

A duality principle and a related convex dual formulation suitable for local non-convex variational optimization

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Abstract

This article develops a duality principle and a related convex dual formulation suitable for the local optimization of a non-convex primal formulation for a large class of models in physics and engineering. The results are based on standard tools of functional analysis, calculus of variations and duality theory. In particular, we develop applications to a Ginzburg-Landau type equation.

Key words: Convex dual variational formulation, duality principle for non-convex local primal optimization, Ginzburg-Landau type equation

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1 Introduction

In this article we establish a duality principle and a related convex dual formulation suitable for the local optimization of the primal formulation for a large class of models in non-convex optimization.

The main duality principle is applied to the Ginzburg-Landau system in superconductivity in the absence of a magnetic field.

Such results are based on the works of J.J. Telega and W.R. Bielski [2, 3, 13, 14] and on a D.C. optimization approach developed in Toland [15].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [9, 5, 6, 7, 12]. Finally, similar models on the superconductivity physics may be found in [4, 11].

Remark 1.1. *It is worth highlighting, we may generically denote*

$$\int_{\Omega} [(-\gamma \nabla^2 + KI_d)^{-1} v^*] v^* dx$$

simply by

$$\int_{\Omega} \frac{(v^*)^2}{-\gamma \nabla^2 + K} dx,$$

where I_d denotes a concerning identity operator.

Other similar notations may be used along this text as their indicated meaning are sufficiently clear.

Finally, ∇^2 denotes the Laplace operator and for real constants $K_2 > 0$ and $K_1 > 0$, the notation $K_2 \gg K_1$ means that $K_2 > 0$ is much larger than $K_1 > 0$.

At this point we start to describe the primal and dual variational formulations.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

For the primal formulation, consider a functional $J : V \rightarrow \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u dx \\ &\quad + \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta)^2 dx - \langle u, f \rangle_{L^2}. \end{aligned} \quad (1)$$

Here $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega) \cap L^\infty(\Omega)$.

Moreover, $V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functionals $F_1 : V \times Y \rightarrow \mathbb{R}$, $F_2 : V \rightarrow \mathbb{R}$ and $G : V \times Y \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_0^*) &= \frac{\gamma}{2} \int_{\Omega} \nabla u \cdot \nabla u dx - \frac{K}{2} \int_{\Omega} u^2 dx \\ &\quad + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2v_0^* u - f)^2 dx + \frac{K_2}{2} \int_{\Omega} u^2 dx, \end{aligned} \quad (2)$$

$$F_2(u) = \frac{K_2}{2} \int_{\Omega} u^2 dx$$

and

$$G(u, v) = \frac{\alpha}{2} \int_{\Omega} (u^2 - \beta + v)^2 dx + \frac{K}{2} \int_{\Omega} u^2 dx - \langle u, f \rangle_{L^2}.$$

We define also $F_1^* : [Y^*]^3 \rightarrow \mathbb{R}$, $F_2^* : Y^* \rightarrow \mathbb{R}$, and $G^* : [Y^*]^2 \rightarrow \mathbb{R}$, by

$$\begin{aligned} &F_1^*(v_2^*, v_1^*, v_0^*) \\ &= \sup_{u \in V} \{ \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_0^*) \} \\ &= \int_{\Omega} \frac{-K_1 f (-\gamma \nabla^2 + K + K_2) f + (v_1^* + v_2^*)^2 - 2K_1 f (-\gamma \nabla^2 + 2v_0^*) (v_1^* + v_2^*)}{2[K_2 + K - \gamma \nabla^2 + K_1 (-\gamma \nabla^2 + 2v_0^*)^2]} dx \end{aligned} \quad (3)$$

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u \in V} \{ \langle u, v_2^* \rangle_{L^2} - F_2(u) \} \\ &= \frac{1}{2K_2} \int_{\Omega} (v_2^*)^2 dx \end{aligned} \quad (4)$$

and

$$\begin{aligned}
G^*(v_1^*, v_0^*) &= \sup_{(u,v) \in V \times Y} \{-\langle u, v_1^* \rangle_{L^2} + \langle v, v_0^* \rangle_{L^2} - G(u, v)\} \\
&= \frac{1}{2} \int_{\Omega} \frac{(v_1^* - f)^2}{2v_0^* + K} dx + \frac{1}{2\alpha} \int_{\Omega} (v_0^*)^2 dx \\
&\quad + \beta \int_{\Omega} v_0^* dx
\end{aligned} \tag{5}$$

if $v_0^* \in B^*$ where

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_{\infty} < K/8 \text{ and } -\gamma \nabla^2 + 2v_0^* < -\varepsilon I_d\},$$

for a small parameter $0 < \varepsilon \ll 1$.

Furthermore, we define

$$D^* = \{v_1^* \in Y^* : \|v_1^* - f\|_{\infty} \leq (3/2)K\}$$

and $J_1^* : Y^* \times D^* \times B^* \rightarrow \mathbb{R}$, by

$$J_1^*(v_2^*, v_1^*, v_0^*) = -F_1^*(v_2^*, v_1^*, v_0^*) + F_2^*(v_2^*) - G^*(v_1^*, v_0^*).$$

Assuming

$$K_2 \gg K_1 \gg K \gg \max\{\|f\|_{\infty}, \alpha, \beta, \gamma, 1/\varepsilon^2\}$$

by directly computing $\delta^2 J_1^*(v_2^*, v_1^*, v_0^*)$ we may obtain that for such specified real constants, J_1^* in convex in v_2^* and it is concave in (v_1^*, v_0^*) on $Y^* \times D^* \times B^*$.

2 The main duality principle and a concerning convex dual formulation

Considering the statements and definitions presented in the previous section, we may prove the following theorem.

Theorem 2.1. *Let $(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \in Y^* \times D^* \times B^*$ be such that*

$$\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in V$ be such that

$$u_0 = \frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*}.$$

Under such hypotheses, we have

$$\delta J(u_0) = \mathbf{0},$$

and

$$\begin{aligned}
J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx \right\} \\
&= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\
&= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*).
\end{aligned} \tag{6}$$

Proof. Observe that $\delta J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}$ so that, since J_1^* is convex in v_2^* and concave in (v_1^*, v_0^*) on $Y^* \times D^* \times B^*$, from the Min-Max theorem, we obtain

$$J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\}.$$

Now we are going to show that

$$\delta J(u_0) = \mathbf{0}.$$

From

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \mathbf{0},$$

and

$$\frac{\partial F_2^*(\hat{v}_2^*)}{\partial v_2^*} = u_0$$

we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, v_0^*)}{\partial v_2^*} + u_0 = \mathbf{0}$$

and

$$\hat{v}_2^* - K_2 u_0 = \mathbf{0}.$$

Observe now that denoting

$$H(v_2^*, v_1^*, v_0^*, u) = \langle u, v_1^* + v_2^* \rangle_{L^2} - F_1(u, v_0^*),$$

there exists $\hat{u} \in V$ such that

$$\frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} = \mathbf{0},$$

and

$$F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) = H(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, \hat{u}),$$

so that

$$\begin{aligned} \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial v_2^*} \\ &\quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_2^*} \\ &= \hat{u}. \end{aligned} \tag{7}$$

Summarizing, we have got

$$u_0 = \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_2^*} = \hat{u}.$$

Also, denoting

$$A(u_0, \hat{v}_0^*) = -\gamma \nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

from

$$\frac{\partial H(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_0^*, u_0)}{\partial u} = \mathbf{0},$$

we have

$$-\hat{v}_1^* + Ku_0 + \gamma\nabla^2 u_0 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) - \hat{v}_2^* + K_2 u_0 = \mathbf{0},$$

so that

$$-\hat{v}_1^* + Ku_0 + \gamma\nabla^2 u_0 + K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = \mathbf{0}. \quad (8)$$

From such results, we may infer that

$$\begin{aligned} & \frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} \\ &= \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial v_1^*} \\ & \quad + \frac{\partial H(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*, \hat{u})}{\partial u} \frac{\partial \hat{u}}{\partial v_1^*} \\ &= \hat{u} \\ &= u_0. \end{aligned} \quad (9)$$

Now observe that from the variation of J_1^* in v_1^* , we have

$$-\frac{\partial F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0}$$

so that

$$-u_0 - \frac{\partial G^*(\hat{v}_1^*, \hat{v}_0^*)}{\partial v_1^*} = \mathbf{0}$$

that is

$$-u_0 - \frac{\hat{v}_1^* - f}{2\hat{v}_0^* + K} = \mathbf{0}.$$

From this and (8), we may infer that

$$\hat{v}_1^* = -\gamma\nabla^2 u_0 - Ku_0 - K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = -(2\hat{v}_0^* + K)u_0 + f,$$

so that

$$-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f - K_1(-\gamma\nabla^2 + 2\hat{v}_0^*)A(u_0, \hat{v}_0^*) = 0.$$

From this and the concerning boundary conditions, since

$$A(u_0, v_0^*) = -\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f,$$

we may obtain

$$-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f = A(u_0, \hat{v}_0^*) = 0.$$

Moreover, from

$$\frac{\partial J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*)}{\partial v_0^*} = \mathbf{0},$$

we have

$$A(u_0, \hat{v}_0^*)2u_0 - \frac{\hat{v}_0^*}{\alpha} + u_0^2 - \beta = \mathbf{0},$$

so that

$$v_0^* = \alpha(u_0^2 - \beta).$$

From such last results we get

$$-\gamma \nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0},$$

and thus

$$\delta J(u_0) = \mathbf{0}.$$

Furthermore, also from such last results and the Legendre transform properties, we have

$$\begin{aligned} F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) &= \langle u_0, \hat{v}_2^* + \hat{v}_1^* \rangle_{L^2} - F_1(u_0, \hat{v}_0^*), \\ F_2^*(\hat{v}_2^*) &= \langle u_0, \hat{v}_2^* \rangle_{L^2} - F_2(u_0), \\ G^*(\hat{v}_1^*, \hat{v}_0^*) &= -\langle u_0, \hat{v}_1^* \rangle_{L^2} + \langle 0, \hat{v}_0^* \rangle_{L^2} - G(u_0, \mathbf{0}), \end{aligned}$$

so that

$$\begin{aligned} &J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ &= -F_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) + F_2^*(\hat{v}_2^*) - G^*(\hat{v}_1^*, \hat{v}_0^*) \\ &= F_1(u_0, \hat{v}_0^*) - F_2(u_0) + G(u_0, \mathbf{0}) \\ &= J(u_0). \end{aligned} \tag{10}$$

Finally, observe that

$$J_1^*(v_2^*, v_1^*, v_0^*) \leq -\langle u, v_2^* \rangle_{L^2} + F_1(u, v_0^*) + F_2^*(v_2^*) + G(u, \mathbf{0}),$$

$\forall u \in V, v_2^* \in Y^*, v_1^* \in D^*, v_0^* \in B^*$.

Thus, we may obtain

$$\begin{aligned} &\inf_{v_2^* \in Y^*} J_1^*(v_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ &\leq \inf_{v_2^* \in Y^*} \{-\langle u, v_2^* \rangle_{L^2} + F_1(u, \hat{v}_0^*) + F_2^*(v_2^*) + G(u, \mathbf{0})\} \\ &= F_1(u, \hat{v}_0^*) - F_2(u) + G(u, \mathbf{0}) \\ &= J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx, \quad \forall u \in V. \end{aligned} \tag{11}$$

From this and (11), we obtain

$$\begin{aligned} &J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*) \\ &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\ &\leq \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx \right\}. \end{aligned} \tag{12}$$

Joining the pieces, from a concerning convexity in u , we have got

$$\begin{aligned} J(u_0) &= \inf_{u \in V} \left\{ J(u) + \frac{K_1}{2} \int_{\Omega} (-\gamma \nabla^2 u + 2\hat{v}_0^* u - f)^2 dx \right\} \\ &= \inf_{v_2^* \in Y^*} \left\{ \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_2^*, v_1^*, v_0^*) \right\} \\ &= J_1^*(\hat{v}_2^*, \hat{v}_1^*, \hat{v}_0^*). \end{aligned} \tag{13}$$

The proof is complete. □

Remark 2.2. *We could have also defined*

$$B^* = \{v_0^* \in Y^* : \|2v_0^*\|_\infty < K/8 \text{ and } -\gamma\nabla^2 + 2v_0^* > \varepsilon I_d\},$$

for a small parameter $0 < \varepsilon \ll 1$. This corresponds to $-\gamma\nabla^2 + 2v_0^*$ be positive definite, whereas the previous case corresponds to $-\gamma\nabla^2 + 2v_0^*$ be negative definite.

3 Conclusion

In this article we have developed a convex dual variational formulation suitable for the local optimization of a non-convex primal formulation.

It is worth highlighting, the results may be applied to a large class of models in physics and engineering.

We also emphasize the duality principle here presented is applied to a Ginzburg-Landau type equation. In a future research, we intend to extend such results for some models of plates and shells and other models in the elasticity theory.

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