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Article

Polyadic Rings of p-adic Integers

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Abstract. In this note we, first, recall that the sets of all representatives of some special ordinary residue classes become (m,n)-rings. Second, we introduce a possible p-adic analog of the residue class modulo a p-adic integer. Then, we find the relations which determine, when the representatives form a (m,n)-ring. At the very short spacetime scales such rings could lead to new symmetries of modern particle models.

1. Introduction

The fundamental conception of p-adic numbers is based on a special extension of the rational numbers which is an alternative to the real and complex numbers. The main idea is the completion of the rational numbers with respect to the p-adic norm, which is non-Archimedean. Nowadays, p-adic methods are widely used in number theory [1,2], arithmetic geometry [3,4] and algorithmic computations [5]. In mathematical physics, a non-Archimedean approach to spacetime and string dynamics at the Planck scale leads to new symmetries of particle models (see, e.g., [6,7] and Refs. therein). For some special applications, see, e.g., [8,9]. General reviews are given in [10–12].

Previously, we have studied the algebraic structure of the representative set in a fixed ordinary residue class [13]. We found that the set of representatives becomes a polyadic or (m,n)-ring, if the parameters of a class satisfy special "quantization" conditions. We have found that similar polyadic structures appear naturally for p-adic integers, if we introduce informally a p-adic analog of the residue classes, and investigate here the set of its representatives along the lines of [13–15].

2. (m,n)-rings of integer numbers from residue classes

Here we recall that representatives of special residue (congruence) classes can form polyadic rings, as was found in [13,14] (see also notation from [15]).

Let us denote the residue (congruence) class of an integer a modulo b by

$$[a]_b = \{\{r_k(a,b)\} \mid k \in \mathbb{Z}, a \in \mathbb{Z}_+, b \in \mathbb{N}, 0 \leq a \leq b-1\}, \tag{2.1}$$

where $r_k(a,b) = a + bk$ is a generic representative element of the class $[a]_b$. The canonical representative is the least nonnegative number among these. Informally, a is the remainder of $r_k(a,b)$ when divided by b. The corresponding equivalence relation (congruence modulo b) is denoted by

$$r = a(\operatorname{mod} b). \tag{2.2}$$

Introducing the binary operations between classes $(+_{cl}, \times_{cl})$, the addition $[a_1]_b +_{cl} [a_2]_b = [a_1 + a_2]_b$ and multiplication $[a_1]_b \times_{cl} [a_2]_b = [a_1 a_2]_b$, the residue class (binary) finite commutative ring $\mathbb{Z}/b\mathbb{Z}$ (with identity) is defined in the standard way (which was named "external" [13]). If $a \neq 0$ and b is prime, then $\mathbb{Z}/b\mathbb{Z}$ becomes a finite field.

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The set of representatives $\{r_k(a,b)\}$ in a given class $[a]_b$ does not form a binary ring, because there are no binary operations (addition and multiplication) which are simultaneously closed for arbitrary a and b. Nevertheless, the following polyadic operations on representatives $r_k = r_k(a,b)$, m-ary addition ν_m

$$\nu_m\left[r_{k_1}, r_{k_2}, \ldots, r_{k_m}\right] = r_{k_1} + r_{k_2} + \ldots + r_{k_m}, \tag{2.3}$$

and n-ary multiplication μ_n

$$\mu_n\left[r_{k_1}, r_{k_2}, \ldots, r_{k_n}\right] = r_{k_1} r_{k_2} \ldots r_{k_n}, \quad r_{k_i} \in [a]_b,\ k_i \in \mathbb{Z}, \tag{2.4}$$

can be closed, but only for special values of $a = a_q$ and $b = b_q$, which defines the nonderived (m,n)-ary ring

$$\mathbb{Z}_{(m,n)}\left(a_q, b_q\right) = \left\langle \left[a_q\right]_{b_q} \mid \nu_m, \mu_n \right\rangle \tag{2.5}$$

of polyadic integers (that was called the "internal" way [13]). The conditions of closure for the operations between representatives can be formulated in terms of the (arity shape [14]) invariants (which may be seen as some kind of "quantization")

$$(m-1)\frac{a_q}{b_q} = I_m\left(a_q, b_q\right) \in \mathbb{N}, \tag{2.6}$$

$$a_q^{n-1}\frac{a_q - 1}{b_q} = J_n\left(a_q, b_q\right) \in \mathbb{N}, \tag{2.7}$$

or, equivalently, using the congruence relations [13]

$$m a_q \equiv a_q \left(\operatorname{mod} b_q\right), \tag{2.8}$$

$$a_q^n \equiv a_q \left(\operatorname{mod} b_q\right), \tag{2.9}$$

where we have denoted by a_q and b_q the concrete solutions of the "quantization" equations (2.6)–(2.9). To understand the nature of the "quantization", we consider in detail the concrete example of nonderived m-ary addition and n-ary multiplication appearence for representatives in a fixed residue class.

Example 1. *Let us consider the following residue class*

$$[[3]]_4 = \{\ldots -45, -33, -29, -25, -21, -17, -13, -9, -5, -1, 3, 7, 11, 15, 19, 23, 27, 31 \ldots\}, \tag{2.10}$$

where the representatives are

$$r_k = r_k(3,4) = 3 + 4k, \quad k \in \mathbb{Z}. \tag{2.11}$$

We first obtain the condition for when the sum of m representatives belongs to the class (2.10). So we compute step by step

$$m = 2, \quad r_{k_1} + r_{k_2} = r_k + 3 \notin [[3]]_4, \quad k = k_1 + k_2, \tag{2.12}$$

$$m = 3, \quad r_{k_1} + r_{k_2} + r_{k_3} = r_k + 6 \notin [[3]]_4, \quad k = k_1 + k_2 + k_3, \tag{2.13}$$

$$m = 4, \quad r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} = r_k + 9 \notin [[3]]_4, \quad k = k_1 + k_2 + k_3 + k_4, \tag{2.14}$$

$$m = 5, \quad r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} + r_{k_5} = \boxed{r_k \in [[3]]_4}, \quad k = k_1 + k_2 + k_3 + k_4 + k_5 + 3. \tag{2.15}$$

$$\vdots$$

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Thus, the binary, ternary and 4-ary additions are not closed, while 5-ary addition is. In general, the closure of m-ary addition holds valid when $4 \mid (m-1)$, *that is for* $m = 5, 9, 13, 17, \ldots$, *such that*

$$\sum_{i=1}^{m} r_{k_i} = r_k \in [[3]]_4, \quad k = \sum_{i=1}^{m} k_i + 3\ell_\nu, \tag{2.16}$$

where $\ell_\nu = \frac{m-1}{4} \in \mathbb{N}$ *is a natural number. The "quantization" rule for the arity of addition (2.6), (2.8) becomes* $m = 4\ell_\nu + 1$.

If we consider the minimal arity $m = 5$, *we arrive at the conclusion that* $\langle \{r_k\} \mid \nu_5 \rangle$, *is a 5-ary commutative semigroup, where* ν_5 *is the nonderived (i.e. not composed from lower arity operations) 5-ary addition*

$$\nu_5 [r_{k_1}, r_{k_2}, r_{k_3}, r_{k_4}, r_{k_5}] = r_{k_1} + r_{k_2} + r_{k_3} + r_{k_4} + r_{k_5}, \quad r_{k_i} \in [3]_4, \tag{2.17}$$

given by (2.15), and total 5-ary associativity follows from that of the binary addition in (2.3) and (2.17). In this case ℓ_ν *is the "number" of composed 5-ary additions (the polyadic power). There is no the neutral element z for 5-ary addition* ν_5 *(2.17) defined by* $\nu_5[z, z, z, z, r_k] = r_k$, *and so the semigroup* $\langle \{r_k\} \mid \nu_5 \rangle$ *is zeroless. Nevertheless,* $\langle \{r_k\} \mid \nu_5 \rangle$ *is a 5-ary group (which is impossible in the binary case, where all groups contain a neutral element, the identity), because each element* r_k *has a unique querelement* $\widetilde{r}_k$ *defined by (see,e.g., [15])*

$$\nu_5 [r_k, r_k, r_k, r_k, \widetilde{r}_k] = r_k, \quad r_k, \widetilde{r}_k \in [3_4], \tag{2.18}$$

and therefore from (2.15) and (2.17) it follows that $\tilde{r}_k = -3r_k = r_{-9-12k}$. *For instance, for the first several elements of the residue class* $[3]_4$ *(2.10), we have the following querelements*

$$\widetilde{7} = -21, \ \widetilde{11} = -33, \ \widetilde{15} = -45, \tag{2.19}$$

$$\widetilde{-1} = 3, \ \widetilde{-5} = 15, \ \widetilde{-9} = 27. \tag{2.20}$$

Note that in the 5-ary group $\left\langle \{r_k\} \mid \nu_5, \widetilde{(\cdot)} \right\rangle$, *the additive quermapping defined by* $r_k \mapsto \widetilde{r}_k$, *is not a reflection (of any order) for* $m \geq 3$, *i.e.* $\widetilde{\widetilde{r}}_k \neq r_k$ *(as opposed to the inverse in the binary case) [15].*

Now we turn to multiplication of n representatives (2.11) of the residue class $[3]_4$ *(2.10). By analogy with (2.12)–(2.15) we obtain, step by step*

$$\begin{aligned}
& n = 2 \quad r_{k_1} r_{k_2} = r_k + 6 \notin [[3]]_4, \quad k = 3k_1 + 3k_2 + 4k_1 k_2, \\
& n = 3 \quad \begin{cases} r_{k_1} r_{k_2} r_{k_3} = \boxed{r_k \in [[3]]_4}, \\ k = 9k_1 + 9k_2 + 9k_3 + 12k_1k_2 + 12k_1k_3 + 12k_2k_3 + 16k_1k_2k_3 + 6, \end{cases} \\
& n = 4 \quad \begin{cases} r_{k_1} r_{k_2} r_{k_3} r_{k_4} = r_k + 2 \notin [[3]]_4, \\ k = 27k_1 + 27k_2 + 27k_3 + 27k_4 + 36k_1k_2 + 36k_1k_3 + 36k_1k_4 \\ +36k_2k_3 + 36k_2k_4 + 36k_3k_4 + 48k_1k_2k_3 + 48k_1k_2k_4 + 48k_1k_3k_4 \\ +48k_2k_3k_4 + 64k_1k_2k_3k_4 + 19, \end{cases} \\
& n = 5 \quad \begin{cases} r_{k_1} r_{k_2} r_{k_3} r_{k_4} r_{k_5} = \boxed{r_k \in [[3]]_4}, \\ k = 81k_1 + 81k_2 + 81k_3 + 81k_4 + 81k_5 + 108k_1k_2 + 108k_1k_3 + 108k_1k_4 \\ +108k_2k_3 + 108k_1k_5 + 108k_2k_4 + 108k_2k_5 + 108k_3k_4 + 108k_3k_5 + 108k_4k_5 \\ +144k_1k_2k_3 + 144k_1k_2k_4 + 144k_1k_2k_5 + 144k_1k_3k_4 + 144k_1k_3k_5 + 144k_2k_3k_4 \\ +144k_1k_4k_5 + 144k_2k_3k_5 + 144k_2k_4k_5 + 144k_3k_4k_5 + 192k_1k_2k_3k_4 + 192k_1k_2k_3k_5 \\ +192k_1k_2k_4k_5 + 192k_1k_3k_4k_5 + 192k_2k_3k_4k_5 + 256k_1k_2k_3k_4k_5 + 60. \end{cases}
\end{aligned} \tag{2.21}$$

By direct computation we observe that the binary and 4-ary multiplications are not closed, but the ternary and 5-ary ones are closed. In general, the product of $n = 2\ell_\mu + 1$ $(\ell_\mu \in \mathbb{N})$ *elements of the residue class* $[3]_4$ *is in the class, which is the "quantization" rule for multiplication (2.7) and*

(2.9). *Again we consider the minimal arity* $n = 3$ *of multiplication and observe that* $\langle \{r_k\} \mid \mu_3 \rangle$ *is a commutative ternary semigroup, where*

$$\mu_3[r_{k_1}, r_{k_2}, r_{k_3}] = r_{k_1} r_{k_2} r_{k_3}, \quad r_{k_i} \in [3]_4 \tag{2.22}$$

is a nonderived ternary multiplication, and the total ternary associativity of μ_3 *is governed by associativity of the binary product in* (2.4) *and* (2.22)*. As opposed to the* 5*-ary addition,* $\langle \{r_k\} \mid \mu_3 \rangle$ *is not a group, because not all elements have a unique querelement. However, a polyadic identity* e *defined by (i.e. as a neutral element of the ternary multiplication* μ_3*)*

$$\mu_3[e, e, r_k] = r_k, \;\; e, r_k \in [3]_4, \tag{2.23}$$

exists and is equal to $e = -1$.

The polyadic distributivity between ν_5 *and* μ_3 *[15] follows from the binary distributivity in* $\mathbb{Z}$ *and* (2.17),(2.22)*, and therefore the residue class* $[3]_4$ *has the algebraic structure of the polyadic ring*

$$\mathbb{Z}_{(5,3)} = \mathbb{Z}_{(5,3)}(3,4) = \left\langle \{r_k\} \mid \nu_5, \widetilde{(\cdot)}, \mu_3, e \right\rangle, \quad e, r_k \in [3]_4, \tag{2.24}$$

which is a commutative zeroless $(5,3)$*-ring with the additive quermapping* $\widetilde{(\cdot)}$ (2.18) *and the multiplicative neutral element* e (2.23)*.*

The arity shape of the ring of polyadic integers $\mathbb{Z}_{(m,n)}(a_q, b_q)$ (2.5) is the (surjective) mapping

$$(a_q, b_q) \Longrightarrow (m, n). \tag{2.25}$$

The mapping (2.25) for the lowest values of a_q, b_q is given in TABLE 1 ($I = I_m(a_q, b_q)$, $J = J_n(a_q, b_q)$).

Table 1. The arity shape mapping (2.25) for the polyadic ring $\mathbb{Z}_{(m,n)}(a_q, b_q)$ (2.5). Empty cells indicate that no such structure exists.

$a_q \setminus b_q$	2	3	4	5	6	7	8	9	10
1	$m=\mathbf{3}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{4}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{5}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{6}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{7}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{8}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{9}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{10}$ $n=\mathbf{2}$ $I=1$ $J=0$	$m=\mathbf{11}$ $n=\mathbf{2}$ $I=1$ $J=0$
2		$m=\mathbf{4}$ $n=\mathbf{3}$ $I=2$ $J=2$		$m=\mathbf{6}$ $n=\mathbf{5}$ $I=2$ $J=6$	$m=\mathbf{4}$ $n=\mathbf{3}$ $I=1$ $J=1$	$m=\mathbf{8}$ $n=\mathbf{4}$ $I=2$ $J=2$		$m=\mathbf{10}$ $n=\mathbf{7}$ $I=2$ $J=14$	$m=\mathbf{6}$ $n=\mathbf{5}$ $I=1$ $J=3$
3			$m=\mathbf{5}$ $n=\mathbf{3}$ $I=3$ $J=6$	$m=\mathbf{6}$ $n=\mathbf{5}$ $I=3$ $J=48$	$m=\mathbf{3}$ $n=\mathbf{2}$ $I=1$ $J=1$	$m=\mathbf{8}$ $n=\mathbf{7}$ $I=3$ $J=312$	$m=\mathbf{9}$ $n=\mathbf{3}$ $I=3$ $J=3$		$m=\mathbf{11}$ $n=\mathbf{5}$ $I=3$ $J=24$
4				$m=\mathbf{6}$ $n=\mathbf{3}$ $I=4$ $J=12$	$m=\mathbf{4}$ $n=\mathbf{2}$ $I=2$ $J=2$	$m=\mathbf{8}$ $n=\mathbf{4}$ $I=4$ $J=36$		$m=\mathbf{10}$ $n=\mathbf{4}$ $I=4$ $J=28$	$m=\mathbf{6}$ $n=\mathbf{3}$ $I=2$ $J=6$
5					$m=\mathbf{7}$ $n=\mathbf{3}$ $I=5$ $J=20$	$m=\mathbf{8}$ $n=\mathbf{7}$ $I=11$ $J=11160$	$m=\mathbf{9}$ $n=\mathbf{3}$ $I=5$ $J=15$	$m=\mathbf{10}$ $n=\mathbf{7}$ $I=5$ $J=8680$	$m=\mathbf{3}$ $n=\mathbf{2}$ $I=1$ $J=2$
6						$m=\mathbf{8}$ $n=\mathbf{3}$ $I=6$ $J=30$			$m=\mathbf{6}$ $n=\mathbf{2}$ $I=3$ $J=3$
7							$m=\mathbf{9}$ $n=\mathbf{3}$ $I=7$ $J=42$	$m=\mathbf{10}$ $n=\mathbf{4}$ $I=7$ $J=266$	$m=\mathbf{11}$ $n=\mathbf{5}$ $I=7$ $J=1680$
8								$m=\mathbf{10}$ $n=\mathbf{3}$ $I=8$ $J=56$	$m=\mathbf{6}$ $n=\mathbf{5}$ $I=4$ $J=3276$
9									$m=\mathbf{11}$ $n=\mathbf{3}$ $I=9$ $J=72$

The binary ring of ordinary integers $\mathbb{Z}$ corresponds to $(a_q = 0, b_q = 1) \Longrightarrow (2,2)$ or $\mathbb{Z} = \mathbb{Z}_{(2,2)}(0,1)$, $I = J = 0$.

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3. Representations of p-adic integers

Let us explore briefly some well-known definitions regarding p-adic integers to establish notations (for reviews, see [10,11,16]).

A p-adic integer is an infinite formal sum of the form

$$\boldsymbol{x} = \boldsymbol{x}(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots + \alpha_{i-1} p^{i-1} + \alpha_i p^i + \alpha_{i+1} p^{i+1} + \ldots, \quad \alpha_i \in \mathbb{Z}, \tag{3.1}$$

where the digits (denoted by Greek letters from the beginning of alphabet) $0 \leq \alpha_i \leq p-1$, and $p \geq 2$ is a fixed prime number. The expansion (3.1) is called standard (or canonical), and α_i are the p-adic digits which are usually written from the right to the left (positional notation) $\boldsymbol{x} = . \ldots \alpha_{i+1}\alpha_i\alpha_{i-1}\ldots\alpha_2\alpha_1\alpha_0$ or sometimes $\boldsymbol{x} = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_i, \alpha_{i+1}\ldots\}$. The set of p-adic integers is a commutative ring (of p-adic integers) denoted by $\mathbb{Z}_p = \{\boldsymbol{x}\}$, and the ring of ordinary integers (sometimes called “rational” integers) $\mathbb{Z}$ is its (binary) subring.

The so called coherent representation of $\mathbb{Z}_p$ is based on the (inverse) projective limit of finite fields $\mathbb{Z}/p^l\mathbb{Z}$, because the surjective map $\mathbb{Z}_p \longrightarrow \mathbb{Z}/p^l\mathbb{Z}$ defined by

$$\alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots + \alpha_i p^i + \ldots \mapsto \left(\alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots + \alpha_{l-1} p^{l-1}\right) \bmod p^l \tag{3.2}$$

is a ring homomorphism. In this case, a p-adic integer is the infinite Cauchy sequence that converges to

$$\boldsymbol{x} = \boldsymbol{x}(p) = \{x_i(p)\}_{i=1}^{\infty} = \{x_1(p), x_2(p), \ldots, x_i(p)\ldots\}, \tag{3.3}$$

where

$$x_i(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots + \alpha_{l-1} p^{l-1} \tag{3.4}$$

with the coherency condition

$$x_{i+1}(p) \equiv x_i(p) \bmod p^i, \quad \forall i \geq 1, \tag{3.5}$$

and the p-adic digits are $0 \leq \alpha_i \leq p-1$.

If $0 \leq x_i(p) \leq p^i - 1$ for all $i \geq 1$, then the coherent representation (3.3) is called reduced. The ordinary integers $x \in \mathbb{Z}$ embed into p-adic integers as constant infinite sequences by $x \mapsto \{x, x, \ldots, x, \ldots\}$.

Using the fact that the process of reducing modulo p^i is equivalent to vanishing the last i digits, the coherency condition (3.5) leads to a sequence of partial sums [16]

$$\boldsymbol{x} = \boldsymbol{x}(p) = \{y_i(p)\}_{i=1}^{\infty} = \{y_1(p), y_2(p), \ldots, y_i(p)\ldots\}, \tag{3.6}$$

where

$$y_1(p) = \alpha_0,\ y_2(p) = \alpha_0 + \alpha_1 p,\ y_3(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2,\ y_4(p) = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \alpha_3 p^3,\ \ldots. \tag{3.7}$$

Sometimes the partial sum representation (3.6) is simpler for p-adic integer computations.

4. (m,n)-rings of p-adic integers

As may be seen from Section 2 and [13,14], the construction of the nonderived (m,n)-rings of ordinary (“rational”) integers $\mathbb{Z}_{(m,n)}(a_q, b_q)$ (2.5) can be done in terms of residue class representatives (2.1). To introduce a p-adic analog of the residue class (2.1), one needs some ordering concept, which does not exist for p-adic integers [16]. Nevertheless, one could informally define the following analog of ordering.

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Definition 1. *A "componentwise strict order" $<_{comp}$ is a multicomponent binary relation between p-adic numbers $\boldsymbol{a} = \{\alpha_i\}_{i=0}^{\infty}$, $0 \le \alpha_i \le p-1$ and $\boldsymbol{b} = \{\beta_i\}_{i=0}^{\infty}$, $0 \le \beta_i \le p-1$, such that*

$$\boldsymbol{a} <_{comp} \boldsymbol{b} \Longleftrightarrow \alpha_i < \beta_i, \quad \text{for all } i = 0, \dots, \infty, \quad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}_p, \quad \alpha_i, \beta_i \in \mathbb{Z}. \tag{4.1}$$

A "componentwise nonstrict order" $\le_{comp}$ is defined in the same way, but using the nonstrict order $\le$ for component integers from $\mathbb{Z}$ (digits).

Using this definition we can define a p-adic analog of the residue class informally by changing $\mathbb{Z}$ to $\mathbb{Z}_p$ in (2.1).

Definition 2. *A p-adic analog of the residue class of $\boldsymbol{a}$ modulo $\boldsymbol{b}$ is*

$$[\boldsymbol{a}]_{\boldsymbol{b}} = \{\{\boldsymbol{r_k}(\boldsymbol{a}, \boldsymbol{b})\} \mid \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{k} \in \mathbb{Z}_p, 0 \le \boldsymbol{a} < \boldsymbol{b}\}, \tag{4.2}$$

and the generic representative of the class is

$$\boldsymbol{r_k}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a} +_p \boldsymbol{b} \bullet_p \boldsymbol{k}, \tag{4.3}$$

where $+_p$ and $\bullet_p$ are the binary sum and the binary product of p-adic integers (we treat them componentwise in the partial sum representation (3.7)), and the ith component of (4.3) r.h.s. is computed by $\bmod p^i$.

As with the ordinary ("rational") integers (2.1), the p-adic integer $\boldsymbol{a}$ can be treated as some kind of remainder for the representative $\boldsymbol{r_k}(\boldsymbol{a}, \boldsymbol{b})$ when divided by the p-adic integer $\boldsymbol{b}$. We denote the corresponding p-adic analog of (2.2) (informally, a p-adic analog of the congruence modulo $\boldsymbol{b}$) as

$$\boldsymbol{r} = \boldsymbol{a}(\mathrm{Mod}_p\, \boldsymbol{b}). \tag{4.4}$$

Remark 1. *In general, to build a nonderived (m, n)-ring along the lines of* SECTION 2, *we do not need any analog of the residue class at all, but only the concrete form of the representative (4.3). Then demanding the closure of m-ary addition (2.3) and n-ary multiplication (2.4), we obtain conditions on the parameters (now digits of p-adic integers), similarly to (2.6)–(2.7).*

In the partial sum representation (3.6), the case of ordinary ("rational") integers corresponds to the first component (first digit α_0) of the p-adic integer (3.7), and higher components can be computed using the explicit formulas for sum and product of p-adic integers [17]. Because they are too cumbersome, we present here the "block-schemes" of the computations, while concrete examples can be obtained componentwise using (3.7).

Lemma 1. *The p-adic analog of the residue class (4.2) is a commutative m-ary group $\langle [\boldsymbol{a}]_{\boldsymbol{b}} \mid \boldsymbol{\nu}_m \rangle$, if*

$$(m-1)\boldsymbol{a} = \boldsymbol{b} \bullet_p \boldsymbol{I}, \tag{4.5}$$

where $\boldsymbol{I}$ is a p-adic integer (addition shape invariant), and the nonderived m-ary addition $\boldsymbol{\nu}_m$ is the repeated binary sum of m representatives $\boldsymbol{r_k} = \boldsymbol{r_k}(\boldsymbol{a}, \boldsymbol{b})$

$$\boldsymbol{\nu}_m[\boldsymbol{r}_{\boldsymbol{k}_1}, \boldsymbol{r}_{\boldsymbol{k}_2}, \dots, \boldsymbol{r}_{\boldsymbol{k}_m}] = \boldsymbol{r}_{\boldsymbol{k}_1} +_p \boldsymbol{r}_{\boldsymbol{k}_2} +_p \dots +_p \boldsymbol{r}_{\boldsymbol{k}_m}. \tag{4.6}$$

Proof. The condition of closure for the m-ary addition $\boldsymbol{\nu}_m$ is $\boldsymbol{r}_{\boldsymbol{k}_1} +_p \boldsymbol{r}_{\boldsymbol{k}_2} +_p \dots +_p \boldsymbol{r}_{\boldsymbol{k}_m} = \boldsymbol{r}_{\boldsymbol{k}_0}$ in the notation of (4.2). Using (4.3) it gives $m\boldsymbol{a} + \boldsymbol{b} \bullet_p (\boldsymbol{k}_1 +_p \boldsymbol{k}_2 +_p \dots +_p \boldsymbol{k}_m) = \boldsymbol{a} +_p \boldsymbol{b} \bullet_p \boldsymbol{k}_0$, which is equivalent to (4.5), where $\boldsymbol{I} = \boldsymbol{k}_0 -_p (\boldsymbol{k}_1 +_p \boldsymbol{k}_2 +_p \dots +_p \boldsymbol{k}_m)$. The querelement $\boldsymbol{r}_{\bar{\boldsymbol{k}}}$ [18] satisfies

$$\boldsymbol{\nu}_m[\boldsymbol{r_k}, \boldsymbol{r_k}, \dots, \boldsymbol{r_k}, \boldsymbol{r}_{\bar{\boldsymbol{k}}}] = \boldsymbol{r_k}, \tag{4.7}$$

which has a unique solution $\bar{\boldsymbol{k}} = (2-m)\boldsymbol{k} - \boldsymbol{I}$. Therefore, each element of $[\boldsymbol{a}]_{\boldsymbol{b}}$ is invertible with respect to $\boldsymbol{\nu}_m$, and $\langle [\boldsymbol{a}]_{\boldsymbol{b}} \mid \boldsymbol{\nu}_m \rangle$ is a commutative m-ary group. □

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Lemma 2. *The p-adic analog of the residue class (4.2) is a commutative n-ary semigroup* $\langle [\boldsymbol{a}]_{\boldsymbol{b}} \mid \boldsymbol{\mu}_n \rangle$, *if*

$$\boldsymbol{a}^n - \boldsymbol{a} = \boldsymbol{b} \bullet_p \boldsymbol{J}, \tag{4.8}$$

where $\boldsymbol{J}$ *is a p-adic integer (multiplication shape invariant), and the nonderived m-ary multiplication* $\boldsymbol{\nu}_m$ *is the repeated binary product of n representatives*

$$\boldsymbol{\mu}_n \left[\boldsymbol{r}_{k_1}, \boldsymbol{r}_{k_2}, \ldots, \boldsymbol{r}_{k_n} \right] = \boldsymbol{r}_{k_1} \boldsymbol{r}_{k_2} \ldots \boldsymbol{r}_{k_n}. \tag{4.9}$$

Proof. The condition of closure for the n-ary multiplication $\boldsymbol{\mu}_n$ is $\boldsymbol{r}_{k_1} \bullet_p \boldsymbol{r}_{k_2} \bullet_p \ldots \bullet_p \boldsymbol{r}_{k_m} = \boldsymbol{r}_{k_0}$. Using (4.3) and opening brackets we obtain $n\boldsymbol{a} + \boldsymbol{b} \bullet_p \boldsymbol{J}_1 = \boldsymbol{a} +_p \boldsymbol{b} \bullet_p \boldsymbol{k}_0$, where $\boldsymbol{J}_1$ is some p-adic integer, which gives (4.8) with $\boldsymbol{J} = \boldsymbol{k}_0 -_p \boldsymbol{J}_1$. □

Combining the conditions (4.5) and (4.8), we arrive at

Theorem 1. *The p-adic analog of the residue class (4.2) becomes a* (m,n)*-ring with m-ary addition (4.6) and n-ary multiplication (4.9)*

$$\mathbb{Z}_{(m,n)}(\boldsymbol{a}_q, \boldsymbol{b}_q) = \left\langle \left[\boldsymbol{a}_q\right]_{\boldsymbol{b}_q} \mid \boldsymbol{\nu}_m, \boldsymbol{\mu}_n \right\rangle, \tag{4.10}$$

when the p-adic integers $\boldsymbol{a}_q, \boldsymbol{b}_q \in \mathbb{Z}_p$ *are solutions of the equations*

$$m\boldsymbol{a}_q = \boldsymbol{a}_q (\mathrm{Mod}_p\, \boldsymbol{b}_q), \tag{4.11}$$

$$\boldsymbol{a}_q^n = \boldsymbol{a}_q (\mathrm{Mod}_p\, \boldsymbol{b}_q). \tag{4.12}$$

Proof. The conditions (4.11)–(4.12) are equivalent to (4.5) and (4.8), respectively, which shows that $\left[\boldsymbol{a}_q\right]_{\boldsymbol{b}_q}$ (considered as a set of representatives (4.3)) is simultaneously an m-ary group with respect to $\boldsymbol{\nu}_m$, and an n-ary semigroup with respect to $\boldsymbol{\mu}_n$, and is therefore a (m,n)-ring. □

If we work in the partial sum representation (3.7), the procedure of finding the digits of p-adic integers $\boldsymbol{a}_q, \boldsymbol{b}_q \in \mathbb{Z}_p$ such that $\left[\boldsymbol{a}_q\right]_{\boldsymbol{b}_q}$ becomes a (m,n)-ring with initially fixed arities is recursive. To find the first digits α_0 and β_0 that are ordinary integers, we use the equations (2.6)–(2.9), and for their arity shape TABLE 1. Next we consider the second components of (3.7) to find the digits α_1 and β_1 of $\boldsymbol{a}_q$ and $\boldsymbol{b}_q$ by solving the equations (4.5) and (4.8) (these having initially given arities m and n from the first step) by application of the exact formulas from [17]. In this way, we can find as many digits $(\alpha_0, , \alpha_{i_{\max}})$ and $(\beta_0, , \beta_{i_{\max}})$ of $\boldsymbol{a}_q$ and $\boldsymbol{b}_q$, as needed for our accuracy preferences in building the polyadic ring of p-adic integers $\mathbb{Z}_{(m,n)}(\boldsymbol{a}_q, \boldsymbol{b}_q)$ (4.10).

Further development and examples will appear elsewhere.

5. Conclusions

The study of "external" residue class properties is a foundational subject in standard number theory. We have investigated their "internal" properties to understand the algebraic structure of the representative set of a fixed residue class. We found that, if the parameters of a class satisfy some special "quantization" conditions, the set of representatives becomes a polyadic ring. We introduced the arity shape, a surjective-like mapping of the residue class parameters to the arity of addition m and arity of multiplication n, which result in commutative (m,n)-rings (see TABLE 1).

We then generalized the approach thus introduced to p-adic integers by defining an analog of residue class for them. Using the coherent representation for p-adic integers as partial sums we defined componentwise the p-adic analog of the "quantization" conditions, for when the set of p-adic representatives form a polyadic ring. Finally, we proposed a recursive procedure to find any desired digits of the p-adic residue class parameters.

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The proposed polyadic algebraic structure of p-adic numbers may lead to new symmetries and features in p-adic mathematical physics and the corresponding particle models.

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