General Relativity, Quantum field theory and Inflation: A review

Santosh Ballav Sapkota

Department of Physics, St.Xavier's College

017bscphy018@sxc.edu.np

November 30, 2022

Abstract

In this article key results in general relativity are studied and the dynamics of quantum fields in curved space-time is reviewed. It has been shown geometrically how dark energy breaks strong energy condition. We study tan-hyperbolic inflation for an open FRWL universe and show that over half and full period scalar field remains in it's initial state.

Keywords— Einstein-Hilbert action; tachyonic scalar field; tan-hyperbolic inflation

1 Introduction

General Relativity is the foundation of modern cosmology which relates the geometric structure of space-time, a four-dimensional Lorentzian manifold, to the matter and energy contained in it. Space-time $\mathcal{M}$ is a smooth, connected, non-compact, Hausdorff manifold whose tangent space at each point is isomorphic to Minkowski space-time and has the metric signature (+,-,-,-) [1]. For a given distribution of mass, coefficients of the metric tensor $g_{\mu\nu}$ and the second-order partial differential equations of general relativity contain some constraints. The total Hamiltonian $\mathcal{H}$ of the universe vanishes [2]. Einstein's field equations describe gravitational force as the geometric property of a space-time manifold. These are non-linear second-order partial differential equations given by:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1}$$

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where $R_{\mu\nu} = R^{\gamma}_{\mu\gamma\nu}$ is Ricci tensor, $R = R^{\mu}_{\mu}$ is Ricci scalar, the stress-energy tensor is $T_{\mu\nu}$, Einstein tensor is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ and Riemann curvature tensor $R^{\sigma}_{\alpha\mu\nu}$ is:

$$R^{\sigma}_{\alpha\mu\nu} = \partial_{\mu}\Gamma^{\sigma}_{\alpha\nu} - \partial_{\nu}\Gamma^{\sigma}_{\alpha\mu} + \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha} - \Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\mu\alpha} \tag{2}$$

which describes the curvature in space-time due to the presence of matter and energy. General relativity sketches the dynamics of large-scale structure of the universe and is one of the two frontiers of modern physics, the other being quantum mechanics. Quantum mechanics describes the evolution of quantum states of particles, their interaction, and the existence of anti-particles as explained by the Dirac equation [3]. In general relativity (diffeomorphism-invariant theory) time is a coordinate on the space-time manifold whereas, in quantum realm time is a fixed background parameter that marks quantum evolution. This implies that general relativity and quantum theory are based on two incompatible notions of time.

General Relativity is a relational theory i.e space-time doesn't exist unless there are sources in it and these sources determine dynamics of space-time. Laws of nature must be expressed without assuming any fixed background geometric structure i.e they must be background independent[4]. Quantum Gravity might provide insights into the early evolutionary phase of our universe. Alan Guth in his paper [5] addressed following problems:

(1) highly homogeneous state of the universe.

(2) the horizon problem and

(3) the monopole problem. Inflation was the period during the early phase of our universe in which entire universe grew immensely driven by hypothetical scalar field called inflaton field [6]. Chaotic inflation even describes the creation of bubble universes. The initial big bang singularity can be resolved by introducing an initial quantum state of the universe and gives some hint of quantum gravity when started with the quantization of the most elementary form of space-time called *space-time atom* plus puts some light on the dynamics of microscopic degrees of freedom and probability distribution of nucleating small closed universes [7].

2 General Theory of Relativity

The theory of general relativity demonstrates that the classical gravitational field is a manifestation of the curvature of space-time. The metric of the Lorentzian manifold determines the length of the path connecting any two points and is formally stated as:

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} \tag{3}$$

Here, $g_{\mu\nu}$ is the symmetric covariant tensor of rank two called the metric tensor. A metric tensor acts as a potential of the gravitational field (due to curvature effect). Riemann curvature tensor, a function of metric tensor $g_{\mu\nu}$ in equation (2) determines the space-time curvature. The equation

 doi:10.20944/preprints202212.0003.v1

of geodesic in a curved space is:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0 \tag{4}$$

The detailed proof can be found in [8]. Gravitational dynamics is described by Einstein's field equations (1) and there is a reciprocation of energy-momentum between matter and gravitation.

2.1 Einstein-Hilbert Action

Space-time is dynamic and the metric is a dynamical variable in the general theory of relativity. In this section, we derive Einstein field equations using variational approach with the assumption that Ricci scalar R completely defines the action. Consider the action of the form [9]:

$$S = \int_{\mathcal{M}} \mathcal{L} d^4V \tag{5}$$

where, $\mathcal{L}$ is the Lagrangian density and V is the 4-volume. Let $g = det(g_{\mu\nu})$ for Riemannian manifold with metric signature (+,+,+,+) and volume element $d^4V = \sqrt{g}d^4x$. For semi-Riemannian manifold the metric signature is (+,-,-,-) and volume element $d^4V = \sqrt{-g}d^4x$.

$$S = \int_{\mathcal{M}} \mathcal{L}\sqrt{-g}d^4x \tag{6}$$

The total Lagrangian density is contributed both by geometry and matter,

$$\mathcal{L} = \mathcal{L}^{(g)} + \mathcal{L}^{(m)} \tag{7}$$

so is the action

$$S_{E-H} = S^{(g)} + S^{(m)} \tag{8}$$

Here,

$$S^{(m)} =^{+}_{-} \int_{\mathcal{M}} d^4x\sqrt{-g}\mathcal{L}^{(m)} \tag{9}$$

$$S^{(g)} = \int_{\mathcal{M}} d^4x\sqrt{-g}\frac{R}{2k} \tag{10}$$

Variation of metric tensor $g^{\mu\nu}$ make the terms $\delta g^{\mu\nu}$ and $\delta g^{\mu\nu}_{,\lambda}$ vanish on the boundary Σ of the manifold $\mathcal{M}$. Variation of the action (9) and (10) w.r.t metric tensor $g_{\mu\nu}$ are respectively:

$$\delta S^{(m)} = \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}T_{\mu\nu}\delta g_{\mu\nu} \tag{11}$$

and

$$\delta S^{(g)} = \frac{1}{2k}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[\delta R + \frac{1}{\sqrt{-g}}R\delta\sqrt{-g}\right] \tag{12}$$

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From appendix (A.2)

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S^{(m)}}{\delta g^{\mu\nu}} = 2\left[\frac{\partial \mathcal{L}(x)}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}(x)\mathcal{L}(x)\right]$$

since,

$$\frac{\delta(\sqrt{-g})}{\delta g^{\mu\nu}} = \frac{-1}{2}\sqrt{-g}g_{\mu\nu}$$

we know

$$\delta R = \delta\left(R_{\mu\nu}g^{\mu\nu}\right) = R_{\mu\nu}\delta(g^{\mu\nu}) + g^{\mu\nu}\delta R_{\mu\nu}$$

Equation (12) becomes

$$\delta S^{(g)} = \frac{1}{2k}\int_M d^4x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\delta g^{\mu\nu} + \frac{1}{2k}\int_{\mathcal{M}} d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} \tag{13}$$

Let us evaluate the second integral of equation (13):

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}g^{\sigma\lambda}\left[(\delta g_{\mu\nu})_{,\sigma\lambda} - (\delta g_{\mu\lambda})_{,\nu\sigma}\right]$$

$$\int_{\mathcal{M}} d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \int_M d^4x\eta^{\mu\nu}\eta^{\sigma\lambda}\left[(\delta g_{\mu\nu})_{,\sigma\lambda} - (\delta g_{\mu\lambda})_{,\nu\sigma})\right]$$

$$\int_{\mathcal{M}} d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \int_\Sigma d^3x\left[\eta^{\mu\nu}\eta^{\sigma\lambda}\left(\delta g_{\mu\nu}\right)_{,\sigma}\hat{n}_\lambda - \eta^{\mu\nu}\eta^{\sigma\lambda}\left(\delta g_{\mu\lambda}\right)_{,\nu}\hat{n}_\sigma\right] \tag{14}$$

Here, $\hat{n}$ is the outward pointing normal, Σ is a 3-manifold which covers 4-manifold $\mathcal{M}$ and is a hypersurface embedded in the 4-Lorentzian manifold containing end points of paths of motion.

Theorem 1. *(Stokes theorem). Let M be an n-dimensional oriented smooth manifold with boundary and give boundary Σ the induced orientation. Then for any $\omega \,\epsilon\, \Omega_c^{n-1}(M)$ we have:*

$$\int_M d\omega = \int_\Sigma \omega$$

For the detailed proof of the above theorem reader can consult [9, 10].

Using Theorem (1) and equation(14) we get:

$$\int_{\mathcal{M}} d^4x\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = 0 \tag{15}$$

Equation (13) reduces to

$$\delta S^{(g)} = \frac{1}{2k}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\delta g^{\mu\nu} \tag{16}$$

Hence, total variation of the Einstein-Hilbert action is:

$$\delta S_{E-H} = \int_M d^4x\sqrt{-g}\left[\frac{1}{2k}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\right]\delta g^{\mu\nu} + \frac{1}{2}\int_M d^4x\sqrt{-g}T_{\mu\nu}\delta g_{\mu\nu} \tag{17}$$

The Principle of least action demands $\frac{\delta S_{E-H}}{\delta g_{\mu\nu}} = 0$ and equation (17) reduces to:

$$\frac{1}{2k}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \frac{1}{2}T_{\mu\nu} = 0$$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -kT_{\mu\nu} \tag{18}$$

Equation (18) is the Einstein's field equation. This is the gravitational equation which is non-linear partial differential equation, essentially hyperbolic and diffeomorphism invariant. In [11] author show how reasonable it is to determine the solution based on initial data on a spacelike hypersurface Σ (Cauchy problem). In locally inertial coordinate exchange of energy and momentum between matter and geometry vanishes but it holds globally.

2.2 Bianchi Identity

The Bianchi identity is important in determining the metric when the form of curvature tensor is given. It expresses the symmetry property of Riemannian curvature tensor [12]. In covariant language, we write the Bianchi identity as:

$$R_{\mu\nu\sigma\lambda;\theta} + R_{\mu\nu\lambda\theta;\sigma} + R_{\mu\nu\theta\sigma;\lambda} = 0 \tag{19}$$

choose locally inertial co-ordinate where $\Gamma^{\mu}_{\nu\sigma} = 0$ holds.

$$\left[R^{\mu}_{\nu\mu\lambda;\theta} + R^{\mu}_{\nu\lambda\theta;\mu} + R^{\mu}_{\nu\theta\mu;\lambda}\right]g^{\nu\lambda} = 0 \tag{20}$$

and metricity condition implies

$$g^{\nu\lambda}_{;\theta} = 0 \tag{21}$$

$$R_{;\theta} + g^{\nu\lambda}R^{\mu}_{\nu\lambda\theta;\mu} - g^{\nu\lambda}R^{\mu}_{\nu\mu\theta;\lambda} = 0 \tag{22}$$

Since Rimemann curvature tensor is:

$$R^{\sigma}_{\alpha\mu\nu} = \partial_\mu\Gamma^{\sigma}_{\alpha\nu} - \partial_\nu\Gamma^{\sigma}_{\alpha\mu} + \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha} - \Gamma^{\sigma}_{\nu\lambda}\Gamma^{\lambda}_{\mu\alpha} \tag{23}$$

we obtain following relations due to the symmetry properties of Riemann curvature tensor

$$R^{\mu}_{\nu\lambda\sigma} + R^{\mu}_{\sigma\lambda\nu} + R^{\mu}_{\lambda\nu\sigma} = 0$$

Further;

$$g^{\nu\lambda}R^{\mu}_{\nu\lambda\theta} = -g^{\nu\lambda}R^{\mu}_{\lambda\theta\nu}$$

$$R_{;\theta} - g^{\nu\lambda}R^{\mu}_{\lambda\theta\nu;\mu} - g^{\nu\lambda}R^{\mu}_{\nu\mu\theta;\lambda} = 0$$

$$R_{;\theta} - 2R^{\mu}_{\theta;\mu} = 0$$

$$\left[R^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu}R\right]_{;\mu} = 0 \tag{24}$$

$$\left[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right]_{;\nu} = 0 \tag{25}$$

$$\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]^{;\nu} = 0 \tag{26}$$

Equations (24), (25), and (26) are the components of Einstein tensor. It follows naturally that the Einstein tensor $G^{\mu\nu}$ has zero divergence and the conservation law given in the appendix (A.3) also follows from these equations. Cosmological models have been studied based on the idea of spatial homogeneity which means the existence of a group of isometries with spacelike generators [13]. One of the simplest solution of Einstein field equations is perfect fluid filled Friedmann Universe. We can extend the results obtained in section (2.2) for a perfect fluid space-time which is shear-free and vorticity-free to get spatial uniformity of physical variables such as pressure, density, etc. *Weyl* postulated that the world lines of particles form a 3-bundle of non-intersecting geodesics which are orthogonal to spacelike hypersurfaces i.e matter defines it's own natural co-ordinates. Here we consider a centrally symmetric gravitational field.

2.3 The Friedmann Universe

Space-time with homogeneous and isotropic spatial sections has the metric which is maximally symmetric called the Robertson-Walker metric of the form [14]:

$$ds^2 = dt^2 - a(t)^2\left[\frac{dr^2}{1-k_o r^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{27}$$

$$(-g)^{\frac{1}{2}} = r^2.\frac{1}{\sqrt{1-k_o r^2}} sin\theta a(t)^3 \tag{28}$$

where (t, r, θ, ϕ) are comoving coordinates, $a(t)$ is the cosmic scale factor and k_o is the spatial curvature. Following [12, 14] we derive the dynamical equations describing the evolution of the scale factor $a(t)$ which follow from the Einstein field equations. For detailed proof, the reader can

 doi:10.20944/preprints202212.0003.v1

refer to [15]. The non-zero components of the Ricci tensor for this highly symmetric metric are:

$$R^0_0 = 3\frac{\ddot{a}}{a}$$

$$R^i_j = -\left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k_o}{a^2}\right] g^i_j$$

The Ricci scalar is:

$$R = 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a} + \frac{k_o}{a^2}\right] \tag{29}$$

The 0-0 component of the equation (1) gives the Friedmann equation :

$$\frac{\dot{a}^2 + k_o}{a^2} = \frac{8\pi G\rho}{3} \tag{30}$$

while i-i component gives

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k_o}{a^2} = -8\pi Gp \tag{31}$$

Here, ρ and p are the density and pressure of the fluid filling the universe with stress-energy-momentum tensor of the form $T^\mu_\nu = diag(\rho, -p, -p, -p)$. We know conservation law implies $T^{\mu\nu}_{;\nu} = 0$. The equation for acceleration is

$$\frac{\ddot{a}}{a} = -\frac{4\pi G(\rho + 3p)}{3} \tag{32}$$

It implies that the universe had point singularity at a finite time in the past.

2.4 The de Sitter spacetime

In an empty de-Sitter spacetime the only source present is the positive cosmological constant and the stress-energy-momentum tensor has the form $T^\mu_\nu = -\rho\delta^\mu_\nu$ and possesses high degree of symmetry [12]. Einstein's equation becomes:

$$G^\mu_\nu = -k\rho\delta^\mu_\nu \tag{33}$$

Here the solutions will be simple inflationary solutions. The Einstein field equation are:

$$3\frac{\dot{a} + k_o^2}{a^2} - \Lambda = 0$$

The equation involving acceleration becomes:

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k_o}{2a^2} - \frac{\Lambda}{2} = 0 \tag{34}$$

We deduce the following result:

$$\dot{a}^2 - \omega^2 a^2 = -k_o \tag{35}$$

 doi:10.20944/preprints202212.0003.v1

Here, $\omega^2 = \frac{\Lambda}{3}$ and K is the curvature of the universe.

$$a(t) = \begin{matrix} \cosh(\omega t), & k_o = +1 \\ \exp(\omega t), & k_o = 0 \\ \sinh(\omega t), & k_o = -1 \end{matrix}$$

For a closed universe, $k_o = +1$, the universe has a minimum size at $t = 0$ which is not a point. The universe bounces and doesn't reach a point singularity. For the universe with $k_o = 0$ the singularity is reached in infinite time and for open universe $k_o = -1$ has coordinate singularity. De Sitter spacetime is not kleinian and is geodesically complete in the future.

Other solutions of Einstein field equations include scwarzschild metric, kerr metric, vaidya metric (for the geometry outside a star which is emitting a great deal of radiation). P.C Vaidya derived the Stress tensor for the mixture of radiation emiited which follow non static radiating distribution in [16]. It is known that a closed ($k_o = +1$) de Sitter spacetime is singularity free but FRWL universe filled with pressure less dust and radiation is singular.

The weak energy condition (WEC)

If the stress-energy-momentum tensor obeys the inequality

$$T^{\mu\nu} u_\mu u_\nu \geq 0 \tag{36}$$

then conditionally energy-momentum tensor obeys weak energy condition for all time-like vectors $\mathbf{u}^\mu$. Here the energy density being positive and is satisfied by most fluids including vacuum fluid [17].

The Strong Energy Condition (SEC)

If the stress-energy-momentum tensor obeys the inequality

$$T^{\mu\nu} u_\mu u_\nu \geq \frac{1}{2} T g^{\mu\nu} u_\mu u_\nu \tag{37}$$

then the energy-momentum tensor obeys strong energy condition for all time-like vectors $\mathbf{u}^\mu$. It implies that spacetime has positive curvature for time-like vector i.e $R^{\mu\nu} u_\mu u_\nu \geq 0$ and if this is satisfied then all geodesics will converge at some point and in some cosmological models this is the initial big bang singularity. Raychaudhuri showed in [18] by simply introducing anisotropy to isotropic cosmological models will not solve the problems in standard cosmology for example cosmological singularity and to solve the initial singularity problem Raychaudhuri introduced spin and shear to the isotropic universe. One key ingredient has been gravitational instanton when it comes to initial singularity. For locally isotropic expansion the space is locally isotropic in the absence of spin. Let Θ be the expansion scalar, then for non-rotating shear free universe we get the inequality

$$\dot{\Theta} \leq -\frac{1}{3} \Theta^2$$

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Dividing by Θ^2 gives

$$\frac{d}{d\tau}\left(\frac{1}{\Theta}\right) \tag{38}$$

On integration from τ_0 to τ yields

$$\frac{1}{\Theta(\tau)} \leq \frac{1}{\Theta(\tau_0)} + \frac{1}{3}(\tau - \tau_0) \tag{39}$$

For an expanding universe, we have at $\tau = \tau_0$ and time like geodesics must have passed through initial singularity. If the matter obeys strong energy conditions and the universe is expanding then the condition for all past-directed geodesics to end at the singularity is: expansion scalar $\Theta = 3\frac{\dot{a}}{a}$ and an FRWL universe is expanding at t_o and $\Theta > 0$ on Cauchy hypersurface. Thus any small perturbation on an expanding FRWL universe following equation (37) will be singular [12, 15].

$$H \geq \sqrt{\frac{3\ddot{a}}{2}} \tag{40}$$

H is Hubble constant and $a(t)$ is scale factor. But this is only sufficient condition not a necessary condition. Equation (40) sets a lower bound on the value of hubble's constant and digressing on equation of state $p = \rho\omega$ implies $\omega \geq -\frac{1}{3}$ for an expanding FRWL universe.

2.5 Lagrangian Formulation

The author in [17] considers $(3 + 1)$ split of the space-time $\mathcal{M}$ with the three-dimensional spatial sections. This is the ADM formalism. Let each hypersurface be denoted by Σ_t, labeled by time parameter t, is space-like. Foliation of manifold $\mathcal{M}$ implies:

$$\mathcal{M} = \mathbb{R} \times \Sigma_t \tag{41}$$

Let the metric on Σ_t be h_{ij} induced by the metric tensor $g_{\mu\nu}$ of Lorentzian manifold. Spatial hypersurface Σ_t evolves dynamically and is governed by time variation of the spatial metric tensor h_{ij}. $\forall\ \Sigma_t\ \exists$ a time-like unit normal vector field $\mathbf{n}$ we can split the time vector $\mathbf{t}$ as:

$$\mathbf{t} = N\mathbf{n}^\mu + \mathbf{N}^i \tag{42}$$

where $\mathbf{N}^i$ is tangent to Σ_t and orthogonal to $\mathbf{n}^\mu$. Lapse function is N, $\mathbf{N}^i$ is shift vector and with freedom to choose time vector $\mathbf{t}$ the shift vector and lapse can be arbitrary, this is *gauge freedom* in GR and reflect it's general covariance.

The components of metric $g_{\mu\nu}$ can be calculated as:

$$g_{tt} = \mathbf{t} \cdot \mathbf{t} = -N^2 + N^i N^i \tag{43}$$

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$$a = \mathbf{t} \cdot \mathbf{e}^i = N^i h_{ij} = N_i \tag{44}$$

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = h_{ij} \tag{45}$$

Instead of considering the metric components $g_{\mu\nu}$ as variables, we take the set of variables (N, N_i, h_{ij}). Determinant of metric tensor $g_{\mu\nu}$ in terms of h_{ij} is $\sqrt{-g} = N\sqrt{h}$ and time derivative is:

$$\dot{h}_{ij} \equiv \pounds_{\mathbf{t}} h_{ij} \tag{46}$$

Let $\mathbf{u}$ be a vector field with special case $\mathbf{u} = \mathbf{n}$, covariant derivative of the vector $\mathbf{u}$ is:

$$n_{\mu;\nu} = u_{\mu;\nu} = K_{\mu\nu}$$

Extrinsic curvature can now be defined as:

$$K_{\mu\nu} = -(\mathbf{e}_\alpha \nabla_\beta \mathbf{n}) h^\alpha_\mu h^\beta_\nu = \frac{1}{2} \pounds_{\mathbf{n}} h_{\mu\nu} \tag{47}$$

and

$$K_{\mu\nu} = \frac{1}{3} K h_{\mu\nu} + \sigma_{\mu\nu} \tag{48}$$

where $K = \theta = K^\mu_\mu$, θ is expansion scalar. We know the vector field $\mathbf{n}$ is normal and non-rotating because we assumed the spatial surface Σ_t to be smooth everywhere and this plane of simultaneity doesn't contain any discontinuity.
The covariant derivative of $\mathbf{e}_\nu$ on the hypersurface Σ_t is given by:

$$\bar{\nabla}_\mu \mathbf{e}_\nu = h^\alpha_\beta h^\beta_\nu \nabla_\alpha \mathbf{e}_\beta \tag{49}$$

Equation (46) can be further expanded as

$$\dot{h}_{\mu\nu} = \pounds_{\mathbf{t}} h_{ij} = N \pounds_{\mathbf{n}} h_{ij} + \pounds_{\mathbf{N}} h_{ij} = 2N K_{\mu\nu} + \bar{\nabla}_\mu N_\nu + \bar{\nabla}_\nu N_\nu$$

We know Einstein-Hilbert action for pure gravity and cosmological constant Λ (i.e contribution due the geometry of space-time) is

$$S_{E-H} = \frac{1}{16\pi} \int_{\mathcal{M}} (R - 2\Lambda)\sqrt{-g} d^4x \tag{50}$$

The Ricci scalar in terms of new variables can be written as

$$R = 2(E_{\mu\nu} n^\mu n^\nu - R_{\mu\nu} n^\mu n^\nu) \tag{51}$$

Theorem 2. *(Theorema Egregium). Gauss curvature is preserved by local isometries.*

Intrinsic geometry of surfaces describes those properties which remain invariant through isome-

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tries, or through maps preserving the first fundamental form [19]. **Scalar Gauss relation**, which relates extrinsic curvature to intrinsic curvature of surface Σ, generalizes **Theorema Egregium**. Twice contracted Gauss' equation [20] gives

$$E_{\mu\nu}n^{\mu}n^{\nu} = \frac{1}{2}(\bar{R} - K^{ij}K_{ij} + K^2) \tag{52}$$

where $\bar{R}$ is the Ricci scalar of the hyper-surface Σ_t. The second term in the equation (51) can be expanded using the Riemann tensor as:

$$R_{\mu\nu}n^{\mu}n^{\nu} = R^{\alpha}_{\mu\alpha\nu}n^{\mu}n^{\nu}R_{\mu\nu}n^{\mu}n^{\nu} = K^2 - K^{ij}K_{ij} - \nabla_{\mu}(n^{\mu}\nabla_{\alpha}n^{\alpha}) + \nabla_{\mu}(n^{\alpha}\nabla_{\alpha}n^{\mu})$$

Last two terms of the above equation yields boundary terms to the action integral (50) and these terms can be omitted from the Lagrangian. The Lagrangian density thus becomes

$$\mathcal{L}_G = (R - 2\Lambda)\sqrt{-g} \tag{53}$$

can be written further in the following form:

$$\mathcal{L}_G = N\sqrt{h}[\bar{R} + K^{ij}K_{ij} - K^2 - 2\Lambda] \tag{54}$$

Extrinsic curvature is given by

$$K_{ij} = \frac{1}{2N}(\dot{h_{ij}} - \bar{\nabla}_i N_j - \bar{\nabla}_j N_i) \tag{55}$$

Hence, the variation of Einstein-Hilbert action with respect to the variables yields Einstein's field equations (N, N_i , h_{ij}).

2.6 Hamiltonian Formulation

Einstein-Hilbert action is:

$$S_{E-H} = -\frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g}\,[R(g) + 2\Lambda] \tag{56}$$

$R(g)$ is Ricci scalar and Λ the cosmological constant, ricci scalar R involves second derivative when a compact manifold with the boundary Σ is considered. In [21] authors stated following theorem:

Theorem 3. *Compact Lorentzian manifold of constant curvature is complete.*

In non flat case, $X(c)$, the universal space, itself is not geodesically connected. It is easy to show that M is itself $X(c)$ and is achieved by analysing the dynamics of holonomy group. But this theorm fails for *locally homogenous* Lorentz manifold modeled on (G, X), G acts trasitively on X and preserves the metric but not necessarily with constant curvature.

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In [14] author adds the boundary term in equation (56) and the action takes the form:

$$S = S_{E-H} + \frac{1}{8\pi G}\int_{\Sigma} d^3x\sqrt{h}K \tag{57}$$

where $K = \mathrm{Tr}(K_{ij})$, $h = \det(h_{ij})$

Consider the set of dynamical variables (N, N_i, h_{ij}). The line element S^3 is:

$$ds^2 = a^2(t)\left[d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{58}$$

We know:

$$h_{ij} = -g_{ij} + n_i n_j \tag{59}$$

The Hamiltonian formulation of general theory of relativity requires (3+1) split of the metric i.e foliation of 4-dimensional Lorentzian spacetime by hypersurface of constant time x^0 and these hypersurfaces Σ_t and Σ_{t+dt} are the leaves of the space-time manifold. The proper distance between the points is $d\tau = Ndt$. These co-planar points are shifted to nearby points because of the shift vector, also responsible for distortion of hypersurfaces. For our closed universe example the normal vector $\mathbf{n}^\mu$ takes us between concentric S^3. The proper distance between these points is:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{60}$$

After foliation of the space-time manifold by hypersurfaces Σ_t the metric on this 4-manifold can be partioned into 3-dimensional metric whose components are simply 3×3 submatrix of g. This matrix contains only spatial parts and is induced by $g_{\mu\nu}$. It can be partitioned as:

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{jt} \\ g_{it} & h_{ij} \end{pmatrix} \tag{61}$$

Using equation (61) in (60) we get the line element in the following form:

$$ds^2 = (Ndt)^2 - h_{ij}(N^i dt + dx^i)(N^j dt + dx^j) \tag{62}$$

Variation with respect to the variables (N, N_i , h_{ij}) immediately yields the constants of motion and this fact becomes more apparent in the Hamiltonian formulation of GR [17]. We know total lagrangian density coincides with the lagrangian density for pure gravity $\mathcal{L}_G$ in an otherwise empty universe. Hamiltonian formulation is not space-time covariant.

The canonical momenta

$$\Pi^{ij} \equiv \frac{\partial \mathcal{L}_G}{\dot{h}_{ij}} = \sqrt{h}(K^{ij} - h^{ij}K) \tag{63}$$

The conjugate momenta Π^N and Π^i vanishes identically i.e

$$\Pi^N \equiv \frac{\partial \mathcal{L}_G}{\dot{N}} = 0$$

$$\Pi^i \equiv \frac{\partial \mathcal{L}_G}{\dot{N}_i} = 0 \tag{64}$$

h_{ij} is the only dynamical variable due to *gauge freedom* and N and N_i are the Lagrange-multipliers. Choice of the Lapse function N, tells how close two subsequent hyper-surfaces Σ_t and Σ_{t+dt} are in time, which is a generator of time evolution. In similar manner, the vector N_i is a generator of the co-ordinate transformations for the hyper-surfaces Σ_t.
Let the Hamiltonian be

$$H_G = \int_{\mathcal{M}} \mathcal{H}_G d^4x \tag{65}$$

Here, $\mathcal{H}_G$ is the Hamiltonian density given by

$$\mathcal{H}_G = \dot{h}_{ij}\Pi^{ij} - \mathcal{L}_G \tag{66}$$

Inserting $\mathcal{L}_G$ in equation (65) from (54), we get

$$\mathcal{H}_G = N\sqrt{h}\left[2\Lambda - \bar{R} + h^{-1}\left(\Pi^{ij}\Pi_{ij} - \frac{1}{2}\Pi^2\right)\right] - N_i 2\sqrt{h}\bar{\nabla}_j\left(h^{-\frac{1}{2}}\Pi^{ji}\right)$$

When

$$\sqrt{h}\left[2\Lambda - \bar{R} + h^{-1}\left(\Pi^{ij}\Pi_{ij} - \frac{1}{2}\Pi^2\right)\right] = 0 \tag{67}$$

and

$$2\sqrt{h}\bar{\nabla}_j\left(h^{-\frac{1}{2}}\Pi^{ji}\right) = 0 \tag{68}$$

When equations (67) (68) are equal to zero then these become twice contracted Gauss' theorema egregium and the contracted Codazzi equation for a vacuum spacetime with a cosmological constant.and The equation (67) is called the *Hamiltonian constraint* and (68) is called *momentum constraint* .In this formalism time is arbitrary and choice of time is unphysical gauge freedom. These constraints hint towards profound problem of time. In Quantum theory of gravity, spacetime is dynamical and problem of time will eventually pop up and has to be resolved. Vacuum Einstein's equation:

$$\dot{h}_{ij} = \frac{\delta H_G}{\delta \Pi^{ij}} = 2h^{-\frac{1}{2}}N\left(\Pi_{ij} - \frac{1}{2}h_{ij}\Pi\right) + 2\bar{\nabla}_{(i}N_{j)} \tag{69}$$

$$\dot{\Pi}^{ij} = -\frac{\delta}{\delta h_{ij}}\left(\int_{\mathcal{M}} \dot{h}_{ij}\Pi^{ij} - \mathcal{L}_G\right) \tag{70}$$

The equations (67), (68), (69) and (70) are equivalent to vacuum Einstein's equations with a cosmological constant Λ.

 doi:10.20944/preprints202212.0003.v1

2.7 Canonical Formulation

Action due to the presence of matter is:

$$S^{(m)} = \int_{\mathcal{M}} \mathcal{L}^{(m)} d^4x \tag{71}$$

Energy momentum tensor involving action term is given by

$$T^{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S^{(m)}}{\delta g_{\mu\nu}} \tag{72}$$

The total Lagrangian density being the sum of Lagrangian density due to the geometry and the matter and energy fields [12],given by:

$$\mathcal{L} = \mathcal{L}^{(g)} + \mathcal{L}^{(m)} \tag{73}$$

Lagrangian density of an electromagnetic field (EM field) is:

$$\mathcal{L}_{E-M} = -\frac{1}{4}\sqrt{-g}F^{\mu\nu}F_{\mu\nu} \tag{74}$$

Here, $F_{\mu\nu}$ is the Electromagnetic tensor [22]. The Lagrangian that leads to Klein-Gordon equation in a space filled with complex quantized field Φ is $\mathcal{L}_{K-G}$ [23] given by:

$$\mathcal{L}_{K-G} = -\frac{1}{2}\sqrt{-g}\left[\nabla_\mu \Phi \nabla^\mu \Phi + m^2\Phi^2\right] \tag{75}$$

Equations of motion can be derived using the same procedure as in vacuum case except we need to add matter degress of freedom. If the lagrangian density contains only a single matter field Φ, then the canonically conjugate momentum is:

$$\Pi^{\Phi} \equiv \frac{\delta \mathcal{L}_T}{\dot{\Phi}} \tag{76}$$

Similarly the Hamiltonian density for the matter and geometry is

$$\mathcal{H}_T = \dot{h}_{ij}\Pi^{ij} + \dot{\Pi}^{\Phi} - \mathcal{L}_T \tag{77}$$

The total Hamiltonian density will be the sum:

$$\mathcal{H}_T = N\Upsilon_T + N_i\Upsilon^i_T \tag{78}$$

where is a sum of contributions from pure gravity (geometry) and the matter

$$\Upsilon_T = \Upsilon_G + \Upsilon_m, \Upsilon^a_T = \Upsilon^a_G + \Upsilon^a_m \tag{79}$$

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There are even more complicated theories for which the total Hamiltonian is not purely the direct sum of individual Hamiltonians. However, total Hamiltonian will always be a constraint due to diffeomorphism invariance of the theory.

$$\Upsilon_T = 0, \Upsilon_T^a = 0 \tag{80}$$

3 Quantum Field Dynamics

In this section we will review some key results in quantum field theory and use it in the context of curved spacetime. The application of quantum mechanics to dynamics of fields, particles as excitation of such underlying fields, is Quantum field theory. The action functional $S[\gamma]$ is relativistic invariant which means Lagrangian method in classical mechanics can be supplemented to relativistic fields [24]. Schrodinger equation is the non-relativistic, linear second order partial differential equation which describes the evolution of a quantum system w.r.t to background time parameter. On relativistic treatment one obtains the Klein-Gordon equation which has the following form:

$$\left(\Box + m^2\right)\Phi(\mathbf{r},\mathbf{t}) = 0 \tag{81}$$

Here, $\Box$ is the de-Alembertian operator $\Box \equiv \nabla^2 - \frac{\partial^2}{\partial t^2}$ which is relativistically invariant and Φ transforms like a scalar. Here, $\Phi(\mathbf{r},\mathbf{t})$ is the amplitude function describing klein-gordon particle with spin-0 and is a scalar field. This equation can also be formed in case of curved space-time.In [23, 24, 25] authors derived the solution using variable separation method in terms of spherical polar coordinates, for spacetime with no torsion. These fields have infinite degress of freedom, described by the amplitude function which can be expanded in terms of orthonormal set of functions $u_k(\mathbf{r}$ and the coefficients $C_k(t)$ which represent field coordinates.

$$\Phi(\mathbf{r}, t) = \sum_k C_k(t) u_k(\mathbf{r}) \tag{82}$$

The Lagrangian density, $\mathcal{L}$ $(\Phi,\ \Phi_{,\mu}, \dot{\Phi}, t)$, is the function of field variables and the action integral, functional of field variable, in a certain domain of space-time V is: $S[\Phi] = \int_V \mathcal{L} d^4x$. In field theoretic description as well lagrangian density

$\mathcal{L}$ must satisfy Euler-Lagrange equation of the following form:

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)}\right] = 0 \tag{83}$$

with conjugate momentum density $\Pi(\mathbf{r}$ and the Hamiltonian density $\mathcal{H}(\mathbf{r})$ are:

$$\Pi(\mathbf{r}) = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \tag{84}$$

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$$\mathcal{H}(\mathbf{r}) = \Pi(\mathbf{r})\dot{\Phi} - \mathcal{L} \tag{85}$$

In quantum field theory the field variables Φ and Π are field operators and satisfy certain commutation relation or anticommutation relation between them. This prescription of algebraic relation are known as second quantization and give rise to field quanta [24, 26]. We have the following algebraic relations:

$$\left[\Phi(\mathbf{r},t), \Pi(\mathbf{r}',t)\right]_{-} = i\hbar\delta(\mathbf{r}-\mathbf{r}')$$

$$\left[\Phi(\mathbf{r},t), \Phi(\mathbf{r}',t)\right]_{-} = 0$$

$$\left[\Pi(\mathbf{r},t), \Pi(\mathbf{r}',t)\right]_{-} = 0$$

These are applicable to boson fields whereas fermion fields satisfy following anticommutation relations.

$$\left[\Phi(\mathbf{r},t), \Pi(\mathbf{r}',t)\right]_{+} = i\hbar\delta(\mathbf{r}-\mathbf{r}')$$

$$\left[\Phi(\mathbf{r},t), \Phi(\mathbf{r}',t)\right]_{+} = 0$$

$$\left[\Pi(\mathbf{r},t), \Pi(\mathbf{r}',t)\right]_{+} = 0$$

3.1 Relativistic Fields

In general relativity we study the dynamics of relativistic fields like: Klein-Gordon field (scalar field), Dirac field (spinor field) etc. Here we will focus on the solution of Klein-Gordon equation and in next section we study Klein-Gordon field in curved space-time manifold and look for it's possible application. These fields are Lorentz covariant. Since the Euler-Lagrange equation (83) gives the equations of motions of field, if we sum up in all repeated indices then we have:

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \sum_{\mu} \partial_{\mu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\Phi)} = 0 \tag{86}$$

Let us find the second rank tensor whose components yield the momentum and energy density of the underlying field. Given the lagrangian density $\mathcal{L}\left(\Phi, \partial_{\mu}\Phi, x_{\mu}\right)$ we know:

$$D_{\mu}\mathcal{L} = \partial_{\mu}\mathcal{L}\frac{\partial \mathcal{L}}{\partial \Phi} + \frac{\partial \mathcal{L}}{\partial(\partial_{\nu}\Phi)} + \partial_{\mu}(\partial_{\nu}\Phi) + \partial_{\mu}\mathcal{L} \tag{87}$$

Using Euler-Lagrange equation in (87) we get:

$$D_{\mu}\mathcal{L} = \partial_{\nu}\frac{\partial \mathcal{L}}{\partial(\partial_{\nu}\Phi)}\partial_{\mu}\Phi + \frac{\partial \mathcal{L}}{\partial_{\nu}(\partial\Phi)}\partial_{\mu}(\partial_{\nu}\Phi) + \partial_{\mu}\mathcal{L}$$

On rearrangement we get:

$$D_{\nu}\left[\frac{\partial \mathcal{L}}{\partial(\partial_{\nu}\Phi)}\partial_{\mu}\Phi - \mathcal{L}\delta_{\mu\nu}\right] = -\partial_{\mu}\mathcal{L} \tag{88}$$

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The term in in the square bracket is the second rank tensor, stress-energy-momentum tensor, $T_{\mu\nu}$.

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \Phi)} - \mathcal{L}\delta_{\mu\nu} \tag{89}$$

We have conditions when there is no explict dependence on covariant vector x_μ i.e $\partial_\mu \mathcal{L} = 0$ and such fields are free fields i.e without any source and sink [24, 27]. For free fields we get

$$\sum_\nu D_\nu T_{\mu\nu} = 0$$

We can verify

$$T_{00} = \frac{\partial \mathcal{L}}{\partial \dot{\Phi}} \Phi - \mathcal{L} \tag{90}$$

This is the energy density similar to Hamiltonian density $\mathcal{H}$. The other components can be written as:

$$T_{i0} = \sqrt{-1}\Pi\partial_i\Phi$$

With momentum density $\mathcal{P}_i = \Pi\partial_i\Phi$. On further enquiry we get;

$$T_{i0} = \sqrt{-1}c\mathcal{P}_i \tag{91}$$

These are the defining features of free field. Next we discuss Klein-Gordon equation as in equation (81).

3.2 Klein-Gordon Field

Klein-Gordon equations desribe particles of spin 0 like π mesons and K mesons. Real fields correspond to uncharged particle fields and complex fields correspond to charged particle fields [23, 24, 27]. While interpreting Klein-Gordon equation in relativistic quantum mechanics we run into trouble because of the occurrence of negative states. But this can be solved by quantizing the Klein-Gordon field, with lagrangian density of the form:

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi + m^2\Phi^2\right)$$

The Euler-Lagrange equation gives:

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \sum_\mu \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = 0 \tag{92}$$

we have:

$$\frac{\partial \mathcal{L}}{\partial \Phi} = -m^2\Phi^2$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = \partial_\mu^2 \Phi$$

Substitute in Euler-Lagrange equation gives the KG equation as:

$$(\Box + m^2)\Phi(\mathbf{r}, t) = 0 \tag{93}$$

$\Phi(\mathbf{r}, t)$ is the Klein-Gordon field, tensor field, representing matter field of particles with spin-0. Following [23, 24, 27, 28] and using proper fourier decomposition of the scalar field, we obtain the solution of equation (93), also satisfied by normalized plane wave solution of real K-G equation:

$$(\Box + m^2)u_{\mathbf{k}}(\mathbf{r}, t) = 0 \tag{94}$$

in terms of quantization volume V and $\omega_k^2 = k^2 + m^2$ is:

$$u_k(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_k}} \exp(i\omega_k t - i\mathbf{k}.\mathbf{r}) \tag{95}$$

When we express the solution $\Phi(\mathbf{r}, t)$ interms of operators then the field$\Phi(\mathbf{r}, t)$ also becomes field operator. The field operator satisfying the real KG field equation is:

$$\hat{\Phi}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\hat{a}_{\mathbf{k}}(t) \exp(-i\mathbf{k}.\mathbf{r}) + \hat{a}_{\mathbf{k}}^+(t) \exp(i\mathbf{k}.\mathbf{r})\right] \tag{96}$$

Since $\Phi(\mathbf{r}, t)$ is real scalar field and represents an assembly of spin-0 particles. We write $\Phi(\mathbf{r}, t)$ as the sum of $\Phi^+(\mathbf{r}, t)$ and $\Phi^-(\mathbf{r}, t)$ which respectively carries annihilation operator $a_k(t)$ and creation operator a_k^+.

$$\Phi(\mathbf{r}, t) = \Phi^+(\mathbf{r}, t) + \Phi^-(\mathbf{r}, t) \tag{97}$$

We apply the transformation of the form

$$\frac{1}{\sqrt{V}} \sum_k \to \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k$$

to make transition from discrete values of $\mathbf{k}$ to continuous value i.e transition from fourier space of discrete functions to fourier space of continuous functions. Fourier modes for $k_0 > 0$:

$$\hat{\Phi}^+(\mathbf{r}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{\sqrt{2\omega_k}} a_{\mathbf{k}}(t) \exp(-ik.\mathbf{r})$$

$$\hat{\Phi}(\mathbf{r}, t) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int \frac{d^3k}{\sqrt{2\omega_k}} \hat{a}_{\mathbf{k}}^+(t) \exp(-i\mathbf{k}.\mathbf{r})$$

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Canonical momentum density $\Phi(\mathbf{r},t)$ is:

$$\Pi(\mathbf{r},t) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi)} = \dot{\Phi}(\mathbf{r},t) \tag{98}$$

in operator formalism developed for KG field canonical momentum density operator becomes

$$\hat{\Pi}(\mathbf{r},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3k}{\sqrt{2\omega_k}}(-i\omega_k)\left[\hat{a}_{\mathbf{k}}(t)\exp(i\mathbf{k}.\mathbf{r}) - \hat{a}^+_{\mathbf{k}}(t)\exp(-i\mathbf{k}.\mathbf{r})\right] \tag{99}$$

Similarly, field operator $\hat{\Phi}(\mathbf{r},t)$ becomes:

$$\hat{\Phi}(\mathbf{r},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3k}{\sqrt{2\omega_k}}\left[a_{\mathbf{k}}(t)\exp(i\mathbf{k}.\mathbf{r}) + \hat{a}^+_{\mathbf{k}}(t)\exp(-i\mathbf{k}.\mathbf{r})\right] \tag{100}$$

For t = 0, inverse fourier transform of equations (99) and (100), which gives creation operator and annihilation operator in the following form:

$$\hat{a}_{\mathbf{k}}(t=0) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3k}{\sqrt{2\omega_k}}\exp(-i\mathbf{k}.\mathbf{r})\left[\omega_k\Phi(\mathbf{r}) + i\Pi(\mathbf{r})\right] \tag{101}$$

$$\hat{a}^+_{\mathbf{k}}(t=0) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{d^3\mathbf{r}}{\sqrt{2\omega_k}}\exp(i\mathbf{k}.\mathbf{r})\left[\omega_k\Phi(\mathbf{r}) - i\Pi(\mathbf{r}\right] \tag{102}$$

The field variables $\Phi(\mathbf{r},t)$ and $Pi(\mathbf{r},t)$ satisfy commutation relation as stated in previous section. The creation $\hat{a}^+_k$ and annihilation $\hat{a}_{k'}$ operators satisfy the following commutation relations:

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}^+_{\mathbf{k'}}\right]_- = \delta_{\mathbf{k},\mathbf{k'}}$$

$$\left[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k'}}\right]_- = \left[\hat{a}^+_{\mathbf{k}}, \hat{a}^+_{\mathbf{k'}}\right]_- = 0$$

Thus, above relation implies quantization of real KG field.

3.3 Quantum Fields in Curved Space-time

General relativity studies about gravitational influence of matter on curved space-time, the propagation of particles and waves on this curved manifold [29]. KG field is real scalar field and is used to model physical phenomena occurring is cosmological scales which include density and pressure perturbations leading to particle creation in a dynamic universe, blackhole evaporation etc. Following [9, 23, 29, 30] we consider the lagrangian density $\mathcal{L}^\Phi$ for the scalar field Φ which takes the form:

$$\mathcal{L}^\Phi = \frac{1}{2}\left[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \xi R\Phi^2 - m^2\Phi^2\right] \tag{103}$$

Then the action for the scalar field Φ takes the form

$$S^{\Phi} = \int d^4x \mathcal{L}^{\Phi} \sqrt{-g} \tag{104}$$

Variation of this action w.r.t metric tensor of the space-time manifold is:

$$\delta S^{\Phi} = \frac{1}{2} \int_M d^4x \sqrt{-g} \left[\delta g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - \xi \delta R \Phi^2 - g_{\mu\nu} \mathcal{L}^{\Phi} \delta g^{\mu\nu} \right] \tag{105}$$

Here, R is the Ricci scalar defined as

$$R = g^{\mu\nu} R_{\mu\nu}$$

Variation of this term is:

$$\delta R = g^{\mu\nu} \delta R_{\mu\nu} + R_{\mu\nu} \delta g^{\mu\nu} \tag{106}$$

We have;

$$g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} \left[\delta \left(\Gamma^{\theta}_{\mu\theta,\nu} \right) - \delta \left(\Gamma^{\theta}_{\mu\nu,\theta} \right) + \delta \left(\Gamma^{\beta}_{\mu\alpha} \Gamma^{\alpha}_{\beta\nu} \right) - \delta \left(\Gamma^{\beta}_{\beta\alpha} \Gamma^{\alpha}_{\mu\nu} \right) \right] \tag{107}$$

In locally inertial co-ordinate we have$\Gamma^{\mu}_{\nu\sigma} = 0$ and we know $g_{\mu\nu,\sigma} = 0$. But Taylor's expansion of $g_{\mu\nu}$ shows second and higher order derivatives are non-zero. $\Gamma^{\theta}_{\mu\theta,\nu} \neq 0$ and $\Gamma^{\theta}_{\mu\nu,\theta} \neq 0$. Hence equation (107) becomes

$$g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} \left[\delta \Gamma^{\theta}_{\mu\theta,\nu} - \delta \Gamma^{\theta}_{\mu\nu,\theta} \right] \tag{108}$$

Simplifying this equation results in the following form:

$$g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\nu} g^{\sigma\theta} \left[\left(\delta g_{\mu\nu} \right)_{,\sigma\theta} - \left(\delta g_{\mu\theta} \right)_{,\sigma\nu} \right] \tag{109}$$

From this equation, we see that the variation of Ricci tensor $R_{\mu\nu}$ can also be evaluated. We evaluate the second term of the equation (105):

$$\int_M d^4x \sqrt{-g} \delta R \Phi^2 = \int_M d^4x \sqrt{-g} \left[R_{\mu\nu} \delta g^{\mu\nu} - \left(g^{\sigma\theta} g_{\mu\nu} \delta g^{\mu\nu}_{,\sigma\theta} - g^{\mu\nu} g_{\mu\theta} \ \delta g^{\sigma\theta} \right) \right] \Phi^2$$

Using generalized Stokes Theorem we get:

$$\int_M d^4x \sqrt{-g} \left(\delta g^{\sigma\theta} \right)_{,\sigma\nu} \Phi^2 = \int_\Sigma d^3x \sqrt{-g} \left(\delta g^{\sigma\theta} \right)_{,\sigma} \hat{n}_\nu \Phi^2 - \int_M d^4x \sqrt{-g} \Phi^2_{,\nu} \delta g^{\sigma\theta}_{,\sigma}$$

Which simplifies into:

$$\int_M d^4x \sqrt{-g} \left(\delta g^{\sigma\theta} \right)_{,\sigma\nu} \Phi^2 = \int_M d^4x \sqrt{-g} \Phi^2_{;\sigma\nu} \delta g^{\sigma\theta} \tag{110}$$

The terms $\delta g^{\sigma\theta}$ and $\delta g^{\sigma\theta}_{,\sigma}$ vanish on the hypersurface Σ with the vector $\hat{n}_\nu$ being components of the vector which is normal to Σ and implies endpoints of the curve in four-dimensional manifold $\mathcal{M}$ to lie on the hypersurface Σ [12, 30]. The further analysis makes our digression simpler and we

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introduce the de-Alembertian operator $\Box$ acting on the square of the tensor field Φ^2.

$$\int_M d^4x\sqrt{-g}g_{\mu\nu}g^{\sigma\theta}\left(\delta g^{\mu\nu}\right)_{,\sigma\theta}\Phi^2 = \int_M d^4x\sqrt{-g}g_{\mu\nu}\Box\Phi^2\delta g^{\mu\nu} \tag{111}$$

Where, $\Box\ \Phi^2 = g^{\sigma\theta}\Phi^2_{;\sigma\theta}$ and equation reduces into the following simpler form:

$$\int_M d^4x\sqrt{-g}\delta R\Phi^2 = \int_M d^4x\sqrt{-g}\left[R_{\mu\nu}\Phi^2 + \Phi^2_{;\mu\nu} - g_{\mu\nu}\Box\Phi^2\right]\delta g^{\mu\nu} \tag{112}$$

The energy-momentum tensor component $T^{\Phi}_{\mu\nu}$ of scalar field Φ takes the form:

$$T^{\Phi}_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \xi\left[R_{\mu\nu}\Phi^2 + \Phi^2_{;\mu\nu} - g_{\mu\nu}\Box\Phi^2\right] - \frac{1}{2}g_{\mu\nu}\left[\partial^\sigma\Phi\partial_\sigma\Phi - \left(\xi R + m^2\right)\Phi^2\right]$$

From equation (105) the variation term takes the form:

$$\delta S^{\Phi} = -\int_M d^4x\sqrt{-g}\left[\Box + \xi R + m^2\right]\Phi\delta\Phi \tag{113}$$

Variation of matter field Φ on the hypersurface Σ always vanishes. Hence employing the variation of the form $(\delta\Phi)_\Sigma = 0$ we get:

$$\frac{\delta S^{\Phi}}{\delta\Phi(x',t')} = -\int_M d^4x\sqrt{-g}\left[\Box + \xi R + m^2\right]\Phi(x)\frac{\delta\Phi(x,t)}{\delta\Phi(x',t')} \tag{114}$$

Since this variation of action has to be minimal for all other laws to hold in curved space-time and least action principle implies:

$$\left[\Box + \xi R + m^2\right]\Phi(x,t) = 0 \tag{115}$$

This is the Klein-Gordon equation in curved space-time. ξ is the coupling constant and takes value $\xi = \frac{1}{6}$ for conformal coupling scalar field. The KG equation for scalar field which is massless and minimally coupled takes the form:

$$\left[\Box + \frac{R}{6}\right]\Phi(x,t) = 0 \tag{116}$$

Further if we define Ricci scalar R interms of Ricci tensor R^μ_ν such that Ricci scalar takes the form of a diagonal matrix then:

$$\left[\Box + \xi R^\mu_\nu\delta^\mu_\nu\right]\Phi(x,t) = 0 \tag{117}$$

Where δ^μ_ν is the kroneckar delta whose value is zero for $\mu \neq \nu$.

$$\Box\Phi = 4\lambda\Phi^3 \tag{118}$$

The potential $V(\Phi) = \lambda\Phi^4$ is that of self-interacting scalar field and this may also be an inflaton field (in some model of inflation) which caused rapid expansion of our universe in the very early phase. This expansion also called the inflationary period made our universe flat and caused all

 doi:10.20944/preprints202212.0003.v1

the magnetic monopoles to vanish [6]. We can generalise equation(118) into the following form involving Potential $V(\Phi)$:

$$\left[\Box + m^2\right]\Phi = V'(\Phi) \tag{119}$$

Here $V'(\Phi)$= -ξRΦ. Occurrence of the potential term in curved space-time makes the KG equation non-linear, presence of Ricci scalar describing curved space-time generates the potential due to a system in gravitational field background. Presence of scalar field in gravitating background gives rise to phenomena of particle creation in an evolving universe [29]. Generally we treat the potential term as small perturbation of real KG field and such perturbations may give rise to kinks or domain walls. Recent discovery suggests that dark energy is accelerating cosmic expansion and breaks the strong energy condition [31].

In string theory dynamics unstable D branes have associated tachyonic scalar field responsible for exponential expansion of the early universe and is a promising candidate as source of dark energy [32]. Let ρ be the density function, $\mathcal{M}$ and $\mathcal{N}$ be the sub-manifolds of Lorentzian Manifold $\bar{\mathcal{M}}$ with atlases $\mathcal{A}_{\mathcal{M}}$and $\mathcal{A}_{\mathcal{N}}$ respectively. Let ρ: $\mathcal{M} \rightarrow \mathcal{N}$ be the mapping of class C^r at $p \; \epsilon \; \mathcal{M}$, if $\exists$ a chart (V, y) from $\mathcal{A}_{\mathcal{N}}$ with $\rho(p)\; \epsilon$ V and a chart (U, x) from $\mathcal{A}_{\mathcal{M}}$ with $p \; \epsilon \; U$ s.t $\rho(U) \subset V$ and $y \; o \; \rho \; o \; x^{-1}$ is of class C^r. Here ρ represents energy density of dark energy and all the partial derivatives

$$\frac{\partial^{\sum_i^r \alpha_i}}{\Pi(\partial x_i)^{\alpha_i}}(\rho)$$

Such partial derivatives of the function ρ doesn't exist up to order r and are discontinuous thus, breaking the strong energy condition. Instead of matter being gravitated towards matter, they move away from each other resulting an expanding universe. Here ρ and ρ^{-1} are not continuous [10, 30, 33]. Scalar fields are displaced the minimum value of potential and leads to different forms of energy which evolve dynamically and such dynamics in the early universe is determined by the form of their potential and Hubble expansion [34]. Scalar field plays significant role in explaining the observed cosmic dynamics, dark energy, inflaton field driven inflation. Consider the Lagrangian density of the form

$$\mathcal{L} = \epsilon\frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi$$

and stress-energy-momentum tensor of the form

$$T_{\mu\nu} = \epsilon\nabla_\mu\Phi\nabla_\nu\Phi - \epsilon\frac{1}{2}g_{\mu\nu}g^{\sigma\tau}\nabla_\sigma\Phi\nabla_\tau\Phi$$

Here, $\epsilon = +1$ for canonical scalar field and $\epsilon = -1$ for Phantom scalar field. Quintessence scalar field is also the canonical scalar field with potential. In an expanding FRLW universe (coupled with scalar field Φ) [35, 36], with the coupled equations of the form:

$$G_{\mu\nu} + k\nabla_\mu\Phi.\nabla_\nu\Phi = \frac{k}{2}g_{\mu\nu}\nabla^\sigma\Phi\nabla_\sigma\Phi \tag{120}$$

 doi:10.20944/preprints202212.0003.v1

satisfying the conservation equation $\nabla_\nu\ T^\nu_\mu = 0$. Here k is the Einstein constant.
For massive scalar field we have:

$$G_{\mu\nu} + k\nabla_\mu\Phi.\nabla_\nu\Phi = \frac{k}{2}g_{\mu\nu}[\nabla^\sigma\Phi.\nabla_\sigma\Phi + m^2\Phi^2] \tag{121}$$

For conventional *big bang cosmology* with $k_o = 0$, $t = 0$ scalar field takes the general form as:

$$\Phi(t) \equiv (constant).sgn(c_1).ln\big|t\big| \tag{122}$$

For massive, differentiable scalar field $\Phi(x)$ satisfying $g^{\mu\nu}(..).\nabla_\mu\ \Phi.\nabla_\nu\ \Phi < 0$ in $D \subset \mathbb{R}^4$. Then, the 4×4 energy-momentum-stress tensor matrix $[\mathrm{T}_{\mu\nu}(\mathrm{x})]$ has the Segre characteristic [(1,1,1),1] in $D \subset \mathbb{R}^4$ i.e if we take the orthonormal components

$$T_{(\alpha)(\beta)}(x_0) = t_{(\alpha)(\beta)}$$

at $x_0 \ \epsilon\ \mathrm{D} \subset \mathbb{R}^4$.

The *Lorentz invariant* eigen values of the matrix $[t_{(\alpha)(\beta)}]$ have more concise physical meaning.

$$det[t_{(\alpha)(\beta)} - \lambda d_{(\alpha)(\beta)}] = 0 \Leftrightarrow det[d^{(\alpha)(\gamma)}t_{(\gamma)(\beta)} - \lambda\delta^{(\alpha)}_{(\beta)}] = 0 \tag{123}$$

Here, the matrix $[d^{(\alpha)(\gamma)}t_{(\gamma)(\beta)}]$ need not be symmetric and has real invariant eigenvalues.

$$\begin{bmatrix} \lambda_{(1)} & 0 & 0 & 0 \\ 0 & \lambda_{(2)} & 0 & 0 \\ 0 & 0 & \lambda_{(3)} & 0 \\ 0 & 0 & 0 & -\lambda_{(4)} \end{bmatrix} \tag{124}$$

We have, $\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)}$ and $\lambda_{(1)} \neq$ - $\lambda_{(4)}$

whose Segre characteristic is $[(1,1,1),1]$. But this Segre characteristic may change from domain to domain if there is the presence of exotic matter. Given initial inequality,

$$g^{\mu\nu}(..).\nabla_\mu\Phi.\nabla_\nu\Phi < 0 \Rightarrow \nabla^\mu\Phi.\frac{\partial}{\partial x^\mu} \neq \vec{O}(x).$$

$$T_{\mu\nu}(..).\nabla^\nu\Phi = \frac{1}{2}.[\nabla^\sigma\Phi.\nabla_\sigma\Phi - m^2.\Phi^2(..)].g_{\mu\nu}(..).\nabla^\mu\Phi$$

for x ϵ $D \subset \mathbb{R}^4$. The four real, invariant eigen values are:

$$\lambda_{(1)} = \lambda_{(2)} = \lambda_{(3)} = \frac{-1}{2}.[\nabla^\sigma\Phi.\nabla_\sigma\Phi + m^2\Phi^2] \tag{125}$$

$$\lambda_{(4)} = \frac{1}{2}[\nabla^\sigma\Phi.\nabla_\sigma\Phi - m^2\Phi^2] \tag{126}$$

These equations imply that $T_{\mu\nu}(x)$ is algebraically compatible to FLRW metric.

 doi:10.20944/preprints202212.0003.v1

4 Tachyonic scalar field and Inflation

In string theory, the classical tachyon effective action implies that around the minimum of the potential there is no open strings excitation and the tachyon potential asymptotically evolves towards the minimum instead of oscillating around the minimum i.e any small perturbation in the field causes it to roll down towards the minimum [37]. The relativistic lagrangian density for tachyonic scalar field [29, 35, 36] gives the effective action as:

$$S = -\frac{1}{16\pi G}\int_M \sqrt{-g}d^4xR + \int_M \sqrt{-g}d^4x\mathcal{L}_{tach} \tag{127}$$

$$\mathcal{L}_{tach} = -V(\Phi)\sqrt{1 - g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi} \tag{128}$$

The energy-momentum tensor is:

$$T^{\mu\nu} = \frac{\partial\mathcal{L}_{tach}}{\partial(\partial_\mu\Phi)}\partial^\nu\Phi - g^{\mu\nu}\mathcal{L}_{tach} \tag{129}$$

Energy density and pressure reads as:

$$\rho = \frac{V(\Phi)}{\sqrt{1 - g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi}} \tag{130}$$

$$P = -V(\Phi)\sqrt{1 - g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi} \tag{131}$$

The equation of motion for tachyonic scalar field Φ in a universe with spatial homogeneity reads as:

$$\frac{\ddot{\Phi}}{1 - \dot{\Phi}^2} + 3\frac{\dot{a}}{a}\dot{\Phi} + \frac{V'(\Phi)}{V(\Phi)} = 0 \tag{132}$$

If scalar field Φ is gravitationally coupled to perfect fluid then we can furnish Raychaudhuri equation as:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\Big[\rho + 3p + V(\Phi)[(1 - \dot{\Phi}^2)^{-\frac{1}{2}}.\dot{\Phi}^2 - 2(1 - \dot{\Phi}^2)^{\frac{1}{2}}]\Big] \tag{133}$$

For ρ = p = 0 consider a special class of tachyonic inflationary phase. The coupled equations for FRW metric gives field equations in the following form:

$$kV(\Phi).(1 - \dot{\Phi}^2)^{\frac{1}{2}} = 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k_o}{a^2} \tag{134}$$

$$kV(\Phi).(1 - \dot{\Phi}^2)^{-\frac{1}{2}} = 3\Big[\frac{\dot{a}^2 + k_o}{a^2}\Big] \tag{135}$$

We take the case for *tan-hyperbolic inflation* for k_o = -1.

$$a(t) = tanh(t) \tag{136}$$

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Employing equations (134) and (135) we get:

$$\dot{\Phi} = \sqrt{\frac{2}{3}}.\frac{tanh(t)}{\sqrt{2 - tanh^2(t)}} \tag{137}$$

If the scalar field evolves from Φ_o to Φ in time 0 to t then:

$$\Phi - \Phi_o = \frac{\sqrt{cosh(2t) + 3}.sech(t).\ln\left[\sqrt{2}cosh(t) + \sqrt{cosh(2t) + 3}\right]}{\sqrt{4 - 2tanh^2(t)}} - \ln(2 + \sqrt{2}) \tag{138}$$

Taylor's expansion at $t = 0$ for the potential function gives:

$$\Phi - \Phi_o = \ln(2 + \sqrt{2}) + \frac{t^2}{2\sqrt{2}} + O(t^4) \tag{139}$$

The integral over half and full period are zero:

$$\int_0^{i\pi} \frac{tanht}{\sqrt{2 - tanh^2 t}} dt = 0 \tag{140}$$

$$\int_0^{2i\pi} \frac{tanht}{\sqrt{2 - tanh^2 t}} dt = 0 \tag{141}$$

Some definite integral for real values of t:

$$\int_0^{\pi} \frac{tanh(t)}{\sqrt{2 - tanh^2(t)}} dt = 2.2639 \tag{142}$$

$$\int_0^{2\pi} \frac{tanh(t)}{\sqrt{2 - tanh^2(t)}} dt = 5.40182 \tag{143}$$

The scalar field potential has the form:

$$V(\Phi) = -\frac{3\sqrt{1 - \dot{\Phi}^2}}{k} \tag{144}$$

We calculate some definite integrals in the argand plane:

$$\Phi - \Phi_o = \int_0^{i\frac{\pi}{2}} \frac{tanh(t)}{\sqrt{2 - tanh^2(t)}} dt = -0.881374 \tag{145}$$

$$\Phi - \Phi_o = \int_0^{i\frac{\pi}{4}} \frac{tanh(t)}{\sqrt{2 - tanh^2(t)}} dt = -0.222895 \tag{146}$$

In the contour c scalar field follows *Cauchy Theorem* i.e $\dot{\Phi}$ is an analytic function.

$$\oint_c \dot{\Phi}(t) dt = 0 \tag{147}$$

5 Quantum Gravity

Theorem 4. *A globally hyperbolic spacetime which satisfy the strong energy conditions and expansion scalar satisfying certain boundary conditions on the 3-geometry, is singular.*

At this singular state of infinite density all laws of general relativity breaks down and a quantum prescription for gravity is required. But singularity theorem due to Hawking and Penrose does not apply to Minkowski space-time, Einstein universe, de Sitter and 2-dimensional Anti-de Sitter space-time [38]. Quantum nature of geometry comes into light through the canonical quantization of the Hamiltonian formalism of general relativity. Canonical Quantization procedure sets a framework for constructing Klein-Gordon equation whose solution is the wavefunction of the universe, a wavefunction which governs both the matter fields and space-time geometry. Third quantization is another aspect of quantum cosmology. Universes complete with second-quantized fields can be created and destroyed by the operation of third-quantized operators [14]. Superspace is the space whose every point represents the hypersurface Σ with the spatial metrics h_{i_j} and each point of this space corresponds to certain spatial metric. Quantization of structureless, homogeneous hunk of space as the first approximation of space-time atom is the starting point for Quantum cosmology [39].Quantum cosmology is an attempt to quantize the gravitational degrees of freedom. When extended to quantum realm trajectories are transformed into wave packets which gives probability of finding the universe in any quantum state. These quantum states are vectors in their respective tangent space [25, 40]. Universes are like quantum foams in superspace which may pop in and out of existence due to the action of some third quantized operators. In quantum cosmology the question of quantum decoherence comes into play when one considers the shift from quantum physics to classical physics [41].

The wavefunction $\Psi[h_{ij}$,Φ,$\Sigma]$ in equation is the wavefunction of the universe, defined in the abstract space of all possible three metrics of cauchy surfaces i.e defined in Σ_t modulo diffeomorphism, called the superspace. In case of infinite dimensional superspace infinitely many solutions exist [42] and this wavefunction is the function of three-metric h_{ij} i.e

$$\Psi = \Psi[h_{ij}]$$

then quantum dynamics is supplied by the functional integral of the form:

$$\Psi[h_{ij}, \Phi, \Sigma] = \mathbf{N} \int_C \delta g(x) \exp\left(iS_{E-H}\right)[g] \tag{148}$$

S_{E-H} is the classical gravitational action with a cosmological constant and the functional is defined over the space spanned by the spatial metric h_{ij}, which also satisfies appropriate conditions to define the state [43]. According to the sum over histories formulation, the universe evolves from, say state A to state B following every possible path. What remains to be analysed is the initial quantum conditions from which the universe evolved. The no boundary proposal formulated in terms of

Euclidean path integral attempts to answer this question. Purely imaginary second fundamental form of the hypersurface Σ_t can be specified to ensure that our universe is Lorentzian and shades light on quantum nucleation of such spacetimes [44]. Third Quantization procedure can be applied in superspace parallel to second quantization for fields that propagate in the space-time manifold.

5.1 Paths Towards Quantum Gravity

Rosenfeld first applied quantum field theory for gravitating fields and noticed technical difficulties to quantize gravity the earliest application was the computation of photons' gravitational self-energy using perturbation theory of lowest order [45]. Let $\Phi(\mathbf{x}$,t)be matter field configuration, h_{ij} be the three metrics on the hypersurface Σ then the probability amplitude of such a universe is described by the functional $\Psi[h_{ij}$,$\Phi(\mathbf{x}, t), \Sigma]$, which lives in an infinite dimensional superspace and is the solution to Wheeler-DeWitt equation(source free):

$$\hat{\mathcal{H}}_{\mathcal{G}}|\Psi\rangle = 0 \tag{149}$$

and in the presence of source

$$\hat{T}|\Psi\rangle = 0 \tag{150}$$

Here,$|\Psi\rangle$ is the physical vacuum state, and $\hat{\mathcal{H}}_{\mathcal{G}}$, $\hat{T}$ are the third-quantized operators acting on the physical vacuum state. In equation (149) there is no time dependence and is equal to zero due to diffeomorphism invariance. But a consistent and unambiguous approach has not been yet known. The wheeler-DeWitt equation has an endlessly wide variety of solutions because it is a second-order hyperbolic functional differential equation.

The wheeler-DeWitt equation is rather complicated because it is defined in an infinite dimensional manifold. Some insights can be developed about the character of the solution of equation (149) when restricted to a submanifold of the superspace i.e by reducing infinite (matter and gravitational) degrees of freedom to finite one [46]. Unique boundary conditions have been proposed which include (i) Tunneling proposal by Alexander Vilenkin [47], in which the de Sitter universe is created out of nothing due to quantum tunneling. Nothing means the absence of space, matter, and time. Vilenkin proposes that the universe, which is spatially closed with the total Hamiltonian equal to zero, was created out of nothing. (ii) No boundary proposal by S.W Hawking, in [48] Hawking stated that he and J.Hartle suggested path integral on a spatially closed universe with positive definite metric determined the quantum state of an evolving universe and the boundary condition implies that the universe has no boundary i.e $\partial(\partial M) = 0$. The universe evolved quantum mechanically from a 4-dimensional Euclidean manifold to (3+1) Lorentzian manifold which is described in terms of euclidean path integral, in the case of a closed universe with no initial boundary [49].

Quantum fluctuations of the isotropic and homogeneous universe probably avoid space-time singularity which is only possible if these are quantum conformal fluctuations corresponding to

 doi:10.20944/preprints202212.0003.v1

classical singular cosmological solution. Evolution of the universe can be viewed as a space-time manifold undergoing geometric flow, for eg. Ricci flow [50].

$$\frac{\partial g_{\mu\nu}}{\partial t} = -R_{\mu\nu} \tag{151}$$

which describes variation of metric tensor w.r.t time variable t, $R_{\mu\nu}$ is Ricci curvature tensor. Authors in [51, 52, 53] applied Ricci flow to study the dynamics of space-time, smoothing out of small irregularities, early universe inhomogeneities and the flow of action. Covariant, Canonical, path integral formalism, string theory, twistor theory, etc are the candidate theory of Quantum Gravity. The unrestricted minisuperspace models can be used to resolve big bang singularity using many different approaches. General covariance determines possible space-time structure [54]. For the quantum description of General relativity split the gravitational field described by complete space-time metric $g_{\mu\nu}$ into a fixed background described by $\eta_{\mu\nu}$ and a dynamical field given by:

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \tag{152}$$

It is convenient to quantize $h_{\mu\nu}$ because of it's symmetry and simplicity[55]. In the canonical approach, field variables in classical equations are replaced by operators on some Hilbert space but this doesn't seem appropriate in the case of gravity because Einstein's equations are non-polynomial in the metric,

Hamiltonian is constructed and a family of space-like surface is introduced [56].

Using Einstein's field equation (in natural units)

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{153}$$

where $G_{\mu\nu}$ describes geometry of space-time, which is purely classical and $T_{\mu\nu}$ is stress-energy-momentum tensor which has an associated quantum operator. Quantum mechanically: This eigen-value equation doesn't make sense because components of $\hat{T}_{\mu\nu}$ do not obey commutation rules. The expectation value is:

$$\langle \hat{G}_{\mu\nu} \rangle = 8\pi \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle \tag{154}$$

leads to semi-classical gravity [57]. Several problems with semi-classical gravity have been identified in literature. One of the strain is that $T_{\mu\nu}$ is function of classical geometric field $g_{\mu\nu}$ and is relates $G_{\mu\nu}$.

This approach towards Quantum gravity is based on the foliation of the Lorentzian manifold by orthogonal space-like surfaces. John Archibald Wheeler first suggested quantization of general relativity based on the Feynman approach and Charles Misner introduced [58] in the following form:

$$\int_{\mathcal{M}} e^{\frac{i\hbar}{S_{E-H}}} d(history) \tag{155}$$

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Other include string theory in which strings (vibrating in seven-dimensional hyperspace) are the building blocks of all forms of matter and radiation in the universe[59].

5.2 Need for Quantum Gravity

The singularity theorem due to Hawking and Penrose imply that for a globally hyperbolic space-time following strong energy condition with non-compact hypersurface is singular [38]. According to the conventional big bang model, space-time and all forms of matter in the universe arose from a singular state of infinite density. To explain singularities that are visible (naked singularity) and not visible (the singularity of a black hole) from future null infinity we need a quantum theory of gravity.

6 Conclusion

We calculated values of scalar field for imaginary and real time interval. For half and full period in the argand plane time derivative of field strength was 0. For $t = 0$ to $i\frac{\pi}{2}$ and $3i\frac{\pi}{2}$ values were equal to -0.881374. For half and full period field stength remained at it's initial value i.e field is unperturbed. But for some values other than above field strength falls below the initial value and for others field strengths. Thus, scalar field behaves dynamically for tan-hyperbolic inflation in an open universe and is responsible for evolution of quantum state of the universe.

References

[1] L.D. Landau. *The Classical Theory of Fields*. COURSE OF THEORETICAL PHYSICS. Elsevier Science, 2013.

[2] Jaume de Haro and Emilio Elizalde. Topics in cosmology—clearly explained by means of simple examples. *Universe*, 8(3):166, 2022.

[3] Paul AM Dirac. Discussion of the infinite distribution of electrons in the theory of the positron. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 30, pages 150–163. Cambridge University Press, 1934.

[4] L. Smolin. *The Trouble with Physics: The Rise of String Theory, the Fall of a Science, and what Comes Next*. A Mariner Book. Houghton Mifflin, 2006.

[5] Alan H. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D*, 23:347–356, Jan 1981.

[6] A. Guth. *The Inflationary Universe*. Basic Books, 1998.

[7] Alexander Vilenkin. Quantum cosmology. *arXiv preprint gr-qc/9302016*, 1993.

[8] M.R. Spiegel and S. Lipschutz. *Schaum's Outline of Vector Analysis, 2ed.* Schaum's Outline Series. McGraw-Hill Education, 2009.

[9] Sean M. Carroll. *Spacetime and Geometry: An Introduction to General Relativity.* Cambridge University Press, 2019.

[10] J. Lee and J.M. Lee. *Manifolds and Differential Geometry.* Graduate studies in mathematics. American Mathematical Society, 2009.

[11] Alan D Rendall. Local and global existence theorems for the einstein equations. *Living Reviews in Relativity*, 3(1):1–40, 2000.

[12] T. Padmanabhan. *Gravitation: Foundations and Frontiers.* Cambridge University Press, 2010.

[13] AK Raychaudhuri and Bijan Modak. Physical approach to cosmological homogeneity. *Physical Review D*, 31(8):1807, 1985.

[14] E.W. Kolb and M.S. Turner. *The Early Universe.* Frontiers in physics. Addison-Wesley, 1990.

[15] C.W. Misner, K.S. Thorne, K.S. Thorne, J.A. Wheeler, W.H. Freeman, and Company. *Gravitation.* Number pt. 3 in Gravitation. W. H. Freeman, 1973.

[16] P. C. Vaidya. Nonstatic solutions of einstein's field equations for spheres of fluids radiating energy. *Phys. Rev.*, 83:10–17, Jul 1951.

[17] O. Gron and S. Hervik. *Einstein's General Theory of Relativity: With Modern Applications in Cosmology.* Springer New York, 2007.

[18] Amalkumar Raychaudhuri. Relativistic cosmology. i. *Phys. Rev.*, 98:1123–1126, May 1955.

[19] S. Montiel, A. Ros, American Mathematical Society, and Real Sociedad Matemática Española. *Curves and Surfaces.* Graduate studies in mathematics. American Mathematical Society, 2005.

[20] É. Gourgoulhon. *3+1 Formalism in General Relativity: Bases of Numerical Relativity.* Lecture Notes in Physics. Springer Berlin Heidelberg, 2012.

[21] Thierry Barbot and Abdelghani Zeghib. Group actions on lorentz spaces, mathematical aspects: a survey. In *The Einstein equations and the large scale behavior of gravitational fields*, pages 401–439. Springer, 2004.

[22] D.J. Griffiths. *Introduction to Electrodynamics.* Pearson Education, 2014.

[23] G. ARULDHAS. *QUANTUM MECHANICS.* PHI Learning, 2008.

[24] V. Devanathan. *Relativistic Quantum Mechanics and Quantum Field Theory*. Alpha Science International, 2011.

[25] Igor Volovich and V. Kozlov. Square integrable solutions to the klein-gordon equation on a manifold. *Doklady Mathematics - DOKL MATH*, 73:441–444, 05 2006.

[26] B.K. AGARWAL and H. PRAKASH. *QUANTAM MECHANICS*. PHI Learning, 1996.

[27] M.E. Peskin. *An Introduction To Quantum Field Theory*. CRC Press, 2018.

[28] D. McMahon. *Quantum Field Theory Demystified*. Demystified. McGraw-Hill Education, 2008.

[29] L. Parker and D. Toms. *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2009.

[30] S.K. SRIVASTAVA. *General Relativity and Cosmology*. PHI Learning, 2008.

[31] Vittorio Gorini, Alexander Kamenshchik, Ugo Moschella, and Vincent Pasquier. Tachyons, scalar fields, and cosmology. *Phys. Rev. D*, 69:123512, Jun 2004.

[32] Kourosh Nozari and Narges Rashidi. Some aspects of tachyon field cosmology. *Phys. Rev. D*, 88:023519, Jul 2013.

[33] L. Godinho and J. Natário. *An Introduction to Riemannian Geometry: With Applications to Mechanics and Relativity*. Universitext. Springer International Publishing, 2014.

[34] Anson Hook, Gustavo Marques-Tavares, and Yuhsin Tsai. Scalars gliding through an expanding universe. *Phys. Rev. Lett.*, 124:211801, May 2020.

[35] A. Das and A. DeBenedictis. *The General Theory of Relativity: A Mathematical Exposition*. Springer New York, 2012.

[36] Murli Manohar Verma and Shankar Dayal Pathak. Cosmic expansion driven by real scalar field for different forms of potential. *Astrophysics and Space Science*, 350(1):381–384, 2014.

[37] Ashoke Sen. Field theory of tachyon matter. *Modern Physics Letters A*, 17(27):1797–1804, 2002.

[38] Leonor Godinho and José Natário. An introduction to riemannian geometry. *With Applications*, 2012.

[39] Martin Bojowald. Quantum cosmology: a review. *Reports on Progress in Physics*, 78(2):023901, 2015.

[40] Salvador J. Robles-Pérez. Quantum cosmology in the light of quantum mechanics. *Galaxies*, 7(2), 2019.

[41] David L Wiltshire et al. An introduction to quantum cosmology. *Cosmology: the Physics of the Universe*, pages 473–531, 1996.

[42] R. Parentani. Interpretation of the solutions of the wheeler-dewitt equation. *Phys. Rev. D*, 56:4618–4624, Oct 1997.

[43] J. B. Hartle and S. W. Hawking. Wave function of the universe. *Phys. Rev. D*, 28:2960–2975, Dec 1983.

[44] Raphael Bousso and Stephen Hawking. Lorentzian condition in quantum gravity. *Phys. Rev. D*, 59:103501, Mar 1999.

[45] Bryce S. DeWitt. Quantum theory of gravity. i. the canonical theory. *Phys. Rev.*, 160:1113–1148, Aug 1967.

[46] S.W. Hawking. The quantum state of the universe. *Nuclear Physics B*, 239(1):257–276, 1984.

[47] Alexander Vilenkin. Creation of universes from nothing. *Physics Letters B*, 117(1):25–28, 1982.

[48] S.W. Hawking. The no-boundary proposal and the arrow of time. *Vistas in Astronomy*, 37:559–568, 1993.

[49] Suddhasattwa Brahma and Dong-han Yeom. On the geometry of no-boundary instantons in loop quantum cosmology. *Universe*, 5(1):22, 2019.

[50] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *Journal of Differential Geometry*, 17(2):255 – 306, 1982.

[51] Aditya Dhumuntarao. Lorentzian einstein-ricci flows. *arXiv preprint arXiv:1807.02731*, 2018.

[52] Matthew Headrick and Toby Wiseman. Ricci flow and black holes. *Classical and Quantum Gravity*, 23(23):6683–6707, oct 2006.

[53] MJ Luo. Ricci flow approach to the very early universe. *arXiv preprint arXiv:2112.00218*, 2021.

[54] Giampiero Esposito. *Quantum gravity, quantum cosmology and Lorentzian geometries*, volume 12. Springer Science & Business Media, 2009.

[55] Martin Bojowald. *Foundations of Quantum Cosmology*. IOP Publishing, 2020.

[56] S. W. Hawking. Quantum gravity and path integrals. *Phys. Rev. D*, 18:1747–1753, Sep 1978.

[57] Steve Carlip. Is quantum gravity necessary? *Classical and Quantum Gravity*, 25(15):154010, 2008.

 doi:10.20944/preprints202212.0003.v1

[58] Charles W. Misner. Feynman quantization of general relativity. *Rev. Mod. Phys.*, 29:497–509, Jul 1957.

[59] B. Greene. *The Elegant Universe: Superstrings, Hidden Dimensions, and the Quest for the Ultimate Theory*. W. W. Norton, 2010.

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A Supplemental Derivation

A.1 Relationship between Jacobian and the Metric

The metric tensor for arbitrary curvilinear coordinates on a flat spacetime is given by [8].

$$g_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\eta_{\alpha\beta} \tag{156}$$

Equation (156) implies matrix form as given below:

$$\mathbf{g} = J^T \eta J \tag{157}$$

where J^T is the transpose of the Jacobian matrix J. Taking the determinant on both sides we get

$$|\mathbf{g}| = |J|^2|\eta| \tag{158}$$

Here,

$$|\eta| = \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1.$$

Here J is the Jacobian matrix and it follows

$$\sqrt{-g} = J$$

A.2 Energy-Momentum tensor for physical fields

In curved spacetime, lagrangian density depends upon $g_{\mu\nu}$ because role of non-canonical form of metric tensor is important.
The action for physical field in curved spacetime is:

$$S = \int_{\mathcal{M}} \sqrt{-g} d^4x \mathcal{L} \tag{159}$$

variation of action w.r.t $g_{\mu\nu}$ is:

$$\delta S = \int_{\mathcal{M}} d^4x \left[\sqrt{-g}\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} + \mathcal{L}\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}}\right]\delta g^{\mu\nu} \tag{160}$$

$$\delta\sqrt{-g} = \frac{-\sqrt{-g}g_{\mu\nu}}{2}\delta g^{\mu\nu}$$

$$\frac{\partial\sqrt{-g}}{\partial g^{\mu\nu}} = \frac{-\sqrt{-g}g_{\mu\nu}}{2}$$

we get,

$$\delta S = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{\partial\mathcal{L}}{\partial g^{\mu\nu}} - \frac{g_{\mu\nu}\mathcal{L}}{2}\right]\delta g^{\mu\nu} \tag{161}$$

Applying co-ordinate transformation in equation(161):

$$\delta S = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{\partial\mathcal{L}(x)}{\partial g^{\mu\nu}} - \frac{1}{2}g_{\mu\nu}(x)\mathcal{L}(x)\right]\delta^4(x-x')$$

Using the property of Dirac delta function:

$$\int_{\mathcal{M}} d^4x f(x)\delta^4(x-x') = f(x') \tag{162}$$

We get,

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}} \tag{163}$$

A.3 Conservation Law

The result obtained in this section is an example of Noether's theorem which relates conservation laws to basic continuous symmetry of the system. Following the assumptions made in [30] and varying the action due to matter:

$$\delta S^{(m)} = -\frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} \tag{164}$$

Transformed metric tensor is:

$$g_{\mu\nu}(x) = g'_{\alpha\beta}(x')\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial x'^{\beta}}{\partial x^{\nu}}$$

On further simplification and using the properties of kronecker delta $\delta_{\mu\nu}$:

$$g_{\mu\nu}(x) = g'_{\mu\nu}(x^{\theta}) + \eta_{\mu;\nu} + \eta_{\nu;\mu} \tag{165}$$

We get following results using the methods in [30]:

$$-\delta g_{\mu\nu} = \eta_{\mu;\nu} + \eta_{\nu;\mu} \tag{166}$$

Equation (164) reduces to the following form;

$$\delta S^{(m)} = \frac{1}{2}\int_{M} d^4x\sqrt{-g}T^{\mu\nu}\left(\eta_{\mu;\nu} + \eta_{\nu;\mu}\right) \tag{167}$$

$$\delta S^{(m)} = \int_{M} d^4x\sqrt{-g}T^{\mu\nu}\eta_{\mu;\nu} \tag{168}$$

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Since $T^{\mu\nu} = T^{\nu\mu}$ is the symmetry property of stress-energy-momentum tensor. The action $S^{(m)}$ is covariant and $(\eta_\mu)_\Sigma = 0$ Further;

$$\delta S^{(m)} = \int_\Sigma d^3x\sqrt{-g}T^{\mu\nu}\eta_\mu \hat{n}_\nu - \int_M d^4x\sqrt{-g}T^{\mu\nu}_{;\nu}\eta_\mu \tag{169}$$

By the principle of least action $\delta S^{(m)} = 0 \Rightarrow T^{\mu\nu}_{;\nu}\eta_\mu = 0$ we obtain our required results:

$$T^{\mu\nu}_{;\nu} = 0 \tag{170}$$

This is the law of conservation of energy and momentum, obeys general covariance and it holds globally.

We can deduce the following result using the conservation principle, which shows there is an exchange of energy-momentum between matter and gravitation. But it vanishes in locally inertial co-ordinate.

$$\frac{1}{\sqrt{-g}}\partial_\nu\left[\sqrt{-g}T^{\mu\nu}\right] = -\Gamma^\mu_{\alpha\nu}T^{\alpha\nu} \tag{171}$$