned} \tag{3.81}$$

Namely, the real valued function $U^{\pm}_{x,\alpha}$ can be regarded as representing the expansion of Planck's law U. □

According to the suggestion 3.12 above, under the condition that the product of the Planck constant h and the frequency ν, that is, $h\nu$ is an approximation of the number of prime numbers $\pi(x)$. Planck's law U is seems to take discrete values and has an approximate spectrum of prime numbers. It is possible that the discrete values of an element of energy are related to the distribution of prime numbers.

3.5. A kind of fine-structure constant.

The distribution function $n^{\pm}(x,\alpha)$ and the Planck distribution $\bar{n}(\nu,\beta)$ is associated by the constant α. The constant $\alpha = e/\sqrt{2\pi h}$ is thought like a *fine-structure constant* that associated with the Planck constant h.

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Let a positive real number α set as follows :

(3.82) $$\alpha = \frac{e}{\sqrt{2\pi h}}.$$

For all sufficiently large $x > 1$, the following inequalities are satisfied :

(3.83) $$\frac{e}{\sqrt{2\pi}}(2 + \log(x)) \geq \alpha.$$

According to the Prime numbers theorem, the following relation is satisfied :

(3.84) $$\pi(x) \sim \frac{x}{\log(x)} \sim \frac{x}{\log(x) + 2}.$$

Thus, the positive real number α is satisfied such that

(3.85) $$\frac{e}{\sqrt{2\pi}} \frac{x}{\pi(x)} \geq \alpha, \quad (\alpha > 0).$$

Hence, the positive real number α can be regard as fine-structure constant by $\sqrt{2}$, π, e and $\pi(x)$. Furthermore, the following inequality is satisfied :

Suggestion 3.13. *The ratio of the Boltzmann constant and the Planck constant. For sufficiently large $x > 1$, the following formulas are satisfied :*

(3.86) $$\frac{h}{k_B} = \frac{e}{\sqrt{2\pi\alpha}} \geq \frac{\pi(x)}{x}.$$

Namely, the ratio of the Boltzmann constant k_B and the Planck constant h is bigger than the ratio of a positive real number x and the number of prime $\pi(x)$ until x.

□

Using discussions above, the constant α can be associated between the Planck distribution function

(3.87) $$\bar{n}(\nu, \beta) = \frac{1}{\exp(h\nu\beta) - 1}$$

and the expansion of the Plank distribution function

(3.88) $$n^{\pm}(x, \alpha) = \frac{1}{\exp(\frac{1}{xR_{\alpha}^{\pm}(x)}) - 1}.$$

Namely, Suppose that a constant $\alpha > 0$ is decided. Specially, the constant h_α and α_h are decided by α, e and π as follows :

(3.89) $$h_\alpha = \frac{e}{\sqrt{2\pi\alpha}},$$

(3.90) $$\alpha_h = \frac{e}{\sqrt{2\pi h}}.$$

Namely, The function $R_{\alpha}^{\pm}(x)$ is changed and depended by a constant $\alpha > 0 (\in \mathbb{R})$. Therefore, the constant h_α can defined for each constant α.

By lemma3.10, Modern physics may be a special case that satisfy the following condition:

(3.91) $$\alpha_h = \alpha = \frac{e}{\sqrt{2\pi h}},$$

(3.92) $$h_\alpha = h = \frac{e}{\sqrt{2\pi\alpha}}.$$

Therefore, the following suggestion is stated :

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Suggestion 3.14. *Let $\alpha > 0$ be a constant.($\alpha \in \mathbb{R}$) The constants h_α can be selected as follows :*

$$h_\alpha = \frac{e}{\sqrt{2\pi}\alpha} \tag{3.93}$$

where the following inequality is satisfied :

$$\frac{e}{\sqrt{2\pi}} \frac{x}{\pi(x)} \geq \alpha, \quad (\alpha > 0). \tag{3.94}$$

□

Namely, if the condition of the equality $\alpha = e/\sqrt{2\pi}h$ is satisfied, therefore the constant h_α becomes the Planck constant h.

Note:
Let me mention here for your attention as follows: The fine-structure constant is a *physical constant* α and is originally expressed using the Planck constant as follows.
In this paper, we describe it as original the fine structure constant α_p to distinguish it from the real number α. Besides, describe it as the elementary charge e_p to distinguish it from the Napier's number e.

$$\alpha_p = \frac{e_p^2}{2h\varepsilon_0 c} = \frac{\mu_0 e_p^2 c}{2h} \tag{3.95}$$

where

$$\begin{aligned} h &: \textit{the Planck constant}, \\ \varepsilon_0 &: \textit{the electric constant}, \\ \mu_0 &: \textit{the magnetic constant}, \\ e_p &: \textit{the elementary charge}, \\ c &: \textit{.the speed of light}. \end{aligned} \tag{3.96}$$

Therefore, the relation the fine structure constant α_p and the real number α in this paper is satisfied as follows :

$$\frac{\alpha_p}{\alpha} = \sqrt{\frac{\pi}{2e^2}} \frac{e_p^2}{\varepsilon_0 c} = \sqrt{\frac{\pi}{2e^2}} e_p^2 \mu_0 c. \tag{3.97}$$

On the following section, using the function $R_\alpha^\pm(x)$, we show that some examples such that the constant $\alpha \neq \frac{e}{\sqrt{2\pi}h}$ as follows :

$$\alpha = \frac{1}{\sqrt{2\pi}}, \tag{3.98}$$

$$\alpha = \frac{2}{\sqrt{2\pi}}, \tag{3.99}$$

$$\alpha = \frac{e}{\sqrt{2\pi}}, \tag{3.100}$$

$$\alpha = \frac{e}{\sqrt{2\pi}\log(\frac{\epsilon+2}{\epsilon+1})}. \tag{3.101}$$

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4. Application the function $R^{\pm}_{\alpha}(x)$ to Number Theory.

4.1. Examples using the function $R^{\pm}_{\alpha}(x)$ for deriving Von Koch's inequality.

We derive Von Koch's inequality using the above the constant α and the function $R^{\pm}_{\alpha}(x)$. (Refer to Fujino[18])

Note : In this paper, we consider an element of energy ε and an arbitrary real number ϵ separately.

Theorem 4.1. *Inequalities for evaluating the number of prime numbers (1). Let $\alpha > 0$ be a positive real number $(\in \mathbb{R})$. There exist a positive real number $C > 1(\in \mathbb{R})$ such that for all sufficiently large real number $x \geq 2(\in \mathbb{R})$, the following conditions are satisfied :*

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{\sqrt{2\pi}\alpha}{48})^{\frac{1}{4}} \exp\big(\frac{e}{\sqrt{2\pi}\alpha}\big)x^{\frac{1}{\sqrt{2\pi}\alpha}}\big(\frac{1}{\log(x)}\big)^{\frac{3}{4}} \exp\big(\frac{1}{\log(x)}\big) \tag{4.1}$$

where the positive real number $\alpha > 0$ is satisfied as follows :

$$1 \leq \frac{1}{\sqrt{2\pi}\alpha} \exp\big(\frac{e}{\sqrt{2\pi}\alpha}\big), \tag{4.2}$$

$$\frac{1}{\sqrt{2\pi}} \leq \alpha \leq C\frac{e}{\sqrt{2\pi}}, \tag{4.3}$$

$$\exp\big(\frac{e}{\sqrt{2\pi}\alpha}\big) = \lim_{x\to\infty} \exp\big(\frac{1}{xR^{\pm}(x)}\big). \tag{4.4}$$

□

Corollary 4.2. *Inequalities for evaluating the number of prime numbers (2). There exist a positive real number $C > 1(\in \mathbb{R})$ such that for all $\epsilon > 0(\in \mathbb{R})$ and for all sufficiently large $x \geq 2(\in \mathbb{R})$, the following conditions are satisfied:*

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{48})^{\frac{1}{4}} \exp(e)x\big(\frac{1}{\log(x)}\big)^{\frac{3}{4}} \exp\big(\frac{1}{\log(x)}\big). \tag{4.5}$$

Proof. Using Theorem(4.1), put a positive real number $\alpha > 0$ as follows:

$$\alpha = \frac{1}{\sqrt{2\pi}}. \tag{4.6}$$

Therefore, the inequality(4.5) are satisfied.

□

The result of Corollary(4.1) is similar to the following result : (Refer to Wladyslaw[1])

$$(\exists C > 0)|\pi(x) - \mathrm{li}(x)| \leq O(x \exp(-C\sqrt{log(x)})). \tag{4.7}$$

Comparing inequalities(4.5) and (4.7), the following condition is satisfied :

$$O\left(x\big(\frac{1}{\log(x)}\big)^{\frac{3}{4}} \exp\big(\frac{1}{\log(x)}\big)\right) \leq O(x \exp(-C\sqrt{log(x)})). \tag{4.8}$$

Namely, the asymptotic of (4.5) gives better than that of (4.7).

Therefore, let be $\alpha = 2/\sqrt{2\pi}$, $(h_\alpha = e/2)$. the theorem above are satisfied as follows :

Corollary 4.3. *Inequalities for evaluating the number of prime numbers (3). There exist a positive real number $C > 1(\in \mathbb{R})$ such that for all $\epsilon > 0(\in \mathbb{R})$ and for all sufficiently large $x \geq 2(\in \mathbb{R})$, the following condition is satisfied:*

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{24})^{\frac{1}{4}} \exp{(\frac{e}{2})} x^{\frac{1}{2}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}). \tag{4.9}$$

Proof. Using Theorem(4.1) and the following conditions are satisfied:

$$1 \leq \frac{1}{\sqrt{2\pi}\alpha} \exp{(\frac{e}{\sqrt{2\pi}\alpha})}. \tag{4.10}$$

Put a positive real number $\alpha > 0$ as follows:

$$\alpha = \frac{2}{\sqrt{2\pi}} (\geq \frac{1}{\sqrt{2\pi}}). \tag{4.11}$$

Hence, the following inequalities is satisfied :

$$\begin{aligned} 1 &\leq \frac{1}{\sqrt{2\pi}\alpha} \exp{(\frac{e}{\sqrt{2\pi}\alpha})} \\ &= \frac{1}{\sqrt{2\pi}\frac{2}{\sqrt{2\pi}}} \exp{(\frac{e}{\sqrt{2\pi}\frac{2}{\sqrt{2\pi}}})} \quad (\because \alpha = \frac{2}{\sqrt{2\pi}}) \\ &= \frac{1}{2} \exp{(\frac{e}{2})} \quad (= 1.946424\cdots). \end{aligned} \tag{4.12}$$

Thus, the positive real number $\alpha > 0$ is satisfied conditions of (4.10) and (4.11). Therefore, there exist a positive real number $C > 1$ such that for all sufficiently large $x \geq 2$ the following condition is satisfied :

$$|\pi(x) - \mathrm{li}(x)| \leq C(\frac{1}{24})^{\frac{1}{4}} \exp{(\frac{e}{2})} x^{\frac{1}{2}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}). \tag{4.13}$$

□

Corollary 4.4. *Von Koch's inequality.*

$$(\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2) \ |\pi(x) - \mathrm{li}(x)| \leq C x^{\frac{1}{2}} \log(x) \tag{4.14}$$

where $C, \epsilon, x \in \mathbb{R}$. Namely,

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{2}} \log(x)). \tag{4.15}$$

Proof. Fixed $\epsilon > 0$. For all sufficient large $x \geq 2$, the following conditions are satisfied :

$$\left(\frac{1}{\log(x)}\right)^{\frac{3}{4}} \exp(\frac{1}{\log(x)}) < \log(x) < x^{\epsilon}. \tag{4.16}$$

Therefore,there exist a positive real number $C > 0$ such that for all sufficiently large $x \geq 2$, the following inequalities are satisfied :

$$\begin{aligned} &|\pi(x) - \mathrm{li}(x)| \\ &\leq C(\frac{1}{24})^{\frac{1}{4}} \exp{(\frac{e}{2})} x^{\frac{1}{2}} (\frac{1}{\log(x)})^{\frac{3}{4}} \exp(\frac{1}{\log(x)}) \\ &\leq C x^{\frac{1}{2}} \log(x). \end{aligned} \tag{4.17}$$

□

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As is well known, Corollary(4.4) is equivalent to the Riemann Hypothesis. (Refer to Wladyslaw[1]) Therefore, the Riemann Hypothesis is considered true.

Furthermore, let $\alpha = e/\sqrt{2\pi}$, $(h_\alpha = 1)$. The following inequality is satisfied :

Corollary 4.5. *The reduction of the upper limit of the inequality that evaluate the number of prime number.*

$$(\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2)\ |\pi(x) - \mathrm{li}(x)| \leq C x^{\frac{1}{e}} \log(x) \tag{4.18}$$

where $C, \epsilon, x \in \mathbb{R}$. *Namely,*

$$|\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{e}} \log(x)). \tag{4.19}$$

Proof. Using theorem(4.1), put a positive real number $\alpha > 0$ as follows:

$$\alpha = \frac{e}{\sqrt{2\pi}} (\geq \frac{e}{\sqrt{2\pi}}). \tag{4.20}$$

Thus, the following conditions are satisfied :

$$\begin{aligned} 1 &\leq \frac{1}{\sqrt{2\pi}\alpha} \exp\big(\frac{e}{\sqrt{2\pi}\alpha}\big) \\ &= \frac{1}{\sqrt{2\pi}\frac{e}{\sqrt{2\pi}}} \exp\big(\frac{e}{\sqrt{2\pi}\frac{e}{\sqrt{2\pi}}}\big) \quad (\because \alpha = \frac{e}{\sqrt{2\pi}}) \\ &= \frac{1}{e} \exp(1) \quad (= 1). \end{aligned} \tag{4.21}$$

Therefore the following condition is satisfied :

$$\begin{aligned} &|\pi(x) - \mathrm{li}(x)| \\ &\leq C(\frac{e}{48})^{\frac{1}{4}} \exp(1) x^{\frac{1}{e}} \big(\frac{1}{\log(x)}\big)^{\frac{3}{4}} \exp\big(\frac{1}{\log(x)}\big) \\ &\leq C x^{\frac{1}{e}} \log(x). \end{aligned} \tag{4.22}$$

□

We have a question that the upper limit of the real number d that satisfies the formula $|\pi(x) - \mathrm{li}(x)| \leq O(x^{1/d} \log(x))$ is e correct or not. That is, the following problem can be considered.

Problem 4.6. *The upper limit of the inequality that evaluate the number of prime number.*

$$e = \sup\{d > 1 \mid (\exists C > 1)(\forall \epsilon > 0)(\forall x >> 2)\ |\pi(x) - \mathrm{li}(x)| \leq C x^{\frac{1}{d}} \log(x)\} \tag{4.23}$$

where $C, \epsilon, x \in \mathbb{R}$. *Namely,*

$$e = \sup\{d > 1 \mid |\pi(x) - \mathrm{li}(x)| \leq O(x^{\frac{1}{d}} \log(x))\}. \tag{4.24}$$

□

We expect the Problem4.6 to be correct. In future, we attempt to solve this problem.

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4.2. Example using the function $R^{\pm}_{\alpha}(x)$ for the abc conjecture.

We derive the weak abc conjecture and the strong abc conjecture using α. Namely, using the constant α, the following theorem are satisfied : (Refer to Fujino[19])

Theorem 4.7. *Let $\alpha > 0$ be a positive real number. For all real number $\epsilon > 0$ and constant $K_\epsilon \geq 1$, there exists countable infinite triples (a, b, c) of coprime positive integers with $a + b = c$ such that the following inequality is satisfied :*

$$K_\epsilon \mathrm{rad}(abc) < c^{\exp(\frac{e}{\sqrt{2\pi\alpha}})-1} \tag{4.25}$$

where the following equation is satisfied :

$$\exp\left(\frac{e}{\sqrt{2\pi\alpha}}\right) = \lim_{x\to\infty} \exp\left(\frac{1}{xR^{\pm}(x)}\right). \tag{4.26}$$

□

Set the constant α as follows:

$$\alpha = \frac{e}{\sqrt{2\pi}\log(\frac{\epsilon+2}{\epsilon+1})}. \tag{4.27}$$

Therefore, the following is satisfied :

$$h_\alpha = \log\left(\frac{\epsilon+2}{\epsilon+1}\right). \tag{4.28}$$

Therefore, the negative consequence of the weak abc conjecture is satisfied as follows :

Theorem 4.8. *The negation of the weak abc conjecture.*
For all real number $\epsilon > 0$ and constant $\bar{K}_\epsilon \geq 1$, there exists countable infinite triples (a, b, c) of coprime positive integers with $a + b = c$ such that the following inequality is satisfied :

$$\bar{K}_\epsilon \mathrm{rad}(abc)^{1+\epsilon} < c. \tag{4.29}$$

Namely, There is a counter-example in the weak abc conjecture. Therefore, the weak abc conjecture is not true. □

Furthermore, let $\epsilon = 1$ and $\bar{K}_\epsilon = 1$. The negative consequence of the strong abc conjecture is satisfied as follows :

Theorem 4.9. *The negation of the strong abc conjecture. There exists countable infinite triples (a, b, c) of coprime positive integers with $a + b = c$ such that the following inequality is satisfied :*

$$\mathrm{rad}(abc)^2 < c. \tag{4.30}$$

Namely, the strong abc conjecture is not true. □

The function $R^{\pm}_{\alpha}(x)$ was used the above discussions of Von Koch's inequality and the abc conjecture. We will investigate the relation Entropy and Number Theorem further.

4.3. Conclusion and the Application to Prime and Number Theory.
In this paper, we attempted to proceed through applying the Boltzmann Principle and the Planck distribution function to the Prime number theory. We first consider to divided natural number x by approximately of the number of prime number $\pi(x)$, that is, the function $x/\log(x)$. Thereby, we obtained that the function $R_\alpha^\pm$. Furthermore, using the function $R_\alpha^\pm$, we derived and define new distribution function $n_\alpha^\pm$.

As mentioned above, the modern physics is considered to be the special condition that the real number α be satisfied as follows :

$$\alpha = \frac{e}{\sqrt{2\pi}h}, \text{ that is, } h_\alpha = h, \tag{4.31}$$

where the constant h is Planck constant.

Furthermore, using new distribution function $R_\alpha^\pm$, we evaluated Von Koch's inequality that equivalent to the Riemann Hypothesis and the abc conjecture.

Namely, we consider the different system that Von Koch's inequality is satisfied

$$\alpha = \frac{2}{\sqrt{2\pi}}, \text{ that is, } h_\alpha = \frac{e}{2}. \tag{4.32}$$

Moreover, the abc conjecture is satisfied

$$\alpha = \frac{e}{\sqrt{2\pi}\log(\frac{\epsilon+2}{\epsilon+2})}, \quad \text{that is, } h_\alpha = \log(\frac{\epsilon+2}{\epsilon+1}). \tag{4.33}$$

Namely, we can take values α different from the value α_h proposed by modern physics as follows :

$$\alpha_h = \frac{e}{\sqrt{2\pi}h}. \tag{4.34}$$

Namely, in future, considering the corresponding above, we would research the existence of *the different system* such that the Planck constant and the Boltzmann constant are defferent from the morden physics, in addition, these constant are *Not constant*.

We considered that there exist the relation between new distribution function $n_\alpha^\pm(x)$ and the Boltzmann principle, furthermore the Planck distribution function. It is meaningful that there exist the relation between statistical mechanics and Number theorem.

5. Issuse for the future.

We would describe some future issues as follows.

5.1. What are $S'_{\pi_f}(x)$ and $S''_{\pi_f}(x)$?
We would like to consider what $S'_{\pi_f}(x)$ and $S''_{\pi_f}(x)$ are.

Let $x > 1$ be real number. The functions above $S_{\pi_f}(x)$, $S'_{\pi_f}(x)$ and $S''_{\pi_f}(x)$ are regarded as follows:

$$\begin{aligned} &S_{\pi_f}(x) : \textit{the entropy divided by } \pi_f(x), \\ &S'_{\pi_f}(x) : \textit{the velocity of entropy } S_{\pi_f}(x), \ (\textit{potential}, \ \textit{rate of change}) \\ &S''_{\pi_f}(x) : \textit{the acceleration of entropy } S_{\pi_f}(x) \ (\textit{field}). \end{aligned} \tag{5.1}$$

where these functions are satisfied as follows:

$$\begin{aligned} f(x) &:= \log(x), \\ Q_f(x) &:= \log(x), \\ \pi_f(x) &:= \frac{x}{f(x)} = \frac{x}{\log(x)}. \end{aligned} \tag{5.2}$$

The first derivative of function $S_{\pi_f}(x)$, that is, $S'_{\pi_f}(x)$ and the second derivative of function $S_{\pi_f}(x)$, that is, $S''_{\pi_f}(x)$ can be describe as follows :

$$\begin{aligned} S'_{\pi_f}(x) &= Q'_f(x)\log\left(1 + \frac{1}{Q_f(x)}\right), \\ S''_{\pi_f}(x) &= k_f(x)\left(\frac{-Q'_f(x)}{Q_f(x)(1+Q_f(x))}\right), \end{aligned} \tag{5.3}$$

where the function $k_f(x)$ is regard as a function decided by x and $\pi_f(x)$. the function $Q_f(x)$ is regard as the position divided by $Q_f(x)$. the first derivative of $Q_f(x)$, that is, $Q'_f(x)$ is regard as the slope of the function $Q_f(x)$ (charge of the position) .

The above content is just under consideration. Therefore, we consider the generalization below.

5.2. Generalize of the function $S''_D(x)$.

We generalize the above equation (3.51) as follows.

Let $x > 1$ be real number. The function $D(x)$ be a positive real valued function such that $D(x) \leq x$. The function $D(x)$ can be thought of as a division of x. Therefore, the above $S_D(x)$, $S'_D(x)$ and $S''_D(x)$ are regarded and defined as follows:

$$S'_D(x) = Q'_D(x)\log\left(1 + \frac{1}{Q_D(x)}\right), \tag{5.4}$$

$$S''_D(x) = k_D(x)\left(\frac{-Q'_D(x)}{Q_D(x)(1+Q_D(x))}\right), \tag{5.5}$$

where the relation between the functions $D(x)$ and $Q_D(x)$ are $D(x) = x/Q_D(x)$. Moreover, the function $k_D(x)$ is regard as a function decided by x and $D(x)$. the function $Q_D(x)$ is regard as the position divided by $Q_D(x)$. the first derivative of $Q_D(x)$, that is, $Q'_D(x)$ is regard as the charge of the position by x. Each functions above are real valued functions.

5.3. New distribution function $R^{\pm}_{\alpha}(x)$.

We examine that the correspondence between generalized the function $S''_D(x)$ of equation (5.5) and $S''_{\pi_f}(x)$ of equation (3.51).

We consider to set as follows:

$$\begin{aligned} S''_D(x) &:= S''_{\pi_f}(x) \ (< 0), \\ Q_D(x) &:= Q_f(x), \\ Q'_D(x) &:= Q'_f(x). \\ k_D &:= R^{\pm}_{\alpha}(x), \end{aligned} \tag{5.6}$$

Therefore, we can obtain the following equation :

$$S''_D(x)dx = R^{\pm}_{\alpha}(x)\frac{-Q'_D(x)}{Q_D(x)(1+Q_D(x))}dx. \tag{5.7}$$

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This equation (5.7) is regarded as the generalization of the equation (3.51).

In the subsections below, we examine some laws can be regarded as the accelerations of $S_D(x)$.

5.4. Logistic equation of the modeling population growth.
We examine the relationship between generalized the function $S''_D(x)$ of equation (5.5) and the logistic equation of the modeling population growth.

Let r and K be positive integer constants. For positive real number $t > 0$, let $N(t)$ be a positive real valued function. We set as follows:

$$\begin{aligned} S''_D(x) &:= -r(<0), \\ Q_D(x) &:= -\frac{N(t)}{K}, \\ Q'_D(x) &:= -\frac{1}{K}\frac{dN(t)}{dt}, \\ k_D &:= 1, \\ x &:= t. \end{aligned} \tag{5.8}$$

Note :
The parameter r, K, t and the function $N(x)$ mean as follows :

$$\begin{aligned} r &: \textit{ intrinsic natural growth rate}, \\ N(x) &: \textit{ the population}, \\ K &: \textit{ carrying capacity}, \\ t &: \textit{ time or step}. \end{aligned} \tag{5.9}$$

Thus, the equation(5.5) becomes the equation of the dynamical system as follows:

$$\int -r dt \geq \int \frac{-(\frac{-1}{K})}{-\frac{N(t)}{K}(1-\frac{N(t)}{K})}\frac{dN(t)}{dt}dt. \tag{5.10}$$

Thus, the formula above are transform as follows :

$$\int r dt \geq \int \frac{K}{N(t)(K-N(t))}dN(t). \tag{5.11}$$

Therefore, the following equation are obtained :

$$\frac{dN(t)}{dt} = rN(t)\frac{(K-N(t))}{K}. \tag{5.12}$$

The equation(5.12) above is the logistic equation of dynamics. In other words, the equation (5.12) derived from the equation (5.7) can be regarded as a generalization of the dynamical system. Therefore, we believe that the entropy and dynamical system are closely related and are studying their applications.

5.5. Correspondence with Coulomb's law.
We examine the relationship between generalized the function $S''_D(x)$ of equation (5.5) and Coulomb's laws.

Let $r > 0$ be a real number. We define the electric field strength as follows:

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$$F_{q_1,q_2,r} = E_{q_1,r} \cdot q_2 = -k_e\left(\frac{q_1 q_2}{r^2}\right), \tag{5.13}$$

$$E_{q_1,r} = -k_e\left(\frac{q_1}{r^2}\right), \tag{5.14}$$

where

$$\begin{aligned} q_1, q_2 &: \textit{ the electric charge}, \\ k_e &: \textit{ the coulomb constant}, \\ r &: \textit{ the distance between } q_1 \textit{ and } q_2. \end{aligned} \tag{5.15}$$

We compare the equation (5.14) and (5.5). It is assumed that the distance between q_1 and q_2 r is sufficiently larger than the electric charge q_1 and q_2. In other words, for sufficiently large $x > 1$, the functions $Q_D(x)$ and $1 + Q_D(x)$ are regarded as the same value. Therefore, put as follows :

$$\begin{aligned} k_D(x) &:= k_e, \\ Q'_D(x) &:= q_1, \\ Q_D(x) &:= r. \end{aligned} \tag{5.16}$$

Hence, for sufficiently large $x > 1$, the following equation is satisfied :

$$E_{q_1,r} = S''_{\pi_D}(x) \tag{5.17}$$

where the condition (5.16) are satisfied.

5.6. Correspondence with classical Gravity theory of Newton's Law. We examine the relationship between generalized the function $S''_D(x)$ of equation (5.5) and classical Gravity theory of Newton's Law.

Let $r > 0$ be a real number. We define the gravitational field as follows:

$$F_{m_1,m_2,R} = g_{m_1,R} \cdot m_2 = -G\left(\frac{m_1 m_2}{R^2}\right), \tag{5.18}$$

$$g_{m_1,R} = -G\left(\frac{m_1}{R^2}\right). \tag{5.19}$$

where

$$\begin{aligned} m_1, m_2 &: \textit{ the mass}, \\ G &: \textit{ the gravitational constant}, \\ R &: \textit{ the distance between } m_1 \textit{ and } m_2. \end{aligned} \tag{5.20}$$

In the same way, we compare the equation (5.19) and (5.5). For sufficiently large $x > 1$, the functions $Q_D(x)$ and $1 + Q_D(x)$ are regarded as the same value. Therefore, it is possible to put as follows :

$$\begin{aligned} k_D(x) &:= G, \\ Q'_D(x) &:= m_1, \\ Q_D(x) &:= R. \end{aligned} \tag{5.21}$$

Hence, for sufficiently large $x > 1$, the following equation is satisfied :

$$g_{m_1,R} = S''_{\pi_D}(x) \tag{5.22}$$

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where the condition (5.21) are satisfied.

The discussion above, we assume that the functions $Q_D(x)$ and $1+Q_D(x)$ are regarded as the same value. However, in the next discussion, we assume that the functions $Q_D(x)$ and $1+Q_D(x)$ are *not* regarded as the same value. Thereby, the following equation is defined :

$$\bar{g}_{m_1,R} = -G\left(\frac{m_1}{R(R+1)}\right). \tag{5.23}$$

We compare equations (5.19) and (5.23). Considering the negative sign, the following inequality is satisfied :

$$-G\left(\frac{m_1}{R(R+1)}\right) \geq -G\left(\frac{m_1}{R^2}\right). \tag{5.24}$$

Namely, the following inequality is satisfied :

$$\bar{g}_{m_1,R} \geq g_{m_1,R}. \tag{5.25}$$

If the functions $Q_D(x)$ and $1+Q_D(x)$ are *not* regarded as the same value, in consequence, it is possible that the gravity $\bar{g}_{m_1,R}$ is greater towards the center than the classical gravity $g_{m_1,R}$.

Suggestion 5.1. *There exists the real-valued function $D(x)$ divides the real number x that satisfies the correspondence the acceleration of $S_D(x)$ and the classical Gravity theory of Newton's Law.*
Let m_1 be a positive real number. For sufficiently large $R>1$, the following condition is satisfied :

$$\bar{g}_{m_1,R} \geq g_{m_1,R}. \tag{5.26}$$

□

In consequence, considering the acceleration of the Entropy $S_D(x)$, the gravity $\bar{g}_{m_1,R}$ is greater towards the center than the classical gravity $g_{m_1,R}$. In other word, the force towards the center of a rotating substance can be slightly greater at sufficient large distance.

5.7. Conclusion.
We considered the following possibilities :
for sufficiently large $x>1$, the equation (5.5)

$$S''_D(x) = k_D(x)\left(\frac{-Q'_D(x)}{Q_D(x)(1+Q_D(x))}\right)$$

is regarded as a generalized expression and approximate representation of the equation (5.12)

$$\frac{dN(t)}{dt} = rN(t)\frac{(K-N(t))}{K},$$

of the equation (5.14)

$$E_{q_1,r} = -k_e\left(\frac{q_1}{r^2}\right),$$

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and of the equation (5.19)

$$g_{m_1,R} = -G\left(\frac{m_1}{R^2}\right)$$

where the function $D(x)$ needs to be chosen appropriately.

According to divide the number of divisions of the Boltzmann principle by the number of prime numbers, the Entropy $S_{\pi_f}(x)$ (the second law of π_f) is related to Quantum mechanics(statistical mechanics) and Number theory.

Besides, the entropy acceleration $S''_D(x)$ (the second derivative of entropy $S_D(x)$) is related to the dynamical systems, and the classical gravity theory of Newton's Law and the electromagnetism of Coulomb's law with the law of inverse squares..

Furthermore, according to the theory of quantum mechanics, an atom can only take discrete spectral values. Similarly, the placement of planetary systems such as the solar system, galaxies, and clusters of galaxies can be considered to only take discrete spectral values. This discrete spectral arrangements seems to be related to the entropy.

In the future, I would like to find out *why* these relationships appear the number of divisions of the Boltzmann principle is divided by the number of prime numbers. We hope that this paper will serve as a bridge for further research in the future.

Increasing entropy (the second law) does not mean becoming disorder. On the contrary, The second law has the potential to cause the movement and the order of phenomena.

6. The proof of Lemma3.6 , Lemma3.7 and Lemma3.8

6.1. The proof of Lemma3.6 and Lemma3.7.

Proof. Let $n \geq 1$ a positive integer. For all sufficiently large positive real number $x > 0$, the following conditions are satisfied :

$$\begin{aligned} |\left(Q_f(x)\right)^{(n)}| &= |\frac{(-1)^{n-1}(n-1)!}{x^n}| \\ &= \frac{(n-1)!}{x^n} \\ &\geq \frac{\sqrt{2\pi}(n-1)^{(n-1+\frac{1}{2})}e^{-(n-1)}}{x^n} \\ &(\because \text{Stirling's formula}: \ n! \geq \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}) \\ &= \left(\frac{\sqrt{2\pi}(n-1)^{n-(\frac{1}{2})}}{e^{(n-1)}x^n}\right) \\ &= \sqrt{2\pi}(n-1)^{n-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n} \quad (*2) \end{aligned} \tag{6.1}$$

Therefore, dividing the end of the formula(6.1), that is (*2), by the number $n^{n-(\frac{1}{2})}$, for sufficiently large $x > 1$, the following condition are satisfied :

Case 1) $x > n$: Because $x \geq x - (\frac{1}{2})$, therefore

$$(*2) \geq \sqrt{2\pi}(\frac{n-1}{n})^{n-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n}$$
$$\geq \sqrt{2\pi}(\frac{x-1}{x})^{x-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n} \quad (\because x > n)$$
$$\geq \sqrt{2\pi}(\frac{x-1}{x})^{x}\frac{1}{e^{(n-1)}x^n} \tag{6.2}$$
$$(\because x \geq x - (\frac{1}{2}) \text{ and } (\frac{x-1}{x})^x \geq \lim_{x\to\infty}(\frac{x-1}{x})^x = e^{-1})$$
$$\geq \sqrt{2\pi}e^{-1}\frac{1}{e^{(n-1)}x^n}$$

Case 2) $n \geq x$:

$$(*2) \geq \sqrt{2\pi}(\frac{n-1}{n})^{n-(\frac{1}{2})}\frac{1}{e^{(n-1)}x^n}$$
$$\geq \sqrt{2\pi}(\frac{n-1}{n})^{n}\frac{1}{e^{(n-1)}x^n} \tag{6.3}$$
$$(\because n \geq n - (\frac{1}{2}) \text{ and } (\frac{n-1}{n})^n \geq \lim_{n\to\infty}(\frac{n-1}{n})^n = e^{-1})$$
$$\geq \sqrt{2\pi}e^{-1}\frac{1}{e^{(n-1)}x^n}$$

Therefore, using Case 1) and Case 2) above, for all sufficiently large $x > 0$, the following inequality is satisfied :

$$|\big(Q_f(x)\big)^{(n)}| \geq \sqrt{2\pi}e^{-1}\frac{1}{e^{(n-1)}x^n} \tag{6.4}$$

Therefore, the following conditions are satisfied :

$$R_m^+(x) \geq \lim_{N\to\infty}\sum_{n=1}^{N}\Big|\big(Q_f(x)\big)^{(n)}\Big|$$
$$\geq \lim_{N\to\infty}\sqrt{2\pi}e^{-1}\sum_{n=1}^{N}\Big|\frac{(-1)^{n-1}}{e^{(n-1)}x^n}\Big| \tag{6.5}$$
$$\geq \lim_{N\to\infty}\sqrt{2\pi}e^{-1}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{e^{(n-1)}x^n}$$

Put the function $A(x)$ as follows:

$$A(x) := \sum_{n=1}^{N}\frac{(-1)^{n-1}}{e^{(n-1)}x^n} \tag{6.6}$$

Therefore,

$$A(x) = \frac{1}{e^0x^1} - \frac{1}{e^1x^2} + \frac{1}{e^2x^3} - \cdots \tag{6.7}$$
$$\frac{1}{ex}A(x) = \frac{1}{e^1x^2} - \frac{1}{e^2x^3} + \frac{1}{e^3x^4} - \cdots + \frac{(-1)^N}{e^Nx^{(N+1)}} \tag{6.8}$$

Add the equation(6.7) and (6.8). Hence, the following conditions are satisfied :

$$\left(1+\frac{1}{ex}\right)A(x) = \lim_{N\to\infty}\left(\frac{1}{x}+\frac{(-1)^N}{e^N x^{(N+1)}}\right) = \frac{1}{x} \tag{6.9}$$

Therefore,

$$\lim_{N\to\infty} A(x) = \frac{1}{x}\left(\frac{1}{1+\frac{1}{ex}}\right) = \frac{e}{ex+1} \tag{6.10}$$

Therefore, the function R_m^+ is approximated as follows: Fixed 20220611 $\tilde{R}$

$$R_m^+(x) \geq \sqrt{2\pi}e^{-1}\lim_{N\to\infty} A(x) = \frac{\sqrt{2\pi}}{ex+1} \tag{6.11}$$

Here, the following conditions are satisfied :

$$R_m^+(x) > \frac{1}{x} > \frac{\sqrt{2\pi}\alpha}{ex+1} = R_\alpha^+(x) \tag{6.12}$$

Put the positive real number $\alpha > 0$ such that as follows:

$$x \geq \frac{-1}{e-\sqrt{2\pi}\alpha} \quad ,(\alpha, x \in \mathbb{R}) \tag{6.13}$$

Consequently, the following conditions are satisfied :

$$R_m^+(x) \geq \sqrt{2\pi}e^{-1}\alpha\lim_{N\to\infty} A(x) = \frac{\sqrt{2\pi}\alpha}{ex+1} = R_\alpha^+(x) \tag{6.14}$$

$$x \geq \frac{-1}{e-\sqrt{2\pi}\alpha}, \text{ that is, } \frac{ex+1}{\sqrt{2\pi}x} \geq \alpha, \quad (\alpha, x \in \mathbb{R}) \tag{6.15}$$

Furthermore, for all sufficiently large $x > 1(\in \mathbb{R})$, the following conditions are satisfied :

$$\frac{1}{x}(2+\log(x)) \geq k_f(x) \geq R_\alpha^+(x) = \frac{\sqrt{2\pi}\alpha}{ex+1}, \tag{6.16}$$

$$\frac{e}{\sqrt{2\pi}}(2+\log(x)) \geq \frac{ex+1}{\sqrt{2\pi}x} \geq \alpha > 0 \tag{6.17}$$

where $\alpha > 0 \in \mathbb{R}$. (The end of the proof of Lemma3.6).

By the same method, the function R_m^- is approximated as follows:

$$\sqrt{2\pi}e^{-1}\alpha\lim_{N\to\infty} A(x) = \frac{\sqrt{2\pi}\alpha}{ex-1} = R_\alpha^-(x) \tag{6.18}$$

$$x \geq \frac{1}{e-\sqrt{2\pi}\alpha} \quad ,(\alpha, x \in \mathbb{R}) \tag{6.19}$$

For all sufficiently large $x > 1(\in \mathbb{R})$, the following conditions are satisfied :

$$\frac{1}{x}(2+\log(x)) \geq k_f(x) \geq R_\alpha^-(x) = \frac{\sqrt{2\pi}\alpha}{ex-1}, \tag{6.20}$$

$$\frac{e}{\sqrt{2\pi}}(2+\log(x)) \geq \frac{ex-1}{\sqrt{2\pi}x} \geq \alpha > 0. \tag{6.21}$$

□

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6.2. The proof of Lemma 3.8.

Proof. Let $n \geq 1$, and $x > 0$ is sufficiently large.

$$
\begin{aligned}
|\big(Q_f(x)\big)^{(n)}| &= |\frac{(-1)^{n-1}(n-1)!}{x^n}| \\
&= \frac{(n-1)!}{x^n} \\
&\leq \frac{e(n-1)^{(n-1+\frac{1}{2})}e^{-(n-1)}}{x^n} \\
&(\because \text{Stirling's formula: } ee^{-n}n^{n+\frac{1}{2}} \geq n! \geq \sqrt{2\pi}e^{-n}n^{n+\frac{1}{2}}) \\
&= \left(\frac{e(n-1)^{n-(1/2)}}{e^{(n-1)}x^n}\right) \\
&= e(n-1)^{n-(1/2)}\frac{1}{e^{(n-1)}x^n} \\
&= e\left(\frac{n-1}{x}\right)^n \frac{(n-1)^{-1/2}}{e^{(n-1)}} \\
&= e\left(\frac{x-1}{x}\right)^x \frac{1}{e^{(n-1)}(x-1)^{1/2}} \quad (x \gg n) \\
&\leq e\left(\frac{x-1}{x}\right)^x \frac{1}{e^{(n-1)}(x-1)^{1/2}} \\
&\leq ee^{-1}\frac{1}{e^{(n-1)}(x-1)^{1/2}} \quad (\because \lim_{x\to\infty}(\frac{x-1}{x})^x = \frac{1}{e}) \\
&\leq \frac{1}{e^{(n-1)}(x-1)^{1/2}}
\end{aligned}
\tag{6.22}
$$

Therefore, the following inequality is satisfied :

$$|\big(Q_f(x)\big)^{(n)}| \leq \frac{1}{e^{(n-1)}(x-1)^{1/2}} \tag{6.23}$$

Furthermore, the inequality

$$|\big(Q_f(x)\big)^{(n)}| > |\big(Q_f(x)\big)^{(n+1)}| \tag{6.24}$$

is satisfied. Besides, the following relation is satisfied :

$$if\ n\ is\ even \quad : \quad 0 < \big(Q_f(x)\big)^{(n)} \leq \frac{1}{e^{(n-1)}(x-1)^{1/2}}, \tag{6.25}$$

$$if\ n\ is\ odd \quad : \quad 0 > \big(Q_f(x)\big)^{(n)} \geq \frac{-1}{e^{(n-1)}(x-1)^{1/2}}, \tag{6.26}$$

The discussion above, the following are satisfied :

$$
\begin{aligned}
R_m^+(x) &= \lim_{N\to\infty}\sum_{n=1}^{N}\left|\big(Q_f(x)\big)^{(n)}\right| \\
&\leq \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{e^{(n-1)}(x-1)^{1/2}}
\end{aligned}
\tag{6.27}
$$

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Put the function A_2 as follows :

$$A_2(x) = \sum_{n=1}^{N} \frac{(-1)^{n-1}}{e^{(n-1)}(x-1)^{1/2}}. \tag{6.28}$$

Hence, we can describe as follows :

$$A_2(x) = \frac{1}{e^0(x-1)^{1/2}} - \frac{1}{e^1(x-1)^{1/2}} + \frac{1}{e^2(x-1)^{1/2}} - \cdots \tag{6.29}$$

$$\frac{1}{e}A_2(x) = \qquad \frac{1}{e^1(x-1)^{1/2}} - \frac{1}{e^2(x-1)^{1/2}} + \cdots + \frac{(-1)^N}{e^N(x-1)^{1/2}} \tag{6.30}$$

Add above the equivalent(6.29) and (6.30), the following equivalent are satisfied :

$$\begin{aligned}\left(1+\frac{1}{e}\right)A_2(x) &= \lim_{N\to\infty}\left(\frac{1}{(x-1)^{1/2}} + \frac{(-1)^N}{e^N(x-1)^{1/2}}\right) \\ &= \frac{1}{(x-1)^{1/2}}\end{aligned} \tag{6.31}$$

Therefore,

$$\begin{aligned}\lim_{N\to\infty} A_2(x) &= \frac{1}{(x-1)^{1/2}}\left(\frac{1}{1+\frac{1}{e}}\right) \\ &= \frac{1}{(x-1)^{1/2}}\frac{e}{e+1}\end{aligned} \tag{6.32}$$

Therefore, the function R_m^+ is satisfied as follows :

$$R_m^+(x) \le \lim_{N\to\infty} A_2(x) = \frac{1}{(x-1)^{1/2}}\frac{e}{e+1} \tag{6.33}$$

where $\alpha > 0(\in \mathbb{R})$ for all sufficiently large $x > 1(\in \mathbb{R})$, the following conditions are satisfied :

$$k_f(x) \le \frac{1}{x}(2+\log(x)) \le \frac{1}{(x-1)^{1/2}}\frac{e}{e+1}. \tag{6.34}$$

By the same method, R_m^+ is approximated as follows :

$$R_m^+ \le \frac{1}{(x-1)^{1/2}}\frac{e}{e-1} \tag{6.35}$$

where $\alpha > 0(\in \mathbb{R})$ for all sufficiently large $x > 1(\in \mathbb{R})$, the following conditions are satisfied :

$$k_f(x) \le \frac{1}{x}(2+\log(x)) \le \frac{1}{(x-1)^{1/2}}\frac{e}{e-1}. \tag{6.36}$$

□

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