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An Innovative Method for Approximating Arcsine Function

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Abstract

This paper presents a new method for approximating the classical arcsine function. The proposed approximating methodology is simpler in its approach than other classical approaches and undeniably innovative. It is based on matrix representation besides the basic interpolation to approximate the inverse trigonometric function. It provides an efficient model which allows for reliable and precise calculations. The results are as per our knowledge unseen results in the previous literature.

Mathematics Subject Classi ication (2010). 26E99, 33B10, 42A10

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1. Introduction

Inequalities, polynomials, series expansions and approximants are largely used in the approximation of trigonometric functions and their inverses. The approaches for approximation of these functions in the present literature are however often based on sharpening and refining bounds by establishing two-sided inequalities which contain obvious complexity as far as possible application is concerned. For example, for the inequalities involving arcsine function we refer reader to [2, 4–7, 9–13] and the references therein. The arcsine function can also be expressed as a Taylor's series or in terms of hypergeometric function [1, 3, 8]. For some applications involving arcsine function all the above mentioned approaches or representations do not sound well and we need a simple algebraic or polynomial approximation for arcsine function. The main purpose of this article is to propose an innovative and unseen method in order to attempt a new way of approximating the arcsine function. The proposed approximation method is relatively simple, elegant, and provides manageable closed-form approximations with reliable results. The rest of the article is organized as follows. Section 2, presents our innovative method for approximating the arcsine function. Section 3 is devoted to the possible application and we conclude in Section 4.

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2. An innovative method

We start with the classical well-known inverse sine function which is formally calculated through a classic textbook trigonometric substitution such as

$$\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}x,\ x \in [0,1] \tag{2.1}$$

with the variable t properly substituted by $\sin(\alpha)$ to get the known classical result. In order to approximate this function, let us observe that by a simple assumption by considering the possibility that the above square root term in (2.1) to be integrated can be somehow written as the following equality with two undetermined coefficients C_1 and C_2 being at this point unknown entities

$$\frac{1}{\sqrt{(1-x)(1+x)}} = \frac{C_1}{\sqrt{1-x}} + \frac{C_2}{\sqrt{1+x}}. \tag{2.2}$$

We are going to propose two new and original approaches for evaluating approximately the inverse sine function. We call these approaches a "Single set of points approximation" and "Multi-set of points approximation". In what follows, our first approach is proposed and elucidated.

2.1. Single set of points approximation approach

Assuming that (2.2) is algebraically possible. It is notable that the variable x cannot be either 1 or -1. It is known that the domain and the range of the arcsine function being respectively $[-1,1]$ and $[-\pi/2,\pi/2]$. Let us choose for calculus' sake one single set of points, $\{x_1, x_2 : x_1 \neq x_2\}$ belonging to the restrained domain $]-1,1[$ of the arcsine function. For example, we take the set $\{-0.5,+0.5\}$ and do some computations at these two chosen points for the equation (2.2). This will lead us straight up to the following linear system of equations:

$$\begin{aligned} \sqrt{\frac{4}{3}} &= C_1\sqrt{\frac{2}{3}} + C_2\sqrt{2} \\ \sqrt{\frac{4}{3}} &= C_1\sqrt{2} + C_2\sqrt{\frac{2}{3}} \end{aligned} \tag{2.3}$$

which in matrix representation is

$$\begin{bmatrix} \sqrt{\frac{2}{3}} & \sqrt{2} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{4}{3}} \\ \sqrt{\frac{4}{3}} \end{bmatrix}. \tag{2.4}$$

Equivalently, $AX = B$, where, the left-hand side matrix A turns out to be of rank 2 invertible matrix as we can see from the determinant

$$\Delta A = |A| = \begin{vmatrix} \sqrt{\frac{2}{3}} & \sqrt{2} \\ \sqrt{2} & \sqrt{\frac{2}{3}} \end{vmatrix} \neq 0. \tag{2.5}$$

So we infer that the linear system is solvable. By applying Cramer's rule and after simple computation, we write real values for our unknown coefficients C_1 and C_2. We have to emphasize here that the choice of the set of points is not unique and we can choose any set of points over the interval $]-1,1[$ with no particular restriction besides the fact that the two points must be distinct in order to have a system of distinct equations. The resulting linear system can be solved eventually by simply applying Cramer's rule to determine the

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coefficients such as for our example (2.3):

$$C_1 = \frac{\Delta C_1}{\Delta A} = \frac{\begin{vmatrix} \sqrt{\frac{4}{3}} & \sqrt{2} \\ \sqrt{\frac{4}{3}} & \sqrt{\frac{2}{3}} \end{vmatrix}}{|A|}; \; C_2 = \frac{\Delta C_2}{\Delta A} = \frac{\begin{vmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{4}{3}} \\ \sqrt{2} & \sqrt{\frac{4}{3}} \end{vmatrix}}{|A|} \tag{2.6}$$

If we do some computation, one interesting fact here is that it turns out that C_1 and C_2 are identical and this is because of the particular "symmetric" choice of the set of points made such as the set $\{-x_i, +x_i\}$. If we choose a random set of points, we will end up with different values for these coefficients. Regardless of the choice, the sum of these coefficients is always expected to be close to one. However, if we look at a compact expression as in (2.2), a wise choice indeed would be a symmetric set of points over the domain. Assuming now that C_1 and C_2 are known coefficients and in fact equal, we can now search for the anti-derivatives which are easily found as follows

$$\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x = C_1(\text{or } C_2)\int\left(\frac{\mathrm{d}x}{\sqrt{1-x}} + \frac{\mathrm{d}x}{\sqrt{1+x}}\right) = 2C_1\left(\sqrt{1+x} - \sqrt{1-x}\right) + C_{nt}, \tag{2.7}$$

where C_{nt} is the integration constant. For our example, and for this particular single set of points $\{-0.5, +0.5\}$, we compute the corresponding identical coefficients as

$$C_1 = C_2 = C_i = \sqrt{\frac{3}{2}} - \sqrt{\frac{1}{2}} \approx 0.517638090205.$$

Now we can approximate properly the arcsine function by the following relatively simple expression with the above computed value and C_{nt} being the expected integration constant:

$$\arcsin(x) \approx 2C_i\left(\sqrt{1+x} - \sqrt{1-x}\right) + C_{nt} = I_{arcsine}. \tag{2.8}$$

Let us recall that the arcsine function is often approximated by the Taylor's series expansion as

$$\arcsin(x) \approx x + \frac{x^3}{6} + 3\frac{x^5}{40} + 5\frac{x^7}{112} + \theta(x). \tag{2.9}$$

We simulate in the next figure the above results (2.8) and (2.9) to compare the approximated expression to the real value of the arcsine function without yet taking into consideration the integration constant ($C_{nt} = 0$).

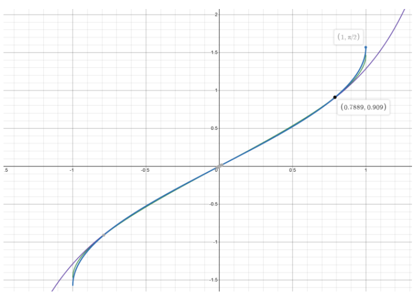

Figure 1. Purple curve: Taylor's series approximation (degree $n = 7$) in (2.9); Green curve: approximation in (2.8); Blue curve: arcsine function

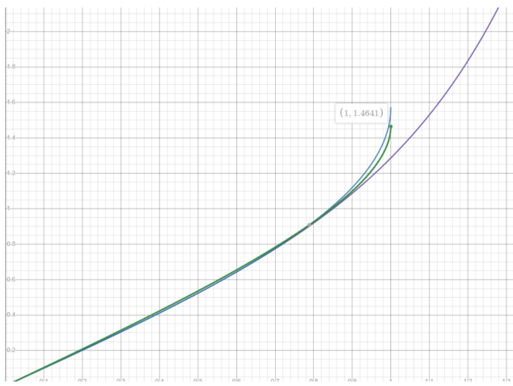

Figure 2. Focus near domain extremities in Figure 1

As we can see especially at the extremities of the domain $[-1, 1]$, the approximation in (2.8) for the arcsine function is much better than the classical Taylor's series expansion.

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This relatively simple conceptual approach gives a remarkable match with the true arcsine function but there is of course a residual error that we deliberately overlook as we did not consider the integration constant in our approximated expression. Let us take first the approximated expression (2.8) without any correction. Let it be the following proposition for the approximation of the inverse sine function.

$$\arcsin(x) \approx I_{arcsine} = 2C_i\left(\sqrt{1+x} - \sqrt{1-x}\right) \tag{2.10}$$

The simulation of the errors given by these two approximations (2.9) and (2.10) comparing with the original function, arcsine is given in the following Figure 3.

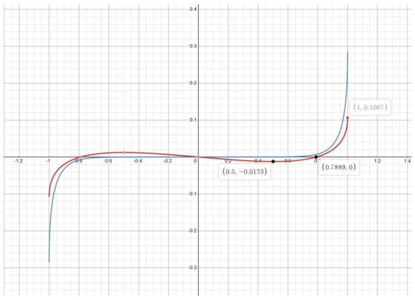

Figure 3. Error's profile for Taylor's expansion(blue curve) and the approximation $I_{arcsine}$(red curve)

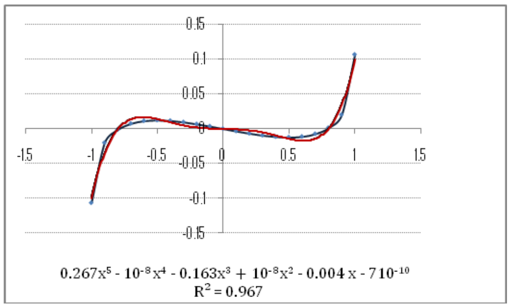

Figure 4. Polynomial interpolation (red graph) of the error (arcsine - $I_{arcsine}$)(blue graph)

An approximation in expression (2.8)(or (2.10)) shows a stable profile of errors, especially at the extremities. Taylor's approximation shows a growing error profile as we get from the origin. Moreover, the approximation (2.8) can be improved if we consider the non-zero integration constant C_{nt} as a residual error corresponding to each particular set of points. So we can use interpolation of these errors to attempt a correction of the values given by $I_{arcsine}$. In order to do that, we have to conceal that C_{nt} is a constant. Thus C_{nt} could be a variable than a constant and in the case of polynomial interpolation, we can write the following interpolation tendency of degree 5.

$$p(x) = 0.265x^5 - 10^{-8}x^4 - 0.163x^3 + 10^{-8}x^2 - 0.004x - 7 \cdot 10^{-10}.$$

We discard the near-zero terms such as $10^{-8}, 10^{-10}$ etc. So we can approximate the errors by the following expression:

$$p(x) = 0.267x^5 - 0.163x^3 - 0.004x.$$

Remark 2.1. The interpolation of the errors above is done with a coefficient of determination $R^2 = 0.967$.

We can finally admit without much presumption that the arcsine function can be approximated by the following corrected expression

$$\arcsin(x) \approx I_{arcsine} = 2C_i\left(\sqrt{1+x} - \sqrt{1-x}\right) + p(x). \tag{2.11}$$

The new profile of the errors given by corrected (2.11) shows indeed an improvement compared to the uncorrected one (2.10).

Comparison with the previous Taylor's series approximation clearly shows that our approximated corrected expression (2.11) gives less error than Taylor's approximation with almost null error at the ends. This is seen especially near the domain extremities

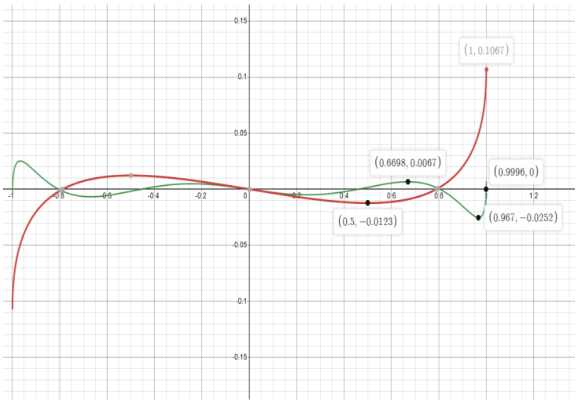

Figure 5. Errors profile uncorrected $I_{arcsine}$ (2.10)(red curve) and corrected with polynomial regression (2.11) (green curve)

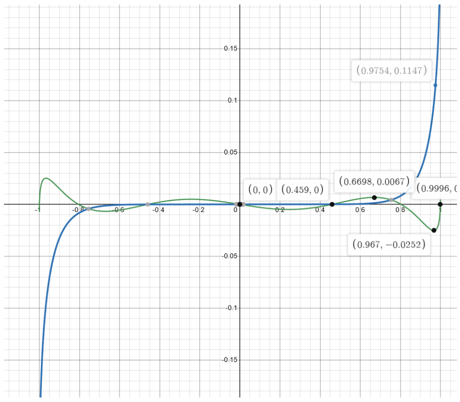

Figure 6. Errors profile of corrected $I_{arcsine}$ (2.11) with polynomial interpolation(green curve) vs. Taylor's series expansion (2.9) (blue curve)

as pointed out in Figure (6). We support our claim by investigating the global L_2 error which is defined as follows:

$$e(u) = \int_{-1}^{1} (\arcsin(x) - u(x))^2 \, \mathrm{d}x$$

where $u(x)$ is either $I_{arcsine}$ or Taylor's series expansion in (2.9). For Taylor's series, the value $e(u)$ is ≈ 0.00212488 whereas for $I_{arcsine}$ in (2.11), the value $e(u)$ is ≈ 0.000130061.

Thus with the single set $\{-0.5, +0.5\}$ of points we propose a simple, affordable and better approximation for the arcsine function as follows:

$$\arcsin(x) \approx \sqrt{2}(\sqrt{3}-1)(\sqrt{1+x}-\sqrt{1-x}) + 0.267x^5 - 0.163x^3 - 0.004x. \qquad (2.12)$$

The corresponding relative error of the above approximation in (2.12) and Taylor's series approximation (2.9) can be found in Table 1.

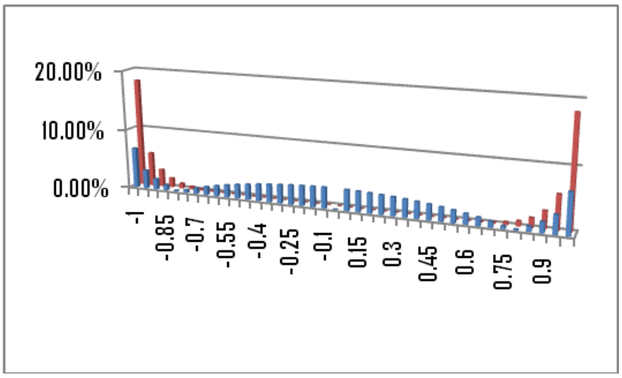

Figure 7. Relative errors of corrected $I_{arcsine}$ (blue graph) vs. Taylor's series expansion (red graph)

2.2. Multi-sets of points approximation approach

We have presented in the previous section a new method for approximating the arcsine function without following the widely known classical methods. In this section, we will attempt to improve the precision of our method of approximation by introducing a multi-sets of points approach over the domain. We have already previously shown that the

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values of the coefficients C_1 and C_2 depend on the choice of the one particular set of points $\{x_1, x_2 : x_1 \neq x_2\}$ over the restricted domain $]-1,1[$. Each and every set of points gives a corresponding set of coefficients C_i and hence the number of calculated coefficients depends on the number of chosen sets of points over the interval. We choose a sampling rate of 0.05 over $]-1,1[$ and for the practical reason we choose sets of symmetric points $\{-x_i, +x_i\}$ over the interval in order to reduce the number of coefficients C_i's. Applying the same approach as before for the single set of points in the previous section, we calculate for each set $\{-x_i, +x_i\}$ of points a single value C_i of the coefficient. The calculated coefficients, in this case, are shown in Table 2. The profile of coefficients C_i's is as shown in Figure 8.

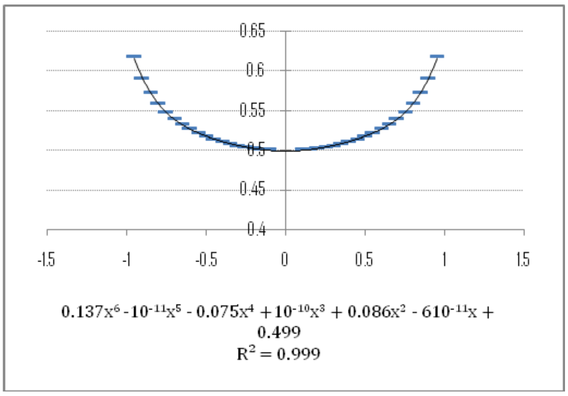

Figure 8. Profile of coefficients C_i over $]-1,1[$

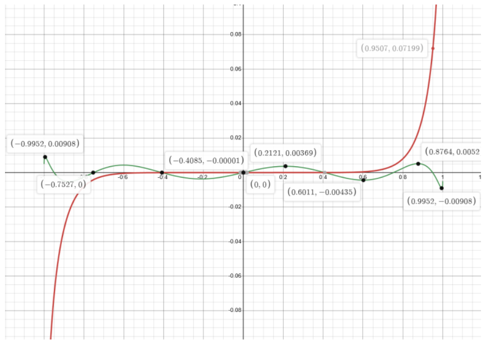

Figure 9. Profile of errors of approximation in (2.15) (green curve) vs. Taylor's series approximation (red curve)

We can easily get the upcoming idea. We are to consider that each coefficient C_i is a variable or function rather than a constant coefficient. As before, we can use polynomial interpolation for describing the coefficient's tendency over the interval such as

$$C_i(x) = 0.137x^6 - 0.075x^4 + 0.086x^2 + 0.499 \tag{2.13}$$

Remark 2.2. We discarded the near-zero terms in (2.13).

We can thus propose this new approximation for the arcsine function based on a multi-sets of points interpolation approach as

$$I_{arcsine} = P(x)(\sqrt{1+x} - \sqrt{1-x}) + C_{nt} \tag{2.14}$$

with $P(x) = 2(0.137x^6 - 0.075x^4 + 0.086x^2 + 0.499)$. and C_{nt} can be as well approximated as before by interpolation of the corresponding errors compared with the true values of the inverse sine function such as

$$C_{nt} \approx Q(x) = -2 \cdot 10^{-8}x^6 - 0.347x^5 + 3 \cdot 10^{-8}x^4 + 0.117x^3 \\ - 8 \cdot 10^{-9}x^2 - 0.027x + 3 \cdot 10^{-10}.$$

So we obtain the final approximation written as

$$I_{arcsine} = P(x)(\sqrt{1+x} - \sqrt{1-x}) + Q(x). \tag{2.15}$$

This expression gives a very good match with the original arcsine function and clearly, it is better than the previous approximation given by (2.11) with a single set of points approach. Taylor's approximation presents definitely a better match for smaller values but soon the errors grow up exponentially. However, the new approach to approximation in (2.15) presents a relatively acceptable and stable profile of errors at the very ends as we can see in Table 3.

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2.3. Evaluation and comparison with other methods

In order to evaluate the new approximated expression for the inverse sine function, we compare it with other approximations given by some researchers via inequalities such as Shafer-Fink inequality and padé's approximants [2, 11, 13]. As compared to Shafer-Fink's boundaries the approximated expression (2.15) presents relatively good behaviour.

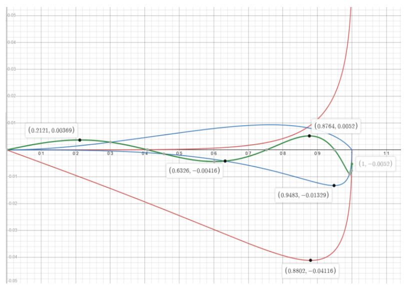

Figure 10. Profile of errors between arcsine and approximation in (2.15) (green curve), Shafer-Fink boundaries (red curves), refined boundaries given by inequality (3.7) of [5] (blue curve) over the interval $[0, 1]$

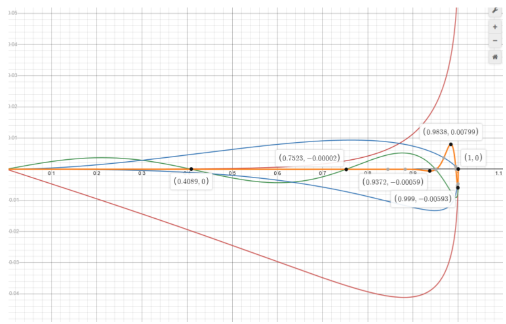

Figure 11. Profile of errors between arcsine and (2.16) (yellow curve), (2.15) (green curve), Shafer-Fink boundaries (red curve), refined boundaries given by inequality (3.7) of [5] (blue curve)

It is clear that the errors given by our method of approximation present a stable behaviour over the interval with a slight rise towards the extremities which in comparison with other approximations is within the acceptable limits.

2.3.1. Higher order and degree interpolation. We established so far that the arcsine function can be approximated fairly with the help of the above methods with an acceptable gap of errors and so far we used a relatively small degree for polynomial interpolation. Some questions arise naturally at this point. What if we attempt a higher order or degree interpolation for our approaches? Would that improve the results of the approximation? In order to provide an answer we actually endeavoured an attempt to interpolate the coefficients C_i's with the help of Newton's polynomial interpolation keeping the same sampling rate of 0.05 over $]-1, 1[$. As before we computed for the same set of points $\{-x_i, +x_i\}$ and this time we performed Newton's polynomial interpolation for the obtained coefficients C_i's in Table 2 and we corrected the corresponding errors also with Newton's polynomial interpolation. We obtained the following approximation

$$I_{arcsine} = P^{35}_{Newton}(x)(\sqrt{1+x} - \sqrt{1-x}) + Q^{40}_{Newton}(x) \tag{2.16}$$

where $P^{35}_{Newton}(x)$ and $Q^{40}_{Newton}(x)$, being polynomials of degrees respectively 35 and 40 as given in Appendix B. Such a high order of interpolation gives a nearly flat errors profile which presents a significant improvement of the results given by the approximation (2.16). This can be seen in above figure 11. The corresponding errors are given in Table 4.

The sharp bounds for the arcsine function due to Chen et. al. [5, Theorem 2.2] are as follows:

$$\frac{b_1 x}{c_1\sqrt{1-x^2} + a_1 x} \le \arcsin(x) \le \frac{b_2 x}{c_2 + \sqrt{1-x^2}};\ 0 \le x \le 1 \tag{2.17}$$

where $a_1 = (4 + 2\pi - \pi^2)/4$, $b_1 = \pi^2/4$, $c_1 = \pi^2/4 - 1$, $b_2 = \pi/(\pi - 2)$, $c_2 = 2/(\pi - 2)$.

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The above bounds are compared in Figures 10 and 11 by the blue curves. As we can clearly see that the errors of our proposed approximations (2.15) and (2.16) fall almost within the boundaries of corresponding errors of those between arcsine and bounds in (2.17).

On the other hand, refined Shafer-Fink inequalities are given by Wu et. al. [12, Theorem 3.3] as

$$\frac{x(80+25\sqrt{1-x^2})}{56-6x^2+\sqrt{1-x^2}} \leq \arcsin(x) \leq \frac{x(19+11\sqrt{1-x^2})}{3(1+\sqrt{1-x^2})(4-\sqrt{1-x^2})};\ x \in (0,1). \tag{2.18}$$

The above inequality is presented as a sharp refinement of Shafer-Fink's inequality. In the following figure, the simulation of our proposed approximation (2.16) and inequalities (2.18) is given.

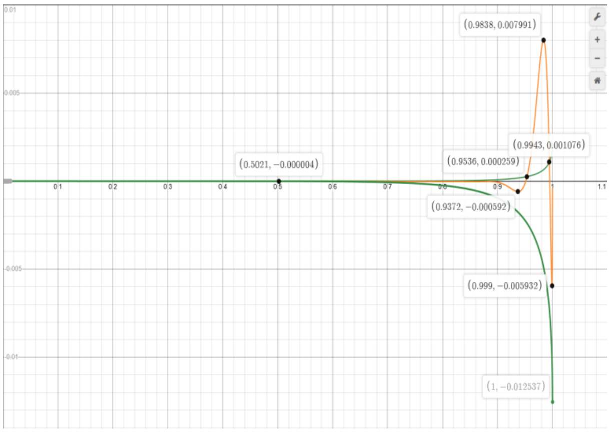

Figure 12. Profile of errors between arcsine and approximation in (2.16) (yellow curve), Refined Shafer-Fink boundaries given by inequality (3.7) of [12] (green curve)

Thus our proposed method of approximation can produce sharp results at least comparable to other's quite sophisticated and complicated methods as given in the above mentioned references.

2.4. Choice of approximation method

As we have seen previously, each choice of one set of points produces corresponding coefficients C_i with a specific profile of errors over $[-1,1]$. In Table 5, we propose a help criterion for the choice of a single set of points such as the 'minimum of the mean of errors' produced by the corrected approximated expression

$$I_{arcsine} = 2C_i(\sqrt{1+x} - \sqrt{1-x}) + Q(x) \text{ over } [-1,1].$$

$$\text{Min (mean (Err)) ; with Err} = |\arcsin(x) - I_{arcsine}| / \arcsin(x).$$

In the following figure, we can see that one particular value stands out.

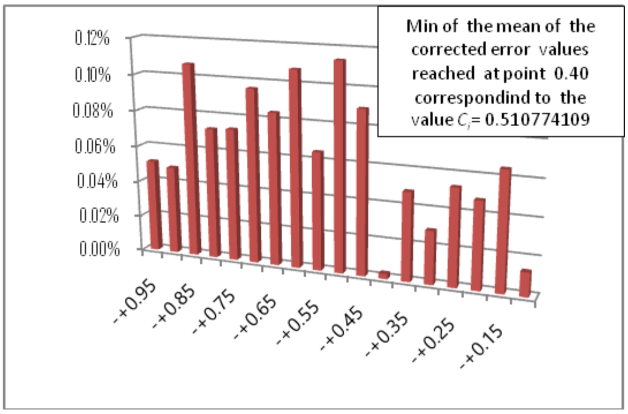

Figure 13. Mean relative errors in between arcsine and corrected $I_{arcsine}$

So one proposition for an acceptable approximation of the arcsine function can be as follows:

$$\arcsin(x) \approx 2(0.510774109)(\sqrt{1+x} - \sqrt{1-x}) + (0.239x^5 - 0.138x^3 + 0.005x).$$

The method of approximation given by the single set approach is limited and can be useful as a coarse approximation but not very suitable for high-precision numerical calculations. The multi-sets approach however presents a more reliable match over the domain and gives better results although there is always a fluctuation of errors near the domain extremities as shown before. We believe however that if we refine the sampling rate over the interval domain we will end up with lower errors but with the cost of a higher degree of interpolation polynomials as seen in Appendix B for an average sampling rate of 0.05.

3. Possible applications

For the possible application, we would like to introduce in this section some examples of integral estimation based on our proposed approximation expressions and compare them to some other more conventional methods. We propose here to estimate the integral of $\arcsin(x)/x$ over $[0, 1]$. Dhaigude and Bagul [7] give the following estimation via inequalities:

$$\int_0^1 \frac{1}{\sqrt{1-\frac{x^2}{3}}}\mathrm{d}x \leq \int_0^1 \frac{\arcsin(x)}{x} \leq \int_0^1 \frac{1}{\sqrt{1-\left(\frac{\pi^2-4}{\pi^2}\right)x^2}}\mathrm{d}x. \tag{3.1}$$

Computation results into

$$1.066042 \leq \int_0^1 \frac{\arcsin(x)}{x}\mathrm{d}x \leq 1.142005.$$

Hence

$$\int_0^1 \frac{\arcsin(x)}{x}\mathrm{d}x \approx \frac{1.066042 + 1.142005}{2} = 1.104023$$

with a relative error of 0.013988 comparing with the reference computer given value of 1.088793.

Our uncorrected approximation $I_{arcsine}$ gives

$$\int_0^1 \frac{\arcsin(x)}{x}\mathrm{d}x \approx \int_0^1 \frac{\sqrt{2}(\sqrt{3}-1)(\sqrt{1+x}-\sqrt{1-x})}{x}\mathrm{d}x = 1.1033$$

with a relative error of 0.013324.

The corrected approximation (2.12) yields

$$\int_0^1 \frac{\arcsin(x)}{x}\mathrm{d}x \approx \int_0^1 \frac{\sqrt{2}(\sqrt{3}-1)(\sqrt{1+x}-\sqrt{1-x})+(0.267x^5-0.163x^3-0.004x)}{x}\mathrm{d}x = 1.09834$$

with a relative improved error of 0.0088. Similarly, the second improved approximation given by expression (2.15) gives the value 1.08322 with a relative error of 0.005119 and the Newton's polynomial interpolated approximation (2.16) gives a sharp 1.08865 with a relative error of only 0.000131. We can see clearly that our approximated expression produces close results to those given by the computer but for the precise computation we should use the approximation that is fully polynomial interpolated such as in (2.15) and (2.16) for better accuracy of results.

4. Conclusion

We have obtained through our new approach an accurate approximation for the arcsine or inverse sine function. We gave acceptable close forms to the approximated functions through versatile single-set and multi-sets of points methods. The proposed approximations present acceptable and reduced errors compared to other classical methods of approximations. We established this innovative way of approximation through general theoretical principles and numerical simulations. To the best of the authors' knowledge, it is the first time that this approach is introduced in the literature.

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Appendix A

Table 1. Relative errors of the approximation in (2.12) vs. Taylor's series approximation

x	$I_{arcsine}$ relative error	Taylor's series relative error
∓1.00	0.43	18.11
∓0.95	1.92	5.68
∓0.90	1.35	2.91
∓0.85	0.61	1.58
∓0.80	0.05	0.87
∓0.75	0.54	0.47
∓0.70	0.83	0.25
∓0.65	0.93	0.13
∓0.60	0.87	0.07
∓0.55	0.66	0.03
∓0.50	0.33	0.01
∓0.45	0.08	0.01
∓0.40	0.54	0.00
∓0.35	1.02	0.00
∓0.30	1.50	0.00
∓0.25	1.95	0.00
∓0.20	2.35	0.00
∓0.15	2.68	0.00
∓0.10	2.92	0.00
0.00	Not calculated	Not calculated

$$\text{Relative error} = \frac{|\text{Actual value of arcsine} - \text{Approximated value}|}{\text{Actual value of arcsine}}$$

Table 2. Calculated coefficients C_i with sampling rate 0.05 over $]-1,1[$

Set points $\{-x_i, +x_i\}$	Calculated coefficients C_i
∓1.00	Not calculated
∓0.95	0.617272214
∓0.90	0.590098394
∓0.85	0.572263951
∓0.80	0.559016995
∓0.75	0.548583770
∓0.70	0.540084231
∓0.65	0.531903496
∓0.60	0.527046277
∓0.55	0.521972333
∓0.50	0.517638090
∓0.45	0.513932899
∓0.40	0.510774109
∓0.35	0.508098898
∓0.30	0.505858998
∓0.25	0.504017170
∓0.20	0.502544810
∓0.15	0.501420279
∓0.10	0.500627751
0.00	Not calculated

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Table 3. Relative errors of the approximation (2.15) vs. Taylor's series approximation

x	Absolute value of errors	$I_{arcsine}$ relative errors	Taylor's series relative
∓ 1.00	0.005196019	0.33	18.11
∓ 0.9952	0.009076043	0.62	13.37
∓ 0.95	$5.67815E-05$	0.00	5.68
∓ 0.90	0.004778319	0.43	2.91
∓ 0.85	0.004792598	0.47	1.58
∓ 0.80	0.002592633	0.28	0.87
∓ 0.75	0.000142151	0.02	0.47
∓ 0.70	0.002455928	0.32	0.25
∓ 0.65	0.003896566	0.55	0.13
∓ 0.60	0.004349533	0.68	0.07
∓ 0.55	0.00391448	0.67	0.03
∓ 0.50	0.002813257	0.54	0.01
∓ 0.45	0.001321664	0.28	0.01
∓ 0.40	0.000280257	0.07	0.00
∓ 0.35	0.001743062	0.49	0.00
∓ 0.30	0.002870633	0.94	0.00
∓ 0.25	0.003532945	1.40	0.00
∓ 0.20	0.003670629	1.82	0.00
∓ 0.15	0.003292834	2.19	0.00
∓ 0.10	0.002469721	2.47	0.00
0.00	$--$	$--$	$--$

Table 4. Relative errors given by the approximated Newton polynomial expression (2.16) vs. approximation in (2.15)

x	Relative errors given by (2.15)	Relative errors given by (2.16)
∓ 1.00	0.33	0.57
∓ 0.9952	0.62	0.58
∓ 0.95	0.00	0.47
∓ 0.90	0.43	0.34
∓ 0.85	0.47	0.24
∓ 0.80	0.28	0.16
∓ 0.75	0.02	0.10
∓ 0.70	0.32	0.07
∓ 0.65	0.55	0.04
∓ 0.60	0.68	0.07
∓ 0.55	0.67	0.02
∓ 0.50	0.54	0.01
∓ 0.45	0.28	0.00
∓ 0.40	0.07	0.00
∓ 0.35	0.49	0.00
∓ 0.30	0.94	0.00
∓ 0.25	1.40	0.00
∓ 0.20	1.82	0.00
∓ 0.15	2.19	0.00
∓ 0.10	2.47	0.00
0.00	$--$	$--$

Table 5. Mean of relative errors $|\arcsin(x) - I_{arcsine}|/\arcsin(x)$ due to corrected $I_{arcsine}$ over $[-1, 1]$ with the sampling rate of 0.05

Set points $\{-x_i, +x_i\}$	Value of C_i	Mean of relative errors	Coefficient of determination R^2
∓ 1.00	Not calculated	Not calculated	Not calculated
∓ 0.95	0.617272214	0.000512169	0.999
∓ 0.90	0.590098394	0.000486896	0.997
∓ 0.85	0.572263951	0.001068988	0.944
∓ 0.80	0.559016995	0.000723846	0.989
∓ 0.75	0.548583770	0.000732384	0.978
∓ 0.70	0.540084231	0.000960610	0.961
∓ 0.65	0.531903496	0.000840648	0.939
∓ 0.60	0.527046277	0.001076747	0.926
∓ 0.55	0.521972333	0.000649635	0.926

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Table 5. Mean of relative errors $|\arcsin(x) - I_{arcsine}|/\arcsin(x)$ due to corrected $I_{arcsine}$ over $[-1, 1]$ with the sampling rate of 0.05 (Continued)

Set points $\{-x_i, +x_i\}$	Value of C_i	Mean of relative errors	Coefficient of determination R^2
∓0.50	0.517638090	0.001137149	0.935
∓0.45	0.513932899	0.000896773	0.944
∓0.40	0.510774109	**0.000027861**	0.951
∓0.35	0.508098898	0.000485566	0.957
∓0.30	0.505858998	0.000291419	0.961
∓0.25	0.504017170	0.000532459	0.964
∓0.20	0.502544810	0.000477488	0.967
∓0.15	0.501420279	0.000649752	0.968
∓0.10	0.500627751	0.000132529	0.969
0.00	Not calculated	Not calculated	Not calculated

Minimum of the mean errors given by the corrected $I_{arcsine}$ with polynomial interpolation ($n = 6$) is 0.000027861 and occurs for $\{-0.4, +0.4\}$ with an interpolation correlation factor of $R = 0.951$.

Appendix B

Newton polynomial interpolation of coefficients C_i over $]-1, 1[$:

$$\begin{aligned}
P(x) =& x^{35} - 3615.0105713x^{34} - 5.27x^{33} + 21963.5192437x^{32} + 12.53537x^{31} \\
&- 60372.5221398x^{30} - 17.8227086x^{29} + 99433.9638945x^{28} + 16.9102457x^{27} \\
&- 109549.252255x^{26} - 11.3218486x^{25} + 85362.451315x^{24} + 5.516599x^{23} \\
&- 48506.38447x^{22} - 1.9889485x^{21} - 20428.4033816x^{20} + 0.534368x^{19} \\
&- 6417.1312555x^{18} - 0.106937x^{17} + 1501.4173469x^{16} + 0.015827x^{15} \\
&- 259.3977988x^{14} - 0.001708x^{13} + 32.5720223x^{12} + 0.0001314x^{11} \\
&- 2.88979x^{10} - 0.000007x^{9} + 0.1872391x^{8} + 0.0000002x^{7} + 0.009141x^{6} \\
&- 0.00x^{5} + 0.0275098x^{4} + 0.00x^{3} + 0.0624979x^{2} - 0.00x + 0.5.
\end{aligned}$$

Newton polynomial interpolation of errors given by the above $P(x)$:

$$\begin{aligned}
Q(x) =& -772.6953x^{40} + 55886.5493x^{39} + 6107.4953x^{38} - 397030.0991x^{37} \\
&- 22116.7082x^{36} + 1294535.0725x^{35} + 48646.8356x^{34} - 2569753.3058x^{33} \\
&- 72655.817x^{32} + 3473833.2023x^{31} + 78037.8169x^{30} - 3389220.9605x^{29} \\
&- 62283.9821x^{28} + 2467578.8956x^{27} + 37643.6463x^{26} - 1367352.2706x^{25} \\
&- 17399.5970x^{24} + 582908.239x^{23} + 6170.079x^{22} - 191966.5143x^{21} \\
&- 1674.4025x^{20} + 48766.8251x^{19} + 345.011x^{18} - 9491.8122x^{17} - 53.2546x^{16} \\
&+ 1397.8822x^{15} + 6.0367x^{14} - 152.8342x^{13} - 0.4888x^{12} + 12.045x^{11} \\
&+ 0.0272x^{10} - 0.6857x^{9} - 0.001x^{8} - 0.0096x^{7} + 0.0x^{6} - 0.0505x^{5} \\
&- 0.0x^{4} - 0.0833x^{3} + 0.0x^{2} - 0.0x.
\end{aligned}$$