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Fractional Taylor Expansions and Derivative Regularization

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Abstract. Power series expansions are useful in approximation theory and mathematical physics. The manuscript presents several types of fractional Taylor expansions of sufficiently smooth functions. This is achieved by employing an incremental regularization procedure to the computation of the derivative. The series are constructed algorithmically, which allows for its implementation in computer algebra systems.

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1. Introduction

Power series expansions are useful in approximation theory. Applications of the fractional power series originate in the works Puiseux and Riemann. The latter demonstrated infinite series using fractional Riemann-Liouville integral. Further studies are due to Odibat and Shawagfeh [15] and Watanabe [23]. Odibat and Shawagfeh derived fractional Taylor series development using repeated application of Caputo's fractional derivative [15]. Lately, the question has gain traction with the work Liu et al. who used Kolwankar-Gangal local fractional derivatives [14]. Examples of fractional Taylor series are the branch-point series of inverses of trigonometric functions.

A fractional Taylor expansion was investigated in the scope of local fractional derivatives in the sense of Kolwankar and Gangal [13]. The existence of a local fractional Taylor is equivalent to existence of one-sided *fractional velocity* of the primitive function [1, 20, 21]. This fact can be used to sequentially extract coefficients of higher orders. The objective of the present work is to demonstrate how fractional velocities and the related scale velocity differential operators can be used to reconstruct the fractional power series of suitable functions.

The approach demonstrated in the present paper differs radically in the sense that a fractional Taylor series is computed only by a limiting operation. Presented approach can be used to characterize and approximate functions about points for which integer-order derivatives diverge. The work has been presented in preliminary form at the ICMMAS 2017 conference, Saint Petersburg, July 24 – 28, 2017.

An advantage of the present approach is the use of derivations, which makes algorithmic implementations in computer algebra systems much easier. Furthermore, the theory which is developed is rather general which is advantageous for unanticipated applications. An interesting result is the external exponent Th. 3,

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which can be used to develop the branch point series of the inverse trigonometric functions and the Lambert W function.

It should be noted that difference quotients of functions of fractional order could be traced back to du Bois-Reymond [8] and Faber [9] in their studies of the point-wise differentiability of functions. While these initial developments followed from purely mathematical interest, later works were inspired from physical questions [6]. Cherbit [6] and later on Ben Adda and Cresson [1] introduced the notion of *fractional velocity* as the limit of the fractional difference quotient. Subsequent results can be traced to [2] and [3], which are surveyed in [21].

The manuscript is structured as follows. Section 3 introduces the notion of F-analytic functions [21]. Section 4 introduces fractional velocities. Section 5 introduces the derivative regularization procedure. Section 6 discusses two types of applications: Itô-Taylor expansions of compound functions and the general algorithm for computation of power series expansion.

2. General definitions and notational conventions

The word *function* denotes the mapping $f : \mathbb{R} \mapsto \mathbb{R}$ or in some cases $\mathbb{C} \mapsto \mathbb{C}$. The notation $f(x)$ refers to the value of the function at the point x. The term *operator* denotes the mapping from one function to another. Conventionally, $\mathcal{C}^0$ denotes the class of continuous functions, considered in the neighborhood of a point. The symbol $\mathcal{C}^n$ – the class of n-times differentiable functions under the same restriction. Square brackets are used for the arguments of operators, while round brackets are used for the arguments of functions. $Dom[f]$ denotes the domain of definition of the function $f(x)$. BVC[I] will mean that the function f is continuous of bounded variation (BV) in the interval I. AC[I] will mean that the function f is absolutely continuous in the interval I.

Definition 1 (Asymptotic $\mathcal{O}$ notation)**.** *The notation $\mathcal{O}(x^\alpha)$ is interpreted as the convention that*

$$\lim_{x\to 0} \frac{\mathcal{O}(x^\alpha)}{x^\alpha} = 0$$

for $\alpha > 0$ is the limit of the expression. The notation $\mathcal{O}_x$ will be interpreted to indicate a Cauchy-null sequence in expression x.

Definition 2. *Let the parametrized difference operators acting on a function $f(x)$ be defined in the following way*

$$\Delta^+_\epsilon[f](x) := f(x+\epsilon) - f(x) \,,$$
$$\Delta^-_\epsilon[f](x) := f(x) - f(x-\epsilon) \,,$$

where $\epsilon > 0$. The first one we refer to as forward difference operator, the second one we refer to as backward difference operator.

3. F-analytic functions as generalizations of the Hölder functions

As a reminder the reader is recalled with the usual definition of Hölder functions.

Definition 3 (Hölder class of order β)**.** *We say that f is of (point-wise) Hölder class $\mathbb{H}^\beta\ (x)$ if for a given x there exist two positive constants $C, \delta \in \mathbb{R}$ that any $y \in Dom[f]$, such that $|x-y| \le \delta$ fulfills the inequality $|f(x) - f(y)| \le C|x-y|^\beta$, where $|\cdot|$ denotes the norm of the argument.*

For simplicity of presentation the reference to the x-variable will be omitted if clear from the context of the statement.

Remark 1. *It will be further assumed that $\beta \leq 1$ unless otherwise stated. The above definition contrasts with the local Hölder exponent where both points x, y are left to vary within an interval around the point x_0.*

Further generalization of this concept is given by introducing the concept of **F-analytic** functions. The definition used in the present paper is based on Oldham and Spanier [16], however, there the fractional exponents were considered to be only rational-valued for simplicity of the presented arguments.

Definition 4 (F-analytic function, [21])**.** *Consider the countable ordered set $\mathbb{E}^{\pm} = \{\alpha_1 < \alpha_2 < \ldots\}$ of positive real constants α. Then F-analytic is a mapping $f : \mathbb{R} \mapsto \mathbb{C}$ which is defined by the convergent (fractional) power series*

$$f(x) := c_0 + \sum_{\alpha_i \in \mathbb{E}^{\pm}} c_i \left(x \pm b_i\right)^{\alpha_i}$$

for some sets of real constants $\{b_i\}_{i=0}^{\infty}$ and $\{c_i\}_{i=0}^{\infty}$. The set $\mathbb{E}^{\pm}$ will be denoted further as the Hölder spectrum of f (i.e. $\mathbb{E}^{\pm}{}_f$).

In the present treatment f can be restricted to $f : \mathbb{R} \mapsto \mathbb{R}$.

So-defined, F-analytic functions present an obvious generalization of the real analytic functions $\mathcal{C}^{\infty}$. At first glance, such a definition can seem contrived but there are important commonly-used special functions that exhibit fractional character. For example the Bessel J_ν function is defined as the infinite power series:

$$J_\nu(x) = \left(\frac{x}{2}\right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu + k + 1)\Gamma(k + 1)} \left(\frac{x}{2}\right)^{2k}$$

so that in this case $\mathbb{E}^{\pm} = \{2k + \nu\}_{k=0}^{\infty}$. Further examples of this kind are the generalized hypergeometric functions of fractional argument, which cover most of the special functions of the mathematical physics [11]. Along these lines, the definition of the Hölder class can be also extended to mixed orders $n + \alpha > 1$ ($\alpha > 0$). In such a case, the Hölder class $\mathbb{H}^{n+\alpha}$ (x) designates the class of functions for which the inequality

$$|f(x) - f(y) - P_n(x - y)| \leq C|x - y|^{n+\alpha} \ ,$$

holds in the interval $[x, y]$, $|x - y| \leq \delta$. $P_n(.)$ designates a real-valued polynomial of degree $n \in \mathbb{N}$ of the form $P_n(z) = \sum_{k=1}^{n} a_k z^k$, where $P_0(z) = 0$ and $\alpha \in (0, 1)$.

From this definition it can be seen that the usual extension of the Hölder class definition towards non-integer orders larger that 1 is a specialization of the F-analytic function definition.

In the subsequent sections of the paper, the class of F-analytic functions will be characterized in terms of regularized fractional velocities and derivatives and will incorporate cases where an F-analytic function is not $\mathcal{C}^{\infty}$. The main applied result of the paper will be to exhibit an algorithm for reconstruction of $\mathbb{E}$ and the corresponding coefficients of the expansion. On a second place, some composition cases for F-analytic functions will be considered.

4. Fractional variation and fractional velocity of functions

Fractional (fractal) variation operators have been introduced in a previous work [17] under the following notation:

Definition 5. *Let the Fractional Variation operators be defined as*

$$\upsilon_{\beta}^{\epsilon+}\left[f\right](x) := \frac{\Delta_{\epsilon}^{+}[f](x)}{\epsilon^{\beta}} = \frac{f(x+\epsilon)-f(x)}{\epsilon^{\beta}}, \tag{1}$$

$$\upsilon_{\beta}^{\epsilon-}\left[f\right](x) := \frac{\Delta_{\epsilon}^{-}[f](x)}{\epsilon^{\beta}} = \frac{f(x)-f(x-\epsilon)}{\epsilon^{\beta}} \tag{2}$$

where $\epsilon > 0$ and $0 < \beta \leq 1$ are real parameters and $f(x)$ is a function.

Definition 6 (Fractional velocities of order β)**.** *Define the forward (resp. backward) fractional velocity of order $\beta \leq 1$ as the limit*

$$\upsilon_{\pm}^{\beta} f\left(x\right) := \lim_{\epsilon \to 0} \upsilon_{\beta}^{\epsilon \pm}\left[f\right](x) . \tag{3}$$

A function for which at least one of the velocities $\upsilon_{\pm}^{\beta} f\left(x\right)$ exists finitely will be called β-differentiable at the point x. This can be denoted also by $\mathcal{C}^{\beta}$.

The set of points where the fractional velocity exists finitely and does not vanish will be denoted as the ***set of change****:*

$$\chi_{\pm}^{\beta}(f) := \left\{ x : \upsilon_{\pm}^{\beta} f\left(x\right) \neq 0 \right\}.$$

The most important property of fractional for the results of the present paper is presented below.

Proposition 1 (Fractional Taylor-Lagrange Property)**.** *Suppose that $x \in \chi^{\beta}$. Then*

$$f(x \pm \epsilon) = f(x) \pm \upsilon_{\pm}^{\beta} f\left(x\right) \epsilon^{\beta} + \mathcal{O}\left(\epsilon^{\beta}\right) \tag{4}$$

Conversely, if Eq. 4 is valid then either of $\upsilon_{\pm}^{\beta} f\left(x\right)$ exists finitely and $x \in \chi^{\beta}$.

The proof is given in [20]. The sign $\pm$ refers to alternative cases and not to logical *and*.

Non-vanishing values of the fractional velocity lead to fractional Taylor series approximations of the functions. Together with the next property, this can be used for extraction of higher-order terms in the fractional Taylor series of the function.

We can make an observation, which will be used throughout the paper: For a β-differentiable function f at $x \in \chi^{\beta}$

$$\frac{f(x \pm \epsilon) - f(x) \mp \upsilon_{\pm}^{\beta} f\left(x\right) \epsilon^{\beta}}{\epsilon^{\beta}} = \mathcal{O}_x$$

This follows immediately from the Fractional Taylor-Lagrange property.

Definition 7. *Denote by MAC[I] the class of locally monotone and absolutely continuous functions in the compact interval I.*

The definition is introduced to exclude the class of singular functions and also the functions which oscillate fast in a given intervals (i.e. the nowhere monotone functions). The term singular function can be traced to Lebesgue and denotes those non-constant continuous functions of bounded variation $f : I \to \mathbb{R}$ such that $f' = 0$ almost-everywhere on the interval I of definition. Under this restriction a simple calculation yields the next result.

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Theorem 1. *Suppose that* $f \in MAC[x, x \pm \epsilon]$. *Then*

$$\upsilon_{\pm}^{\beta} f(x) = \lim_{\epsilon \to 0} \frac{\epsilon^{1-\beta}}{\beta} f'(x \pm \epsilon)$$

if the right-hand side one-sided limit exists.

Proof. Denote by $I = [x, x+\epsilon]$ and assume that ϵ is arbitrary. Denote by $D_I := \{f'(x) = \pm\infty\} \bigcap I$ the discontinuity set of the derivative and suppose that the set is non-empty. Since f is MAC in I then f' exists a.e. in I by Lebesgue's differentiation Theorem and it is continuous wherever it exists. Then by Th. 5 D_I can consist only of isolated points at the boundaries of I. So let us suppose that x is such a point. Hence, we can take $x + \epsilon$ to be a generic point in the open interval $D_I^c = (x, x+\epsilon)$, such that $f'(x+\epsilon)$ is bounded. Under such conditions, application of l'Hôpital's rule [12] leads to

$$\lim_{\epsilon \to 0} \upsilon_{\beta}^{\epsilon+} [f](x) = \lim_{\epsilon \to 0} \frac{\Delta_{\epsilon}^{+} f(x)}{\epsilon^{\beta}} = \lim_{\epsilon \to 0} \frac{f'(x+\epsilon)}{\beta \epsilon^{\beta-1}} = \lim_{\epsilon \to 0} \frac{\epsilon^{1-\beta}}{\beta} f'(x+\epsilon),$$

provided the last limit exists.

On the other hand, by the Mean Value Theorem [12] $\Delta_{\epsilon}^{+} f(x) = \epsilon f'(\xi)$ for some $\xi \in D_I^c$. Therefore,

$$\upsilon_{\beta}^{\epsilon+} [f](x) = \epsilon^{1-\beta} f'(\xi)$$

and

$$L_{\epsilon} = \left| \frac{\epsilon^{1-\beta}}{\beta} f'(x+\epsilon) - \epsilon^{1-\beta} f'(\xi) \right| = \epsilon^{1-\beta} \left| \frac{f'(x+\epsilon)}{\beta} - f'(\xi) \right| \leq \epsilon^{1-\beta} K$$

for some $K > 0$. Therefore, L_{ϵ} is also continuous in ϵ and moreover $L_{\epsilon} = \mathcal{O}\left(\epsilon^{1-\beta}\right)$. Therefore, for a generic point x both limits vanish as $\epsilon \to 0$.

The backward case can be proven in an identical manner by reflexion $\epsilon \mapsto -\epsilon$. □

Remark 2. *A brief remark about the applicability of L'Hoṕotal's rule is in order. The simplest statement of the rule works under the assumption that functions in the numerator and denominator are differentiable everywhere around the point of interest. This restriction was lifted in* [12, Prop. 3] *who demonstrated that for a finite L*

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L$$

provided that $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ *or* $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$. *The result was derived from a generalized, inequality, version of the Mean Value Theorem proven in the same article. The result is proven under the assumption that g is increasing locally, that is* g' *does not change sign in the neighborhood of the point* a. *On the other hand, L'Hoṕotal's rule can fail to provide the correct value of the limit for functions for which the zero set accumulates towards a* [5]. *Therefore, one needs the hypothesis of local monotonicity in Th. 1.*

4.1. **Point-wise Hölder exponents.**

Definition 8. *Consider the continuous function* $f(x)$. *Define the right point-wise Hölder exponent* β_x *as*

$$\beta_x := \limsup_{\epsilon \to 0} \frac{\log |\Delta_{\epsilon}^{+} f(x)|}{\log \epsilon}$$

where as usual $\epsilon > 0$.

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Proposition 2. *Suppose that f is β-differentiable at x and $f \in MAC[x, x+\epsilon]$. Then the value of the point-wise Hölder exponent β is given by the limit*

$$\beta = \lim_{\epsilon \to 0} \frac{\epsilon f'(x \pm \epsilon)}{\Delta^{\pm}_{\epsilon} f(x)} \tag{5}$$

Proof. We observe that $\beta \leq 1$ by hypothesis. Let $I = [x, x+\epsilon]$ and suppose that $x \in \chi^{\beta}$. By hypothesis f is differentiable in the open interval I^{o}. We evaluate the limit

$$L_x = \lim_{\epsilon \to 0} \frac{\log |\Delta^{+}_{\epsilon} f(x)|}{\log \epsilon}.$$

This has the form ∞/∞, which is suggestive for a potential application of L'Hôpital's rule. On the other hand, let us evaluate the limit

$$M_x = \lim_{\epsilon \to 0} \frac{\epsilon f'(x+\epsilon)}{\Delta^{+}_{\epsilon} f(x)}$$

According to Th. 1

$$M_x = \beta \lim_{\epsilon \to 0} \frac{\epsilon^{1-\beta}}{\beta} \frac{f'(x+\epsilon)}{\frac{\Delta^{+}_{\epsilon} f(x)}{\epsilon^{\beta}}} = \beta$$

Therefore, $L_x = M_x$. □

Remark 3. *A left Holder exponent can be defined in analogous way and proven by reflection arguments. Since equality of forward and backward fractional velocities is not required one may observe a situation where $\beta_{+} \neq \beta_{-}$ for some set of points.*

5. Derivative regularization

5.1. **Fractional order regularization.** We extend the fractional velocity notation for a continuous function f as $\upsilon^{0}_{+} f(x) \equiv 0$ since for a continuous function f $\lim_{\epsilon \to 0} \Delta^{+}_{\epsilon}[f](x)] = \lim_{\epsilon \to 0} \Delta^{-}_{\epsilon}[f](x)] = 0$.

The composition of variation definition proceeds in two steps:

Definition 9 (Composition of variations)**.** *Consider a function f continuous about the point x. Define the composition of fractal variations as*

$$\upsilon^{\alpha \circ \beta}_{\pm} f(x) := \lim_{\epsilon \to 0} \frac{\Delta^{\pm}_{\epsilon} f(x) - \upsilon^{\beta}_{\pm} f(x)\, \epsilon^{\beta}}{\epsilon^{\alpha+\beta}}, \tag{6}$$

where $0 < \alpha, \beta \leq 1$.

By induction, for a set of multiple exponents $0 < \alpha_1 < \alpha_2 < \ldots \leq 1$ define the composition recursively as:

$$\upsilon^{\alpha_1 \circ \ldots \circ \alpha_n}_{\pm} f(x) := \lim_{\epsilon \to 0} \frac{\Delta^{\pm}_{\epsilon}[f](x) - \sum\limits_{\{\alpha_k\}} \upsilon^{\prod_i^k \circ \alpha_i}_{\pm} f(x)\, \epsilon^{\sum_i^k \alpha_i}}{\epsilon^{\sum_i^n \alpha_i}} \tag{7}$$

So-composed fractional velocities will be referred to as ***regularized*** *velocities, respectively regularized derivatives for $\alpha_k = 1$.*

The basis for further computations will be the equation

$$\upsilon^{\Delta \alpha_1 \circ \alpha_1}_{\pm} f(x) = \lim_{\epsilon \to 0} \frac{\Delta^{\pm}_{\epsilon}[f](x) - \upsilon^{\alpha_1}_{\pm} f(x)\, \epsilon^{\alpha_1}}{\epsilon^{\alpha_2}}, \quad \Delta\alpha_1 = \alpha_2 - \alpha_1 \tag{8}$$

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For a β-differentiable function we can use the composition of variations to regularize the first derivative with respect to the fractional velocity. This process will be used to extract the values of the coefficients in the fractional Taylor series.

Definition 10. *Suppose that f is β-differentiable function in $I = [x, x+\epsilon]$ and $x \in \chi^\beta$. Denote the regularized derivative by*

$$\frac{\bar{d}^{\,\beta\pm}}{dx} f(x) := \upsilon_{\pm}^{1-\beta\circ\beta} f(x) . \tag{9}$$

We will require as before that the forward and backward regularized derivatives be equal for a uniformly continuous function.

We will use further also a shortened notation $\bar{d}\, f(x)$ when the value of β is fixed.

5.2. **Mixed order regularization.** The regularization procedure can be extended also towards mixed orders in the following way:

Definition 11. *Let us define formal fractional Taylor polynomials for an increasing sequence α_i as:*

$$T^{+}_{\alpha,n}(x,\epsilon) = f(x) + \sum_{\alpha_i} c_i \epsilon^{\alpha_i} + \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\epsilon^k ,$$

$$T^{-}_{\alpha,n}(x,\epsilon) = f(x) - \sum_{\alpha_i} c_i^{\star} \epsilon^{\alpha_i} + \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}(-\epsilon)^k ,$$

where α denotes the multi-index and c_i are arbitrary constants. Then α-regularized derivatives are defined as

$$\frac{\bar{d}^{\,n+}}{dx^n} f(x) = (n+1)! \lim_{\epsilon\to 0} \frac{f(x+\epsilon) - T_{\alpha,n}(x,\epsilon)}{\epsilon^{n+\beta}} ,$$

$$\frac{\bar{d}^{\,n-}}{dx^n} f(x) = (-1)^n (n+1)! \lim_{\epsilon\to 0} \frac{T^{-}_{\alpha,n}(x,\epsilon) - f(x-\epsilon)}{\epsilon^{n+\beta}} ,$$

where $\beta = \sup \alpha_i$. We will require as usually that the forward and backward regularized derivatives be equal a uniformly continuous function.

Proposition 3. *According to Prop. 1 we have $c_i = \upsilon_{+}^{\alpha_i} f(x)$ and $c_i^{\star} = \upsilon_{-}^{\alpha_i} f(x)$.*

The proof follows by induction considering that the sequence $\{\alpha_i\}$ is increasing. Then according to Prop. 1 a new regularization term is added for every α_i.

6. Applications

6.1. **Compound differential (Taylor – Itô) rules.** The differentiation rule for compositions of functions can be derived using the regularization of the derivatives. The statement has been given in [19] without proof. The differentiation rule for compositions of functions provides a result formally analogous to Itô's Lemma.

Definition 12 (Fractal co-variation)**.** *Define the fractal co-variation as the limit*

$$[w^q]^{\pm}(x) := \lim_{\epsilon\to 0} \left(\upsilon_{1/q}^{\epsilon\pm} [w](x) \right)^q .$$

The notation borrows from the one used in stochastic calculus due to the analogy of the results.

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Theorem 2 (Generalized Itô –Taylor expansion)**.** *Let $q \in \mathbb{N}$ and $q \geq 2$. Suppose that $f(x,w) \in \mathcal{C} \times \mathcal{C}^q$ is a composition with a function w, which is $1/q$-differentiable at x, then*

$$\frac{\text{đ}^{\pm}}{dx} f = \frac{\partial f}{\partial x} + \frac{\text{đ}^{\pm}}{dx} w \cdot \frac{\partial f}{\partial w} + \frac{(\pm 1)^{q-1}}{\Gamma(q+1)} [w^q]^{\pm} \cdot \frac{\partial^q f}{\partial w^q} \tag{10}$$

Proof. The theorem will be proven for $q = 2$. The other cases follow by the same arguments taking note of the polarity of the differences.

We will prove first the forward case.

Forward case: By second order Taylor's expansion we get

$$f(w(x+\epsilon), x+\epsilon) = f(w,t) + \frac{\partial f}{\partial x}\epsilon + \frac{\partial f}{\partial w}\Delta_\epsilon^+ w(x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\epsilon^2 + \frac{\partial^2 f}{\partial x \partial w}(x)\Delta_\epsilon^+ w(x)\,\epsilon + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}\left(\Delta_\epsilon^+ w(x)\right)^2 + \mathcal{O}\left(\epsilon^3\right)$$

It follows that two cases have to be considered:

Case 1 ($x \notin \chi^\beta$)**.** *when $\upsilon_+^{1/2} w(x) = 0$ and the ordinary derivative is well-defined:*

$$\frac{\Delta_\epsilon^+ f(x)}{\epsilon} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial w}\frac{\Delta_\epsilon^+ w(x)}{\epsilon} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\epsilon + \frac{\partial^2 f}{\partial x \partial w}(x)\Delta_\epsilon^+ w(x) + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}[w,w]_\epsilon^+ + \mathcal{O}\left(\epsilon^2\right)$$

Taking the limit gives the expected result

$$\frac{d f}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial w}\frac{dw}{dx}.$$

Case 2 ($x \in \chi^\beta$)**.** *when $\upsilon_+^{1/2} w(x) = K \neq 0$ and the ordinary derivative is not defined: In this case $w(x+\epsilon) = w(x) + K\sqrt{\epsilon} + \mathcal{O}(\sqrt{\epsilon})$. We substitute partially in the Taylor expansion:*

$$f(w(x+\epsilon), x+\epsilon) - f(w,x) = \frac{\partial f}{\partial x}\epsilon + \frac{\partial f}{\partial w}\left(K\sqrt{\epsilon} + \mathcal{O}\left(\sqrt{\epsilon}\right)\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\epsilon^2 + \frac{\partial^2 f}{\partial x \partial w}(x)\Delta_\epsilon^+ w(x)\,\epsilon + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}\left(\Delta_\epsilon^+ w(x)\right)^2 + \mathcal{O}\left(\epsilon^3\right)$$

Let $\upsilon_{1/2}^{\epsilon\pm}[w](x)^2 = [w,w]_\epsilon^\pm$ for notational convenience. Then after rearrangement we get

$$\frac{f(w(x+\epsilon), x+\epsilon) - f(w,x) - \frac{\partial f}{\partial w}\left(K\sqrt{\epsilon} + \mathcal{O}(\sqrt{\epsilon})\right)}{\epsilon} = \frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\epsilon + \frac{\partial^2 f}{\partial x \partial w}(x)\Delta_\epsilon^+ w(x) + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}[w,w]_\epsilon^+ + \mathcal{O}\left(\epsilon^3\right)$$

But the LHS is the forward regularized derivative of $f(w,x)$. Therefore, we obtain

$$\frac{\text{đ}^+ f}{dx} = \frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}[w,w]^+ \tag{11}$$

Therefore, we can rewrite the expansion using the uniform notation

$$\frac{\text{đ}^+ f}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial w}\frac{\text{đ}\, w}{dx} + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}[w,w]^+ \tag{12}$$

Backward case: By second order Taylor's expansion we get

$$f(w(x-\epsilon), x-\epsilon) = f(w,x) - \frac{\partial f}{\partial x}\,\epsilon - \frac{\partial f}{\partial w}\Delta_\epsilon^- w\,(x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,\epsilon^2 + \frac{\partial^2 f}{\partial x\,\partial w}(x)\Delta_\epsilon^- w\,(x)\,\epsilon + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}\,\left(\Delta_\epsilon^+ w\,(x)\right)^2 - \mathcal{O}(\epsilon^3)$$

The proof technique is completely analogous.

Therefore, we can rewrite the expansion using the uniform notation

$$\frac{đ^- f}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial w}\,\frac{đ\,w}{dx} - \frac{1}{2}\frac{\partial^2 f}{\partial w^2}\,[w,w]^- \tag{13}$$

For $q > 2$ the proof proceeds in the same way by Taylor expansion of the second argument up to order q.

In the expansion for the backward case above we notice that the $\mathcal{O}(\epsilon^3)$ term has a negative sign, which by reversing polarity in the finite difference $\Delta_\epsilon^- f\,(x)$ transforms into positive. Therefore, for $q = 3$ the sign in front of the fractal co-variation will be positive. The same reasoning holds for all odd q, while all even q will have negative polarities. □

Remark 4. *The Itô-Taylor expansion can be reformulated in a homogeneous form considering that formally* $[x,x] = 0$ *and* $[w,x] = 0$ *because* $f(x) = x \in \mathcal{C}^1$.

$$\frac{đ^+}{dx} f(x,w) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{đ^+ w}{dx} + \frac{1}{2}\frac{\partial^2 f}{\partial w^2}[w,w]^+$$

$$\frac{đ^-}{dx} f(t,w) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{đ^- w}{dx} - \frac{1}{2}\frac{\partial^2 f}{\partial w^2}[w,w]^-$$

6.2. **Extremal series.** In another application let us consider a regular function $F(y) = x$, having an extremum at $y = a$. Let us compute the fractional expansion of its inverse $f(x)$. For a regular point, the inverse function theorem holds. On the other hand, at the extremum $F'(y) = 0$. Then

$$\Delta_\epsilon^+ F\,(y) = F'_y dy + \frac{1}{2} F''_y dy^2 + o\left(dy^2\right) = \epsilon \to \frac{1}{2} F''_y dy^2 + o\left(dy^2\right) = \epsilon.$$

Therefore,

$$\lim_{\epsilon\to 0} \frac{1}{2} F''_y \frac{\Delta_\epsilon^+ f\,(a)^2}{\epsilon} = 1 \to [f,f]^+ = \frac{2}{F''_y(a)}$$

Therefore, it can be claimed that

Theorem 3 (Extremal exponent)**.** *Suppose that* $F(y)$ *has a local inverse* $y = f(x)$ *in* $I = [a, a\pm\epsilon]$ *and is continuous there. Let* $F(a)$ *be an extremum. Then* $f(x)$ *has a point-wise Hölder exponent* $1/2$ *and*

$$\upsilon_\pm^{1/2} f\,(a) = \pm\sqrt{\frac{2}{|F''_y(a)|}}, \quad [f,f]^+ = \frac{2}{F''_y(a)}$$

where the sign agrees with the sign of the derivative $f'(x)$ *about* $a^\pm$*, respectively.*

Example 1. *A non-trivial example of the above theorem is the Lambert W function, which is defined as the solution of the equation* [7]

$$e^{W_\pm(x)} W_\pm(x) = x, \quad y = W_\pm(x)$$

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Therefore, for the derivative we have

$$\frac{dy}{dx} = \frac{e^{-y}}{1+y}, \quad y \neq -1 \tag{14}$$

by the Inverse Function Theorem. However, for $x = -e^{-1}$ $W(x) = -1$ *and the function can be expanded in branch-point series. By the proof of Th. 3*

$$e^y \left(dy(1+y) + \frac{y+2}{2} dy^2 \right) + \mathcal{O}\left(dy^2\right) = dx \to \frac{e^y}{2} dy^2 + \mathcal{O}\left(dy^2\right) = dx, \quad y = -1$$

or $F(y) = ye^y$, $F''_y = (y+2)e^y$. *Therefore,* $[y, y] = 2e^{-y} = 2e$ *and*

$$W_{\pm}(x) = -1 \pm \sqrt{2e(x+e^{-1})} + \mathcal{O}\left(\sqrt{x+e^{-1}}\right)$$

around the branch point. Calculation of the subsequent coefficients is a substantially more complicated exercise because the derivatives of $W(x)$ *are of mixed exponential-rational form.*

The regularized derivative at the branch point can be calculated from the limit

$$\frac{d\!\!\!/^{\,+}}{dx} W_{\pm}(-e^{-1}) = \lim_{y \to -1^+} \frac{-\sqrt{2ye^{y+1}+2} + y + 1}{ye^y + e^{-1}} = -\frac{2e}{3},$$

so that

$$W_{\pm}(x) = -1 \pm \sqrt{2e(x+e^{-1})} - \frac{2e}{3}(x+e^{-1}) + \mathcal{O}\left((x+e^{-1})\right)$$

around $x = -e^{-1}$. *A plot is presented in Fig. 1.*

The last theorem can be generalized for the case of vanishing derivatives up to order k.

Corollary 1. *Suppose that* $F(y)$ *has a local inverse* $y = f(x)$ *in* $I = [a, a \pm \epsilon]$ *and the first* $k-1$ *partial derivatives* $F_y^{(k-1)}$ *vanish at* a.

$$\upsilon_+^{1/k} f(a) = \pm \sqrt[k]{\frac{k!}{|F_y^k(a)|}}, \quad [f^k]^+ = \frac{k!}{F_y^{(k)}(a)} \tag{15}$$

where the sign agrees with the sign of the derivative $f'(x)$ *about* a.

Proof. Suppose that $F_y^{(j)} = 0, \quad j \leq k$. Then in a similar way

$$\Delta_\epsilon^+ F(y) = F'_y dy + \ldots + \frac{1}{k!} F_y^{(k)} dy^k + \mathcal{O}\left(dy^k\right) = \epsilon \to \frac{1}{k!} F_y^{(k)} dy^k + \mathcal{O}\left(dy^k\right) = \epsilon.$$

Therefore,

$$\lim_{\epsilon \to 0} \frac{1}{k!} F_y^{(k)} \frac{\Delta_\epsilon^+ f(a)^k}{\epsilon} = 1 \to [f^k]^+(a) = \frac{k!}{F_y^{(k)}(a)}$$

and

$$\upsilon_+^{1/k} f(a) = \pm \sqrt[k]{\frac{k!}{|F_y^k(a)|}}, \quad [f^k]^+ = \frac{k!}{F_y^{(k)}(a)}$$

where the sign agrees with the sign of the derivative $f'(x)$ about a^+. □

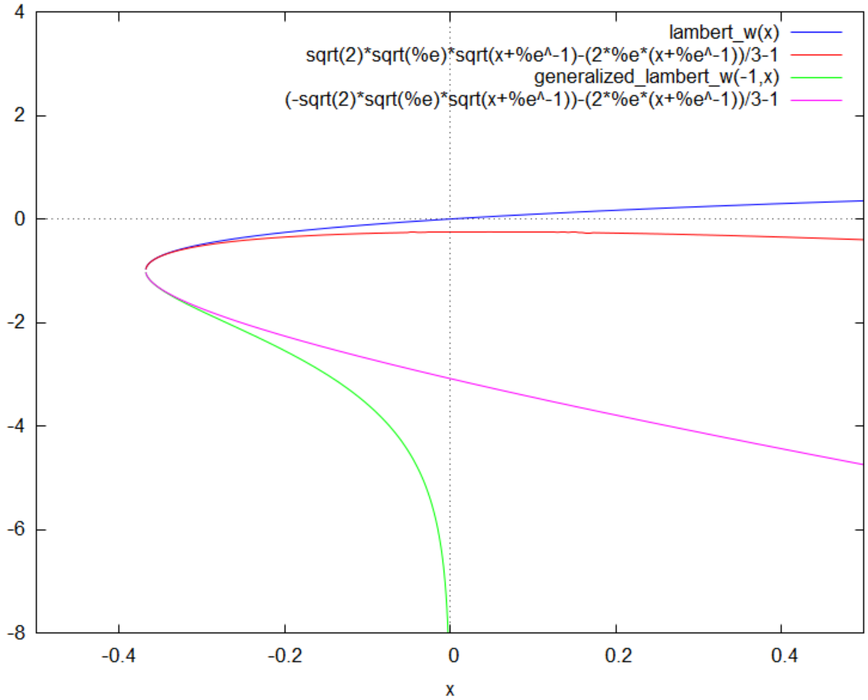

FIGURE 1. Lambert W function and its branch point expansion.

6.3. **Fractional power series.** In the following section we demonstrate a method for explicit computation of the coefficients of the higher order monomial of an F-analytic function. One can define two types of **scale-dependent differential operators** acting on F-analytic functions.

Definition 13. *Define the left (resp. right) scale velocity operators* [18]*:*

$$\mathcal{S}_{\beta}^{\epsilon\pm}\left[f\right](x) := \frac{1}{1-\{\beta\}}\epsilon^{\beta}\frac{\partial}{\partial\epsilon}f(x\pm\epsilon), \quad \beta \leq 1 \tag{16}$$

and their composition rules as

$$\mathcal{S}_{\alpha}^{\epsilon\pm}\circ\mathcal{S}_{\beta}^{\epsilon\pm}f := \frac{\epsilon^{\alpha}}{1-\{\alpha\}}\frac{\partial}{\partial\epsilon}\left(\frac{\epsilon^{\beta}}{1-\{\beta\}}\frac{\partial}{\partial\epsilon}f(x\pm\epsilon)\right) \tag{17}$$

Observe that for a MAC function f:

$$\lim_{\epsilon\to 0}\mathcal{S}_{1-\beta}^{\epsilon\pm}\left[f\right](x) = \upsilon_{\pm}^{\beta}f\left(x\right)$$

The proof follows directly from application of Th. 1. The form of the scale velocity operators facilitates the algebraical manipulations since the image of the function does not vanish. Therefore, one does not need to specify the set of change χ^{β}. Furthermore, logarithmic singularities in the derivative can also be accounted for since

$$\lim_{\epsilon\to 0}\mathcal{S}_{1}^{\epsilon\pm}\left[f\right](x) = \lim_{\epsilon\to 0}\epsilon\, f'(x+\epsilon) = \frac{\partial}{\partial(\log\epsilon)}f(x+\epsilon)\Big|_{\epsilon=0}$$

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Theorem 4 (Fractional Series Approximation Theorem)**.** *Suppose that f is F-analytic in the neighborhood of x, that is in the interval $I = [x, x+\delta)$ and choose $\epsilon \leq \delta$.*

Suppose that the Hölder exponent spectrum $\mathbb{E}$ is non-lacunar, that is $\sup_i \alpha_{i+1} - \alpha_i \leq 1$. Let the fractional Taylor series of f be given as

$$f(x+\epsilon) = \sum_{n=0}^{\infty} c_n \epsilon^{\alpha_n}, \quad c_0 = f(x)$$

Then

$$c_n = \lim_{\epsilon \to 0} \left(\prod_{k=1}^{n-1} \circ \frac{\Delta_{n-k}(\alpha)}{\alpha_n - \alpha_k} \mathcal{S}^{\epsilon}_{1-\Delta_{n-k}\alpha} \right) f, \quad \Delta_k(\alpha) = \alpha_k - \alpha_{k-1}$$

where the product is understood as composition and the limit is taken in the end of the computation.

Proof. For the technique of the proof we denote the equivalence in limit as $\cong$. Denote the fractional Taylor polynomial by

$$T_{k,n}(\epsilon) := \sum_{j \geq k}^{n} c_j \epsilon^{\alpha_j}$$

and assume that its coefficients c_j are indeterminate but its Hölder spectrum is fixed. We are set to compute the limit

$$L_N = \frac{f(x+\epsilon) - T_{0,n}(\epsilon)}{\epsilon^{\alpha}_{N}} \cong \mathcal{O}_\epsilon$$

By induction for $N = 1$ by L'Hôpital's rule

$$L_1 \cong \frac{f'(x+\epsilon) - c_1 \alpha_1 \epsilon^{\alpha_1 - 1}}{\alpha_1 \epsilon^{\alpha_1 - 1}} = \mathcal{S}^{\epsilon\pm}_{1-\alpha_1}[f](x) - c_1 = \mathcal{O}_\epsilon$$

Therefore, $c_1 = \lim_{\epsilon \to 0} \mathcal{S}^{\epsilon+}_{1-\alpha_1}[f](x)$. For $N > 1$ we observe that

$$L_N = \frac{f(x+\epsilon) - T_{k,N-1}(\epsilon) - c_N \epsilon^{\alpha_N}}{\epsilon^{\alpha_N}} = \frac{f(x+\epsilon) - T_{k,N-1}(\epsilon)}{\epsilon^{\alpha_N}} - c_N \cong \mathcal{O}_\epsilon.$$

By L'Hôpital's rule

$$\frac{f'(x+\epsilon) - T'_{1,N-1}(\epsilon)}{\alpha_N \epsilon^{\alpha_N - 1}} \cong c_N.$$

Therefore the expression can be rearranged as:

$$c_N \cong \frac{\alpha_1}{\alpha_N} \frac{\mathcal{S}^{\epsilon}_{1-\alpha_1} f(x) - c_1 - T_{2,N-1}(\epsilon)}{\epsilon^{\alpha_N - \alpha_1}}, \quad T_{2,N-1}(\epsilon) = \sum_{k=2}^{N-1} c^{(2)}_k \epsilon^{\alpha_k - \alpha_1}, \quad c^{(2)}_k = \frac{c_k \alpha_k}{\alpha_1}$$

Denote for convenience $a_1 = \alpha_1/\alpha_N$. We can apply another L'Hôpital step so that

$$c_N \cong a_1 \frac{\left(\mathcal{S}^{\epsilon}_{1-\alpha_1} f(x)\right)' - c^{(2)}_2 (\alpha_2 - \alpha_1)\epsilon^{\alpha_2 - \alpha_1 - 1} - T_{3,N-1}(\epsilon)}{(\alpha_N - \alpha_1)\epsilon^{\alpha_N - \alpha_1 - 1}} =$$

$$a_2 \frac{\mathcal{S}^{\epsilon}_{1-\Delta\alpha_1} \circ \mathcal{S}^{\epsilon}_{1-\alpha_1} f(x) - T_{3,N-1}(\epsilon)}{\epsilon^{\alpha_N - \alpha_2}}, \quad T_{3,N-1}(\epsilon) = \sum_{k=3}^{N-1} c^{(3)}_k \epsilon^{\alpha_k - \alpha_1},$$

$$\Delta\alpha_1 = \alpha_2 - \alpha_1, \quad c^{(3)}_k = \frac{c^{(2)}_k (\alpha_k - \alpha_1)}{\Delta\alpha_1}, \quad a_2 = a_1 \frac{\Delta\alpha_1}{\alpha_N - \alpha_1}$$

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Therefore by reduction.

$$c_N \cong \left(\prod_{k=1}^{N-1} \circ \frac{\Delta_{N-k}(\alpha)}{\alpha_N - \alpha_{N-k}} \mathcal{S}^{\epsilon}_{1-\Delta_{N-k}\alpha} \right) f, \quad \Delta_k(\alpha) = \alpha_k - \alpha_{k-1}$$

and formally $\alpha_0 = 0$. Therefore, for any N the same result will follow by induction. □

We can further give some concrete implementations of the algorithm revealed by the Fractional Series Approximation Theorem.

First, we will consider the case when $\Delta_k \alpha = \alpha$. In this case, the following result holds:

Proposition 4. *Suppose that the fractional Taylor polynomials are of the form*

$$T^{\pm}_{\alpha,n}(x,\epsilon) = f(x) + \sum_{k=1}^{n} c_k (\pm 1)^k \epsilon^{\alpha k},$$

where α denotes the multi-index and c_i are arbitrary constants. Then the following expansion holds:

$$f(x+\epsilon) = f(x) + \sum_{k=1}^{n} c_k \epsilon^{\alpha k} + \mathcal{O}\left(\epsilon^{n\,\alpha}\right),$$

for

$$c_k = \frac{1}{k!\ \alpha^k} \lim_{\epsilon \to 0} \prod^{k} \circ \left(\epsilon^{1-\alpha} \frac{\partial}{\partial \epsilon} \right) f(x+\epsilon)$$

and

$$f(x-\epsilon) = f(x) + \sum_{k=1}^{n} c_k \epsilon^{\alpha k} + \mathcal{O}\left(\epsilon^{n\,\alpha}\right)$$

for

$$c_k = \frac{(-1)^k}{k!\ \alpha^k} \lim_{\epsilon \to 0} \prod^{k} \circ \left(\epsilon^{1-\alpha} \frac{\partial}{\partial \epsilon} \right) f(x-\epsilon).$$

Proof. We will establish the relationship to the growth of the function at $x \pm \epsilon$, respectively. The technique of the proof is similar to the previous case.

Obviously for $k = 1$ holds $c_1 = \upsilon^{\alpha}_{+} f(x)$. Then the second coefficient can be calculated as

$$\begin{aligned} \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x) - \upsilon^{\alpha}_{+} f(x)\, \epsilon^{\alpha}}{\epsilon^m} &= \lim_{\epsilon \to 0} \frac{T^{+}_{\alpha,n}(x,\epsilon) - f(x) - \upsilon^{\alpha}_{+} f(x)\, \epsilon^{\alpha}}{\epsilon^m} \\ &= \lim_{\epsilon \to 0} \frac{c_1 \epsilon^{2\,\alpha} + \mathcal{O}(\epsilon^{2\,\alpha})}{\epsilon^m} \\ &= \lim_{\epsilon \to 0} c_1 \epsilon^{2\alpha - m} + \lim_{\epsilon \to 0} \mathcal{O}(\epsilon^{2\,\alpha - m}) \end{aligned}$$

Therefore, in order for the RHS to be finite we must have $m = 2\alpha$. Then for the LHS we have

$$\lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x) - \upsilon^{\alpha}_{+} f(x)\, \epsilon^{\alpha}}{\epsilon^{2\,\alpha}} = \frac{1}{2\,\alpha} \lim_{\epsilon \to 0} \frac{f'(x+\epsilon) - \alpha\, \upsilon^{\alpha}_{+} f(x)\, \epsilon^{\alpha-1}}{\epsilon^{2\,\alpha-1}}.$$

The argument of last limit is then

$$\frac{\epsilon^{\alpha-1}}{\epsilon^{2\,\alpha-1}} \left(\epsilon^{1-\alpha} f'(x+\epsilon) - \upsilon^{\alpha}_{+} f(x) \right) = \frac{\epsilon^{1-\alpha} f'(x+\epsilon) - \upsilon^{\alpha}_{+} f(x)}{\epsilon^{\alpha}}$$

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The limit can be evaluated by application of L'Hôpital's rule and rationalization.

$$\lim_{\epsilon \to 0} \frac{\epsilon^{1-\alpha} f'(x+\epsilon) - \upsilon_{+}^{\alpha} f(x)}{\epsilon^{\alpha}} = \frac{1}{\alpha} \lim_{\epsilon \to 0} \epsilon^{1-\alpha} \frac{\partial}{\partial \epsilon} \left(\epsilon^{1-\alpha} f'(x+\epsilon) \right).$$

Therefore,

$$c_2 = \frac{1}{2\,\alpha^2} \lim_{\epsilon \to 0} \epsilon^{1-\alpha} \frac{\partial}{\partial \epsilon} \left(\epsilon^{1-\alpha} \frac{\partial}{\partial \epsilon} f(x+\epsilon) \right),$$

Therefore by induction

$$c_k = \frac{(\pm 1)^k}{k!\ \alpha^k} \lim_{\epsilon \to 0} \prod^{k} \circ \left(\epsilon^{1-\alpha} \frac{\partial}{\partial \epsilon} \right) f(x \pm \epsilon)$$

where the product has to be understood as composition.

□

Remark 5. *The same result can be established in a different way. Let us suppose that*

$$f = F \circ u, \quad u(x) = x^{\alpha}$$

Then

$$\left(\frac{d\,u(x)}{dx} \right)^{-1} \frac{d\,F}{dx} = \frac{\partial F}{\partial u}$$

and

$$\frac{\partial F}{\partial u} = \frac{x^{1-\alpha}}{\alpha} \frac{d\,F}{dx}$$

and we can recognize the derivative of $u \sim x^{\alpha}$ *evaluated at* $x = \epsilon$. *Similar arguments can be demonstrated to hold for all* $k \leq n$ *by induction. Therefore, in the general case we would have as expected expansion of the function* $f(x)$ *in Taylor series w.r.t* $u(x) = x^{\alpha}$ *followed by substitution. Therefore,*

$$f(x+\epsilon) = f(x) + \sum_{k=1}^{n} c_k \epsilon^{\alpha k} + \mathcal{O}\left(\epsilon^{n\,\alpha}\right)\ ,$$

for

$$c_k = \frac{1}{k!} \frac{\partial^{\,k}}{\partial u^k}\ F(u)$$

and

$$f(x-\epsilon) = f(x) + \sum_{k=1}^{n} c_k \epsilon^{\alpha k} + \mathcal{O}\left(\epsilon^{n\,\alpha}\right)\ ,$$

for

$$c_k = \frac{(-1)^k}{k!} \frac{\partial^{\,k}}{\partial u^k}\ F(u)\ .$$

We have established that in this case the function of interest is a composition of an analytic function and a power function that is

$$f = F \circ u, \quad u(x) = x^{\alpha}.$$

This was the particular case which was considered by Odibat and Shawagfeh [15].

Example 2. *Consider the function* $f(x) = \cos(x^{1/3})$. *We will develop the fractional Taylor expansion about* $x = 0$. *The first derivative of the function is*

$$f'(x) = -\frac{1}{3}\frac{\sin x^{\frac{1}{3}}}{x^{\frac{2}{3}}} \; ,$$

which is undefined for $x = 0$. *According to Th. 5 for* $\alpha = \frac{1}{3}$ *we would have*

$$c_1 = -\lim_{\epsilon \to 0} \sin \epsilon^{1/3} = 0$$

Continuing for other degrees

$$c_2 = -\frac{1}{2}\lim_{\epsilon \to 0} \cos \epsilon^{1/3} = -\frac{1}{2}$$

and so on. Therefore, in agreement with the previous proposition we would have

$$f(x) = 1 - \frac{1}{2}x^{\frac{2}{3}} + \frac{1}{24}x^{\frac{4}{3}} - \frac{1}{720}x^2 + \frac{1}{40320}x^{\frac{8}{3}} + o\left(x^{8/3}\right)$$

Proposition 5 (Mixed order Taylor expansion)**.** *Suppose that the fractional Taylor polynomials are of the form*

$$T^{+}_{\alpha,n}(x,\epsilon) = f(x) + \sum_{k=0}^{n} c_k \epsilon^{\alpha+k} \; ,$$

$$T^{-}_{\alpha,n}(x,\epsilon) = f(x) + \sum_{k=0}^{n} c_k (-1)^k \epsilon^{\alpha+k} \; ,$$

Then the following expansions hold:

$$f(x+\epsilon) = f(x) + \sum_{k=0}^{n} c_k \epsilon^{\alpha+k} + \mathcal{O}\left(\epsilon^{n+\alpha}\right) \; ,$$

where

$$c_k = \frac{1}{k!\,(k+\alpha)} \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial \epsilon}\right)^k \epsilon^{1-\alpha} f'(x+\epsilon)$$

and

$$f(x-\epsilon) = f(x) + \sum_{k=0}^{n} c_k \epsilon^{\alpha+k} + \mathcal{O}\left(\epsilon^{n+\alpha}\right) \; ,$$

where

$$c_k = \frac{(-1)^k}{k!\,(k+\alpha)} \lim_{\epsilon \to 0} \left(\frac{\partial}{\partial \epsilon}\right)^k \epsilon^{1-\alpha} f'(x-\epsilon) \; .$$

Proof. We will establish the relationship to the growth of the function at $x \pm \epsilon$, respectively. We will look for $\mathcal{O}(\epsilon^m)$ equivalence. That is

$$\lim_{\epsilon \to 0} \frac{f(x+\epsilon) - T^{+}_{\alpha,n}(x,\epsilon)}{\epsilon^m} = 0$$

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Obviously for $k = 0$ holds $c_0 = \upsilon_+^\alpha f(x)$. Then the second coefficient can be calculated as

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{f(x+\epsilon)-f(x)-\upsilon_+^\alpha f(x)\,\epsilon^\alpha}{\epsilon^m} &= \lim_{\epsilon\to 0}\frac{T^+_{\alpha,n}(x,\epsilon)-f(x)-\upsilon_+^\alpha f(x)\,\epsilon^\alpha}{\epsilon^m}\\ &= \lim_{\epsilon\to 0}\frac{c_1\epsilon^{\alpha+1}+\mathcal{O}(\epsilon^{2+\alpha})}{\epsilon^m}\\ &= \lim_{\epsilon\to 0} c_1\epsilon^{\alpha+1-m}+\lim_{\epsilon\to 0}\mathcal{O}(\epsilon^{2+\alpha-m}).\end{aligned}$$

Therefore, in order for the RHS to be finite we must have $m = 1+\alpha$. Then for the LHS we have

$$\lim_{\epsilon\to 0}\frac{f(x+\epsilon)-f(x)-\upsilon_+^\alpha f(x)\,\epsilon^\alpha}{\epsilon^{1+\alpha}} = \frac{1}{1+\alpha}\lim_{\epsilon\to 0}\frac{f'(x+\epsilon)-\alpha\,\upsilon_+^\alpha f(x)\,\epsilon^{\alpha-1}}{\epsilon^\alpha}$$

The limit can be evaluated by application of L'Hôpital's rule and rationalization.

$$\begin{aligned}\frac{1}{1+\alpha}\lim_{\epsilon\to 0}\frac{f'(x+\epsilon)-\alpha\,\upsilon_+^\alpha f(x)\,\epsilon^{\alpha-1}}{\epsilon^\alpha} &= \frac{1}{1+\alpha}\lim_{\epsilon\to 0}\frac{\epsilon f'(x+\epsilon)-\alpha\,\upsilon_+^\alpha f(x)\,\epsilon^\alpha}{\epsilon^{1+\alpha}}\\ &= \frac{1}{1+\alpha}\lim_{\epsilon\to 0}\frac{\epsilon^{1-\alpha}f'(x+\epsilon)-\alpha\,\upsilon_+^\alpha f(x)}{\epsilon}\\ &= \frac{1}{1+\alpha}\lim_{\epsilon\to 0}\frac{\partial}{\partial\,\epsilon}\,\epsilon^{1-\alpha}f'(x+\epsilon).\end{aligned}$$

Therefore,

$$c_1 = \frac{1}{1+\alpha}\lim_{\epsilon\to 0}\frac{\partial}{\partial\,\epsilon}\,\epsilon^{1-\alpha}f'(x+\epsilon)\,.$$

The same procedure can be extended for the general case by induction. For an arbitrary $k \leq n$ we will have

$$c_k = \lim_{\epsilon\to 0}\frac{f(x+\epsilon)-T^+_{\alpha,n}(x,\epsilon)}{\epsilon^{k+\alpha}} = \lim_{\epsilon\to 0}\frac{f(x+\epsilon)-T^+_{\alpha,k}(x,\epsilon)}{\epsilon^{k+\alpha}}$$

Therefore, we would have

$$\frac{1}{k+\alpha}\lim_{\epsilon\to 0}\frac{f'(x+\epsilon)-\left(T^+_{\alpha,k}(x,\epsilon)\right)'_\epsilon}{\epsilon^{k+\alpha}}$$

$$= \frac{1}{k+\alpha}\lim_{\epsilon\to 0}\frac{\epsilon^{\alpha-1}\left(\epsilon^{1-\alpha}f'(x+\epsilon)-\sum_{j=0}^{k}c_j(\alpha+j)\epsilon^{\alpha+j-1-(\alpha-1)}\right)}{\epsilon^{k+\alpha}}$$

$$\frac{1}{k+\alpha}\lim_{\epsilon\to 0}\frac{\epsilon^{1-\alpha}f'(x+\epsilon)-\sum c_k(\alpha+k)\epsilon^k}{\epsilon^{k+1}}$$

By applying k times L'Hôpital's rule the denominator can be evaluated to give $k!$ in order to eliminate the Taylor polynomial. Therefore, finally

$$c_k = \frac{1}{k!\,(k+\alpha)}\lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\right)^k\,\epsilon^{1-\alpha}f'(x+\epsilon)\,.$$

Applying similar procedure to the backward case would yield

$$c_k = \frac{(-1)^k}{k!\,(k+\alpha)}\lim_{\epsilon\to 0}\left(\frac{\partial}{\partial\epsilon}\right)^k\,\epsilon^{1-\alpha}f'(x-\epsilon)\,.$$

□

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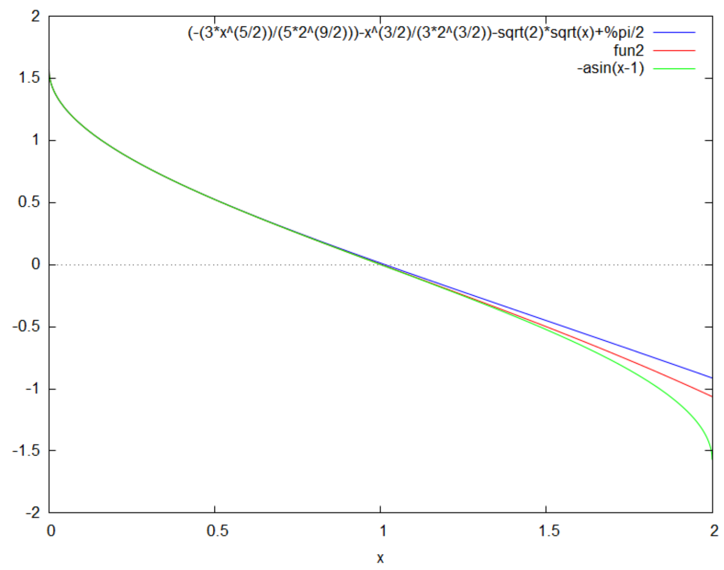

FIGURE 2. Mixed-order Taylor expansion of $\arcsin(1-x)$ about the origin The plot shows two and four fractional term expansions compared to $\arcsin(1-x)$, respectively.

Example 3. *Consider the right fractional velocity of* $f(x) = \arcsin(1-x)$ *about* $x = 0$. *It is a solution of the differential equation*

$$f' + \frac{x^2 - 2x}{x-1} f'' = 0 \tag{18}$$

The first derivative of the function is

$$f'(x) = -\frac{1}{\sqrt{2\,x - x^2}} \; ,$$

which is undefined about 0. *Therefore, the function does not posses an integer-order Taylor expansion about 0. On the other hand,*

$$\upsilon_{+}^{\beta} f\,(0) = -\frac{1}{\beta} \lim_{\epsilon \to 0} \frac{\epsilon^{\beta - \frac{1}{2}}}{\sqrt{2-\epsilon}}.$$

Therefore, for $\beta = 1/2$ $\upsilon_{+}^{\beta} f\,(0) = -\sqrt{2}$. *The fractional Taylor expansion about 0 then is*

$$f(x) = \frac{\pi}{2} - \sqrt{2}\,\sqrt{x} + \mathcal{O}(\sqrt{x})$$

The regularized derivative then is

$$đ\, f(0) = \lim_{\epsilon \to 0} \frac{\arcsin(1-\epsilon) - \frac{\pi}{2} + \sqrt{2}\,\sqrt{\epsilon}}{\epsilon} =$$

$$\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2-\epsilon}} \right) = -\lim_{\epsilon \to 0} \frac{\sqrt{\epsilon}}{\sqrt{2-\epsilon}^{\,3}} = 0$$

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Applying Th. 5 gives the following approximation:

$$f(x) = \frac{\pi}{2} - \sqrt{2}\sqrt{x} - \frac{x^{\frac{3}{2}}}{3 \cdot 2^{\frac{3}{2}}} - \frac{3 \cdot x^{\frac{5}{2}}}{5 \cdot 2^{\frac{9}{2}}} - \frac{5 \cdot x^{\frac{7}{2}}}{7 \cdot 2^{\frac{13}{2}}} - \frac{35 \cdot x^{\frac{9}{2}}}{9 \cdot 2^{\frac{21}{2}}} + \mathcal{O}\left(x^{9/2}\right)$$

The approximation can be appreciated in Fig. 2. It is also apparent that the result also follows from Th. 3 since the inverse function $f^{-1}(x) = 1 - \sin(x)$ has an extremum at $x = \pi/2$.

7. Mixed order extensions of fractional velocity for lacunar series

The scale differential operator can be extended also for mixed fractional orders $k + \beta$. The operator will be extended as

$$\mathcal{S}^{\epsilon}_{k,\beta}\big|_{k=0} f(x) = \mathcal{S}^{\epsilon}_{\beta} f(x)$$

Therefore, the action on the power function must retain the property

$$\mathcal{S}^{\epsilon}_{k,1-\beta}\, x^{k+\beta}\big|_{\epsilon=0} = 1$$

To compute the coefficient we proceed from the n-the order derivative of the power function $f(x) = x^p$ [16] :

$$D^{k+1} x^p = \frac{\Gamma(p+1)}{\Gamma(p-k)} x^{p-k-1}, \quad p > -1$$

so that for $k = [p]$ and $\beta = p - k$

$$\mathcal{S}^{\epsilon}_{k,1-\beta} f(x) := \frac{\Gamma(\beta)}{\Gamma(\beta+k+1)} \epsilon^{k+1-\beta} f^{(k+1)}(x+\epsilon)$$

However, this is the composition law for the scale differential operators up to a multiplicative factor

$$\mathcal{S}^{\epsilon}_{k,1-\beta} f(x) = \frac{\Gamma(\beta+1)}{\Gamma(\beta+k+1)} \mathcal{S}^{\epsilon}_{1-\beta} \circ \prod^{k} \circ\, \mathcal{S}^{\epsilon}_{0} f(x)$$

Therefore,

$$\mathcal{S}^{\epsilon}_{k,\alpha} f(x) = \frac{(1-\alpha)\Gamma(1-\alpha)}{(1-\alpha+k)\Gamma(1-\alpha+k)} \mathcal{S}^{\epsilon}_{\alpha} \circ \prod^{k} \circ\, \mathcal{S}^{\epsilon}_{0} f(x)$$

which reduces to $\mathcal{S}^{\epsilon}_{\alpha}$ for $k = 0$.

Equipped with this operator the main Theorem 4 can be extended to lacunar series.

7.1. Power Series Computation Algorithm.

Based on the result from the preceding section, the following iterative algorithm can be proposed for the computation of the series

(1) initialize $D_1 \leftarrow f(x \pm \epsilon)$, $n = 1$, $T_1 = f(x)$
(2) compute the Hölder exponent

$$\beta_n = \lim_{\epsilon \to 0} \epsilon \frac{\partial_\epsilon D_n}{D_n}, \quad n = 1$$

(3) compute the fractional part $\alpha_n = \beta_n - k = \lfloor \beta_n \rfloor$, where $k = [\beta_n]$ is the integral part of the number,
(4) compute $a_n = \lim_{\epsilon \to 0} \mathcal{S}^{\epsilon}_{k,1-\alpha_n} D_1$
(5) assign $D_{n+1} \leftarrow D_n - a_n \epsilon^{\beta_n}$

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(6) assign $T_{n+1} \leftarrow T_n + a_n \epsilon^{\beta_n}$
(7) go to item (2)

The algorithm is implemented in the Computer Algebra System Maxima.

Appendix A. Essential properties of fractional velocity

The reader is reminded about the essential properties of fractional velocity[17, 21]. In this section we assume that the functions are BVC in the neighborhood of the point of interest.

- Product rule

$$\upsilon_{+}^{\beta}[f\,g]\,(x) = \upsilon_{+}^{\beta} f\,(x)\,g(x) + \upsilon_{+}^{\beta} g\,(x)\,f(x) + [f,g]_{\beta}^{+}(x)$$
$$\upsilon_{-}^{\beta}[f\,g]\,(x) = \upsilon_{-}^{\beta} f\,(x)\,g(x) + \upsilon_{-}^{\beta} g\,(x)\,f(x) - [f,g]_{\beta}^{-}(x)$$

- Quotient rule

$$\upsilon_{+}^{\beta}[f/g]\,(x) = \frac{\upsilon_{+}^{\beta} f\,(x)\,g(x) - \upsilon_{+}^{\beta} g\,(x)\,f(x) - [f,g]_{\beta}^{+}}{g^2(x)}$$
$$\upsilon_{-}^{\beta}[f/g]\,(x) = \frac{\upsilon_{-}^{\beta} f\,(x)\,g(x) - \upsilon_{-}^{\beta} g\,(x)\,f(x) + [f,g]_{\beta}^{-}}{g^2(x)}$$

where

$$[f,g]_{\beta}^{\pm}(x) := \lim_{\epsilon \to 0} \upsilon_{\beta/2}^{\epsilon\pm}[fg]\,(x)$$

wherever the limit exists finitely.

For compositions of functions

- f β-differentiable and $g \in \mathcal{C}^1$

$$\upsilon_{+}^{\beta} f \circ g\,(x) = \upsilon_{+}^{\beta} f\,(g)\,(g'(x))^{\beta}$$
$$\upsilon_{-}^{\beta} f \circ g\,(x) = \upsilon_{-}^{\beta} f\,(g)\,(g'(x))^{\beta}$$

- $f \in \mathcal{C}^1$ and g β-differentiable

$$\upsilon_{+}^{\beta} f \circ g\,(x) = f'(g)\,\upsilon_{+}^{\beta} g\,(x)$$
$$\upsilon_{-}^{\beta} f \circ g\,(x) = f'(g)\,\upsilon_{-}^{\beta} g\,(x)$$

Reflection formula

For $f(x) + f(a-x) = b$

$$\upsilon_{+}^{\beta} f\,(x) = \upsilon_{-}^{\beta} f\,(a-x)$$

Appendix B. Totally disconnected sets

The following definition is given in Bartle (2001)[4, Part 1, Ch. 2]:

Definition 14 (Null sets)**.** *A* ***null set*** $Z \subset \mathbb{R}$ *(or a set of measure 0) is called a set, such that for every* $0 < \epsilon < 1$ *there is a countable collection of sub-intervals* $\{I_k\}_{k=1}^{\infty}$*, such that*

$$Z \subseteq \bigcup_{k=1}^{\infty} I_k, \quad \sum_{k=1}^{\infty} |I_k| \leq \epsilon$$

where $|.|$ *is the interval length. Then we write* $|Z| = 0$*.*

Definition 15 (Totally disconnected space). *A metric space M is totally disconnected if every non-empty connected subset of M is a singleton* [24, p. 210]*. That is, for every $S \subset M$, S non-empty and connected implies $\exists p \in M$ with $S = \{p\}$.*

The next theorem was proven in [22].

Theorem 5 (Null set disconnectedness). *Suppose that E is a null set. Then E is totally disconnected. Conversely, suppose that E is totally disconnected and countable. Then E is a null set.*

Proof. **Forward statement:** Suppose that $Z \subset E$ is connected and open. Then there exist 3 numbers $x_1 < z < x_2$, such that $[x_1, x_2] \subset Z$. Then $|[x_1, x_2]| = x_2 - x_1 > 0$. Therefore, $\exists \epsilon$, such that $0 < \epsilon \leq z - x_1 < x_2 - x_1$; so that $\epsilon < |Z| \leq |E|$, which is a contradiction. Therefore, $x_2 = x_1$ and hence Z is singleton. Therefore, by induction E is totally disconnected.

Converse statement: The countability requirement in the statement of the theorem comes from the fact that there are sets that are totally disconnected, uncountable and non-null [10]. Since E is totally disconnected for every $z, w \in E$, trivially, there is a number h, such that $[z - h/2, z + h/2] \cap [w - h/2, w + h/2] = \emptyset$. Therefore, there is a collection of such intervals, $\{I_k\}_{k=1}^{\infty}$

$$I_k = [z_k - h/2^{k+1}, z_k + h/2^{k+1}]$$

of length $|I_k| = 1/2^k$. Therefore,

$$\sum_{k=1}^{\infty} |I_k| = h$$

for any such a number h. Since h can be chosen arbitrarily small the claim follows.

□

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