A duality principle and a concerning convex dual formulation suitable for non-convex variational optimization

Fabio Silva Botelho

Department of Mathematics

Federal University of Santa Catarina

Florianópolis - SC, Brazil

Abstract

This article develops a duality principle and a related convex dual formulation suitable for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a model in non-linear elasticity.

Key words: Convex dual variational formulation, duality principle for non-convex optimization, model in non-linear elasticity

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1 Introduction

In this section we establish a duality principle and a related convex dual formulation for a large class of models in non-convex optimization.

More specifically, the main duality principle is applied to a model in non-linear elasticity.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 10, 11] and on a D.C. optimization approach developed in Toland [12].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [7, 4, 5, 6, 9]. Finally, the model in non-linear elasticity here presented may be found in [8].

Remark 1.1. *In this text we adopt the standard Einstein convention of summing up repeated indices unless otherwise indicated.*

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \to \mathbb{R}$ where

$$J(u) \;=\; \frac{1}{2} \int_{\Omega} H_{ijkl} e_{ij}(u) e_{kl}(u) \; dx - \langle u_i, f_i \rangle_{L^2}. \tag{1}$$

Here $\{H_{ijkl}\}$ is a fourth order symmetric positive definite tensor and

$$\{e_{ij}(u)\} = \left\{ \frac{1}{2}(u_{i,j} + u_{j,i}) + \frac{1}{2}(u_{m,i} u_{m,j}) \right\},$$

where

$$u = (u_1, u_2, u_3) \in V = W_0^{1,4}(\Omega; \mathbb{R}^3)$$

denotes the field of displacements resulting from the action of the external forces $f = (f_1, f_2, f_2) \in L^2(\Omega; \mathbb{R}^3)$ on the elastic solid comprised by $\Omega \subset \mathbb{R}^3$.

Moreover, denoting $Y = Y^* = L^2(\Omega; \mathbb{R}^{3 \times 3})$, the stress tensor $\sigma \in Y^*$ is defined by

$$\{\sigma_{ij}(u)\} = \{H_{ijkl} e_{kl}(u)\}.$$

At this point we define the functionals $F_1 : V \times Y \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $G : V \to \mathbb{R}$ by

$$
\begin{aligned}
F_1(u, \sigma) \;=\; & -\sum_{i=1}^{3} \frac{K}{2} \int_{\Omega} (u_{i,i})^2 \; dx \\
& + \sum_{i=1}^{3} \frac{K_1}{2} \int_{\Omega} (\sigma_{ij,j} + (\sigma_{im} u_{m,j})_{,j} + f_i)^2 \; dx \\
& + \frac{K_2}{2} \int_{\Omega} u_{i,j} u_{i,j} \; dx - \langle u_i, f_i \rangle_{L^2},
\end{aligned} \tag{2}
$$

for appropriate positive real constants, K, K_1, K_2 to be specified,

$$F_2(u) = \frac{K_2}{2} \int_{\Omega} u_{i,j} u_{i,j} \; dx,$$

and

$$G(u) = \frac{1}{2} \int_{\Omega} H_{ijkl} e_{ij}(u) e_{kl}(u) \; dx + \sum_{i=1}^{3} \frac{K}{2} \int_{\Omega} (u_{i,i})^2 \; dx.$$

Here, it is worth highlighting that

$$F_1(u, \sigma) - F_2(u) + G(u) = J(u) + \sum_{i=1}^{3} \frac{K_1}{2} \int_{\Omega} (\sigma_{ij,j} + (\sigma_{im} u_{m,j})_{,j} + f_i)^2 \; dx, \forall u \in V, \; \sigma \in Y^*.$$

Furthermore, we define the functionals $F_1^* : [Y^*]^3 \to \mathbb{R}$, $F_2^* : Y^* \to \mathbb{R}$ and $G^* : [Y^*]^2 \to \mathbb{R}$ by

$$F_1^*(\sigma, Q, \tilde{Q}) = \sup_{u \in V} \{ -\langle u_{i,j}, \sigma_{ij} \rangle_{L^2} - \langle u_{i,j}, Q_{ij} \rangle_{L^2} + \langle u_{i,j}, \tilde{Q}_{ij} \rangle_{L^2} - F_1(u, \sigma) \},$$

2

$$F_2^*(\tilde{Q}) = \sup_{v_2 \in Y} \left\{ \langle (v_2)_{ij}, \tilde{Q}_{ij} \rangle_{L^2} - \frac{K_2}{2} \int_\Omega (v_2)_{ij}(v_2)_{ij} \, dx \right\}$$

$$= \frac{1}{2K_2} \int_\Omega \tilde{Q}_{ij} \tilde{Q}_{ij} \, dx, \tag{3}$$

and

$$G^*(\sigma, Q) = \sup_{(v_1, v_2) \in Y \times Y} \{ \langle (v_1)_{ij}, \sigma_{ij} \rangle_{L^2} + \langle (v_2)_{ij}, Q_{ij} \rangle_{L^2} - \hat{G}(v_1, v_2) \},$$

where

$$\hat{G}(v_1, v_2) = \frac{1}{2} \int_\Omega H_{ijkl} \left((v_1)_{ij} + \frac{1}{2}(v_2)_{mi}(v_2)_{mj} \right) \left((v_1)_{kl} + \frac{1}{2}(v_2)_{mk}(v_2)_{ml} \right) \, dx$$

$$+ \sum_{i=1}^3 \frac{K}{2} \int_\Omega ((v_2)_{ii})^2 \, dx, \tag{4}$$

so that

$$G^*(\sigma, Q) = \frac{1}{2} \int_\Omega (\overline{\sigma_{ij}^K}) Q_{mi} Q_{mj} \, dx + \frac{1}{2} \int_\Omega \overline{H}_{ijkl} \sigma_{ij} \sigma_{kl} \, dx,$$

if $\sigma \in B^*$ where

$$B^* = \{ \sigma \in Y^* \; : \; \|\sigma_{ij}\|_\infty \leq K/8, \; \forall i, j \in \{1, 2, 3\} \text{ and } \{\sigma_{ij}\} < -\varepsilon I_d \},$$

for some small parameter $\varepsilon > 0$ and where I_d denotes the 3×3 identity matrix. Observe that such a definition for B^* corresponds to the case of negative definite stress tensors, which refers to compression in a solid mechanics context.

Here

$$\{\sigma_{ij}^K\} = \begin{bmatrix} \sigma_{11} + K & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} + K & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} + K \end{bmatrix} \tag{5}$$

$$\overline{\sigma_{ij}^K} = \{\sigma_{ij}^K\}^{-1},$$

and

$$\{\overline{H}_{ijkl}\} = \{H_{ijkl}\}^{-1}$$

in an appropriate tensor sense.

At this point we define

$$J^*(\sigma, Q, \tilde{Q}) = -F_1^*(\sigma, Q, \tilde{Q}) + F_2^*(\tilde{Q}) - G^*(\sigma, Q).$$

Specifically for

$$K_2 \gg K_1 \gg K \gg \max\{1/\varepsilon, 1, K_3, \|H_{ijkl}\|\},$$

we define

$$D^* = \{ Q \in Y^* \; : \; \|Q_{ij}\|_\infty \leq K_3, \; \forall i, j \in \{1, 2, 3\} \}.$$

By direct computation, we may obtain

$$\left\{ \frac{\partial^2 J^*(\sigma, Q, \tilde{Q})}{\partial \sigma \partial Q} \right\} < \mathbf{0},$$

and

$$\left\{ \frac{\partial^2 J^*(\sigma, Q, \tilde{Q})}{\partial \tilde{Q}^2} \right\} > \mathbf{0},$$

on $B^* \times D^* \times Y^*$, so that J^* is concave in (σ, Q) and convex in $\tilde{Q}$ on $B^* \times D^* \times Y^*$.

2   The main duality principle and a related convex dual variational formulation

Our main duality principle is summarized by the following theorem.

Theorem 2.1. *Considering the statements and definitions of the previous section, suppose* $(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) \in B^* \times D^* \times Y^*$ *is such that*

$$\delta J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) = \mathbf{0}.$$

Let $u_0 \in V$ be such that

$$(u_0)_{i,j} = \frac{\partial F_2^*(\tilde{Q})}{\partial \tilde{Q}_{ij}}.$$

Under such hypotheses, we have

$$
\begin{aligned}
J(u_0) &= \min_{u \in V} \left\{ J(u) + \sum_{i=1}^{3} \frac{K_1}{2} \int_\Omega (\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im} u_{m,j})_{,j} + f_i)^2 \, dx \right\} \\
&= \inf_{\tilde{Q} \in Y^*} \left\{ \sup_{(\sigma, Q) \in B^* \times D^*} J^*(\sigma, Q, \tilde{Q}) \right\} \\
&= J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}).
\end{aligned}
\tag{6}
$$

Proof. Observe that there exists $\hat{u} \in V$ such that, defining

$$H(u, \sigma, Q, \tilde{Q}) = -\langle u_{i,j}, \sigma_{ij} \rangle_{L^2} - \langle u_{i,j}, Q_{ij} \rangle_{L^2} + \langle u_{i,j}, \tilde{Q}_{ij} \rangle_{L^2} - F_1(u, \sigma),$$

we have

$$\frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial u} = \mathbf{0}$$

and

$$F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) = H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}).$$

Moreover, from the variation in of J^* in $\tilde{Q}$, we obtain

$$-\frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} + \frac{\partial F_2^*(\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} = \mathbf{0},$$

4

where

$$\frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}}$$

$$= \frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}}$$

$$+ \frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial u} \frac{\partial \hat{u}}{\partial \tilde{Q}_{ij}}$$

$$= \hat{u}_{i,j}. \tag{7}$$

From such last two equations we get

$$(u_0)_{i,j} = \frac{\partial F_2^*(\hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} = \frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial \tilde{Q}_{ij}} = \hat{u}_{i,j},$$

so that from the concerning boundary conditions,

$$u_0 = \hat{u}.$$

On the other hand, from the variation of J^* in Q we have

$$-\frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial Q_{ij}} - \frac{\partial G^*(\hat{\sigma}, \hat{Q})}{\partial Q_{ij}} = \mathbf{0},$$

so that

$$(u_0)_{i,j} - \overline{\hat{\sigma}_{im}^K} \hat{Q}_{mj} = 0,$$

and therefore

$$\hat{Q}_{ij} = \hat{\sigma}_{im}(u_0)_{m,j} + K\delta_{ij}(u_0)_{i,j}.$$

Finally, from the variation of J^* in σ we obtain

$$-\frac{\partial F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial \sigma_{ij}} - \frac{\partial G^*(\hat{\sigma}, \hat{Q})}{\partial \sigma_{ij}} = \mathbf{0},$$

so that

$$(u_0)_{i,j} + \frac{1}{2}(u_0)_{m,i}(u_0)_{m,j} - \overline{H}_{ijkl}\hat{\sigma}_{kl} = 0.$$

Thus, since $\{H_{ijkl}\}$ is symmetric, we get

$$\hat{\sigma}_{ij} = H_{ijkl}e_{kl}(u_0).$$

From these last results and from

$$\frac{\partial H(\hat{u}, \hat{\sigma}, \hat{Q}, \hat{\tilde{Q}})}{\partial u} = \mathbf{0}$$

we obtain

$$\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im}(u_0)_{m,j})_{,j} + f_i = 0, \ \forall i \in \{1, 2, 3\},$$

so that
$$\delta J(u_0) = \mathbf{0}.$$

Finally, from such last results and the Legendre transform properties, we have

$$
\begin{aligned}
F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) &\\
= -\langle (u_0)_{i,j}, \hat{\sigma}_{ij} \rangle_{L^2} &- \langle (u_0)_{i,j}, \hat{Q}_{ij} \rangle_{L^2} + \langle (u_0)_{i,j}, \hat{\tilde{Q}}_{ij} \rangle_{L^2} - F_1(u_0, \hat{\sigma}),
\end{aligned}
\tag{8}
$$

$$F_2^*(\hat{\tilde{Q}}) = \langle (u_0)_{i,j}, \hat{\tilde{Q}}_{ij} \rangle_{L^2} - F_2(u_0),$$

and

$$G^*(\hat{\sigma}, \hat{Q}) = \langle (u_0)_{i,j}, \hat{\sigma}_{ij} \rangle_{L^2} + \langle (u_0)_{i,j}, \hat{Q}_{ij} \rangle_{L^2} - G(u_0).$$

From these results, we obtain

$$
\begin{aligned}
J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) &= -F_1^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}) + F_2^*(\hat{\tilde{Q}}) - G^*(\hat{\sigma}, \hat{Q}) \\
&= F_1(u_0, \hat{\sigma}) - F_2(u_0) + G(u_0) \\
&= J(u_0).
\end{aligned}
\tag{9}
$$

Joining the pieces, we have got

$$
\begin{aligned}
J(u_0) &= \min_{u \in V} \left\{ J(u) + \sum_{i=1}^{3} \frac{K_1}{2} \int_{\Omega} (\hat{\sigma}_{ij,j} + (\hat{\sigma}_{im} u_{m,j})_{,j} + f_i)^2 \, dx \right\} \\
&= \inf_{\tilde{Q} \in Y^*} \left\{ \sup_{(\sigma, Q) \in B^* \times D^*} J^*(\sigma, Q, \tilde{Q}) \right\} \\
&= J^*(\hat{\sigma}, \hat{Q}, \hat{\tilde{Q}}).
\end{aligned}
\tag{10}
$$

The proof is complete.

$\square$

Remark 2.2. *A similar result is valid if we would define*

$$B^* = \{ \sigma \in Y^* \; : \; \|\sigma_{ij}\|_\infty \le K/8, \; \forall i,j \in \{1,2,3\} \text{ and } \{\sigma_{ij}\} > \varepsilon I_d \}.$$

This case refers to a positive definite tensor $\{\sigma_{ij}\}$ and the previous case to a negative definite one.

3   Conclusion

In this article we have developed a convex dual variational formulation suitable for non-convex variational primal formulations.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principle here presented is applied to a model in non-linear elasticity. In a future research, we intend to extend such results for other related models of plates and shells.

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