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Novel recurrence relations for volumes and surfaces of *n*-balls, regular *n*-simplices, and *n*-orthoplices in integer dimensions

Szymon Łukaszyk *
Łukaszyk Patent Attorneys, ul. Głowackiego 8, 40-052 Katowice, Poland

New recurrence relations for *n*-balls, regular *n*-simplices, and *n*-orthoplices in integer dimensions are submitted. They remove indefiniteness present in known formulas. In negative, integer dimensions volumes of *n*-balls are zero if *n* is even, positive if $n = -4k - 1$, and negative if $n = -4k - 3$, for natural *k*. Volumes and surfaces of *n*-cubes inscribed in *n*-balls in negative dimensions are complex, wherein for negative, integer dimensions they are associated with integral powers of the imaginary unit. The relations are continuous for $n \in \mathbb{R}$ and show that the constant of π is absent for $0 \leq n < 2$. For $n < -1$ self-dual *n*-simplices are undefined in negative, integer dimensions and their volumes and surfaces are imaginary in negative, fractional ones, and divergent with decreasing *n*. In negative, integer dimensions *n*-orthoplices reduce to the empty set, and their real volumes and imaginary surfaces are divergent in negative, fractional ones with decreasing *n*. Out of three regular, convex polytopes present in all non-negative dimensions, only *n*-orthoplices, *n*-cubes (and *n*-balls) are defined in negative, integer dimensions.

1. Introduction

Natural dimensions are of particular importance, as they define a minimum number of independent parameters (coordinates) needed to specify a point within an Euclidean space $\mathbb{R}^n$, where $n = -1$ is the dimension of the empty set, the void, having zero volume and undefined surface.

However, a spectrum, topological generalization of the notion of space allows for negative dimensions [1, 2]. Natural dimensions refer to sizes, negative ones refer to densities [2]. Negatively dimensional *spaces* are defined by analytic continuations from positive dimensions [3].

In $\mathbb{R}^2$ there are countably infinite number of regular, convex polygons, in $\mathbb{R}^3$ there are five regular, convex Platonic solids, in $\mathbb{R}^4$ there are six regular, convex polytopes. For $n > 4$, there are only three: self-dual *n*-simplex, and *n*-cube dual to *n*-orthoplex [4]. Furthermore, any natural dimension is also equipped with perfectly regular, and obviously also convex, *n*-ball.

Volume of an *n*-ball is known to be

$$V_n(R)_B = \frac{\pi^{n/2}}{\Gamma(n/2+1)} R^n, \qquad (1)$$

where Γ is the Euler's gamma function and *R* is the *n*-ball radius. This becomes

$$V_{2k}(R)_B = \frac{\pi^k R^{2k}}{k!}, \qquad (2)$$

if *n* is even and

$$V_{2k+1}(R)_B = \frac{2(k!)(4\pi)^k}{(2k+1)!} R^{2k+1}, \qquad (3)$$

if *n* is odd. Another known recurrence relation expresses the volume of an *n*-ball in terms of the volume of an (*n*-2)-ball of the same radius

$$V_n(R)_B = \frac{2\pi R^2}{n} V_{n-2}(R)_B, \qquad (4)$$

where $V_0(R)_B = 1$ and $V_1(R)_B = 2R$. It is also known that *n*-1 dimensional surface of an *n*-ball can be expressed as

$$S_n(R)_B = \frac{n}{R} V_n(R)_B. \qquad (5)$$

Volume of a regular *n*-simplex is known to be

$$V_n(A)_S = \frac{\sqrt{n+1}}{n!\sqrt{2^n}} A^n, \qquad (6)$$

where *A* is the edge length. A regular *n*-simplex has *n*+1 (*n*-1)-facets so its surface is

$$S_n(A)_S = (n+1) V_{n-1}(A)_S. \qquad (7)$$

Volume of *n*-orthoplex is known to be

$$V_n(A)_O = \frac{\sqrt{2^n}}{n!} A^n. \qquad (8)$$

As *n*-orthoplex has 2^n facets [5] being (*n*-1)-simplices, its surface is

$$S_n(A)_O = 2^n V_{n-1}(A)_S. \qquad (9)$$

Relations (1)-(3) and (6)-(9) are undefined in negative, integer dimensions, since factorial is undefined for negative integers, while gamma function is undefined for non-positive integers. Relations (4), (5) are undefined if $n = 0$.

The aim of this study was to examine *n*-balls, *n*-simplices and *n*-orthoplices in negative dimensions using novel recurrence relations that remove indefiniteness present in known formulas.

2. Novel recurrence relations

A radius recurrence relation

$$f_n \doteq \frac{2}{n} f_{n-2}, \qquad (10)$$

for $n \in \mathbb{N}_0$, where $f_0 := 1$ and $f_1 := 2$, allows to express the volumes and, using (5), surfaces of *n*-balls as

* szymon@patent.pl

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$$V_n(R)_B = f_n \pi^{\lfloor n/2 \rfloor} R^n, \tag{11}$$

$$S_n(R)_B = n f_n \pi^{\lfloor n/2 \rfloor} R^{n-1}. \tag{12}$$

The sequence (10) allows to present n-balls volume and surface recurrence relations (11), (12) as a product of a rational factor f_n or nf_n, an irrational factor $\pi^{\lfloor n/2 \rfloor}$ (for $n \neq 0$ and $n \neq 1$), and a metric (radius) factor R^n or R^{n-1}. The relation (10) can be then extended into negative dimensions as

$$f_n \doteq \frac{n+2}{2} f_{n+2}, \tag{13}$$

solving for f_{n-2} and assigning new $n \in \mathbb{Z}$ as old n-2.

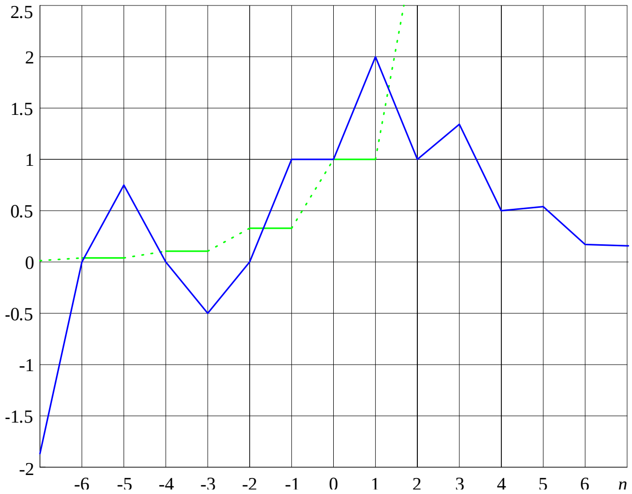

Fig. 1: Radius n-ball recurrence relation (blue) with the π^⌊n/2⌋ factor (green) for n = -7, -6,…,7.

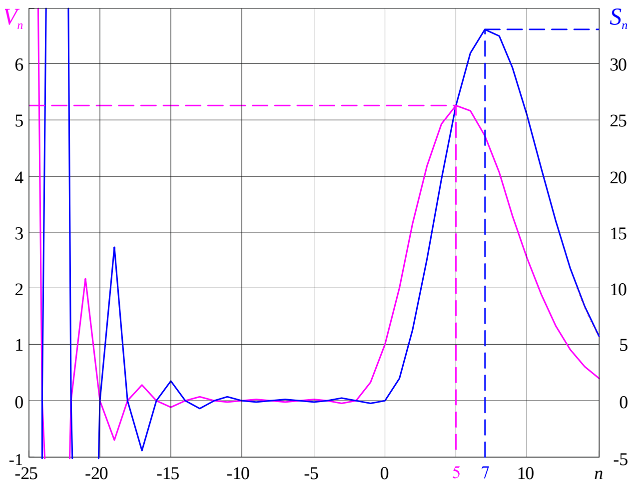

Fig. 2: Graphs of volumes (V) and surface areas (S) of n-balls of unit radius for n = -25, -6,…,15.

The same assignment of new $n \in \mathbb{Z}$ as old n - 2 can be made in (4) solved for $V_{n-2}(R)_B$ yielding

$$V_n(R)_B = \frac{n+2}{2\pi R^2} V_{n+2}(R)_B. \tag{14}$$

This enables to avoid the indefiniteness of factorial and gamma function in negative dimensions present in relations (1)-(3) and removes singularity present in (4).

Radius recurrence relations (10), (13) are shown in Fig. 1 along with the π^⌊n/2⌋ factor and listed in Table 1.

Volumes and surfaces of n-balls calculated with relations (11) and (12) are shown in Fig. 2 and Fig. 3.

For positive dimensions n = 5 (the largest unit radius n-ball volume) is the last odd n where $f_n > f_{n-1}$, while n = 7 (the largest unit radius n-ball surface) is the first odd n where $f_n < f_{n-1}$. It is sufficient to define $f_{-1} = f_0 := 1$ (for the empty set and point dimension) to initiate (10) and (13).

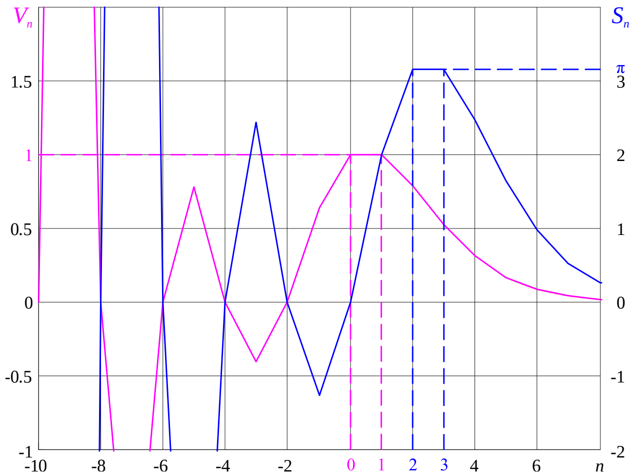

Fig. 3: Graphs of volumes (V) and surface areas (S) of n-balls of unit diameter for n = -10, -9,…,8.

If n < -1 and odd

$$f_n = -(-1)^{\lfloor n/2 \rfloor} \frac{\prod_{k=0}^{\lfloor |n/2| \rfloor - 1} (2k+1)}{2^{\lfloor |n/2| \rfloor}}. \tag{15}$$

If n > 0 and odd

$$f_n = \frac{2^{(n+1)/2}}{\prod_{k=0}^{\lfloor n/2 \rfloor} (2k+1)}. \tag{16}$$

If n > 0 and even

$$f_n = \left(\prod_{k=1}^{n/2} k \right)^{-1}. \tag{17}$$

Also if n is odd

$$-(-1)^{\lfloor n/2 \rfloor} f_{-(n+2)} f_n = 1. \tag{18}$$

Table 1: Volumes and surfaces of n-balls for -11 ≤ n ≤ 9.

n	f_n	g_n	$V_n(R=1)_B$	$S_n(R=1)_B$	$V_n(D=1)_B$	$S_n(D=1)_B$
-11	-945/32	-60480	-0.031	0.338	-62.909	1383.997
-9	105/16	3360	0.021	-0.193	10.980	-197.634
-7	-15/8	-240	-0.019	0.135	-2.464	34.494
-5	3/4	24	0.024	-0.121	0.774	-7.7404
-3	-1/2	-4	-0.051	0.152	-0.405	2.432
-1	1	2	0.318	-0.318	0.637	-1.273
0	1	1	1	0	1	0
1	2/1	1	2	2	1	2
2	1/1	1/4	3.142	6.283	0.785	3.142
3	4/3	1/6	4.189	12.566	0.524	3.142
4	1/2	1/32	4.935	19.739	0.308	2.467
5	8/15	1/60	5.264	26.319	0.164	1.645
6	1/6	1/384	5.168	31.006	0.081	0.969
7	16/105	1/840	4.725	33.073	0.037	0.517
8	1/24	1/6144	4.059	32.470	0.016	0.254
9	32/945	1/15120	3.299	29.687	0.006	0.116

One can also express the volumes and, using (5), surfaces of n-balls in terms of their diameters D as

$$V_n(D)_B = g_n \pi^{\lfloor n/2 \rfloor} D^n \,, \tag{19}$$

$$S_n(D)_B = 2n g_n \pi^{\lfloor n/2 \rfloor} D^{n-1} \,, \tag{20}$$

defining diameter recurrence relation

$$g_n \doteq \frac{1}{2n} g_{n-2} \tag{21}$$

having inverse

$$g_n \doteq 2(n+2) g_{n+2} \,, \tag{22}$$

for $n \in \mathbb{Z}$, where $g_{-1} := 2$ and $g_0 := 1$.

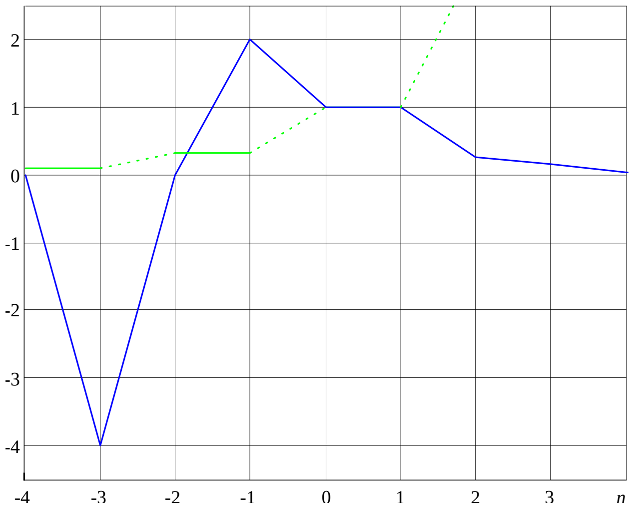

Fig. 4: Diameter n-ball recurrence relation (blue) with the π^⌊n/2⌋ factor (green) for n = -4, -3,…,4.

Diameter recurrence relation (21), (22) is shown in Fig. 4 along with the π^⌊n/2⌋ factor and listed in Table 1.

If $n \geq 0$

$$g_n = \frac{\pi^{n/2}}{2^n \Gamma(n/2+1) \pi^{\lfloor n/2 \rfloor}} \,, \tag{23}$$

which is the reciprocal of OEIS A087299 sequence. If $n < 0$ and odd

$$g_n = -(-1)^{\lfloor n/2 \rfloor} \frac{2(2\lfloor -n/2 \rfloor)!}{(\lfloor -n/2 \rfloor)!} \,, \tag{24}$$

which corresponds to OEIS A151817 (for $k = \lfloor -n/2 \rfloor$ and excluding the sign factor) and OEIS A052718 sequence (for $n < -2$ and excluding the sign factor).

Also, if n is odd

$$-(-1)^{\lfloor n/2 \rfloor} g_{-(n+2)} g_n = 4 \,. \tag{25}$$

Since $f_{-2} = 0$ (13) and $g_{-2} = 0$ (22), in negative, even dimensions n-balls have zero (void-like) volumes and zero (point-like) surfaces.

In the case of regular n-simplices, equation (6) can be written as a recurrence relation

$$V_n(A)_S = A V_{n-1}(A)_S \sqrt{\frac{n+1}{2n^3}} \,, \tag{26}$$

with $V_0(A)_S := 1$. This removes indefiniteness of factorial for n = -1 present in (6). Solving (26) for V_{n-1}, and assigning new $n := n - 1 \in \mathbb{Z}$ yields

$$V_n(A)_S = \frac{V_{n+1}(A)}{A} \sqrt{\frac{2(n+1)^3}{n+2}} \,, \tag{27}$$

which shows that n-simplices are indefinite only for integer $n < -1$, as shown in Fig. 5. The volume of an empty or void (-1)-simplex is $V_{-1}(A)_S = 0$, while its surface $S_{-1}(A)_S$ (7) is undefined, as the void itself.

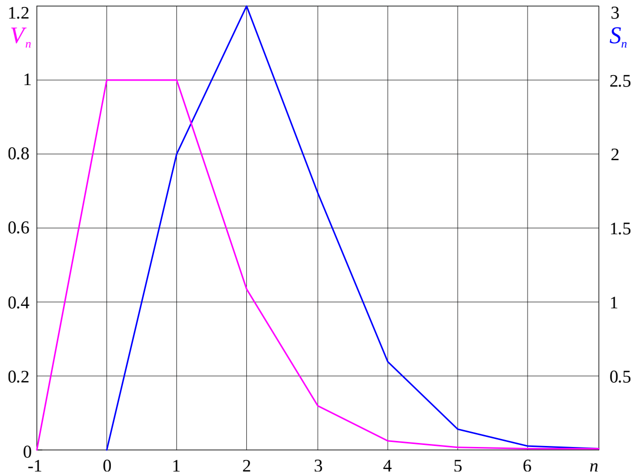

Fig. 5: Graphs of volumes (V) and surface areas (S) of regular n-simplices of unit edge length for n = -1,…,7.

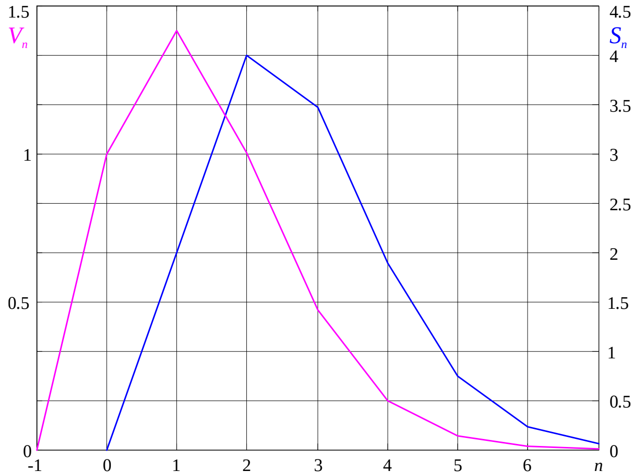

Fig. 6: Graphs of volumes (V) and surface areas (S) of n-orthoplices of unit edge length for n = -1,…,7.

In the case of n-orthoplices, equation (8) can be written as a recurrence relation

$$V_n(A)_O = A V_{n-1}(A)_O \frac{\sqrt{2}}{n} \,, \tag{28}$$

with $V_0(A)_O := 1$ and reversed solving for n-1 as

$$V_n(A)_O = V_{n+1}(A)_O \frac{n+1}{A\sqrt{2}} \,, \tag{29}$$

which removes singularity from (28) and is zero for integer $n \leq -1$ showing that for negative, integer dimensions

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volumes of n-orthoplices are zero, while their surfaces (9) are undefined, as shown in Fig. 6. It is not surprising: n-orthoplex has facets being simplexes of the previous dimension, and these are undefined for integer $n \le -1$. Peculiarly $V_1(A)_O := A\sqrt{2}$ not A, as in the case of 1-simplex and 1-cube.

3. *n*-balls circumscribed about and inscribed in *n*-cubes

The edge length A_C of n-cube circumscribed about n-ball corresponds to the diameter D of this n-ball. The ratio of volume or surface of n-ball to volume or surface of n-cube circumscribing this n-ball can be expressed using diameter recurrence relation (21)-(24) as

$$\frac{V_{nB}}{V_{nC}} = \frac{S_{nB}}{S_{nC}} = g_n \pi^{\lfloor n/2 \rfloor}. \tag{30}$$

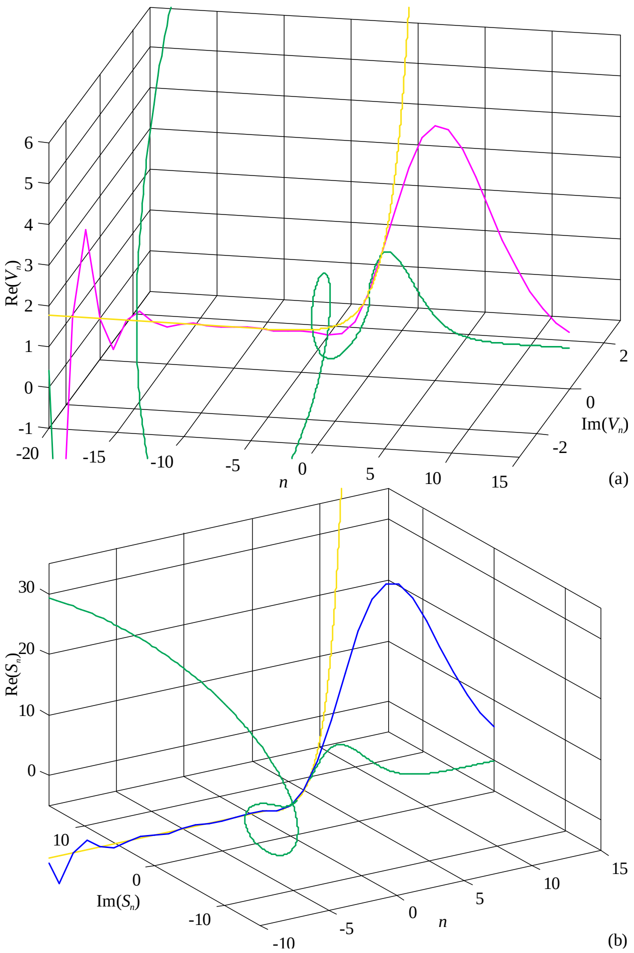

Fig. 7: Graphs of volumes (a, pink) and surface areas (b, blue) of n-balls of radius 1, along with volumes and surface areas of n-cubes circumscribed about (yellow) and inscribed in (green) these n-balls. In negative dimensions the latter are complex.

The edge length A_I of n-cube inscribed inside the n-ball of diameter D is

$$A_I = \frac{D}{\sqrt{n}}, \tag{31}$$

which is singular for $n = 0$ and complex for $n < 0$. The volume of n-cube inscribed in n-ball is

$$V_n(D)_C = A_I^n = D^n n^{-n/2}, \tag{32}$$

and the surface is

$$S_n(D)_C = 2nA_I^{n-1} = 2D^{n-1}n^{(3-n)/2}. \tag{33}$$

The volume (32) is real if n is negative and even, and imaginary if n is negative and odd. The opposite holds true for the surface (33). In negative, integer dimensions volumes (32) are associated with a coefficient i^n, while surfaces (33) with a coefficient i^{n+1}. By convention $0^0 := 1$.

Volumes and surfaces of n-cubes given by formulas (32) and (33) are shown in Fig. 7 and listed in Table 2. They are drawn as continuous lines as formulas (32) and (33) admit fractional dimensions.

Table 2: Volumes V_n and surfaces S_n of n-cubes inscribed in n-balls of unit radius and diameter for $-8 \le n \le 2$.

n	$V_n(R=1)_C$	$S_n(R=1)_C$	$V_n(D=1)_C$	$S_n(D=1)_C$
-8	16	362.039i	4096	185363.8i
-7	7.090i	-16807/128	907.493i	-33614
-6	-216/64	-49.602i	-216	-6349.077i
-5	-1.747i	19.531	-55.902i	1250
-4	1	8i	16	256i
-3	0.650i	-216/64	5.196i	-54
-2	-1/2	-$i\sqrt{2}$	-2	-8$i\sqrt{2}$
-1	-i/2	1/2	-i	2/1
0	1	0	1	0
1	2	2	1	2
2	2	4$\sqrt{2}$	1/2	2$\sqrt{2}$

4. Summary

The value of π as n-ball volume and surface irrational factor appears only for $n < 0$ and $n \ge 2$ (π^$\lfloor n/2 \rfloor = 1$ for $0 \le n < 2$). For negative dimensions radius and diameter recurrence relations and also volumes and surfaces of n-balls are divergent with decreasing n. Radius and diameter recurrence relations are rational numbers, while all π^$\lfloor n/2 \rfloor$ (for $n < 0$ and $n \ge 2$) are most likely transcendental numbers. Doubled maxima for unit diameter n-balls (volume for $n = 0$, 1 and surface for $n = 2$, 3) are also interesting.

Relations (4), (10)-(14), (19)-(22), (26)-(29) are continuous for $n \in \mathbb{R}$. The starting points for fractional dimensions can be provided e.g. using spline interpolation between two (or three in the case of n-balls) subsequent integer dimensions.

n-simplices are undefined in negative, integer dimensions if $n < -1$. This is congruent with the fact that every simplicial n-manifold inherits a natural topology from Euclidean space $\mathbb{R}^n$ [6] and by researching Euclidean space $\mathbb{R}^n$ as a simplicial n-manifold topological (metric-independent) and geometrical (metric-dependent) content of the modeled quantities are disentangled [6]. Thus, lack of n-simplices in negative, integer dimensions excludes the notion of negatively dimensional Euclidean space $\mathbb{R}^n$ for $n < -1$. Volumes and surfaces and surfaces of regular n-simplices are imaginary in negative, fractional dimensions for $n < -1$ (surfaces also for $n < 0$) and are divergent with decreasing n.

Volumes of n-orthoplices are zero in negative, integer dimensions, but are divergent in negative, fractional ones with decreasing n. Surfaces of n-orthoplices are undefined for integer $n < -1$, imaginary for fractional $n < 0$, and also divergent with decreasing n.

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In negative dimensions *n*-simplices, *n*-orthoplices, and *n*-balls have different properties than their positively dimensional counterparts. *n*-cube is an exception. A volume $V_n(A)_C = A^n$ and surface $S_n(A)_C = 2nA^{n-1}$ of *n*-cube are defined for any $n \in \mathbb{R}$ and are real if $A \in \mathbb{R}$.

Out of three regular, convex polytopes (and *n*-balls) present in all non-negative dimensions [4] only *n*-cubes, *n*-orthoplices, and *n*-balls are defined in negative, integer dimensions with *n*-cubes being dual to the void.

5. Discussion

Once upon a time there was a (-1)-dimensional void of volume zero and undefined surface. A 0-dimensional point of unit volume and zero surface somehow appeared in this void. This first point is now called primordial Big Bang singularity. An existence of the first point implied countably infinite number of other labelled points forming various relations among each other. And thus the void expanded into real and imaginary dimensionalities.

The results of this study could perhaps be applied in molecular physics. There are countably infinitely many spherical harmonics but nature uses only the first four as subshells of s, p, d, and f electron shells that can hold 2, 6, 10, and 14 electrons respectively. Further subshells are not populated in ground states of all the observed elements. The first element that would require a g subshell (18 electrons) would have an atomic number of 121, while the heaviest element synthesized is Oganesson, with an atomic number of 118 and a half-life of about 1/1000 of a second. Perhaps this is linked with properties of the unit radius *n*-balls in negative dimensions as illustrated in Fig. 2. The “flattening” occurring between dimensions -14 and -2 is intriguing. Dimensions -2, -6, -10, and -14 are bounded from both sides, with -14, that would represent the f subshell, already at the onset of divergence. In nature, the f subshell occurs essentially only in lanthanides and actinides.

Acknowledgments

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