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Article

Fermat's Last Theorem Proved in Hilbert Arithmetic. III. The Quantum-Information Unification of Fermat's Last Theorem and Gleason's Theorem

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Abstract: The previous two parts of the paper (correspondingly, https://philpapers.org/rec/PENFLT-2 and https://philpapers.org/rec/PENFLT-3) demonstrate that the interpretation of Fermat's last theorem (FLT) in Hilbert arithmetic meant both in a narrow sense and in a wide sense can suggest a proof by induction in *Part I* and by means of the Kochen - Specker theorem in *Part II*. The same interpretation can serve also for a proof FLT based on Gleason's theorem and partly similar to that in *Part II*. The concept of (probabilistic) measure of a subspace of Hilbert space and especially its uniqueness can be unambiguously linked to that of partial algebra or incommensurability, or interpreted as a relation of the two dual branches of Hilbert arithmetic in a wide sense. The investigation of the last relation allows for FLT and Gleason's theorem to be equated in a sense, as two dual counterparts, and the former to be inferred from the latter, as well as vice versa under an additional condition relevant to the Gödel incompleteness of arithmetic to set theory. The qubit Hilbert space itself in turn can be interpreted by the unity of FLT and Gleason's theorem. The proof of such a fundamental result in number theory as FLT by means of Hilbert arithmetic in a wide sense can be generalized to an idea about "quantum number theory". It is able to research mathematically the origin of Peano arithmetic from Hilbert arithmetic by mediation of the "nonstandard bijection" and its two dual branches inherently linking it to information theory. Then, infinitesimal analysis and its revolutionary application to physics can be also re-realized in that wider context, for example, as an exploration of the way for physical quantity of time (respectively, for time derivative in any temporal process considered in physics) to appear at all. Finally, the result admits a philosophical reflection of how any hierarchy arises or changes itself only thanks to its dual and idempotent counterpart.

Keywords: completeness; Gleason's theorem; Fermat's last theorem; Hilbert arithmetic; idempotency and hierarchy; Kochen and Specker theorem; nonstandard bijection; Peano arithmetic; quantum information

1. An interpretation of "probabilistic measure", its uniqueness, and the sense of gleason's theorem

Gleason's theorem (1957) inspires a series of:

- generalizations (e.g. Benavoli, Facchini, Zaffalon 2017; Flatt, Barnett, Croke 2017; De Zela 2016; Edalat 2004; Rudolph, Wright 1998; Drisch 1979; Eilers, Horst 1975);
- alternative proofs (e.g. Busch 2003; Richman, Bridges 1999; Cooke, Keane, Moran 1985), reformulations (e.g. Billinge 1997; Nishimura 1994);
- interpretations (e.g. Buhagiar, Chetcuti, Dvurečenskij 2009; Held 2009; Richman 2000; Tarrach 1997; Hellman 1993);
- applications (e.g. Dvurečenskij 1993);
- testing experiments (e.g. Campos, Gerry 2002; Peres 1992);

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- or only connotations, analogies and allusions in quantum mechanics and information (e.g. Wright, Weigert 2019; Rieder, Svozil 2007; Hrushovski, Pitowsky 2004; Isham, Linden, Schreckenberg 1994; Chevalier, Dvurečenskij, Svozil 2000; Carruthers 1984);

- as well as in other, mainly mathematical sciences (e.g. Moretti, Oppio 2018; Marlow 2006; Mushtari 1998; 1998a; Matvejchuk 1997; Dvurečenskij 1996; Dvurečenskij 1993; Dvurečenskij, Mišík 1988; Dvurečenskij 1987).

So, it is a considerable scientific and cultural fact. Literally, Gleason's theorem is a statement about the uniqueness of any probabilistic measure in Hilbert spaces of dimension greater than two.

The introduction section[1] of the present *Part III* is dedicated to clarifying the meaning and philosophical sense of those concepts involved in Gleason's theorem in the context of FLT by the mediation of Hilbert arithmetic (e.g. Penchev 2021 August 24) as well as by an interpretation of metaphysical transcendentalism as a scientific hypothesis admitting to be refuted (e.g. Penchev 2021 August 31)[2].

"Probabilistic measure" means any mapping of an "event space", e.g. in the context of the Kolmogorov (1933) axioms of probability theory, into the interval [0,1] whether real or rational. "Event space" forces any of its elements to be either "successful" or "unsuccessful" therefore dividing disjunctively all elements into two classes interpretable correspondingly as the internality or as the externality of the same wholeness furthermore idempotent to each other.

Thus, the concept of "event space" is quite similar or mathematically isomorphic to that of wholeness as it is meant by scientific transcendentalism. Indeed, it doubles the wholeness (originating from the totality, e.g. for scientific transcendentalism) either as "successful" or as "unsuccessful" events "depicting" all events into their "successful" part. So, the mathematical meanings (or structures) of scientific transcendentalism's "wholeness" and probability theory's "event space" can be thought to be the same.

In fact, even set theory's "set" can be interpreted as isomorphic to that structure after restricting Cantor's "loose use"[3] of "set" to a more rigorous meaning able to prevent paradoxes such as Russell's (1902; 1903) as far as it is to be a subset of another set. Then, the complement of the subset to the other set, on the one hand, and the subset itself, on the other hand, can be immediately interpreted whether as the externality versus the internality of a certain wholeness or not worse, as the unsuccessful events versus the successful events of a certain event space relevant to the set at issue. Thus, scientific transcendentalism's "wholeness" and probability theory's "event space" can be immediately related to set theory's "set" being one of the most fundamental notions for mathematics at all and especially for its foundations.

That "cure" of set-theoretical paradoxes is alleged to be "empirical" as far as the restriction of the meaning of the "set" is added "ad hoc"[4] eventually preventing all known paradoxes, but without proof about whether it would be sufficient to avoid any paradox, for example, which can appear in the future (e.g. Sean 2018; Mayberry 1977; Orey 1956). The discussion which is forthcoming during the present *Part III* of the paper may furthermore supply the sketch of a proof for that sufficiency due to the completeness of Hilbert arithmetic in turn exploiting the completeness of quantum mechanics because of the

[1] The numeration of the sections in Part *III* continues following that in *Part II*.

[2] Both "Hilbert arithmetic" and "scientific transcendentalism" are discussed in detail in the context of FLT in the previous parts of the paper and will be granted to be familiar to the reader.

[3] That is: an unlimited use of the concept of set, e.g. any collection can be interpreted as a set of the same elements (e.g. Sean 2018; Srivastava 2014; Jane 2010; Bussotti, Tapp 2009; Dauben 2005; 1983; 1978; Tiles 2004).

[4] The logical necessity of any eventual solution to be only "ad hoc" relevant to the relative consistency of set theory (if it is meant just to arithmetic) originates (and even can be inferred rigorously) from the Gödel incompleteness. Indeed, it would be contradictory not to be always "ad hoc", since all solutions being always "ad hoc" can be interpreted to be a synonym of incompleteness.

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theorems of the absence of hidden variables (Neumann 1932; Kochen and Specker 1967). In other words, one is able to trace that restriction of a set, after being isomorphic to "event space" and "wholeness" as a sufficient condition of completeness including even to the absence of hidden variables in quantum mechanics.

The concept of probabilistic measure means that any event space once supplied by that double and idempotent structure of successful versus unsuccessful events can be mapped onto the numerical interval [0,1]. In fact, that numerical interval can be in turn interpreted as an area of successful events versus its complement to the real or rational abscisa (or at least its positive semi-axis). Indeed, the abscissa is interpretable again as an event space, in which successful events are all probabilities (in a narrow or classical meaning of "probability"[5]).

As well as vice versa: transcendentality can be interpreted by means of possibility, scientific transcendentality by probability, and quantum mechanics itself involves inherently the probabilistic description of the world, therefore implicitly introducing the principle of scientific transcendentality.

Furthermore, one can trace the pathway able to link scientific transcendentality, after it is interpreted as a probabilistic description of the world in quantum mechanics, to quantum neo-Pythagoreanism advocated in all three parts of the paper. A step following that pathway consists in involving a certain mathematical structure such as Hilbert space (in the case of Gleason's theorem) exemplifying the arbitrary set mappable into the interval [0, 1] according to the condition for probabilistic measure.

Then, the philosophical sense of Gleason's theorem can be realized by means of the relation of idempotency and hierarchy already discussed in detail in the previous two parts, and especially, in the introduction *Section I* (i.e. in *Part I*). Indeed, the general concept of dimension (also embodied in the structure of Hilbert space being arbitrarily dimensional) can allow for that of infinity in set theory (therefore the Cantorian hierarchy of infinities) to be reorganized as many copies of the same finiteness, each of which gapped from the next one by the same kind of gaps as between two successive dimensions such as those of Hilbert space, which can be reinterpreted as a mathematical model of set theory as to the rest part of mathematic, supplied furthermore by the additional advantage to involves consistently Peano arithmetics in relation to its dimensions.

The relevant interpretation of infinity as the gap between two copies of finiteness (respectively two anti-isometric copies of Peano arithmetic as in Hilbert arithmetic) can be equivalently represented by the counterparts of the axiom of choice and the well-ordering "theorem" as follows. Any choice among the elements of any infinite set admissible according to the axiom of choice can be decomposed into a well-ordered series of binary choices each of which means a bit of information and thus as a whole: an infinite binary string. Then, the gap between the options of a bit of information would correspond to any two successive dimensions following immediately one after another, and the well-ordered string of bits, to all successive dimensions of the Hilbert space at issue.

Meaning that difference between two dimensions of Hilbert space versus more than two of its dimensions, furthermore being involved in the statement of Gleason's theorem, it can be directly interpreted in terms of set theory and even only in those supposed by the equivalent counterparts of the axiom of choice and the well-ordering "theorem". Dimensions "$n = 1,2$" can be related both to the states of a bit of information and to the first two bits of the binary string of bits of information. On the contrary, the rest dimensions "$n = 3, 4, 5, \ldots$ " can refer only to the string, but not to a single bit of it, therefore generating an asymmetry between a binary choice and the series of binary choices therefore, maybe eventually violating the equivalency of the axiom of choice and the well-ordering "theorem" furthermore interpretable as "incommensurability" between them.

[5] Indeed, quantum information and thus, the tool of Hilbert arithmetic in a wide sense allows for the generalization of the concept of probability also to any complex values (particularly, negative real values as well as positive ones, but greater than a unit) discussed in detail in another paper (Penchev 2012). However, that generalization seems not to relate to the application of Hilbert arithmetic for proving FLT.

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Utilizing the tool of Hilbert arithmetic in a wide sense, one can transfer the same asymmetry described above from the "branch" of Hilbert arithmetic in a narrow sense into its dual counterpart of the qubit Hilbert space. Then, Gleason's theorem (as well as the Kochen - Specker theorem, in fact) means just the last asymmetry (but without the context of Hilbert arithmetic in a wide sense) or that of FLT therefore restricting the proper consideration only to the separable complex Hilbert space of quantum mechanics (though generalized further to any Hilbert space as to Gleason's theorem).

Set in the same context of Hilbert arithmetic, FLT[6] means the analogical difference between two and more than two dimension due to the distinction of idempotency from hierarchy, but restricting its explicit statement only to the opposite branch of Hilbert arithmetic in a narrow sense similarly (or "dually") as the two rest theorems limiting themselves to the qubit Hilbert space.

As the previous *Part II* demonstrates, the analogy between two dual counterparts of Hilbert arithmetic in a wide sense can be absolutely formalized to an isomorphism or a homomorphism under the definitive condition of Hilbert arithmetic and thus this allows for the Kochen - Specker theorem meaning a statement referring to the one branch to be nonetheless utilized for the proof of the gap between two and more than two dimensions on the other branch, as FLT can be reinterpreted.

The same pathway between both counterparts of Hilbert arithmetic in a wide sense pioneered from the Kochen - Specker theorem applied to FLT in the previous part will be now exploited starting from Gleason's theorem as far as it and the Kochen - Specker theorem mean the essential distinction between two and more than two dimensions in the separable complex Hilbert space (though the Kochen - Specker theorem does this in a rather implicit way).

Another preliminary notice, furthermore generalizing the unusual way of representing mathematical considerations in the previous two parts, refers to the philosophical distinction of "understanding" and "logical syllogism" since the latter is the standard for any mathematical proof to be represented in a scientific paper:

The correctness of the logical syllogism is sufficient for the corresponding proof, but it by itself does not need any *understanding*: for example, it can be repeated and thus checked by a Turing machine. Unlike logical syllogism being "objective" in the sense that can be repeated by anyone, even by a contemporary computer in principle, any proper understanding is "subjective" depending crucially on a relevant insight accomplishable in one's brain-mind, but not by any Turing machine.

Any real human mathematician is forced to understand the proof at issue, otherwise he or she would not be able to pioneer its syllogism for the specific way of working of the human brain and mind being absolutely different from that of a Turing machine. That requirement of objectiveness as to any mathematical proof as far as all mathematicians are human beings therefore necessarily supplying and applying their own individual understandings to the objective logical syllogism of the proof deforming the logical syllogism of the proof at issue by the human understanding and even: in a unique way in general.

In fine, that permanently acting "force" of objectiveness in all mathematicians' mind-brains has fatally influenced what a mathematical understanding should be causing it to be sharply restricted professionally: at that, the subject within which any understanding permitted for a mathematician become more and more narrow therefore excluding even its extension to neighboring or adjacent mathematical areas since the mathematician trying to understand a problem or its proof is not competent about them: she or he is ignorant beyond the very narrow professional subject, perhaps more or less not suspecting those close mathematical considerations or the corresponding wider insight on the investigated problem.

On the contrary, the present paper tends to demonstrate that FLT has become one of the greatest mathematical puzzles of all times just for that organization of mathematical

[6] The idea that FLT can be inferred from Gleason's theorem is articulated for the first time in: *Penchev 2015*.

cognition reflecting the analogical organization of human cognition at all. As even Wiles's proof demonstrates sophisticatedly, complicatedly, indirectly and implicitly, its solution is quite impossible being restricted to a narrow "Fach"[7] since it passes through a series of rather different mathematical disciplines.

However, one may go much further reflecting on all troubles about FLT to be proved as originating from the fundamental organization of cognition in Modernity, according to which for example, mathematics can build only models of reality so that the crucial gap between any mathematical model and reality meant by it is not removable, in fact being established in philosophy since Descrates's age and his dualism. In other words, the restriction of "Fach" limiting the possibility of understanding of each mathematician is generalized (in the present paper) to the boundary of the Cartesian "mind" and (versus) "body" and then, realized to be the main obstacle for FLT to be proved.

The mathematical tool of Hilbert arithmetic allows for both disjunctive areas to be merged utilizing the proved completeness of quantum mechanics due to the theorems of the absence of hidden variables. Transcending the limits of what condition should be in Modernity, the syllogism for proving FLT turns out to be shockingly simple and elementary, literally within a page therefore confirming again that the incredible difficulty which has been insurmountable for all mathematicians for almost half a millennium, is due to what Michel Faucalt (1966) called "episteme" rather than to FLT by itself being easily provable in another episteme.

Even more, once that elementary proof has been revealed after the "Gestalt change", it can be translated into mathematical concepts of the standard Peano arithmetic or into those accessible in Fermat's age (more or less speculatively alleging what Fermat's lost proof should be). However, the demonstration of that extremely short syllogism in Peano arithmetic, though spectacular as a magician to pull a rabbit out of his or her cylinder, is not especially instructive or the syllogism at issue to be important for mathematics.

On the contrary, the Gestalt change able to transcend our organization of cognition is what is really essential, but it cannot be represented as a syllogism, crucially depending on the individual reader's understanding. An author such as any philosopher cannot represent that meant understanding in an absolutely formal way (for example, as a logical syllogism also accomplishable by a Turing machine) as far as the human being's understanding is an *individual* and *unique* act definitively: this is a leap in *one*'s mind-brain.

Instead of demonstrating sophisticated syllogisms accessible really only to several dozen mathematicians, the current paper calls, invites, and begs for understanding just like any other philosophical paper and unlike any mathematical study. What one should understand is the way of transcending our modern "episteme" to a conjectural "quantum neo-Pythagoreanism", from the viewpoint of which the research can make sense.

2. Gleason's theorem and its proof

Gleason's theorem (1957) is well-known, widely discussed, interpreted and generalized in many ways, but never[8] as to number theory or to FLT in particular. The reason is the absence of Hilbert arithmetic (or other relevant tool for such an application), which supplies the context of their almost obvious link as this is suggested in the previous *Section XVIII*. It literally states the following (Gleason 1957: 892-893):

"Let μ be a measure on the closed of a separable (real or complex) Hilbert space $\mathcal{H}$ of dimension at least three. There exists a positive semi-definite self-adjoint operator T of the trace class for all closed subspaces of A of $\mathcal{H}$

$$\mu(A) = trace\ (TP_A)$$

where P_A is the orthogonal projection of $\mathcal{H}$ onto A."

Then, the uniqueness of $\mu(A)$ follows immediately for the uniqueness of the operator of "trace" applied to any square matrix of arbitrary dimension and can be visualized

[7] "Fach" is to refer to Richard Rorty's paper (1982), to its context and way of use.

[8] At least as far as I know.

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by the uniqueness of the diagonal of any square matrix (indeed, the "left"[9] diagonal of any square is single). Gleason means $\mu(A)$ to be a measure, i.e. (p. 885): "A measure on the closed subspaces means a function μ which assigns to every closed subspace a nonnegative real number such that if $\{A_i\}$ is a countable collection of mutually orthogonal subspaces having closed linear span B, then $\mu(B) = \Sigma\, \mu(A_i)$."

Obviously, the probabilistic measure "$\frac{A_i}{B}$" (according to the common interpretation of Gleason's theorem) corresponds unambiguously to the measure "A_i". Then, the measure "$\mu(A_i)$" of any subspace "A_i" is mapped bijectively on "P_A" by the operator "T". That is: the unique measure $\mu(A_i)$ is absolutely determined by the projection of "A_i" onto "Ⓗ"**.**

So, one can notice, that Gleason's theorem means an almost obvious statement following immediately from the orthogonality of all subspaces "A_i" (including "Ⓗ" itself) to each other, after the countable additivity of which is defined as "measure".

Gleason's way of proving is not necessary for the application of the theorem to FLT. Instead of that, one can demonstrate the "projection" or its meaning by Hilbert arithmetic in a narrow sense in the framework of Hilbert arithmetic in a wide sense being much more useful and relevant to the intended objective:

Then, each subspace "A_i" of "Ⓗ" is mapped bijectively onto a certain substring "S_i" of "Ⓢ" where that "Ⓢ" can be interpreted either *set-theoretically* as the set, the set of all subsets of which is the set of all possible binary strings, or *arithmetically* as the binary strings of any length. Interpreting *unambiguously* (by virtue of the fact that mapping of "Ⓗ" onto "Ⓢ" is a bijection) Gleason's theorem also to binary strings, one can notice that it is transformed into a trivial statement: the length of any[10] binary string being substring of "Ⓢ" is unique. Meaning that obvious observation and under the crucial condition of "Ⓢ" to be *bijectively*[11] mapped onto "Ⓗ", a new and elementary proof of Gleason's theorem is demonstrated, therefore again heralding how powerful the tool of Hilbert arithmetic is.

One can check what that method would state about a Hilbert space of dimension one or two: the length of binary string really make sense neither about any alternative of a bit of information (notatable as "0" or "1" and corresponding to Hilbert space of dimension one) nor about its two alternatives together (corresponding to Hilbert space of dimension two). The latter consists of a single cell, interpretable also as an "empty" qubit, i.e. a unit three-dimesional ball obviously needing three dimensions to be defined.

One can reflect on the *bijection* between the consecutive dimensions of Hilbert space and the corresponding units of Hilbert arithmetic for proving Gleason's theorem in the latter quite briefly. The first two dimensions seem to be shareable by all dimensions greater or equal than three if that bijection with the string of consecutive bits has been established. Then, that first two dimensions would be more relevant to all "number-sake" pairs of both dual Hilbert spaces[12].

So, the eventual proof of FLT based on the counterpart of Gleason's theorem in Hilbert arithmetic in a narrow sense relies on the identity of the two possible interpretations of the first two dimensions of Hilbert space whether dual and thus idempotent, or as the

[9] "Trace" is standardly defined by the "left" diagonal of a square matrix, eventually of infinite dimension.

[10] In fact, it is to consist of at least two cells, but the reason for that precision originates from the confusion after defining a bit of information since two independent oppositions, respectively binary cells, correspond to a bit of information. The confusion at issue is made clear in detail in another paper (Penchev 2021 July 8).

[11] The relevant bijection needs the following peculiarity: the *first three* dimensions correspond to the *first* bit, however the first two bits need only four rather than six dimensions and any following bit adds a single dimension so that "n" bits correspond to "n+2" dimensions rather than to "3n" dimensions.

[12] This means: the one member of the n^{th} pair corresponds to the n^{th} dimension of Hilbert space, and the other member of the same pair, to the n^{th} dimension of the dual Hilbert space.

first two ones in the well-ordered sequence of all natural numbers constituting therefore a "hierarchy".

Then, the only puzzle would be about why that equivalence cannot be extended to any other dimensions, i.e. different than two first ones in their well-ordering. Properly, the sense in the meaning of FLT in the sketched contexts refers to that privilege of those two first dimensions in relation to the "equivalence of idempotency and hierarchy" (discussed in *Part I*, *Section I*). That is: why can only the first dimensions be considered to be idempotent (dual) to each other unlike any other pair of subsequent dimensions?

The natural and intuitive answer could be the following. If the idempotency starts from the least element, i.e. "1", once the relevance of the well-ordering of all natural numbers has been in advance granted, e.g. in virtue of the Peabo axioms of arithmetic, it is not "able" to continue to any dimension greater than two:

$$(1 = 1)_{mod(2)}; (2 = 2)_{mod(2)}; (3 = 1)_{mod(2)}; (4 = 2)_{mod(2)}; \ldots; (2n - 1 = 1)_{mod(2)}; (2n = 2)_{mod(2)}$$

This means that just the well-ordering is what privileges the first two dimensions after it has definitively privileged the first element as the least one. In other words, if one considers an auxiliary well-ordering of idempotent pairs, the first pair is privileged to be the least one as a direct corollary from the well-ordering of all elements constituting the idempotent pairs. The consideration in the last several paragraphs is already able to refer to FLT as follows (though it will be discussed in detail in the next Section XX):

Once Gleason's theorem is granted in advance, one can transform it to Hilbert arithmetic since it is valid also to the class of equivalence what any unit in Hilbert arithmetic is: even more, Gleason's theorem being formulated to subspaces of Hilbert space relates directly to classes of equivalence which any subspace represents in fact. One notices that the first two dimensions are not meant after that mapping.

Then, the uniqueness of probabilistic measure to each member of a pair of Hilbert spaces is to be investigated to be proved so: it implies for them to be incommensurable to each other even in the rigorous arithmetic meaning (i.e. its ratio cannot be any rational number) if one translates it in terms of Hilbert arithmetic. The same statement has been deduced in the previous *Part II* starting from the Kochen - Specker: now, after it is again available, one might refer to the following exposition in *Part II* literally.

Of course, a proper pathway to further proof, relevant to the intentions and connotations of Gleason's theorem, would be much more interesting and just this is the sketched idea in the previous several paragraphs. Then, one needs a quite rigorous proof that the property of idempotency can be related only to the first two dimensions (respectively, exponents in Fermat's equation). Besides the aforementioned rather intuitive tenet, one can utilize the Gödel incompleteness of arithmetic (the axiom of induction) to set theory (the axiom of infinity) in the following sense:

Starting from the dimension (exponent) of three, i.e. for "$n \geq 3$", only the axiom of induction is relevant therefore being inherently incomplete to the gap between dimensions (exponents) "1" and "2" obeying the axiom of infinity since no finite series of natural numbers is able to overcome it. Then, the problem is: why cannot the same argument be applied to the gap between "2" and "3" (and then, etc.) following from the idea of the Gödel incompleteness?

This would imply that the inconsistency of set theory and arithmetic is preventable only by their complementarity to each other as e.g. in Hilbert arithmetic. The observation suggests that the area related to set theory and that of arithmetic should be distinguished in a way not to admit their simultaneous utilization to avoid the direct contradiction (analogically to the prohibition in quantum mechanics for two conjugate quantities to be simultaneously measured, i.e. in a single experiment).

The necessity for the "natural" order of those two areas to be exchanged in comparison with common sense's one is emphasized a few times in the paper until now. Common sense postulates "naturally" and unquestionably that infinity is "more" than "finiteness" and accordingly their innate order cannot be other than that: first, finiteness, then,

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infinity[13]. However, meaning their duality only being able to avoid their direct contradiction if the incompleteness of arithmetic to set theory has been rejected in advance for one reason or another, the reverse order (first, infinity, then finiteness) is admissible and even necessary to avoid the Gödel incompleteness in a sense:

This means that one can interpret the Gödel incompleteness as a tenet by *reductio ad absurdum* that only the counterintuitive order for infinity to be "less" than infinity is consistent with the condition for them to be dual to each other. That is: their order is opposite to common sense's natural one, and the Gödel incompleteness is interpreted as a contradiction under the additional condition for completeness, valid e.g. in Hilbert mathematics. So, the order is to be just infinity first in virtue of *reductio ad absurdum.*

One can demonstrate that the counterintuitive order at issue does not generate the Gödel incompleteness: the structure of Boolean algebra alone (i.e. allowing for only two idempotent dimensions, or exponents in Fermat's equation: either "1" or "2") interpretable further whether as propositional logic or as set theory does not implies incompleteness as Gödel (1930) himself showed in a sense[14]. Indeed, the Gödel incompleteness cannot be placed within the idempotent pair of "1" and "2": even repeated an actually infinite set of times, the operation of logical negation (as an example of idempotency) cannot transcend the only two options to be able to generate incompleteness.

Returning back to the interpretation of FLT in the context of Gleason's theorem, one pays attention that duality (complementarity) does not admit "infinity" to be situated anywhere among finiteness besides either in its absolute beginning (as the first two elements) or in its absolute end, however implying the Gödel incompleteness in the latter case, being to be rejected under the additional condition of completeness, e.g. as in Hilbert mathematics based on Hilbert arithmetic. So, only the order of "infinity first, then finiteness" (though extremely counterintuitive) can be relevant in the framework of Hilbert arithmetic therefore proving FLT exceptionally on the foundation of Gleason's theorem once it has been translated from the branch of the qubit Hilbert space itho that of Hilbert arithmetic in a narrow sense in the general framework of Hilbert arithmetic in a wide sense.

3. *FLT* proved in hilbert arithmetic by gleason's theorem

As the previous *Part II* demonstrates, the "quantum" proof tends to verify the incommensurability of the arithmetic variables y^n and z^n first for $n = 3$ and then, for $n \geq 3$. An additional check shows their commensurability for $n = 1, 2$. This is the approach to be proved FLT starting from the Kochen - Specker theorem after making clear that the newly introduced by them concept of partial algebra (particularly, partial Boolean algebra being relevant to the case of FLT) relied on the binary relation of commensurability (notated by them as "⚲") is only a "conservative" generalization of arithmetic commensurability, respectively incommensurability brought the ancient Pythagoreanism to a crisis after proving the incommensurability of the length of the diagonal of a square to its side if the side is a natural or rational number.

So, involving the tool of Hilbert arithmetic in a wide sense, one can reversely reduce Kochen and Specker's proof of the absence of hidden variables in quantum mechanics to its ancient fount being relevant to FLT. Indeed, if y^n and z^n for $n \geq 3$ are arithmetically incommensurable to each other, their sum is necessarily an irrational number and no rational solution at all for Fermat's equation meaning just their sum.

In fact, the Kochen - Specker theorem and Gleason's theorem are relative and this is even more obvious after the joint degeneration to the case of Hilbert arithmetic in a narrow

[13] Cantor's hierarchy of infinities (also conserved in the contemporary axioms of set theory) obeys the same "natural" and ostensibly unquestionable order originating from common sense.

[14] Only set theory (unlike the case where arithmetic is added to set theory) as a first-order logic to propositional logic does not contain any insoluble statement as the main result of Gödel's earlier paper can be interpreted in terms of the later one.

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sense. Both mean in the final analysis the same observation however interpreted in two different contexts though not too remote from each other. The shared observation consists in the following equivalence: the way for an "empty" qubit to be seen "inside" as an empty tridimensional unit ball with its volume of "$\frac{4}{3}\pi$" in the relevant three physical dimensions, i.e. in volume units, on one hand, and (respectively, "versus") "outside": as a usual arithmetic unit and thus, one-dimensional, on the other hand.

Then, that equivalence in virtue of Hilbert arithmetic in a wide sense implies for a pair of natural numbers, each of which consisting of one-dimensional units considered equivalently "outside", i.e. as tridimensional unit balls, turn out to be necessarily incommensurable to each other. The last statement is equivalent to FLT(3) and suggests a corresponding approach to prove FLT (in the general case for any arithmetic exponent greater than two).

Accordingly, the context of the Kochen - Specker theorem for the observation at issue rather emphasizes that their incommensurability excludes any finite common divisor (common denominator) different from "1". In other words, the concept of hidden variables in quantum mechanics can be interpreted as a generalization of "finite common divisors" in the case of arbitrary qubits if Kochen and Specker's approach has been granted.

In comparison with the above paragraph, the context of Gleason's theorem highlights alternatively sooner the analogical statement but in relation to the qubits "inside", i.e. as tridimensional unit balls rather than as usual arithmetic, thus one-dimesional units, meaning in the final analysis (as to the latter arithmetic units) the trivial statement that any irrational number is equivalent to a unique infinite series of "digits" in any positional number system (such as binary or decimal).

So, one is to infer the incommensurability of the commutative arithmetic pair "(y^n, z^n)" for "$n \geq 3$" from Gleason's theorem following the already pioneered pathway started from the Kochen - Specker theorem in the previous *Part II* and mediated by Hilbert arithmetic in a wide sense. It is an instant corollary, e.g. by *reductio ad absurdum*:

Let "(y^n, z^n)" for "$n \geq 3$" (where y, z are two variables on all natural numbers) be commensurable. This means that their ratio (i.e. each of both possible ratios: $\frac{y^n}{z^n}; \frac{z^n}{y^n}$) is a certain rational number. Applying the transition from the branch of Hilbert arithmetic in a narrow sense to the qubit Hilbert space in the framework of Hilbert arithmetic in a wide sense, the assumption for them to be commensurable implies that there exists two different measures distinguishing from each other for $max(y, z)$, and Gleason's theorem excludes that to be possible. One checks the option for "y" and "z" to coincide, which implies by virtue of Fermat's equation: "$x = (\sqrt[n]{2})y$", which cannot be satisfied for any pair of natural numbers since "$\sqrt[n]{2}$" for "$n \geq 3$" is an irrational number. Then, "(y^n, z^n)" for "$n \geq 3$" is incommensurable due to the rejected opposite assumption.

As aforementioned, all ways for the proof of FLT after the incommensurability of that pair has been proved can be literally borrowed from the consideration in *Part II* remaining valid to Gleason's theorem equally well as to the Kochen - Specker theorem. What would be interesting now are other pathways for proving the general case of FLT which originate from proper connotations of Gleason's theorem. Those are relied on the opposition of dimensions "$n = 1,2$" versus those of "$n \geq 3$" in Hilbert space and implying their opposition also in Hilbert arithmetic in a narrow sense: while the same opposition in the proper limitation of Peano arithmetic seems to be mysterious and inexplicable, particularly conditioning all troubles for proving FLT.

Utilizing again the "extraordinary" ("2:1") bijection of Hilbert arithmetic in Peano arithmetic:

$$(P^- \otimes P^+ \rightarrow P^0) \rightarrow P$$

discussed in detail in *Part II*, one can state that the opposition in question is due to that bijection itself just remaining even ridiculous in the framework of Peano arithmetic alone. In other words, it is concentrated in "$P^- \otimes P^+ \rightarrow P^0$" and only transferred in Peano arithmetic as if outside in virtue of the "second mapping" notated by "$\rightarrow P$". Indeed, the first mapping "$P^- \otimes P^+ \rightarrow P^0$" means that any natural number (in "P^0", literally) originates from a corresponding consecutive bit of information in an unlimited binary string (i.e. of any length). That bit being furthermore a universal condition for the nonstandard bijection is what privileges the first two natural numbers (or "dimensions" in the qubit Hilbert space or Gleason's theorem), after which their privilege is only repeated in Peano arithmetic therefore remaining mysterious and inexplicable *within* itself since it originates *out of* it, rather even *beyond* it, gifting FLT with its puzzling and bewildering "transcendent charm of Mona Lisa's smile".

So, the own connotations of Gleason's theorem for proving FLT consist in the investigation of the privilege of the first two natural numbers (or dimensions of Hilbert space) satisfying furthermore the relation of idempotence corresponding to the function successor for all greater natural numbers (or dimensions of Hilbert space). One can express this so: Gleason's theorem offers a newly invented sense (or Frege's "Sinn") under the same "Bedeutung" of FLT: if the general case of incommensurability in Fermat's equation seems natural, obvious, and even trivial, what needs a proof is the exception for "$n = 1,2$": why it takes place.

Indeed, Fermat's equation needs at least two measures to be possible for its solution. For example, if one considers the triple (3, 4, 5) satisfying $y^2 + z^2 = x^2$, that solution requires as its necessary condition the availability of more than two measures of Hilbert space of dimension 2 meant as a subspace to itself if the solution is translated from the branch of Hilbert arithmetic in a narrow sense (where it can coincide with Peano arithmetic, in term of which Fermat's equation and last theorem are formulated) into the counterpart of the qubit Hilbert space, respectively the separable complex Hilbert space, to which Gleason's theorem refers directly, within the shared framework of Hilbert arithmetic in a wide sense. However, if the cases of the equation $y^n + z^n = x^n$ is for $\forall n \geq 3$, Gleason's theorem is what excludes the necessary ambiguity of the measure and thus any solution in natural numbers. Thus, it is able to supply a proof for the general case of FLT without induction or any additional arguments unlike the Kochen - Specker theorem relatable immediately to FLT(3) properly.

Then, the application of Gleason's theorem rather generates the question how the exception of one or two dimensions can be interpreted and explained as to the dual branch of Hilbert arithmetic in a narrow sense resulting in Peano arithmetic. As aforementioned, it is due to the nonstandard bijection able to map Hilbert arithmetic into Peano arithmetic therefore remaining beyond the latter (just to which FLT not to be proved only arithmetically during a few centuries is owing). The same circumstance can be also revealed in the structure of Hilbert space (particularly, in that of the qubit Hilbert space) and then repeated in Hilbert arithmetic (also allowing for it to overcome the Gödel incompleteness without any contradiction) and absolutely missing in Peano arithmetic itself: this is the pair of two dual counterparts, furthermore reproduced the formal structure of a bit of information.

4. Both mutual interpretations of gleason's theorem into *flt* and vice versa

The previous *Paragraph XX* shows that Gleason's theorem implies *FLT* as its dual counterpart in the other branch within the framework of Hilbert arithmetic in a wide sense. The question whether the converse implication is also valid is natural morerver in virtue of the meaning the corresponding Hilbert subspaces as whole sets, each of which mappable just as a whole into a certain number set whether the interval [0,1], or positive real numbers, or positive rational numbers, or natural numbers in the final analysis as the translation into the dual branch of Hilbert arithmetic in a narrow sense is necessary for proving FLT.

A preliminary notice is to clarify the link and consistency of the enumerated four different cases, onto the number set of each which all subspaces of Hilbert space are supposed to be mapped. The first case of the real or rational interval [0. 1] means just a probabilistic measure, in terms of which Gleason's theorem is often formulated.

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Gleason's proof deals directly with the mapping of Hilbert subspaces into the set of all positive real numbers, from which the probabilistic measure at issue follows unambiguously in relation to the Hilbert space itself, to which all subspaces are meant, simply as a ratio. So, what remains to be considered are the three cases of number sets, into which Hilbert subspaces are mapped: positive real numbers, positive rational numbers, natural numbers.

Gleason's theorem formulated as to a unique measure of the separable Hilbert space therefore means a countable set: so, the continuum of real numbers would be irrelevant. Thus, one can grant the set of all natural numbers as quite sufficient and might even establish the dimensionality of any subspace to be its measure as far as the theorem identifies the subspaces of the same dimensions (for example as translatable collinear vectors of the same length).

Meaning that, one can interpret Gleason's theorem as referring to Hilbert arithmetic in a narrow sense, in fact, though literally formulated in terms of the separable Hilbert space (including the complex one of quantum mechanics and relevant to its epistemological puzzles). Then, FLT can be considered as an exact analogue of it after involving the nonstandard bijection already as a kind of "nonstandard homomorphism" of Hilbert arithmetic into Peano arithmetic their suggesting the converse implication, which can be demonstrated in detail as follows:

One regards the dual (reverse) bijection or homomorphism of Peano arithmetic in Hilbert arithmetic, that is: $P \to (P^0 \to P^- \otimes P^+)$, which can be furthermore visualized to each bit of the "universal arithmetic binary string"[15], in which conventionally and for example all "zeros" (the one alternative state of a bit) are ascribed to " P^-" and accordingly, all units, to " P^+", and they are interpreted as the corresponding classes of equivalence of qubits of the qubit Hilbert space as the definition of Hilbert arithmetic needs, e.g. as tridimensional unit balls.

So, an arithmetical unit of Peano arithmetic ("1") is mapped onto an axis of the qubit Hilbert space (corresponding to two axes of the separable complex Hilbert space), however in way so that all qubits share the first two "missing" or "vanishing" dimensions. This can be schematized admissibly so: a Peano unit → a bit of information → both states of a bit of information shared as the same for each bit of the universal arithmetic string also relevant to Hilbert arithmetic.

Then both those states are identified as the first two "missing" or "vanishing" dimensions of Hilbert space (which is not necessary to be limited only to the qubit Hilbert space though the mediation of Hilbert arithmetic implies just this), for which Gleason's theorem does not establish a unique measure. That absence of a unique measure for them, by the by, follows obviously from sharing the same states ("0", "1") of a bit, independently of its consecutive number in the string, for all of them therefore excluding to be assigned unambiguously by virtue of the discussed bijection and thus revealed in the underlying nonstandard bijection as two first dimensions, furthermore relevant to all the rest, and which are necessary to distinguish the same natural numbers as belonging either to "P^-" or "P^+" accordingly.

Then, the incommensurability equivalent to FLT (as this is explained above) implies the unique measure being traceable in detail (as a little further in the text of this section) but needing two preliminary notices meaning: (1) the distinction of FLT as an *arithmetic* statement versus Gleason's theorem referring to Hilbert space and thus implicitly, to *set theory*; (2) the generalization to real and complex Hilbert spaces, to which Gleason's theorem has been formulated in original.

FLT as an eventual equivalent of Gleason's theorem needs the proviso about its incompleteness forced by the Gödel one. That is: it can imply Gleason's theorem only to all

[15] That "universal arithmetic string" has not to be identified with a doubling of the set of all natural numbers as far as it is defined exceptionally within the framework of Peano arithmetic in order to avoid the Gödel incompleteness after an eventual addition of set theory.

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dimensions being any natural numbers, but not to the *set* of all dimensions being equivalent to the *set* of all natural numbers.

The latter notice is to infer the case of the separable real Hilbert space from that of the separable complex Hilbert space as a particular case after the latter is deduced from the qubit Hilbert space as far as Hilbert arithmetic means just it definitively. Indeed, the deduction of the real case from the complex one is obvious since any real number can be interpreted to be a complex with a zero proper imaginary part. In turn, the separable complex Hilbert space is also a particular case of the qubit Hilbert case if the orthogonal subspaces meant in the later are reduced to be "successive axes" of the former, i.e. $e^{in\omega}$; $e^{i(n+1)\omega}$ after standard notations.

The proof of Gleason's theorem from FLT under the above explicit restrictions or elucidations seems to be transparent as well. One interprets FLT as the incommensurability of the pair of any arithmetic variables, or y^n and z^n if $n \geq 3$. Arithmetic in default is meant to be Hilbert one after the the *dual* nonstandard homomorphism of Peano arithmetic into Hilbert arithmetic: "$P \to (P^0 \to P^- \otimes P^+)$", after which it is related to the qubit Hilbert space in the framework of Hilbert arithmetic in a wide sense. Then, the incommensurability equivalent to FLT implies the uniqueness of the (probabilistic) measure for any subspace of Hilbert space of dimension $n \geq 3$ since if it were ambiguous it would imply the commensurability at issue (i.e. within Fermat's equation) and thus solutions for which FLT states not to exist.

So, FLT and Gleason's theorem can be interpreted to mean the same (with the proviso above) once Hilbert arithmetic in a wide sense has been involved in advance.

5. *FLT* and the uniqueness of a single probability measure in the qubit hilbert space

The partial equivalence of FLT and Gleason's theorem can be interpreted furthermore physically after the latter, inspired from an idea for Born's rule for probabilities in quantum mechanics to be proved rigorously from the standard formalism of the separable complex Hilbert space[16], is translated in terms of probabilities for quantum events to be observed or not.

Indeed, the "atom" of that equivalence, a qubit can be expounded both as "empty", i.e. as the class of equivalence of all values which one can measure as its possible values, or said otherwise "before measurement", and as any of those values after a certain measurement, i.e. "after measurement". One may also say that the measurement of a qubit is equivalent to the record of a certain value "within it" or that the "reading" and "writing" of a qubit are equivalent in a sense (just as those of a bit).

Then, Gleason's theorem (just as Born's rule itself) establishes the new and generalized way for quantum mechanics to be an objective experimental science though being quite different from that of classical physics. The latter sets that a unique value for any physical quantity in a certain moment of time exists and just it can be called to be the real value of that quantity. Quantum mechanics dare not state the same postulating a manifold of values measurable by the apparatus to any quantum quantity (maybe with the single exception of time). So, the objectivity of classical physics is inapplicable to quantum mechanics.

Nonetheless, Gleason's theorem proves that its empirical substitution by virtue of Born's rule is consistent and can be discussed to be a relevant generalization: though a

[16] Born's rule determining a certain probability for a quantum event to be measured can be realized in two ways:

(1) as the statement that any quantum event is featured by a certain value to be observed (or to "happen by itself" in corresponding "realist" interpretations of quantum mechanics) thus delivering the generalized scientific objectivity applicable to classical mechanics (after the classical one is not valid);

(2) as the formula assigning that unambiguous probability to any wave function inherent for a quantum state. Already in terms of Hilbert arithmetic, the former realization is to refer to Hilbert arithmetic in a narrow sense, and the latter, to the qubit Hilbert space, therefore both remaining in Hilbert arithmetic in a wide sense.

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manifold of measurable values is the case meant by quantum mechanics, the probability of any "point" of it is unique and can be postulated to be the unique "objective probability". As to a single qubit, both reading and writing are unambiguous and can be considered to be "objective" in that sense (though the read or written quantity is ambiguous in general).

If FLT is equivalent to Gleason's theorem in a sense, it should share the same transformation or generalization of what objectivity in science is, forced particularly by quantum mechanics. All epistemological or ontological troubles of the interpretation of the latter are well-known and the debates continue. That vanity around quantum mechanics allows for one to penetrate why FLT is so difficult to be proved if it shares that fundamental change of what objectivity is though in an implicit way able to be manifested only in virtue of its (partial) equivalence to Gleason's theorem.

One can repeat the essence of that change especially as to the proof of FLT in proper philosophical terms. It needs the unambiguous distinction of "subject" and "object" (respectively, "body" versus "mind"; physics versus mathematics) to be abandoned since the unification of discreteness and continuity, for example implied by the modularity theorem if FLT is proved as a corollary from it as Wiles did, but not less necessary for quantum mechanics to make them consistent to each other (namely the "discrete" quantum entity and the "continuous" apparatus measuring it and obeying the smooth laws of classical mechanics) is relevant to the inherent link or links of the enumerated above concepts in the scope of philosophy.

In fact, the present paper and especially its first part demonstrate that the transcendence over the Cartesian "abyss" can be omitted as a "Wittgenstein ladder" in the ultimate text of the proof at the cost of inexplicable otherwise artificiality, for which the heuristic pathway to it turns out be absolutely and intentionally hidden just as a magician who hides the real series of the actions thanks to which the "rabbit has been taken out of the cylinder" to the delight of the audience.

Even more, one can conjecture (also in *Part II* in detail) that the option for any trancendendence over that abyss to be abandonable is universal in the final analysis so that the ultimate result is representable "classically", e.g. in terms of "classical physical" or in those of mathematics "granted to be standard". Then, the "journey over the abyss" would be necessary only heuristically, but not formally and logically.

One can trace that "magician art" in its "atom" of a single qubit either "empty" (in a coherent state) or absolutely determined by a certain value (read or written). In fact, both FLT and Gleason's theorem as ultimate results can be related only to the former option of an "empty" qubit (though "outside" or "inside", accordingly). However, the heuristic pathway for both to be proved needs the former option to be transcended by its relation to the latter, that of an absolutely determined value and thus needing the dual branch of the qubit Hilbert space, nevertheless FLT can be thoroughly formulated within Peano arithmetic, respectively in Hilbert arithmetic in a narrow sense. Furthermore, the proof of each of both theorems can be accomplished in its "native" branch as, for example, Gleason's original proof or the proof of FLT in "Fermat's arithmetic" in *Part I* can convincingly show.

If the hypothesis about the absolute representability of any statement alleged to be non-classical (such as those of quantum mechanics or FLT in the case) can be anyway demonstrated formally and logically "classically", though at the cost of some artificiality, quantum neo-Pythagoreanism advocated during the paper is only a method of thinking rather heuristic than logically necessary. However, nobody knows whether that conjecture is a true statement in any case.

As to mathematics properly, the same tenet can be articulated even more discernibly. A constructive proof for any true statement exists always in Hilbert mathematics (besides no Gödel insoluble statements in it), but it is to be represented as a proof of existence passing into Gödel mathematics (for example if the proof is necessary to cross the dual area of Gödel insoluble statements).

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One can stare at the relation of two probabilities which any determined value of a qubit means and alleged to remain hidden in each arithmetic unit ("1") after the interpretation of Peano arithmetic by Hilbert arithmetic. Gleason's theorem implying only a single probabilistic measure for any string of qubits and thus also for one qubit means that it links two probabilities unambiguously. If one follows the usual interpretation of information as relative entropy (i.e. entropy to the entropy of another probability distribution), each qubit may mean just an elementary relative entropy: that is a probability in relation to another probability eventually granted to be a standard to the former probability deviating more or less from the latter.

If the "standard"[17] probability has been ascribed to be "objective probability" and thus, referring to the Cartesian "body" ("object"), the other probability linked in any qubit is to mean "subjective probability" of the Cartesian "mind" ("subject"), or respectively, our knowledge about the former, ostensibly real probability. Interpreted so, any qubit turns out to be a generalization of the usual episteme of Modernity and what is generalized can be enumerated as follows:

Two fundamental philosophical entities are distinguished, but which is "body" and which is "mind" is absolutely conventional: only its relation embedded in each qubit is what can be rigorously and unambiguously defined. The same relativity can be seen also as a universal invariance: already invariant even to the pair of subject and object finishing the modern development in physics to more and more general invariances (a direction especially discernible in Einstein's theories of relativity), and thus that of objectivity being understood as that extending invariance.

Any qubit is furthermore a relation just of probabilities therefore substituting the classical idea of "fact" with that of the corresponding idea about the probability of a fact and as if "undermining" what "fact" is as far as the classical fact needs always the absolute reliability of being tautologically true. Any qubit can be considered to be an elementary fact if "fact" has been generalized in advance as above.

Kronecker's slogan that "Natural numbers were created by God, everything else is the work of men" (Weber 1893: 15; Kneser 1925: 221) can be already justified otherwise rather than as "God's creation". Indeed, if one continues the direction of generalization followed to be constituted the concept of "qubit" meant as an elementary ("postmodern") fact, the next generalization would be to be the independence of any given fact of that kind, i.e. the fact at all, which can be identified as an empty qubit, and following the intentions and objectives of Hilbert arithmetic, as a natural number in fine as well.

Summarizing those considerations, one can reveal the reason for proving FLT so difficulty in the misunderstanding of what the natural numbers are (since it is a statement in terms of them), for example, as Kroneker postulating them to be God's creation therefore "closing the door" for investigating their real origin. Speaking rather loosely or aphoristically, FLT needs the reason for natural numbers to appear not by virtue of God's will but following scientific methods. They originate from the generalized "facts at all" after the latter have been realized to be qubits after quantum information.

Said even more metaphorically, one needs a few aphorisms to precede the first two ones in Wittgenstein's "Logisch-Philosophische Abhandlung" (1921, in English "Tractatus Logico - Philosophicus"), which are: "1. Die Welt ist alles, was der Fall ist. 1.1. Die Welt ist die Gesamtheit der Tatsachen, nicht der Dinge"[18] since the natural numbers should be before "facts" meaning the same as "the world is all": or at least, should mediate after the former, but before the latter statement. If, on the contrary, the origin of the natural

[17] It is a question of convention: which of both probabilities to be proclaimed as a "standard".

[18] "The world is everything that is the case.The world is the totality of facts, not of things" or "The world is all that is the case. The world is the totality of facts, not of things" in English according to the bilingual edition (2018) available at: http://people.umass.edu/klement/tlp/ (2022.06.07 accessed).

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numbers is "7. Wovon man nicht sprechen kann, darüber muss man schweigen"[19], FLT cannot be proved during a few centuries.

6. The idea of *quantum* number theory inspired by the proof of *flt* from the kochen - specker theorem or gleason's theorem

Hilbert arithmetic involved for proving FLT as the paper demonstrates can be also realized as a method for problems in number theory, which can be called "quantum number theory" for the utilization of the qubit Hilbert space as well as the separable complex Hilbert space as relevant tools. The reasons can be divided into two groups: (1) Hilbert arithmetic is suitable for investigating statements for which one can prove that they belong to the class of the Gödel insoluble proposition if set theory along with arithmetic is forced to be used for their solution; (2) Hilbert arithmetic in a wide sense suggests an instrument similar to infinitesimal calculation, but generalizing it[20] in a way quite relevant to arithmetic being inherently discrete so that both discrete and continuous (smooth) description are unified.

(1) As one can see in *Part I*, FLT obeys Yablo's paradox and thus, it is a Gödel insoluble statement if one needs set theory together with arithmetic for its proof, on the one hand. On the other hand, set theory seems to be inevitable for the proof as it is to mean the infinite set of all cases of Fermat's equation for exponents greater than two. The only way for Peano arithmetic to prove so general statements is induction in virtue of the axiom of induction but conflicting with the axiom of infinity in set theory just in relation to *all* natural numbers: each of which is *finite* according to the former, but the set of them being *infinite* according to the latter. In the final analysis, this contradiction itself is what generates the Gödel incompleteness (to be avoided) in the framework of the Gödel dichotomy of arithmetic to set theory: either incompleteness or contradiction.

However, the application of the axiom of induction to FLT seems to be impossible at first glance since Fermat's equation involves the sum of two exponents though they are equal. Hilbert arithmetic hints how that obstacle can be overcome even remaining thoroughly in the framework of Peano arithmetic.

Summarizing, one can utilize Hilbert arithmetic and therefore its inherent completeness where the idea of a set-theoretical proof of an arithmetic proposition has to be "retold" only in terms of induction, e.g. for getting rid of the Gödel incompleteness.

(2) A strange and rather inexplicable empirical fact is that many very difficult problems in number theory can be elegantly resolved involving the infinitesimal analysis of complex variables not having any relation to their formulation only in terms of arithmetic. Furthermore, quantum mechanics turned out to be forced to reveal a general method for unifying the discrete description of any quantum entity "by itself" due to the fundamental Planck constant with the smooth description of the apparatus and its readings obeying classical mechanics or physics.

The separable complex Hilbert space able to unify Schrödinger "ondulatory mechanics" with Heisenberg's "matrix mechanics" can be interpreted to be that general method, thus relevant rather to mathematics than to physics (just as Newton's infinitesimal calculation is relevant to mathematics though he used it for its physical theory of gravitation). As this is very well known, infinitesimal analysis is the main method nowadays for creating mathematical models of various processes taking place over time, not only physical ones.

One can mull the way for any discrete experimental or empirical data (as far as the result of any measurement is a rational number) to be always thought as interpretable by

[19] "Whereof one cannot speak, thereof one must be silent" or "What we cannot speak about we must pass over in silence" in English according to according to the bilingual edition (1918) available at (2022.06.07 accessed): http://people.umass.edu/klement/tlp/ (2022.06.07 accessed).

[20] The viewpoint to Hilbert arithmetic as a generalization of infinitesimal calculus is suggested in more detail in another paper (Penchev 2021 February 25); then applied to the problem of mathematical history (Penchev 2020 December 14).

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a relevant system of smooth (thus continuous) differential equations, to which the mathematical model built by intestinal analysis can be usually reduced. Indeed, any transition from those data to the model at issue would encounter the Gödel incompleteness, which can be discovered in any limit (a real or complex number in general usually) approached by an enumerated series of experimental data though becoming more and more precise:

The observed data are always a finite series of finite (rational) numbers. Even if one ascribes to the model as an infinitely close real number as a limit of the data, this cannot prevent the gap of the Gödel incompleteness between them. The transition to any model cannot be ever justified. It needs a human being to establish just this model rather than another since no model follows logically and rigorously from the data[21].

However, that abyss between data and the corresponding model is never considered to be a problem since it is granted in the organization of any possible knowledge or cognition in Modernity therefore guaranteeing the figure of human as an unavoidable arbiter to judge whether a model and reality (data) correspond to each other or not.

In other words, the Gödel incompleteness or dichotomy is only a mathematical paraphrase of the modern episteme relying rather on the conviction of its validity than on logical and mathematical arguments[22]. The Gödel incompleteness can be also revealed in any physical motion or in the unfoundedness of any logical conclusion as, for example, the ancient aporia about Achilles and the Tortoise or its interpretation by Lewis Carroll can demonstrate (Penchev 2021 November 18).

There exists even the temptation to suggest that the initial and fundamental unfoundedness of modern cognition is intentionally supported to provide the dominance of human society and it hierarchy over nature and the world as far as a human decision independent of any logical justification is obligatory for any statement originating from data to be heralded for truth. One can admit further that FLT needing for its proof the boundary of the modern organization of cognition to be transcended is a relevant symbol of modern episteme underlain by the figure of human and "capability of judgment" embedded in the foundation of objectivity: objective cognition and knowledge.

However, quantum mechanics has been forced to invent an implicit alternative approach to what condition should hold in order to be able to resolve the main problem in its framework: how both discrete and continuous descriptions of quantum reality and its data can be unified. The "capability of judgment" cannot be more a privilege only of human beings. It has be shared even with any electron, a corollary from quantum mechanics generated the famous words of Einstein (1926) in a letter to Born that "God does not play dice", and why the thought about an electron deciding its behavior is so unbearable for him so better to be a "croupier in a gambling house or a shoemaker rather than a physicist" if that is the case.

Alas, the "free will theorems" (Conway, Kochen 2006; 2009) deduce rigorously logically and mathematically from a few standard statements of quantum mechanics and special relativity that if the experimenter, a human being, is gifted by free will, he or she has to share that "valuable commodity" with the observed quantum entity, for example an electron as meant by Einstein.

The revolution of quantum mechanics in relation to what cognition is to be consists just in the deprivation of the privilege of all human beings to be the only judges able to decide which is true and which is not. The capability of judgment should be "democratized" and extended to cover even "electrons". Einstein is absolutely correct that the

[21] This is a widely exploited motif in Western philosophy, for example as a criticism of whether induction as "Hume's problem" (e.g.: Belkind 2019; Jackson 2019; Biondi, Groarke, eds. 2014; Qu 2014; Steel 2010; Tucker 2009; Hetherington 2008; Weintraub 2008; 1995; Loeb 2006; Okasha 2005; 2003; Boulter 2002; Lipton 2002; Helm 1993; Jakobson 1987; Parush 1977) or causality as "Hume's theory" of it (e.g.: Henschen 2018; McBreen 2007; Watkins 2005; 2004; Jakobson 1986; Röd 1983; Ducasse 1966).

[22] The idea is formulated for the first time in an earlier paper (Penchev 2010).

formalism of quantum mechanics (granting his own theory of special relativity) implies the "free will of electron" in fine and just that is precisely inferred by Conway and Kochen.

Also one can stare at the separable complex Hilbert space to reveal where exactly is embedded "the electron's free will" to show that FLT has not been proved for a few centuries just because the "gifting of all electrons with free will" has been so "unbearable" for all scientists as for Einstein, indeed contradicting fundamentally the modern organization of cognition. On the contrary, if one dare gift the "electrons" with free will, the proof of FLT turns out to be sooner elementary and therefore needing rather philosophical efforts for changing the Gestalt than mathematical skills for a long enough syllogism however within common sense's Gestalt.

A drama of ideas can be embodied in a plot involving Descartes and Fermat (being ten years younger) as personages. For example, Descartes created Cartesianism establishing the gap between "body" and "mind" able to be overcome only by God, and practically, by God's vicegerent on earth, i.e. any human being. Once that had been done, Fermat's proof of his last theorem should be lost for its sense contradicted the modern episteme as it had been founded by Descartes before that. That is the present paper tends to clarify why the thought "unbearable" to Einstein for the electron's free will along with that of any human being contributes the proof of FLT; as well as it even conjectures a hypothetical proof of FLT, accessible to Fermat (in *Part I*).

The ultimate human decision eventually changeable by some next human decision is replaced by an abstract choice (also mathematically guaranteed by the axiom of choice) as if by the universe itself as a whole and thus by any electron not less than by any human being. The dominance of humankind is dethroned. Human chauvinism is not more than another illusion rejected by science as ridiculous nonsense due to ignorance. So, if one abandons the myth of human uniqueness in comparison with e.g. an electron (i.e. according to the thought so unbearable to Einstein) FLT is easily provable, extending and generalizing that conclusion of quantum mechanics.

In other words, the alleged difficulty for proving FLT is rather ostensible or "ideological" since the only obstacle is the "ideology of human uniqueness" underlying and supporting furthermore all the building of modern cognition and the refusal of which is "unbearable" even for one of the greatest minds as Einstein. On the other hand, that circumstance can explain rather the philosophical (op "anti-ideological") style of the present paper since a relatively simple proof of FLT also accessible even to Fermat needs the philosophical reflection of common sense's prejudice and human self-admiration.

The "atom" of that abstract choice originating from all entities in the universe, and also from humankind only within that framework, is any qubit, the unit of the generalized, quantum information linking unambiguously two probabilities and interpretable e.g. as the relation of a probability originating from any entity in the universe to the same probability where all entities or the universe as a whole contribute to it.

That realization of a qubit can elucidate why its idea is inconsistent with the "ideology of human chauvinism" privileging all human decisions at the expense of the (only fictional) deprivation of any entity's irrevocable participation in the right to choose the "Tao of the universe". The universe is democratic and humankind is only a "crazy" entity among all, madly imagining to be unique by the domination over others. In fine, humankind is the "Nude King" dressed in the missing clothes of his majesty.

Once, the conception of qubit has been postulated in the foundations of the universe, furthermore coinciding with those of both mathematics and physics in quantum neo-Pythagoreanism, that of an arithmetic unit can be naturally understood as originating from that of qubit, e.g as an "empty" qubit and the pathway to Fermat's "lost proof" can be already pioneered, however necessarily "sacrificing" humankind's "insanity" described above, or really rather, "recovering" from it.

Accordingly, quantum number theory utilizing the qubit Hilbert space (or respectively, the separable complex Hilbert space of quantum mechanics) investigates rigorously logically and mathematically what has been mythologized and tabooed by the ideology of human chauvinism as the ostensibly unrepealed "prerogative of human

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decision" and representable, for example and in particular, as the Gödel incompleteness of arithmetic to set theory. The elementary proof of FLT needs that area (then defined as non-arithmetic, but set-theoretical) to be involved at least heuristically since it can be abandoned finally in the ultimate result as a "Wittgenstein ladder".

If one compare the tool of Hilbert arithmetic with that of infinitesimal analysis invented by Leibniz and Newton, and as to the latter, in order to be applied to his physical theory published as "Philosophiæ Naturalis Principia Mathematica", the corresponding generalization can be realized as follows, consisting in the substitution of the transition to the infinitesimal limit (model) of observable data by the quantum complementarity or mathematical duality of the model and data:

The transition from the discrete data to the classical smooth model being, logically and consistently impossible for the Gödel incompleteness and needing always a human arbiter, is replaced by their complementarity and duality, both being complete and involving a universal and omnipresent concept of "choice" and "decision" not originating from human beings. Properly and mathematically, the substitution consists in a probability (eventually, density) distribution mediating between the discrete data (such as data referring to quantum entities and studied by quantum mechanics) and representable by a wave function, which is the characteristic function of the corresponding probability distribution (known and studied for a long time in probability theory).

Speaking more or less figuratively, the Gödel incompleteness in the case (i.e. that situated between quantitative data and any mathematical model) is described quite rigorously, but probabilistically thus not needing any human "judge" to decide for the eventual conflict between them. This can be visualized rather instructively by the application of infinitesimal analysis to model any temporal process implying a certain function of time derivative and thus the quantity of "time", whether to the process at issue or universally, as a necessary condition for any time derivative.

On the contrary, quantum mechanics cannot consistently involve time as an operator in Hilbert space at least to the investigated quantum entity "by itself" and that fact is well known being expressly articulated yet e.g. by Pauli (1980: 63, footnote 2). The reason for that additional distinction from classical mechanics is fundamental. Quantum mechanics can be interpreted as that it investigates how time appears "ex nihilo" unlike classical mechanics, for which time "ready-for-use" is supposed to be available in advance.

Then, the mathematical formalism of the relevant Hilbert space, furthermore tested and corroborated in all experiments of quantum mechanics, can represent the way for time to come into being as to any process modeled to be temporal by classical physics or other exact sciences. For example, the "Big Bang", the temporal beginning of the universe, in fact as it suggested by classical physics, enumerating within which also general relativity, by means of that time "ready-for-use", being furthermore a necessary condition for all models offered by it, can be consistently realized only by quantum mechanics alone, by a probabilistic description in the mysterious and contradictory moment of the "zero time". That time necessary for any representation of classical physics can be relevant only "later", i.e. after a *finite* interval of time after the "zero time" of the Big Bang[23].

However, the present paper demonstrates furthermore, that the same mechanism (though slightly modified as the qubit Hilbert space after quantum information) able to make clear the genesis of time itself, can be applied (and this is maybe the main idea of quantum number theory sketched now) to the genesis of (Peano) arithmetic itself (a problem relevant to Husserl's intellectual background and especially, to his quite newly approach shown in "Logical investigations": *Penchev 2021 July 26; Penchev 2020 June 29*).

[23] In fact, if one wishes to investigate the "zero time" of the universe by classical infinitesimal methods (for example, the "right" time derivative unlike the "left" one exists anyway), the Big Bang would rather represent an integral of all decoherence in the universe, a viewpoint discussed in more detail in another paper (Penchev 2020 August 31).

Though FLT is thoroughly formulated in a kind of arithmetic "ready-for-use" (for example, as it is axiomatized by Peano), its proof (at least heuristically) needs a penetration into how arithmetic can appear "ex nihilo" (though that understanding can be abandoned in the ultimate proof as long as one comes to terms with the artificiality of the latter), but by means of Hilbert arithmetic in a wide sense and following the methods and thesaurus elaborated by quantum mechanics for quite different objectives, but only at first glance.

This implies furthermore a reinterpretation and re-realization of quantum mechanics able to deliver a *general theory of genesis* even "before time", the concept of which suggests that the genesis at issue had been over "before" it appeared. Indeed, quantum mechanics has identified itself only with a particular application of that general theory of genesis and referring to the genesis of the readings of the macroscopic apparatus reflecting a microscopic quantum system by itself.

7. Instead of conclusion: both idempotency and hierarchy after *flt* proved in hilbert arithmetic

The introduction of the paper (in *Part I*) considers the fundamental relation of idempotency and hierarchy and interprets it philosophically, by means of scientific transcendentalism, as the most general and initial relation originating from the postulate of the totality. To the viewpoint sketched in the introduction, the entire text after it makes clear that the proof of FLT needs as a necessary condition, whether explicitly or implicitly, that relation of completeness; or vice versa: in Gödel mathematic being inherently incomplete and just as still one embodiment of Cartesian dualism, it is unprovable, as one can show by Yablo's paradox, and any real proof such as Wiles's transcends obligatorily though eventually secretly and shyly in order the "Boeotians" or common sense not to feel and notice that it is inconsistent with the general organization of cognition in Modernity therefore announcing in fact one of the most fundamental scientific revolutions of all times (though its beginning has been already heralded for about a century by quantum mechanics, but now extended to mathematics and philosophy).

The last *Section XXIII*, seen as the ultimate result of the same course of thought penetrating the entire text, discusses the generalization of Hilbert arithmetic as a research of how Peano arithmetic appears absolutely "ready-for-use", hiding its origin and the way to be elaborated. This section being "instead of a conclusion" intends to link the beginning of the paper with its last section as a circular structure claiming to be a "hermeneutic circle". In other words, the idea is the process in which arithmetic comes into view to be reflected and re-interpreted in terms of idempotency and hierarchy and its whole as completeness.

The creation of arithmetic can be simultaneously realized properly philosophically, as a general doctrine of how hierarchy arises in a way to hide its origin and elaboration, i.e. as if "ready-for-use" but only by the mediation of idempotency, and speaking metaphorically, by the "murder of its twin" been equally possible and idempotent to it. This can be expressed also otherwise, mathematically and by means of the nonstandard bijection though it has been formulated only in relation to arithmetic and its dual branches as above: "$P \to (P^0 \to P^- \otimes P^+)\ \&\ (P^+ \otimes P^- \to P^0) \to P$" where P (also modified by the several indexes) means "Peano arithmetic", which now is to be generalized rather philosophically as the concept of hierarchy though remaining homomorphic to Peano arithmetic. This can be illustrated only by the replacement with a symbol called to mean "hierarchy", e.g. by "H": "$H \to (H^0 \to H^- \otimes H^+)\ \&\ (H^+ \otimes H^- \to H^0) \to H$".

Then one can notice that the homomorphism of hierarchy to Peano arithmetic is "rudimentary", accidental and redundant in that sense because the arithmetic operations of addition and multiplication though formally definable for a hierarchy are rather meaningless to its essence. This can be traced back to its origin in Dedekind's paper "Was sind und was sollen die Zahlen?" (1988), by the by, cited expressly by Peano (1989: 5) in his study about the axioms of arithmetics[24].

Then, all natural numbers can be understood to be an abstract well-ordering following from the axiom of choice in virtue of its equivalency to the well-ordering theorem. Indeed, that well-ordering in turn implies the usual arithmetic and just that fact had been inspired Peano for his famous concise list of axioms, which are to be rather related to that abstract well-ordering meant by Dedekind and from which arithmetic can be deduced as a "side effect" or an "artifact" once that abstract well-ordering has be introduced in advance.

So, the concept of hierarchy is to be related to Dedekind's abstract well-ordering though it is homeomorphic to Peano arithmetic, but the last circumstance is inessential to hierarchy though being formally relevant. Thus, the sense of how (Peano) arithmetic appears from Hilbert arithmetic is to be reduced to its philosophical core of the way of any hierarchy to come into being, relevant furthermore to its change, after which hierarchy doubles itself therefore vanishing and then "crystallizes" into a new one. Only those segments of hierarchy, which will be changed, need be doubled, passing into an intermediate amorphous state of "missing hierarchy" to the new one.

If one restricts the consideration only to Hilbert arithmetic in a narrow sense the counterpart of the hierarchy at issue is unambiguously determined to be the anti-isometric one, thus both being idempotent to each other. Any pair constituted in this way can be interpreted as a bit of information and the state of vanishing hierarchy to be substituted by another corresponds to erasing the value (e.g. either "0" or "1") in a binary cell in order to record a new one.

However, if one generalizes the discussion in the framework of Hilbert arithmetic in a wide sense, the counterpart of the hierarchy at issue cannot be more determined unambiguously therefore being able to be any different hierarchy under an additional condition to be different enough from the former one and needing moreover to be further formulated quantitatively and mathematically in a future work.

The latter case admits a very essential and important visualization by the way for arising a new biological entity after a sexual fusion of two different enough DNA (possibly RNA) originating from both female and male parents. Indeed, a male hierarchy and a female one constitute a child DNA (RNA) consisting of only four "letters" (adenine, cytosine, guanine, and thymine; or uracil, and for thymine in RNA) being supposedly the minimal sufficient number of letters for sexual reproduction. This can be even proved rigorously (Penchev 2020 July 17) and mathematically inherently linked to the famous mathematical problem about the four colors for any (geographical) map (respectively, "mapping" in the exact mathematical meaning).

The same example can make clear the meaning of "different enough" as to the two interacting hierarchy to create a new one: indeed, both parents should belong to the same biological species or eventually, to two different, but sufficiently close and able to create a viable generation. So the sense of "two different enough hierarchy" has to be understood as an admissible *interval* of difference therefore possessing both lower and upper limits rather than only a lower one.

Meaning the last observation heuristically and in relation to the corresponding operator in the qubit Hilbert space, able to transform the DNA of the female parent in that of any DNA belonging to an admissible male partner for a viable generation (not necessarily being a Hermitian one), one can define "continuity" even to discrete transformations in a generalized (again probabilistic) mathematical meaning furthermore reducible to the standard definition of mathematical continuity, e.g. by the limit of an enumerable sequence of members.

The suggested example by the sexual fusion of two different, male and female DNA (RNA) illustrates how wide the horizon suggested by the context of the proof of FLT in

[24] Another paper (Penchev 2020 August 25) traces back that link in detail.

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Hilbert arithmetic can be. Indeed, its framework is able to include even areas seeming to be rather remote from scientific transcendentalism immediately applicable to physics: namely that of sexual reproduction (or even: mutations due to reproduction) belonging to genetics and more loosely, to biology.

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