PROOF OF TWO-DIMENSIONAL JACOBIAN CONJECTURE

YUCAI SU
Department of Mathematics, Jimei University, Xiamen, 361021, China, and
School of Mathematical Science, Tongji University, Shanghai, 200092, China
ycsu@tongji.edu.cn

Abstract. We give a proof of two-dimensional Jacobian conjecture.
Keywords: Jacobian conjecture; Keller maps; injectivity of Keller maps

1. Main theorem

Let $(F,G)\in\mathbb{C}[x,y]^2$ be a *Jacobian pair*, i.e., a pair of polynomials on two variables x,y with a nonzero constant *Jacobian determinant* $J(F,G):=\begin{vmatrix}\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y}\\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y}\end{vmatrix}\in\mathbb{C}_{\neq 0}$. The corresponding *Keller map* σ is defined by $\sigma:\mathbb{C}^2\to\mathbb{C}^2$, $p\mapsto (F(p),G(p)):=(F(a,b),G(a,b))$ for $p=(a,b)\in\mathbb{C}^2$. Then the main result of this paper is the following.

Theorem 1.1. *The Keller map σ is injective. In particular, the 2-dimensional Jacobian conjecture holds, i.e., F,G are generators of $\mathbb{C}[x,y]$.*

To prove this theorem, we start with any Jacobian pair (F,G) such that the corresponding Keller map σ is not injective, i.e., there exist $p_1=(x_1,y_1),\ p_2=(x_2,y_2)\in\mathbb{C}^2$ such that

$$\sigma(p_1)=\sigma(p_2),\quad p_1\neq p_2. \tag{1.1}$$

For convenience, we denote $(p_1,p_2)=\big((x_1,y_1),(x_2,y_2)\big)\in\mathbb{C}^2\times\mathbb{C}^2\cong\mathbb{C}^4$, and

$$V=\big\{(p_1,p_2)=\big((x_1,y_1),(x_2,y_2)\big)\in\mathbb{C}^4\,\big|\,\sigma(p_1)=\sigma(p_2),\ p_1\neq p_2\big\}. \tag{1.2}$$

Then $V\neq\emptyset$ by assumption (1.1) (note from the proof of Lemma 4.3 that V is in fact a closed subset of $\mathbb{C}^4$). We define the *height* of $(p_1,p_2)=\big((x_1,y_1),(x_2,y_2)\big)\in V$ to be

$$h_{p_1,p_2}=|x_1|+|y_1|+|x_2|+|y_2|. \tag{1.3}$$

Then Theorem 1.1 will be obtained from the following 3 results.

Theorem 1.2. *There exists a Jacobian pair, which for convenience is still denoted by (F,G), such that for $(p_1,p_2)\in V$, when $h_{p_1,p_2}\to\infty$, we have $|y_1|+|y_2|=o(h_{p_1,p_2})$.*

We define the projection $\pi_1:V\to\mathbb{C}^2$ by

$$\pi_1:(p_1,p_2)\ \mapsto\ \pi_1(p_1,p_2):=(x_1,x_2)\quad\text{for } (p_1,p_2)=\big((x_1,y_1),(x_2,y_2)\big)\in V. \tag{1.4}$$

Theorem 1.3. *The projection π_1 is not surjective.*

Theorem 1.4. *Fix $\xi=(\xi_1,\xi_2)\in\mathbb{C}^2\backslash\pi_1(V)$. We define a continuous function ℓ_{p_1,p_2} on V by*

$$\ell_{p_1,p_2}=d\big(\pi_1(p_1,p_2),\xi\big)^2=|x_1-\xi_1|^2+|x_2-\xi_2|^2\quad\textit{for}\quad (p_1,p_2)\in V, \tag{1.5}$$

Date: December 2, 2022
Supported by NSF grant 11971350 of China
Mathematics Subject Classification (2000): 14R15, 14E20, 13B10, 13B25, 17B63.

 doi:10.20944/preprints202212.0171.v1

where $d(\alpha,\beta)$ denotes the distance between α and β for $\alpha,\beta\in\mathbb{C}^2$. Then for any $(p_1,p_2)\in V$, there exists $(q_1,q_2)=\big((\dot{x}_1,\dot{y}_1),(\dot{x}_2,\dot{y}_2)\big)\in V$ such that

$$\ell_{q_1,q_2}<\ell_{p_1,p_2}. \tag{1.6}$$

2. Some preparations

For any ring R, we use $R((y))$ to denote the ring whose elements have the form $\sum_{i=-\infty}^{\infty}a_iy^i$ with $a_i\in R$ such that $a_i=0$ when $i\ll -1$. We need some conventions and notations, which, for easy reference, are listed as follows.

Convention 2.1. (1) For $a\in\mathbb{C}$, we write $a=a_{\rm re}+a_{\rm im}\boldsymbol{i}$ for some $a_{\rm re},a_{\rm im}\in\mathbb{R}$, where $\boldsymbol{i}=\sqrt{-1}$. If a^b appears somewhere, then we always assume $b\in\mathbb{R}$, and in case $a\neq 0$, we interpret a^b as the unique complex number $r^be^{b\theta\boldsymbol{i}}$ by writing $a=re^{\theta\boldsymbol{i}}$ for some $r\in\mathbb{R}_{>0}$, $0\le\theta<2\pi$.

(2) Let $P=\sum_{i\in\mathbb{Z}_{\ge0}}p_iy^{\alpha+i}=p^\alpha\big(1+\sum_{i=1}^{\infty}p_iy^i\big)\in\mathbb{C}[x]((y))$ with $\alpha\in\mathbb{Z}$, $p_i\in\mathbb{C}[x]$ and $p_0=1$.

(i) Let $\beta\in\mathbb{Q}$ with $\alpha\beta\in\mathbb{Z}$. Then we can uniquely define P^β to be an element in $\mathbb{C}[x]((y))$ as follows,

$$P^\beta=y^{\alpha\beta}\Big(1+\sum_{j=1}^{\infty}\binom{\beta}{j}\Big(\sum_{i=1}^{\infty}p_iy^i\Big)^j\Big)\in\mathbb{C}[x]((y)), \tag{2.1}$$

where, we denote the *multi-nomial coefficient* $\binom{k}{\lambda_1,\lambda_2,\dots,\lambda_i}=\frac{k(k-1)\cdots(k-(\lambda_1+\lambda_2+\cdots+\lambda_i)+1)}{\lambda_1!\lambda_2!\cdots\lambda_i!}$.

(ii) For $Q_1,Q_2\in\mathbb{C}[x]((y))$, we use $P(Q_1,Q_2)$ and $P|_{(x,y)=(Q_1,Q_2)}$ to denote the following element [as long as it is algebraically a well-defined element in $\mathbb{C}[x]((y))$],

$$P(Q_1,Q_2)=P|_{(x,y)=(Q_1,Q_2)}=\sum_i p_i(Q_1)Q_2^{\alpha+i}. \tag{2.2}$$

(iii) If $Q_1,Q_2\in\mathbb{C}$, we also use (2.2) to denote a well-defined complex number as long as the series (2.2) converges absolutely.

(iv) Assume $\alpha\neq0$. For any $Q=\sum_{i\in\mathbb{Z}_{\ge0}}q_iy^{\alpha_1+i}\in\mathbb{C}[x]((y))$ with $\alpha_1\in\mathbb{Z}$, by comparing coefficients of y^{α_1+i} for $i\ge0$, there exists uniquely $b_i\in\mathbb{C}[x]$ such that

$$Q=\sum_{i=0}^{\infty}b_iP^{\frac{\alpha_1+i}{\alpha}}. \tag{2.3}$$

We call b_i the *coefficient of* $P^{\frac{\alpha_1+i}{\alpha}}$ *in* Q, and denote by $\mathrm{C}_{\rm oeff}(Q,P^{\frac{\alpha_1+i}{\alpha}})$. If $Q=\sum_{i,j}q_{ij}x^iy^j$ with $q_{ij}\in\mathbb{C}$, we also denote $\mathrm{C}_{\rm oeff}(Q,x^iy^j)=q_{ij}$.

(3) Let $\mathcal{E}\to0$ be a variable. Sometimes we need to consider elements in $\mathbb{C}[x]((y))$ which may depend on $\mathcal{E}$. We use $O(\mathcal{E})^i$ for $i\in\mathbb{Q}_{\ge0}$ to denote any element $P\in\mathbb{C}[x]((y))$ (or especially in $\mathbb{C}$) which may depend on $\mathcal{E}$ such that $P(a,b)$ converges absolutely and $|\mathcal{E}^{-i}P(a,b)|<\boldsymbol{s}$ for some fixed $\boldsymbol{s}$, where (a,b) is in some required region.

We also denote $a=o(b)$ if a,b are some variables depending on another variable $c\to c_0$ (for some $c_0\in\mathbb{C}\cup\{\infty\}$) such that $\lim_{c\to c_0}\frac{a}{b}=0$.

Let $P=\sum_j p_jy^j\in\mathbb{C}[x]((y))$, $p_j\in\mathbb{C}[x]$ and $(x_0,y_0)\in\mathbb{C}^2$. If $z_0=\sum_j|p_j(x_0)y_0^j|$ converges, then z_0 is denoted by $A_{(x_0,y_0)}(P)$ [or by $A_{(y_0)}(P)$ if P is independent of x].

 doi:10.20944/preprints202212.0171.v1

Definition 2.2. (1) Let $Q=\sum_i q_i y^i\in\mathbb{C}((y))$, $q_i\in\mathbb{R}_{\ge0}$, $x_0\in\mathbb{C}$. If $|p_i(x_0)|\le q_i$ for all possible i, then we say Q is a *controlling function* for P on y at point x_0, and denote $P\trianglelefteq_y^{x_0}Q$, or $P\trianglelefteq_y Q$ when there is no confusion. In particular if P,Q are independent of y then we write $P\trianglelefteq^{x_0}Q$ (thus $a\trianglelefteq b$ for $a,b\in\mathbb{C}$ simply means that $|a|\le b$ with $b\ge 0$).

(2) An element in $\mathbb{C}((y))$ with non-negative coefficients is called a *controlling function* on y.

(3) If $Q=q_0y^\alpha+\sum_{j>0}q_jy^{\alpha+j}\in\mathbb{C}((y))$ is a controlling function on y with $q_0>0$, then we always use the same symbol with subscripts "$_{\rm ign}$" and "$_{\rm neg}$" to denote the elements

$$Q_{\rm ign}=q_0^{-1}\sum_{j>0}q_jy^j,\quad Q_{\rm neg}:=q_0y^\alpha\Big(1-q_0^{-1}\sum_{j>0}q_jy^j\Big)=q_0y^\alpha(1-Q_{\rm ign})=2q_0y^\alpha-Q. \tag{2.4}$$

We call $Q_{\rm ign}$ the *ignored part* of Q, and $Q_{\rm neg}$ the *negative correspondence of* Q [in sense of (2.6) and (2.7), where $a,-k$ are nonpositive].

Lemma 2.3. (1) *If*

$$P=p_0y^\alpha+\sum_{j>0}p_jy^{\alpha+j}\in\mathbb{C}[x]((y)),\qquad Q=q_0y^\alpha+\sum_{j>0}q_jy^{\alpha+j}\in\mathbb{C}((y)), \tag{2.5}$$

with $P\trianglelefteq_y^{x_0}Q$, $x_0\in\mathbb{C}$ *and* $|p_0(x_0)|=q_0\in\mathbb{R}_{>0}$, *then for* $a,b,k\in\mathbb{Q}$ *with* $a\alpha,b\alpha,k\alpha\in\mathbb{Z}$,

$$\text{(a)}\ \frac{\partial P}{\partial y}\ \trianglelefteq_y^{x_0}\ \pm\frac{dQ}{dy},\qquad \text{(b)}\ P^a\ \trianglelefteq_y^{x_0}\ Q_{\rm neg}^a\ \trianglelefteq_y\ (q_0y^\alpha)^{-b}Q_{\rm neg}^{a+b}\ \text{for } a,b\in\mathbb{Q}_-, \tag{2.6}$$

$$Q^k\ \trianglelefteq_y\ (q_0y^\alpha)^{2k}Q_{\rm neg}^{-k}\ \trianglelefteq_y\ \begin{cases}\dfrac{(q_0y^\alpha)^k}{1-kQ_{\rm ign}} & \text{if } k\in\mathbb{Z}_{\ge1},\\ (q_0y^\alpha)^k\Big(1+\dfrac{kQ_{\rm ign}}{1-Q_{\rm ign}}\Big) & \text{if } k\in\mathbb{Q}_{\ge0} \text{ with } k<1.\end{cases} \tag{2.7}$$

where (2.6) (*a*) *holds under the condition: either both* P, Q *are power series of* y (*in this case the sign is* "$+$"), *or else both are polynomials on* y^{-1} (*in this case the sign is* "$-$").

(2) *If* $x_0,y_0\in\mathbb{C}$ *and* $P_1\trianglelefteq_y^{x_0}Q_1$, $P_2\trianglelefteq_y^{x_0}Q_2$, *then*

$$A_{(x_0,y_0)}(P_1P_2)\le A_{(y_0)}(Q_1)A_{(y_0)}(Q_2)=Q_1(|y_0|)Q_2(|y_0|). \tag{2.8}$$

Proof. One can see that (2) and (2.6) (a) are obvious, and (2.6) (b), (2.7) are obtained by noting that for $a,b\in\mathbb{Q}_-$ and $i\in\mathbb{Z}_{>0}$, one has

$$(-1)^i\binom{a}{i}=\Big|\binom{a}{i}\Big|\le\Big|\binom{a+b}{i}\Big|=(-1)^i\binom{a+b}{i},\qquad \binom{k}{i}\le\Big|\binom{-k}{i}\Big|\le\begin{cases}k^i & \text{if } k\in\mathbb{Z}_{\ge1},\\ k & \text{if } 0<k\in\mathbb{Q}_{<1}.\end{cases}$$

This proves the lemma. □

Take, where $\tilde f_i\in\mathbb{C}[x]$ with $\tilde f_1\ne0$,

$$\tilde F=\tilde f_1y+\sum_{i=2}^\infty\tilde f_iy^i\in\mathbb{C}[x][[y]]. \tag{2.9}$$

Regarding $\tilde F$ as a formal function on y (with parameter x being regarded as fixed), we have the *formal inverse function* denoted by $y_{\tilde F}\in\mathbb{C}[x,\tilde f_1^{-1}][[\tilde F]]\subset\mathbb{C}[x][[\tilde F]]$ such that [cf. (2.3)]

$$y\ =\ y_{\tilde F}(\tilde F)=\boldsymbol{b}_1\tilde F+\sum_{i=2}^\infty\boldsymbol{b}_i\tilde F^i, \tag{2.10}$$

 doi:10.20944/preprints202212.0171.v1

with $\boldsymbol{b}_i = \mathrm{C}_{\mathrm{oeff}}(y, \tilde{F}^i) \in \mathbb{C}[x, \tilde{f}_1^{-1}]$ being determined by $\boldsymbol{b}_1 = \tilde{f}_1^{-1} \in \mathbb{C}[x, \tilde{f}_1^{-1}]$ and (we do not need to use the following explicit expression of $\boldsymbol{b}_i$, we only want to present that $\boldsymbol{b}_i$'s exist),

$$\boldsymbol{b}_i \ = \ -\sum_{j=1}^{i-1} \boldsymbol{b}_j \tilde{f}_1^{j-i} \sum_{\ell=0}^{j} \binom{j}{\ell} \sum_{\substack{n\in\mathbb{Z}_{\geq 0},\, \lambda_1,\lambda_2,\dots,\lambda_n\geq 0 \\ \lambda_1+2\lambda_2+\dots+n\lambda_n=i-j}} \binom{\ell}{\lambda_1, \lambda_2, ..., \lambda_n} \tilde{f}_1^{-\lambda_1-\lambda_2-\dots-\lambda_n} \tilde{f}_2^{\lambda_2} \tilde{f}_3^{\lambda_3} \cdots \tilde{f}_n^{\lambda_n}, \quad (2.11)$$

for $i \geq 2$, which is obtained by comparing the coefficients of y^i in (2.10).

Lemma 2.4. *For $\hat{a}_i \in \mathbb{R}_{\geq 0}$ with $\hat{a}_1 > 0$, let*

$$\hat{\boldsymbol{F}} = \hat{a}_1 y + \sum_{i=2}^{\infty} \hat{a}_i y^i \in \mathbb{C}[[y]] \quad \textit{and} \quad \hat{\boldsymbol{F}}_{\mathrm{neg}} = \hat{a}_1 y - \sum_{i=2}^{\infty} \hat{a}_i y^i, \quad (2.12)$$

be a controlling function on y and its negative correspondence [cf. (2.4)], and let

$$y = y^{\mathrm{neg}}(\hat{\boldsymbol{F}}_{\mathrm{neg}}) = \hat{\boldsymbol{b}}_1 \hat{\boldsymbol{F}}_{\mathrm{neg}} + \sum_{i=2}^{\infty} \hat{\boldsymbol{b}}_i \hat{\boldsymbol{F}}^i_{\mathrm{neg}}, \quad (2.13)$$

be the formal inverse function of $\hat{\boldsymbol{F}}_{\mathrm{neg}}$, where $\hat{\boldsymbol{b}}_1 = \hat{a}_1^{-1}$ and $\hat{\boldsymbol{b}}_i = \mathrm{C}_{\mathrm{oeff}}(y, \hat{\boldsymbol{F}}^i_{\mathrm{neg}}) \in \mathbb{C}$. Then

(1) *$y^{\mathrm{neg}}(\hat{\boldsymbol{F}}_{\mathrm{neg}})$ is a controlling function on $\hat{\boldsymbol{F}}_{\mathrm{neg}}$, i.e., for $i \geq 1$,*

$$\hat{\boldsymbol{b}}_i = \mathrm{C}_{\mathrm{oeff}}(y, \hat{\boldsymbol{F}}^i_{\mathrm{neg}}) \geq 0. \quad (2.14)$$

(2) *If $\tilde{F} \trianglelefteq_y^{x_0} \hat{\boldsymbol{F}}$ with $\tilde{F}$ as in (2.9) and $|\tilde{f}_1(x_0)| = \hat{a}_1$, then*

$$y = y_{\tilde{F}}(\tilde{F}) \trianglelefteq_{\tilde{F}}^{x_0} y^{\mathrm{neg}}(\tilde{F}), \quad \textit{i.e.,} \quad \boldsymbol{b}_i \trianglelefteq^{x_0} \hat{\boldsymbol{b}}_i, \quad (2.15)$$

where $\boldsymbol{b}_i = \mathrm{C}_{\mathrm{oeff}}(y, \tilde{F}^i)$ is as in (2.10), and $\boldsymbol{b}_i \trianglelefteq^{x_0} \hat{\boldsymbol{b}}_i$ means that $|\boldsymbol{b}_i(x_0)| \leq \hat{\boldsymbol{b}}_i$. In particular

$$y \trianglelefteq_y y^{\mathrm{neg}}(\hat{\boldsymbol{F}}), \quad (2.16)$$

where the right side of "$\trianglelefteq_y$" is regarded as a function on y by substituting $\hat{\boldsymbol{F}}$ by (2.12).

Proof. Note that (1) follows from (2) by simply taking $\tilde{F} = \hat{a}_1 y$. Thus we prove (2). We want to prove, for $i \geq 1$,

$$\frac{\partial^i y}{\partial \tilde{F}^i} \trianglelefteq_y^{x_0} \frac{d^i y}{d\hat{\boldsymbol{F}}^i_{\mathrm{neg}}}, \quad (2.17)$$

where the left-hand side is understood as that we first use (2.10) to regard y as a function on $\tilde{F}$ (with parameter x) and apply $\frac{\partial^i}{\partial \tilde{F}^i}$ to it, then regard the result as a function on y (and the like for the right-hand side, which does not contain the parameter x). By (2.6) (a), we have $\frac{\partial \tilde{F}}{\partial y} \trianglelefteq_y^{x_0} \frac{d\hat{\boldsymbol{F}}}{dy}$, and thus by (2.6) (b),

$$\left(\frac{\partial \tilde{F}}{\partial y}\right)^{-1} \trianglelefteq_y^{x_0} \left(\left(\frac{d\hat{\boldsymbol{F}}}{dy}\right)_{\mathrm{neg}}\right)^{-1} = \left(\frac{d\hat{\boldsymbol{F}}_{\mathrm{neg}}}{dy}\right)^{-1},$$

i.e., $\frac{\partial y}{\partial \tilde F} \trianglelefteq_y^{x_0} \frac{dy}{d\hat{\boldsymbol F}_{\rm neg}}$ and (2.17) holds for $i=1$. Inductively, by Lemma 2.3,

$$\begin{aligned}\frac{\partial^i y}{\partial \tilde F^i} &= \frac{\partial}{\partial \tilde F}\Big(\frac{\partial^{i-1} y}{\partial \tilde F^{i-1}}\Big) = \frac{\partial}{\partial y}\Big(\frac{\partial^{i-1} y}{\partial \tilde F^{i-1}}\Big)\Big(\frac{\partial \tilde F}{\partial y}\Big)^{-1}\\ &\trianglelefteq_y^{x_0} \frac{d}{dy}\Big(\frac{d^{i-1} y}{d\hat{\boldsymbol F}_{\rm neg}^{i-1}}\Big)\Big(\frac{d\hat{\boldsymbol F}_{\rm neg}}{dy}\Big)^{-1} = \frac{d^i y}{d\hat{\boldsymbol F}_{\rm neg}^i}.\end{aligned} \tag{2.18}$$

This proves (2.17). Using (2.17) and noting from (2.10) and (2.13), we have

$$\boldsymbol b_i = \frac{1}{i!}\frac{\partial^i y}{\partial \tilde F^i}\Big|_{\tilde F=0} = \frac{1}{i!}\frac{\partial^i y}{\partial \tilde F^i}\Big|_{y=0} \trianglelefteq^{x_0} \frac{1}{i!}\frac{d^i y}{d\hat{\boldsymbol F}_{\rm neg}^i}\Big|_{y=0} = \frac{1}{i!}\frac{d^i y}{d\hat{\boldsymbol F}_{\rm neg}^i}\Big|_{\hat{\boldsymbol F}_{\rm neg}=0} = \hat{\boldsymbol b}_i.$$

This proves (2.15). Since $\tilde F \trianglelefteq_y^{x_0} \hat{\boldsymbol F}$ and $y^{\rm neg}$ is a controlling function, we have $y^{\rm neg}(\tilde F) \trianglelefteq_y^{x_0} y^{\rm neg}(\hat{\boldsymbol F})$. This together with (2.15) proves (2.16). □

3. Proof of Theorem 1.2

First we need to reformulate (F,G). If necessary by replacing (F,G) by $\big(F+(G+F^k)^k, G+F^k\big)$ for some k, we may assume $2 \le \deg_y G$ (where $\deg_y G$ denotes the degree of G with respect to the variable y) and $\deg_y G \,|\, \deg_y F$ (i.e., $\frac{\deg_y F}{\deg_y G} \in \mathbb Z_{>0}$). Fix a sufficiently large $\ell \in \mathbb Z_{>0}$. Applying the following variable change,

$$(x,y) \mapsto (y,\, y^\ell + x), \tag{3.1}$$

and rescaling F, G, we can assume, for some $m,n \in \mathbb Z_{>0}$, $f_{jk}, g_{jk} \in \mathbb C$ with $2 \le n$ and $n\,|\,m$,

$$\text{(i) } F = y^m + F_1,\ F_1 = \sum_{j=0}^{m-1}\sum_{k=0}^{m-1-j} f_{jk} y^j x^k, \qquad \text{(ii) } G = y^n + G_1,\ G_1 = \sum_{j=0}^{n-1}\sum_{k=0}^{n-1-j} g_{jk} y^j x^k. \tag{3.2}$$

Note that $\deg F_1 \le m-1$, $\deg G_1 \le n-1$ (where $\deg F_1$ denotes the total degree of F_1).

In the following we consider elements as in the ring $\mathbb C[x]((y^{-1}))$. By (3.2), we rewrite F, G as

$$\text{(i) } F = y^m\Big(1+\sum_{i=1}^{\infty} f_i y^{-i}\Big), \quad \text{(ii) } G = y^n\Big(1+\sum_{i=1}^{\infty} g_i y^{-i}\Big) \quad \text{for some } f_i, g_i \in \mathbb C[x] \text{ with} \tag{3.3}$$

(iii) $\deg_x f_i \le i-1$ if $i \le m$ and $f_i = 0$ if $i > m$, (iv) $\deg_x g_i \le i-1$ if $i \le n$ and $g_i = 0$ if $i > n$.

Set $f_0 = g_0 = 1$, and denote the set $A = \frac{n-\mathbb Z_{\ge 0}}{m}$. By (3.3), we can write,

$$G = \sum_{\alpha \in A} c_\alpha F^\alpha = \sum_{i=0}^{\infty} c_{\frac{n-i}{m}} F^{\frac{n-i}{m}} \quad \text{for some } c_\alpha \in \mathbb C[x], \tag{3.4}$$

where as in (2.11), by comparing the coefficients of y^{n-i}, we can inductively determine $c_{\frac{n-i}{m}} \in \mathbb C[x]$ for $i \ge 0$ as follows:

$$c_{\frac{n-i}{m}} = g_i - \sum_{j=1}^{i-1} c_{\frac{n-j}{m}} f_1^{j-i} \sum_{\ell=0}^{j}\binom{j}{\ell} \sum_{\substack{n\in\mathbb Z_{\ge0},\,\lambda_1,\lambda_2,\dots,\lambda_n\ge0\\ \lambda_1+2\lambda_2+\cdots+n\lambda_n=i-j}} \binom{\ell}{\lambda_1,\lambda_2,\dots,\lambda_n} f_1^{-\lambda_1-\lambda_2-\cdots-\lambda_n} f_2^{\lambda_2} f_3^{\lambda_3}\cdots f_n^{\lambda_n}.$$

Lemma 3.1. *We have*

$$\text{(i) } c_\alpha \in \mathbb C \text{ if } \alpha > \tfrac{1-m}{m}, \qquad \text{(ii) } c_{-1} = 0, \qquad \text{(iii) } \deg_x c_{\frac{-m-j}{m}} \le j+2 \text{ if } j \in \mathbb Z_{\ge -1}. \tag{3.5}$$

Proof. Taking the Jacobian of F with G, we obtain, where $J_0 = J(F,G)$,

$$\text{(i) } J_0\Big(\frac{\partial F}{\partial y}\Big)^{-1} = -\sum_{\alpha\in A}\frac{dc_\alpha}{dx}F^\alpha, \qquad \text{(ii) } G\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\Big(\sum_{-1\neq\alpha\in A}\frac{c_\alpha F^{\alpha+1}}{\alpha+1}\Big) + c_{-1}F^{-1}\frac{\partial F}{\partial y}. \tag{3.6}$$

We immediately obtain (3.5) (i), (ii). By (3.3) (iii), (iv), we have

$$\deg_x \mathrm{C}_{\mathrm{oeff}}(F, y^{m-j}) \le j, \quad \deg_x \mathrm{C}_{\mathrm{oeff}}(G, y^{n-j}) \le j \ \text{ for } j \in \mathbb{Z}_{\ge 0}. \tag{3.7}$$

Then by (2.1), (2.3), we can prove by induction on $j, k \in \mathbb{Z}_{\ge 0}$,

$$\deg_x \mathrm{C}_{\mathrm{oeff}}\Big(\Big(\frac{\partial F}{\partial y}\Big)^{-1}, y^{1-m-j}\Big) \le j, \qquad \deg_x \mathrm{C}_{\mathrm{oeff}}(F^{\frac{n-j}{m}}, y^{n-j-k}) \le k. \tag{3.8}$$

From this and (3.6) (i), we obtain (3.5) (iii) by induction on j. □

Lemma 3.2. *By some reformulation of (F,G), we may assume $c_{\frac{4-m}{m}}c_{\frac{3-m}{m}} \neq 0$.*

Proof. Let $a_0 \in \mathbb{C}$ be chosen later, and we denote, for some $\bar f_{jk} \in \mathbb{C}$,

$$\text{(i) } \bar F := F(y, y^3 + a_0y^2 - x) = y^{\bar m}\Big(1 + \sum_{j=1}^{\bar m}\sum_{k=0}^{j-1}\bar f_{jk}y^{-j}x^k\Big), \qquad \text{(ii) } \bar G = G(y, y^3 + a_0y^2 - x), \tag{3.9}$$

and we use same symbols with a bar to denote elements associated to the pair $(\bar F, \bar G)$, in particular, $\bar n = 3n$, $\bar m = 3m$, and the last equality of (3.9) (i) follows from (3.2) (i). Note from (3.9) (i) that $y \in \sum_{k=0}^{\infty}\mathbb{C}[x]\bar F^{\frac{1-k}{\bar m}}$. Thus by (3.5) (iii), we have, for $j \in \mathbb{Z}_{\ge -1}$,

$$\Big(c_{\frac{-m-j}{m}}F^{\frac{-m-j}{m}}\Big)\Big|_{(x,y)=(y,y^3+a_0y^2-x)} \in \Big(\sum_{k=0}^{\infty}\mathbb{C}[x]\bar F^{\frac{j+2-k}{\bar m}}\Big)\bar F^{\frac{-\bar m-3j}{\bar m}} = \sum_{k=0}^{\infty}\mathbb{C}[x]\bar F^{\frac{2-2j-k-\bar m}{\bar m}}. \tag{3.10}$$

From this and (3.5), we see that the right-hand side of (3.4) is a well-defined element in $\mathbb{C}[x]((y^{-1}))$ when (x,y) is set to $(y, y^3+a_0y^2-x)$ (as $\bar F^{\frac{\bar n-j}{\bar m}}$ only appears finitely many times for each $j \in \mathbb{Z}_{>0}$). Thus we can set (x,y) to $(y, y^3+a_0y^2-x)$ in (3.4) to obtain [using (3.5)],

$$\begin{aligned}\bar G &= \Big(\sum_{j=0}^{\infty}c_{\frac{n-j}{m}}F^{\frac{n-j}{m}}\Big)\Big|_{(x,y)=(y,y^3+a_0y^2-x)} \\ &= \sum_{j=0}^{m+n-2}c_{\frac{n-j}{m}}\bar F^{\frac{\bar n-3j}{\bar m}} + \Big(c_{\frac{1-m}{m}}\Big|_{(x,y)=(y,y^3+a_0y^2-x)}\Big)\bar F^{\frac{3-\bar m}{\bar m}} + \sum_{j=1}^{\infty}\Big(c_{\frac{-m-j}{m}}F^{\frac{-m-j}{m}}\Big)\Big|_{(x,y)=(y,y^3+a_0y^2-x)}.\end{aligned} \tag{3.11}$$

Note that the right-hand side of (3.11) can be written as the form $\sum_{j=0}^{\infty}\bar c_{\frac{\bar n-j}{\bar m}}\bar F^{\frac{\bar n-j}{\bar m}}$ for some $\bar c_{\frac{\bar n-j}{\bar m}} \in \mathbb{C}[x]$. One can use (3.10) to see that only the middle term in the right-hand side of (3.11) can contribute to $\bar c_{\frac{4-\bar m}{\bar m}}$ or $\bar c_{\frac{3-\bar m}{\bar m}}$. By (3.5) (iii), (3.6) (i), we can write $c_{\frac{1-m}{m}} = -J_0m^{-1}(x + a_1)$ for some $a_1 \in \mathbb{C}$. Then

$$c_{\frac{1-m}{m}}\Big|_{(x,y)=(y,y^3+a_0y^2-x)} \in -J_0m^{-1}\Big(\bar F^{\frac{1}{\bar m}} + a_1 - \frac{a_0}{3} + \sum_{j=1}^{\infty}\mathbb{C}[x]\bar F^{\frac{-j}{\bar m}}\Big), \tag{3.12}$$

which with (3.11) gives that $\bar c_{\frac{4-\bar m}{\bar m}} = -J_0m^{-1}$, $\bar c_{\frac{3-\bar m}{\bar m}} = -J_0m^{-1}(a_1 - \frac{a_0}{3}) \neq 0$ (by choosing $a_0 \neq 3a_1$). Thus by replacing (F,G) by $(\bar F, \bar G)$ [by (3.9) we still have (3.2) after the replacement], we have the lemma. □

We slightly generalize notions and notations in Definition 2.2. In the rest of this section we regard all elements as in the ring $\mathbf{R} := \mathbb{C}[x^{\frac{1}{m}}]((y^{-1}))$.

Definition 3.3. Let $R = \sum_{i,j} r_{ij} x^i y^j$, $Q = \sum_{i,j} q_{ij} x^i y^j \in \mathbf{R}$ with $r_{ij} \in \mathbb{C}$, $q_{ij} \in \mathbb{R}_{\geq 0}$.

(i) If $|r_{ij}| \leq q_{ij}$ for all possible i, j, then we say R is controlled by Q with respect to x, y, and denote $R \trianglelefteq_{x,y} Q$.

(ii) For $x_0, y_0 \in \mathbb{C}$, if $\sum_{i,j} |r_{ij} x_0^i y_0^j|$ converges, then we say the series R with respect to x, y converges *strongly* when (x, y) is set to (x_0, y_0), and denote $R|_{(x,y)=(x_0,y_0)} = \sum_{i,j} r_{ij} x_0^i y_0^j$.

Denote $t = (1+x^{\frac{1}{m}})^{m-1} \in \mathbf{R}$. By (3.3) (iii), we easily obtain (in the following, $\boldsymbol{s}_j \in \mathbb{R}_{>0}$ is some fixed number for all possible j; for instance, we can take $\boldsymbol{s}_0 = \sum_{j,k} |f_{jk}|$),

$$F \trianglelefteq_{x,y} y^m \Big(1 + \boldsymbol{s}_0 \sum_{j=1}^{m} (ty^{-1})^j\Big) \trianglelefteq_{x,y} \boldsymbol{F}, \quad \text{where} \quad \boldsymbol{F} = y^m \Big(1 + \frac{\boldsymbol{s}_0 t y^{-1}}{1 - ty^{-1}}\Big). \tag{3.13}$$

We have $\boldsymbol{F}_{\text{ign}} = \frac{\boldsymbol{s}_0 t y^{-1}}{1-ty^{-1}}$ [cf. (2.4)] and by Lemma 2.3, for $j \in \mathbb{Z}_{\geq 0}$,

$$F^{\pm \frac{j}{m}} \trianglelefteq_{x,y} y^{\pm j} \Big(1 - \boldsymbol{F}_{\text{ign}}\Big)^{-\frac{j}{m}} = y^{\pm j} \Big(\frac{1 - ty^{-1}}{1 - (1 + \boldsymbol{s}_0) t y^{-1}}\Big)^{\frac{j}{m}}. \tag{3.14}$$

Denote $w = y^{-1}$. We can write [using (3.14) and Lemma 2.3], for some $p_j \in \mathbb{C}[x]$,

$$\text{(i) } P := F^{-\frac{1}{m}} = w\Big(1 + \sum_{j=1}^{\infty} p_j w^j\Big) \trianglelefteq_{x,w} \frac{w}{(1 - \boldsymbol{F}_{\text{ign}})^{\frac{1}{m}}} \trianglelefteq_{x,w} \boldsymbol{P}, \quad \text{where,}$$

$$\text{(ii) } \boldsymbol{P} := \frac{w}{1 - \boldsymbol{F}_{\text{ign}}} = \frac{w(1 - tw)}{1 - (1 + \boldsymbol{s}_0) t w} = w(1 + \boldsymbol{P}_{\text{ign}}) \text{ with } \boldsymbol{P}_{\text{ign}} = \frac{\boldsymbol{s}_0 t w}{1 - (1 + \boldsymbol{s}_0) t w}. \tag{3.15}$$

Thus

$$\boldsymbol{P}_{\text{neg}} = w\Big(1 - \frac{\boldsymbol{s}_0 t w}{1 - (1 + \boldsymbol{s}_0) t w}\Big) = \frac{w\big(1 - (1 + 2\boldsymbol{s}_0) t w\big)}{1 - (1 + \boldsymbol{s}_0) t w}. \tag{3.16}$$

Denote $\tilde{c}_j = c_{-\frac{j}{m}} = \tilde{c}_{0j} + \tilde{c}_{1j}$ with $c_{0j} \in \mathbb{C}$, $\tilde{c}_{1j} \in x\mathbb{C}[x]$. By (3.5) and Lemma 3.2,

$$\text{(i) } \tilde{c}_{1j} = 0 \quad \text{for} \quad j \leq m - 2, \qquad \text{(ii) } \tilde{c}_{m-4} \tilde{c}_{m-3} \neq 0. \tag{3.17}$$

Lemma 3.4. *We have, for some $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbb{R}_{>0}$, where $\bar{\tilde{c}}_{1j} \in x\mathbb{R}_{\geq 0}[x]$ is obtained from $\tilde{c}_{1j}$ by replacing all coefficient $\mathrm{C_{oeff}}(\tilde{c}_{1j}, x^k)$ by its absolute value,*

$$\text{(i) } G = G_0 + G_1, \quad \text{(ii) } G_0 := \sum_{j=-n}^{\infty} \tilde{c}_{0j} P^j \trianglelefteq_{x,P} P^{-n}\Big(1 + \frac{\boldsymbol{s}_1 P}{1 - \boldsymbol{s}_2 P}\Big) \trianglelefteq_{x,y} \boldsymbol{P}_{\text{neg}}^{-n}\Big(1 + \frac{\boldsymbol{s}_1 \boldsymbol{P}}{1 - \boldsymbol{s}_2 \boldsymbol{P}}\Big),$$

$$\text{(iii) } G_1 := \sum_{j=-n}^{\infty} \tilde{c}_{1j} P^j \trianglelefteq_{x,P} \sum_{j=m-1}^{\infty} x \frac{d\bar{\tilde{c}}_{1j}}{dx} P^j \trianglelefteq_{x,P} J_0 m^{-1} x P^{m-1} \Big(1 - \frac{\boldsymbol{s}_1 t P}{1 - \boldsymbol{s}_2 t P}\Big)^{-1}. \tag{3.18}$$

Proof. Solving w from (3.16) we obtain the formal inverse function of $\boldsymbol{P}_{\text{neg}}$ is,

$$w = w^{\text{neg}}(\boldsymbol{P}_{\text{neg}}) = \frac{1 - (1 + \boldsymbol{s}_0) t \boldsymbol{P}_{\text{neg}} - B}{2(1 + 2\boldsymbol{s}_0) t}, \quad B = \sqrt{(1 - \beta_+ t \boldsymbol{P}_{\text{neg}})(1 - \beta_- t \boldsymbol{P}_{\text{neg}})}, \tag{3.19}$$

where $\beta_\pm = 3 + 5\boldsymbol{s}_0 \pm 2(2 + 7\boldsymbol{s}_0 + 6\boldsymbol{s}_0^2)^{\frac{1}{2}}$. Noting that

$$B \trianglelefteq_{x,\boldsymbol{P}_{\rm neg}} (1 - \beta_+ t\boldsymbol{P}_{\rm neg})^{-\frac{1}{2}}(1 - \beta_- t\boldsymbol{P}_{\rm neg})^{-\frac{1}{2}} \trianglelefteq_{x,\boldsymbol{P}_{\rm neg}} (1 - \beta_+ t\boldsymbol{P}_{\rm neg})^{-1}, \tag{3.20}$$

we can deduce from (3.19) that $w^{\rm neg}(\boldsymbol{P}_{\rm neg}) \trianglelefteq_{x,\boldsymbol{P}_{\rm neg}} \boldsymbol{P}_{\rm neg}(1 + \frac{\boldsymbol{s}_3 t\boldsymbol{P}_{\rm neg}}{1-\boldsymbol{s}_4 t\boldsymbol{P}_{\rm neg}})$ for some $\boldsymbol{s}_3, \boldsymbol{s}_4 \in \mathbb{R}_{>0}$. Thus by (2.6), (2.15), we have, for some $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbb{R}_{>0}$ (we can always assume $\boldsymbol{s}_1 > \boldsymbol{s}_3$, $\boldsymbol{s}_2 > \boldsymbol{s}_4$),

$$\begin{aligned} &\text{(i) } y^{-1} = w \trianglelefteq_{x,P} w^{\rm neg}(P) \trianglelefteq_{x,P} P\Big(1 + \frac{\boldsymbol{s}_3 tP}{1 - \boldsymbol{s}_4 tP}\Big) \trianglelefteq_{x,P} P\Big(1 + \frac{\boldsymbol{s}_1 tP}{1 - \boldsymbol{s}_2 tP}\Big), \\ &\text{(ii) } y = w^{-1} \trianglelefteq_{x,P} P^{-1}\Big(1 - \frac{\boldsymbol{s}_3 tP}{1 - \boldsymbol{s}_4 tP}\Big)^{-1} \trianglelefteq_{x,P} P^{-1}\Big(1 + \frac{\boldsymbol{s}_1 tP}{1 - \boldsymbol{s}_2 tP}\Big). \end{aligned} \tag{3.21}$$

From this, (3.2) (ii) (with the fact that $n|m$) and (3.15), we can obtain, for some $\boldsymbol{s}_5 \in \mathbb{R}_{>0}$,

$$G \trianglelefteq_{x,P} P^{-n}\Big(1 + \frac{\boldsymbol{s}_5 tP}{1 - \boldsymbol{s}_4 tP}\Big) \trianglelefteq_{x,y} \boldsymbol{P}_{\rm neg}^{-n}\Big(1 + \frac{\boldsymbol{s}_5 t\boldsymbol{P}}{1 - \boldsymbol{s}_4 t\boldsymbol{P}}\Big). \tag{3.22}$$

From this, (3.4) and definition of G_0 in (3.18) (ii), we obtain (note that $t|_{x=0} = 1$),

$$G_0 \trianglelefteq_{x,P} P^{-n}\Big(1 + \frac{\boldsymbol{s}_5 (t|_{x=0}) P}{1 - \boldsymbol{s}_4 (t|_{x=0}) P}\Big) \trianglelefteq_{x,y} \boldsymbol{P}_{\rm neg}^{-n}\Big(1 + \frac{\boldsymbol{s}_5 (t|_{x=0}) \boldsymbol{P}}{1 - \boldsymbol{s}_4 (t|_{x=0}) \boldsymbol{P}}\Big), \tag{3.23}$$

i.e., we have (3.2) (ii). Note from (3.13) that $m^{-1}(\frac{\partial F}{\partial y}) \trianglelefteq_{x,y} y^{m-1}\big(1 + \boldsymbol{s}_0 \sum_{j=1}^{m-1} (ty^{-1})^j\big)$, we obtain,

$$\begin{aligned} \sum_{j=-n}^{\infty} \frac{d\tilde{c}_{1j}}{dx} P^j &= \sum_{\alpha \in A} \frac{dc_\alpha}{dx} F^\alpha = -J_0 \Big(\frac{\partial F}{\partial y}\Big)^{-1} \trianglelefteq_{x,y} J_0 m^{-1} y^{-(m-1)} \Big(1 - \boldsymbol{s}_0 \sum_{j=1}^{m-1} (ty^{-1})^j\Big)^{-1} \\ &\trianglelefteq_{x,P} J_0 m^{-1} y^{-(m-1)} \Big(1 - \boldsymbol{s}_0 \sum_{j=1}^{m-1} (ty^{-1})^j\Big)^{-1}\Big|_{y^{-1} = P(1 + \frac{\boldsymbol{s}_3 tP}{1 - \boldsymbol{s}_4 tP})} \\ &\trianglelefteq_{x,P} J_0 m^{-1} x P^{m-1} \Big(1 - \frac{\boldsymbol{s}_1 tP}{1 - \boldsymbol{s}_2 tP}\Big)^{-1}, \end{aligned} \tag{3.24}$$

where the first "$\trianglelefteq_{x,P}$" is obtained from (3.21) (i) and the second is obtained by choosing sufficiently large $\boldsymbol{s}_1, \boldsymbol{s}_2$. Now the first "$\trianglelefteq_{x,P}$" of (3.18) (iii) is obvious and the second "$\trianglelefteq_{x,P}$" is obtained from (3.24). This proves the lemma. □

Proposition 3.5. *There exists $\boldsymbol{s}_5 \in \mathbb{R}_{>0}$ such that for any $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in V$ with $h_{p_1,p_2} \geq \boldsymbol{s}_5$, we must have*

$$|y_1| < h_{p_1,p_2}^{\frac{m}{m+1}}, \qquad |y_2| < h_{p_1,p_2}^{\frac{m}{m+1}}. \tag{3.25}$$

Proof. Assume conversely that there exists $(p_{1i}, p_{2i}) = \big((x_{1i}, y_{1i}), (x_{2i}, y_{2i})\big) \in V$ for any $i \in \mathbb{Z}_{>0}$ satisfying $h_{p_{1i},p_{2i}} \geq i$, such that at least one of the following (i) and (ii) does not hold:

$$\text{(i)} \quad |y_{1i}| < h_{p_{1i},p_{2i}}^{\frac{m}{m+1}}, \qquad \text{(ii)} \quad |y_{2i}| < h_{p_{1i},p_{2i}}^{\frac{m}{m+1}}. \tag{3.26}$$

Thus we obtain a sequence (p_{1i}, p_{2i}), $i = 1, 2, ...$ Since $x_{1i} \neq x_{2i}$ or $y_{1i} \neq y_{2i}$ for all i, if necessary by replacing (F, G) by $(\bar{F}, \bar{G}) = \big(F(x + \alpha y, y), G(x + \alpha y, y)\big)$ for some sufficiently small $\alpha > 0$ (one can easily observe that all previous results in this section still hold after the replacement) and by replacing the sequence by a subsequence [if the sequence (p_{1i}, p_{2i}) is replaced by the subsequence (p_{1,i_j}, p_{2,i_j}), then we always have $i_j \geq j$; thus we still have $h_{p_{1i},p_{2i}} \geq i$ after the replacement], we may assume $x_{1i} \neq x_{2i}$ for all i. Since at least one of the conditions in (3.26) cannot hold for infinite many i's, by replacing the sequence by a subsequence, we may assume one of the conditions in (3.26) does not hold for all i. If necessary by switching p_{1i} and p_{2i}, we can assume (3.26) (i) cannot hold for all i, i.e.,

$$|y_{1i}| \geq h_{p_{1i},p_{2i}}^{\frac{m}{m+1}} \to \infty, \tag{3.27}$$

for all $i \gg 1$. We need to use the following notations:

$$a_i \sim b_i, \quad a_i \prec b_i, \quad a_i \preceq b_i, \tag{3.28}$$

which mean respectively $\boldsymbol{s}_1 < |\frac{a_i}{b_i}| < \boldsymbol{s}_2$, $\lim_{i\to\infty} \frac{a_i}{b_i} = 0$, $|\frac{a_i}{b_i}| \leq \boldsymbol{s}_1$ for some fixed $\boldsymbol{s}_1, \boldsymbol{s}_2 \in \mathbb{R}_{>0}$. For $k = 1, 2$, since $|x_{ki}| \leq h_{p_{1i},p_{2i}}$, $|y_{ki}| \leq h_{p_{1i},p_{2i}}$, by (3.2) (i), we have

$$\text{(i) } F_1(x_{ki}, y_{ki}) \preceq h_{p_{1i},p_{2i}}^{m-1} \prec h_{p_{1i},p_{2i}}^{\frac{m^2}{m+1}} \preceq |y_{1i}|^m, \qquad \text{(ii) } F(x_{1i}, y_{1i}) \sim y_{1i}^m. \tag{3.29}$$

Thus,

$$1 = \frac{F(x_{2i}, y_{2i})}{F(x_{1i}, y_{1i})} = \lim_{i\to\infty} \frac{F(x_{2i}, y_{2i})}{F(x_{2i}, y'_{2i})} = \lim_{i\to\infty} \frac{\frac{y_{2i}^m}{y_{1i}^m} + \frac{F_1(x_{2i},y_{2i})}{y_{1i}^m}}{1 + \frac{F_1(x_{2i},y_{2i})}{y_{1i}^m}} = \lim_{i\to\infty} \Big(\frac{y_{2i}}{y_{1i}}\Big)^m. \tag{3.30}$$

Therefore, by replacing the sequence by a subsequence, we have

$$\lim_{i\to\infty} \frac{y_{2i}}{y_{1i}} = \omega, \quad \text{where } \omega \text{ is some } m\text{-th root of unity.} \tag{3.31}$$

Denote $\mathcal{E} = h_{p_{1i},p_{2i}}^{-\frac{1}{m(m+1)}} \to 0$ (when $i \to \infty$). Let $a \in \mathbb{C}$ with $|a| \leq h_{p_{1i},p_{2i}}$. By (3.27), (3.31), we have

$$tw|_{(x,y)=(|a|,|y_{ki}|)} = (1 + |a|^{\frac{1}{m}})^{m-1}|y_{ki}|^{-1} \preceq h_{p_{1i},p_{2i}}^{\frac{m-1}{m}} |y_{1i}|^{-1} \leq h_{p_{1i},p_{2i}}^{\frac{m-1}{m}-\frac{m}{m+1}} = \mathcal{E} \text{ for } k = 1, 2. \tag{3.32}$$

To continue the proof of Proposition 3.5, we need the following.

Lemma 3.6. *Let $k = 1, 2$ and $a \in \mathbb{C}$ with $|a| \leq h_{p_{1i},p_{2i}}$.*

(i) *In (3.15) (i), the series P with respect to x, y converges strongly when (x, y) is set to (a, y_{ki}), and*

$$P_{a,k} := P|_{(x,y)=(a,y_{ki})} = y_{ki}^{-1}\Big(1 + O(\mathcal{E})^1\Big). \tag{3.33}$$

(ii) *In (3.21) (ii), the series y with respect to x, P converges strongly when (x, P) is set to $(a, P_{a,k})$, and*

$$Y_{a,k} := y|_{(x,P)=(a,P_{a,k})} = y_{ki}\Big(1 + O(\mathcal{E})^1\Big). \tag{3.34}$$

(iii) *In (3.18), the series G with respect to x, P converges strongly when (x, P) is set to $(a, P_{a,k})$, and*

$$\text{(a) } A_{a,k,\ell} := \sum_{j=\ell}^{\infty} \tilde{c}_{0j} P_{a,k}^j \preceq P_{a,k}^{-\ell} \sim y_{1k}^{-\ell} \quad \textit{for} \quad \ell \geq -n,$$

$$\text{(b) } B_{a,k} := G_1|_{(x,y)=(a,y_{ki})} \preceq h_{p_{1i},p_{2i}} y_{ki}^{-(m-1)} \prec y_{1i}^{-(m-3)}. \tag{3.35}$$

(iv) *The series* $\left(\frac{\partial F}{\partial y}\right)^{-1}$ *with respect to* x, P *converges strongly when* (x, P) *is set to* $(a, P_{a,k})$, *and*

$$m\Big(\frac{\partial F}{\partial y}\Big)^{-1}\Big|_{(x,P)=(a,P_{a,k})} = y_{ki}^{-(m-1)}\Big(1 + O(\mathcal{E})^1\Big). \tag{3.36}$$

(v) $P_1 := P_{x_{1i},0} = P(x_{1i}, y_{1i}) = P(x_{2i}, y_{2i}) = P_{x_{2i},1}$.

Proof. (i)–(iii) follow from (3.15), (3.21), (3.18), while (iv) can be observed from (3.24). To prove (v), we have $P(x_{1i}, y_{1i})^{-m} = F(x_{1i}, y_{1i}) = F(x_{2i}, y_{2i}) = P(x_{2i}, y_{2i})^{-m}$. Thus $P(x_{2i}, y_{2i}) = \omega' P(x_{1i}, y_{1i})$ for some m-th root ω' of unity. Assume there exists $j \le m-3$ such that $\tilde c_j = \tilde c_{0j} \ne 0$ and $P(x_{1i}, y_{1i})^j \ne P(x_{2i}, y_{2i})^j = \omega'^j P(x_{1i}, y_{1i})^j$. Let $j_0 \le m-3$ be the minimal such j. Then $|1-\omega'^{j_0}| > \delta$ for some fixed $\delta > 0$. By (3.18) and Lemma 3.6 (iii), we have [note that $\tilde c_{0,j_0} \in \mathbb{C}_{\ne 0}$ is a number independent of i and $P(x_{1i}, y_{1i}) \sim y_{1i}^{-1}$ by (3.33)],

$$\begin{aligned} 0 &= G(x_{1i}, y_{1i}) - G(x_{2i}, y_{2i}) \\ &= \tilde c_{0,j_0}(1-\omega'^{j_0})P(x_{1i}, y_{1i})^{j_0} + A_{x_{1i},0,j_0+1} - A_{x_{2i},1,j_0+1} + B_{x_{1i},0} - B_{x_{2i},1} \sim y_{1i}'^{-j_0}, \end{aligned} \tag{3.37}$$

which is a contradiction. This proves that $\omega'^j = 1$ for all $j \le m-3$ with $\tilde c_j \ne 0$. In particular by (3.17) (ii), $\omega'^{m-4} = 1, \omega'^{m-3} = 1$, which implies that $\omega' = 1$. This proves (v) and the lemma. □

Now we denote

$$\bar F = F(x_{1i} + \beta_i x, y), \quad \bar G = \beta_i^{-1} G(x_{1i} + \beta_i x, y), \quad \bar c_j = \beta_i^{-1}\tilde c_j|_{x=x_{1i}+\beta_i x}, \quad \beta_i = x_{2i} - x_{1i}. \tag{3.38}$$

Then $\bar F(0, y_{1i}) = \bar F(1, y_{2i})$, $\bar G(0, y_{1i}) = \bar G(1, y_{2i})$, and $J(\bar F, \bar G) = J(F, G) = J_0$. By (3.18) and Lemma 3.6 (v), we have

$$\begin{aligned} 0 &= \bar G(0, y_{1i}) - \bar G(1, y_{2i}) = -\sum_{j=-n}^{\infty}\big(\bar c_j(1) - \bar c_j(0)\big)P_1^j = -\sum_{j=-n}^{\infty}\int_0^1 \frac{d\bar c_j}{dx}P_1^j dx \\ &= -\int_0^1 \sum_{j=-n}^{\infty}\frac{d\bar c_j}{dx}P_1^j dx = \int_0^1 J_0\Big(\frac{\partial \bar F}{\partial y}\Big)^{-1}\Big|_{P=P_1} dx = J_0 m^{-1} y_{1i}^{-(m-1)}\Big(1 + O(\mathcal{E})^1\Big), \end{aligned} \tag{3.39}$$

where the third equality follows from the fact that $\bar c_j$'s are polynomials on x, the fourth follows from the fact that the series there converges absolutely and uniformly for $x \in [0,1] := \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ by Lemma 3.6 (note that $|x_{1i} + \beta_i x| \le h_{p_{1i},p_{2i}}$ when $x \in [0,1]$), and the last two equalities can be observed from (3.24). We obtain a contradiction in (3.39). This proves Proposition 3.5. □

Now Theorem 1.2 follows from Proposition 3.5.

4. Proof of Theorem 1.3

We start the section by assuming conversely Theorem 1.3 is not true, i.e.,

$$V_{\xi_1,\xi_2} := \Big\{(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in V \;\Big|\; x_1 = \xi_1,\, x_2 = \xi_2\Big\} \ne \emptyset \text{ for any } (\xi_1, \xi_2) \in \mathbb{C}^2. \tag{4.1}$$

Proposition 4.1. *Assume we have* (4.1).

(i) *Denote by V_0 some connected component of the subset of V such that all its elements $(p_1,p_2)=\big((x_1,y_1),(x_2,y_2)\big)$ satisfy* (4.2). *Then $V_0\neq\emptyset$.*

$$\text{(a) } \kappa_0\le|f_1|\le|f_2|\le|f_3|\le\kappa_1, \qquad \text{(b) } |f_4|\le|x_1|\le|f_5|,$$
$$\text{(c) } |f_6|\le|x_2|\le|f_7|, \quad \text{(d) } \ell_{p_1,p_2}:=|f_8|+|x_2|+|x_2+y_2|\ge\kappa_2. \tag{4.2}$$

Here f_i's are some locally holomorphic functions on x_1,x_2,y_2, and $\kappa_i\in\mathbb{R}_{>0}$, which will be chosen such that there exist $\theta_i\in\mathbb{R}_{>0}$ satisfying: when conditions (4.2) *hold, we have*

$$\theta_0\le|x_1|,|x_2|\le\theta_1, \qquad |f_8|,|x_2+y_2|>0. \tag{4.3}$$

(ii) *For any $(p_1,p_2)\in V_0$, no equality can occur in the first or last inequality of* (4.2) (a), *or in any inequality of* (4.2) (b) *or* (c)*; further, two equalities cannot simultaneously occur in the second and third inequalities of* (4.2) (a).

To prove Proposition 4.1, let us make the following assumption.

Assumption 4.2. *Assume Proposition* 4.1 *is not true.*

Lemma 4.3. *Assume we have* (4.1). *The following subset of V is a nonempty compact subset of $\mathbb{C}^4$ for any $k_1,k_2\in\mathbb{R}_{\ge0}$,*

$$A_{k_1,k_2}=\big\{(p_1,p_2)=\big((x_1,y_1),(x_2,y_2)\big)\in V\;\big|\;|x_1|=k_1,\,|x_2|=k_2\big\}. \tag{4.4}$$

Proof. By assumption (4.1), A_{k_1,k_2} is nonempty. Let, for $i=1,2,...,$

$$(p_{1i},p_{2i})=\big((x_{1i},y_{1i}),(x_{2i},y_{2i})\big)\in A_{k_1,k_2}, \tag{4.5}$$

be a sequence such that $h_{p_{1i},p_{2i}}\to\infty$. By definition, we have $|x_{1i}|=k_1$, $|x_{2i}|=k_2$. Thus $h_{p_{1i},p_{2i}}\sim\max\{|y_{1i}|,|y_{2i}|\}$. We see that at least one inequation of (3.25) is violated. Hence A_{k_1,k_2} is bounded. Now assume (4.5) is a sequence converging to some $(p_1,p_2)=\big((x_1,y_1),(x_2,y_2)\big)\in\mathbb{C}^4$. Then $\sigma(p_1)=\sigma(p_2)$ and $|x_1|=k_1$, $|x_2|=k_2$. We must have $p_1\neq p_2$ (otherwise the local bijectivity of σ does not hold at the point p_1), i.e., $(p_1,p_2)\in A_{k_1,k_2}$, and so A_{k_1,k_2} is a closed set in $\mathbb{C}^4$. □

By Lemma 4.3, the following is a well-defined function on $k_1,k_2\in\mathbb{R}_{\ge0}$,

$$\gamma_{k_1,k_2}=\max\big\{|x_2+y_2|\;\big|\;(p_1,p_2)=\big((x_1,y_1),(x_2,y_2)\big)\in A_{k_1,k_2}\big\}. \tag{4.6}$$

Lemma 4.4. *Assume we have* (4.1). *The γ_{k_1,k_2} is a strictly increasing function on $k_1\in\mathbb{R}_{\ge0}$ when $k_2\in\mathbb{R}_{\ge0}$ is fixed, i.e.,*

$$\gamma_{k_1',k_2}>\gamma_{k_1,k_2} \quad \text{if } \; k_1'>k_1\ge0,\;k_2\ge0. \tag{4.7}$$

Proof. For any $k_1'>0$, let

$$\begin{aligned}\beta &= \max\{\gamma_{k_1,k_2}\,|\,0\le k_1\le k_1'\}\\ &= \max\big\{\,|x_2+y_2|\;\big|\;(p_1,p_2)=\big((x_1,y_1),(x_2,y_2)\big)\in V,\,0\le|x_1|\le k_1',\,|x_2|=k_2\,\big\}.\end{aligned} \tag{4.8}$$

Assume conversely that there exists $k_1<k_1'$ with $k_1\ge0$, $\gamma_{k_1,k_2}=\beta$. We use the local bijectivity of Keller maps to obtain a contradiction. Let

$$(\tilde p_1,\tilde p_2)=\big((\tilde x_1,\tilde y_1),(\tilde x_2,\tilde y_2)\big)\in V \quad \text{with} \quad |\tilde x_1|=k_1,\quad |\tilde x_2|=k_2,\quad |\tilde x_2+\tilde y_2|=\beta. \tag{4.9}$$

Set (and define $\tilde G_1, \tilde G_2$ similarly),

$$\tilde F_1 = F(\tilde x_1 + x, \tilde y_1 + y), \qquad \tilde F_2 = F(\tilde x_2 + x, \tilde y_2 + y). \tag{4.10}$$

Denote

$$\tilde a_1 = \mathrm{C_{oeff}}(\tilde F_1, x^1y^0), \quad \tilde b_1 = \mathrm{C_{oeff}}(\tilde F_1, x^0y^1), \quad \tilde a = \mathrm{C_{oeff}}(\tilde F_2, x^1y^0), \quad \tilde b = \mathrm{C_{oeff}}(\tilde F_2, x^0y^1). \tag{4.11}$$

We use $\tilde c_1, \tilde d_1, \tilde c, \tilde d$ to denote the corresponding elements for $\tilde G_1, \tilde G_2$. Then $A_1 = \left(\begin{smallmatrix} \tilde a_1 & \tilde c_1 \\ \tilde b_1 & \tilde d_1 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} \tilde a & \tilde c \\ \tilde b & \tilde d \end{smallmatrix}\right)$ are invertible 2×2 matrices such that $\det A_1 = \det A = J(F, G)$. For the purpose of proving Lemma 4.4, we can replace (F, G) by $(F, G)A_1^{-1}$, then A_1 becomes $A_1 = I_2$ (the 2×2 identity matrix), and AA_1^{-1} becomes the new A. Then we can write [here "$\equiv$" means equal modulo terms with degrees ≥ 3]

$$\begin{aligned} &\tilde F_1 \equiv x + \tilde\beta_1 x^2 + \tilde\beta_2 xy + \tilde\beta_3 y^2, && \tilde F_2 \equiv \tilde a x + \tilde b y + \tilde\alpha_1 x^2 + \tilde\alpha_2 xy + \tilde\alpha_3 y^2, \\ &\tilde G_1 \equiv y + \tilde a_1 x^2 + \tilde a_2 xy + \tilde a_3 y^2, && \tilde G_2 \equiv \tilde c x + \tilde d y + \tilde a_4 x^2 + \tilde a_5 xy + \tilde a_6 y^2, \end{aligned} \tag{4.12}$$

for some $\tilde a_i, \tilde\alpha_i, \tilde\beta_i \in \mathbb{C}$, where, by subtracting $\tilde F_i$ (resp., $\tilde G_i$) by the constant $\alpha_F = F(\tilde x_1, \tilde y_1)$ [resp., $\alpha_G = G(\tilde x_1, \tilde y_1)$], we have assumed $\tilde F_i$, $\tilde G_i$ do not contain constant terms.

Let $\mathcal{E} > 0$ be a parameter such that $\mathcal{E} \to 0$. For any $s, t, u, v \in \mathbb{C}$, denote

$$q_1 := (\dot x_1, \dot y_1) = (\tilde x_1 + s\mathcal{E}, \tilde y_1 + t\mathcal{E}), \qquad q_2 := (\dot x_2, \dot y_2) = (\tilde x_2 + u\mathcal{E}, \tilde y_2 + v\mathcal{E}). \tag{4.13}$$

The local bijectivity of Keller maps says that for any $u, v \in \mathbb{C}$ (cf. Remark 4.5), there exist $s, t \in \mathbb{C}$ such that $(q_1, q_2) \in V$, where (s, t) is uniquely determined from (u, v) by the equation

$$\Big(\tilde F_1(s\mathcal{E}, t\mathcal{E}), \tilde G_1(s\mathcal{E}, t\mathcal{E})\Big) = \Big(\tilde F_2(u\mathcal{E}, v\mathcal{E}), \tilde G_2(u\mathcal{E}, v\mathcal{E})\Big). \tag{4.14}$$

Remark 4.5. When considering the local bijectivity of Keller maps, we always assume $u, v \in \mathbb{C}$ are bounded by some fixed $\boldsymbol{s} \in \mathbb{R}_{>0}$ (independent of $\mathcal{E}$) and assume $\mathcal{E} > 0$ is as small as we wish.

In fact we can easily use (4.12) to solve s up to $O(\mathcal{E})^1$ as follows,

$$s = s_1 + O(\mathcal{E})^1, \qquad s_1 = \tilde a u + \tilde b v. \tag{4.15}$$

It is straightforward to see that we can choose suitable u, v such that

$$\text{(i)}\ \ |\dot x_2| = |\tilde x_2 + u\mathcal{E}| = |\tilde x_2| = k_2, \quad \text{(ii)}\ \ |\dot x_2 + \dot y_2| = |\tilde x_2 + \tilde y_2 + (u+v)\mathcal{E}| > |\tilde x_2 + \tilde y_2| = \beta. \tag{4.16}$$

Since $|x_1| = k_1 < k_1'$, we automatically have $|\dot x_1| < k_1'$ [as $\mathcal{E} \ll 1$, cf. (4.13)]. This means that we can choose $(q_1, q_2) \in V$ with $|\dot x_1| < k_1'$, $|\dot x_2| = k_2$, but $|\dot x_2 + \dot y_2| > \beta$, which is a contradiction with the definition of β in (4.8). This proves the lemma. □

Lemma 4.6. *Assume we have* (4.1) *and Assumption* 4.2. *The* γ_{k_1,k_2} *is a weakly increasing function on* $k_2 \in \mathbb{R}_{>0}$ *when* $k_1 \in \mathbb{R}_{>0}$ *is fixed, i.e.,*

$$\gamma_{k_1,k_2} \ge \gamma_{k_1,k_2'} \quad \text{if}\ \ k_1 > 0,\ \ k_2 > k_2' > 0. \tag{4.17}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 December 2022 doi:10.20944/preprints202212.0171.v1

Proof. Assume $\gamma_{k_1,k_2} < \gamma_{k_1,k_2'}$ for some $k_1 > 0$, $k_2 > k_2' > 0$. Take $\boldsymbol{k} \gg 1$. We can choose sufficiently small $\delta \in \mathbb{R}_{>0}$ (independent of $\boldsymbol{k}$) satisfying (the following holds when $\delta = 0$ and $\boldsymbol{k} \to \infty$, thus also holds when $\delta > 0$ is sufficiently small)

$$\alpha := (k_2^{-1}k_2')^{\delta}\gamma_{k_1,k_2'} + (k_2' + \gamma_{k_1,k_2'})\boldsymbol{k}^{-4} > \gamma_{k_1,k_2} + (k_2 + \gamma_{k_1,k_2})\boldsymbol{k}^{-4}. \tag{4.18}$$

We define V_0 to be some connected component of the subset of V consisting of elements $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big)$ satisfying,

$$\text{(a) } 1 \le \big(k_1^{-1}|x_1|\big)^{\boldsymbol{k}} \le k_2|x_2|^{-1} \le \big(k_1^{-1}|x_1|\big)^{\boldsymbol{k}+1} \le \boldsymbol{k}, \quad \text{(b) } \boldsymbol{k}^{-3} \le |x_1| \le \boldsymbol{k}^3,$$
$$\text{(c) } \boldsymbol{k}^{-3} \le |x_2| \le \boldsymbol{k}^3, \quad \text{(d) } (k_2^{-1}|x_2|)^{\delta}|x_2 + y_2| + (|x_2| + |x_2 + y_2|)\boldsymbol{k}^{-4} \ge \alpha. \tag{4.19}$$

Remark 4.7. Note that when we define (4.19), $\boldsymbol{k}$ is simply some fixed positive real number. When we say $\boldsymbol{k} \gg 1$, it means that we may need to choose sufficiently large $\boldsymbol{k}$ such that the system (4.19) can satisfy our requirement. This will also apply to some similar situations later.

Then we can rewrite (4.19) as the form in (4.2), and we have (4.3) [if $x_2 + y_2 = 0$ then (4.19) (d) shows that $|x_2| \ge \alpha\boldsymbol{k}^4 \sim \boldsymbol{k}^4$ by (3.28) and by noting that $\alpha > (k_2^{-1}k_2')^{\delta}\gamma_{k_1,k_2'} > (k_2^{-1}k_2')^{\delta}\gamma_{k_1,k_2} \ge 0$, a contradiction with (4.19) (c)]. Denote $k_1' := (k_2 k_2'^{-1})^{\boldsymbol{k}^{-1}} k_1 > k_1$. By definition, there exists,

$$(\check{p}_1, \check{p}_2) = \big((\check{x}_1, \check{y}_1), (\check{x}_2, \check{y}_2)\big) \in V \quad \text{with } |\check{x}_1| = k_1', \ |\check{x}_2| = k_2', \ |\check{x}_2 + \check{y}_2| = \gamma_{k_1',k_2'}. \tag{4.20}$$

Noting that $\gamma_{k_1',k_2'} > \gamma_{k_1,k_2'}$ by Lemma 4.4, one can verify

$$1 < k_2 k_2'^{-1} = (k_1^{-1}|\check{x}_1|)^{\boldsymbol{k}} = k_2|\check{x}_2|^{-1} < (k_2 k_2'^{-1})^{1+\boldsymbol{k}^{-1}} = (k_1^{-1}|\check{x}_1|)^{\boldsymbol{k}+1} < \boldsymbol{k},$$
$$(k_2^{-1}|\check{x}_2|)^{\delta}|\check{x}_2 + \check{y}_2| + (|\check{x}_2| + |\check{x}_2 + \check{y}_2|)\boldsymbol{k}^{-4} = (k_2^{-1}k_2')^{\delta}\gamma_{k_1',k_2'} + (k_2' + \gamma_{k_1',k_2'})\boldsymbol{k}^{-4} > \alpha, \tag{4.21}$$

i.e., (4.19) (a), (d) hold for $(\check{p}_1, \check{p}_2)$. Thus we can choose V_0 to be the connected component such that $(\check{p}_1, \check{p}_2) \in V_0$ by (4.19), (4.20), i.e., $V_0 \ne \emptyset$.

Let $(p_1, p_2) \in V_0$. In (4.19) (a), assume the equality occurs in the first inequality, or two equalities simultaneously occur in the second and third inequalities. Then we obtain that $|x_2| = k_2$, $|x_1| = k_1$ (and thus $|x_2 + y_2| \le \gamma_{k_1,k_2}$). By (4.19) (d), we have

$$\gamma_{k_1,k_2} + (k_2 + \gamma_{k_1,k_2})\boldsymbol{k}^{-4} \ge (k_2^{-1}|x_2|)^{\delta}|x_2 + y_2| + (|x_2| + |x_2 + y_2|)\boldsymbol{k}^{-4} \ge \alpha, \tag{4.22}$$

which is a contradiction with (4.18). Assume the equality occurs in the last inequality of (4.19) (a), i.e., $|x_1| = k_1\boldsymbol{k}^{\frac{1}{\boldsymbol{k}+1}} \sim 1$. Then by the second and third inequalities of (4.19) (a), $k_2|x_2|^{-1} = (k_1^{-1}|x_1|)^{\boldsymbol{k}(1+O(\boldsymbol{k}^{-1})^1)} \sim \boldsymbol{k}$, i.e., $|x_2| \sim \boldsymbol{k}^{-1} \prec 1$. Then by (4.19) (d), $|x_2 + y_2| \succeq |x_2|^{-\delta} \sim \boldsymbol{k}^{\delta} \succ 1$, and thus $|y_2| \sim |x_2 + y_2| \succ \boldsymbol{k}^{\delta}$. We obtain that $h_{p_1,p_2} \sim |y_2|$ if $|y_2| \ge |y_1|$ or $h_{p_1,p_2} \sim |y_1|$ if $|y_2| < |y_1|$. In any case we obtain a contradiction with Theorem 1.2.

By (4.19) (a), we easily see that no equality can hold in any inequality of (4.19) (b), (c). This shows that Proposition 4.1 holds, a contradiction with Assumption 4.2. This proves the lemma. □

Lemma 4.8. *Assume we have (4.1) and Assumption 4.2. For any $\delta \in \mathbb{R}_{\ge 0}$, $k, k_1, k_2 \in \mathbb{R}_{>0}$ with $k > 1$, $\delta < \frac{1}{m}$, we have $\gamma_{k^{1+\delta}k_1,kk_2} < k\gamma_{k_1,k_2}$.*

 doi:10.20944/preprints202212.0171.v1

Proof. Assume the result is not true, then by choosing δ' with $\delta<\delta'<\frac{1}{m}$ and by Lemma 4.4, we may assume $\gamma_{\bar k^{1+\delta'}k_1,\bar kk_2}>\gamma_{\bar k^{1+\delta}k_1,\bar kk_2}\ge\bar k\gamma_{k_1,k_2}$ for some $\bar k,k_1,k_2\in\mathbb{R}_{>0}$ with $\bar k>1$. Take $\boldsymbol{k}\gg1$. As in the proof of the previous lemma, we can choose sufficiently small $\delta_1\in\mathbb{R}_{>0}$ with $\delta_1<\delta'$ satisfying,

$$\alpha:=(\bar kk_2)^{-(1+\delta_1)}\gamma_{\bar k^{1+\delta'}k_1,\bar kk_2}+(\bar kk_2+\gamma_{\bar k^{1+\delta'}k_1,\bar kk_2})\boldsymbol{k}^{-4}>k_2^{-(1+\delta_1)}\gamma_{k_1,k_2}+(k_2+\gamma_{k_1,k_2})\boldsymbol{k}^{-4}. \quad (4.23)$$

We define V_0 to be some connected component of the subset of V consisting of elements $(p_1,p_2)=\big((x_1,y_1),(x_2,y_2)\big)$ satisfying,

$$\text{(a) } 1\le\big(k_2^{-1}|x_2|\big)^{1+\delta'-\boldsymbol{k}^{-3}}\le k_1^{-1}|x_1|\le\big(k_2^{-1}|x_2|\big)^{1+\delta'+\boldsymbol{k}^{-3}}\le\boldsymbol{k}^{1+\delta'+\boldsymbol{k}^{-3}},\quad\text{(b) } \boldsymbol{k}^{-3}\le|x_1|\le\boldsymbol{k}^3,$$
$$\text{(c) } \boldsymbol{k}^{-3}\le|x_2|\le\boldsymbol{k}^3,\quad\text{(d) } \frac{|x_2+y_2|}{|x_2|^{1+\delta_2}}+(|x_2|+|x_2+y_2|)\boldsymbol{k}^{-4}\ge\alpha. \quad (4.24)$$

Then we can rewrite the above as the form in (4.2), and we have (4.3) (as in the proof of the previous lemma). Further, by definition, there exists

$$(\check p_1,\check p_2)=\big((\check x_1,\check y_1),(\check x_2,\check y_2)\big)\in V\text{ with }|\check x_1|=\bar k^{1+\delta'}k_1,\ |\check x_2|=\bar kk_2,\ |\check x_2+\check y_2|=\gamma_{\bar k^{1+\delta'}k_1,\bar kk_2}. \quad (4.25)$$

One can verify that (4.24) holds for $(\check p_1,\check p_2)$. Thus we can choose V_0 such that $(\check p_1,\check p_2)\in V_0$, i.e., $V_0\ne\emptyset$.

Let $(p_1,p_2)\in V_0$. In (4.24) (a), if the equality occurs in the first inequality, or two equalities simultaneously occur in the second and third inequalities, then we obtain that $|x_2|=k_2$, $|x_1|=k_1$, but by (4.24) (d) and the definition of γ_{k_1,k_2}, we have

$$k_2^{-(1+\delta_1)}\gamma_{k_1,k_2}+(k_2+\gamma_{k_1,k_2})\boldsymbol{k}^{-4}\ge\frac{|x_2+y_2|}{|x_2|^{1+\delta_1}}+(|x_2|+|x_2+y_2|)\boldsymbol{k}^{-4}\ge\alpha, \quad (4.26)$$

which is a contradiction with (4.23). If the equality occurs in the last inequality of (4.24) (a), then one obtains that $|x_2|\sim\boldsymbol{k}$, $|x_1|\sim\boldsymbol{k}^{1+\delta'}$. Note from (3.25) that $h_{p_1,p_2}\sim\max\{|x_1|,|x_2|\}\sim\boldsymbol{k}^{1+\delta'}$, but by (4.24) (d), we have $|x_2+y_2|\succeq|x_2|^{1+\delta_1}\sim\boldsymbol{k}^{1+\delta_1}$, and thus $|y_2|\sim|x_2+y_2|\succ\boldsymbol{k}\succ\boldsymbol{k}^{\frac{(1+\delta')m}{1+m}}\sim h_{p_1,p_2}^{\frac{m}{m+1}}$, a contradiction with (3.25). By (4.24) (a), we easily see that no equality can hold in any inequality of (4.24) (b), (c). This shows that Proposition 4.1 holds, a contradiction with Assumption 4.2. □

Lemma 4.9. *Assume we have* (4.1) *and Assumption* 4.2*. For any* $k_1,k_2\in\mathbb{R}_{>0}$*, we have* $\gamma_{k_1,k_2}>k_2$.

Proof. Assume the result is not true, then by choosing $k_1'\in\mathbb{R}_{>0}$ with $k_1'<k_1$, by Lemma 4.4, we can assume $\gamma_{k_1',k_2}<k_2$ for some $k_1',k_2>0$. Then $\alpha:=\frac{\gamma_{k_1',k_2}}{k_2}<1$. By Lemma 4.8, we have $\gamma_{\boldsymbol{k}k_1',\boldsymbol{k}k_2}<\boldsymbol{k}\gamma_{k_1',k_2}=\boldsymbol{k}k_2\alpha$ for all $\boldsymbol{k}\gg1$. Let

$$(p_1,p_2)\in V\quad\text{with}\quad|x_1|=\boldsymbol{k}k_1',\quad|x_2|=\boldsymbol{k}k_2,\quad|x_2+y_2|=\gamma_{\boldsymbol{k}k_1',\boldsymbol{k}k_2}<\boldsymbol{k}k_2\alpha. \quad (4.27)$$

Then as in the proof of the previous lemma, we have $h_{p_1,p_2}\sim\boldsymbol{k}$, but then

$$|y_2|\ge|x_2|-|x_2+y_2|>(1-\alpha)\boldsymbol{k}k_2>h_{p_1,p_2}^{\frac{m}{m+1}}, \quad (4.28)$$

which is a contradiction with (3.25). □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 December 2022 doi:10.20944/preprints202212.0171.v1

Now we fix sufficiently large $\boldsymbol{k} \gg 1$. Take

$$(\bar p_1, \bar p_2) = \big((\bar x_1, \bar y_1), (\bar x_2, \bar y_2)\big) \in A_{\boldsymbol{k},\boldsymbol{k}} \quad \text{with} \quad |\bar x_1| = |\bar x_2| = \boldsymbol{k}, \quad |\bar x_2 + \bar y_2| = \gamma_{\boldsymbol{k},\boldsymbol{k}} > \boldsymbol{k}, \tag{4.29}$$

where the inequality follows from Lemma 4.9. Similar to (4.10) (but not exactly), we define

$$F_1 = F\big(\bar x_1(1+x), \bar y_1 + y\big), \qquad F_2 = F\big(\bar x_2(1+x), \bar y_2 + y\big), \tag{4.30}$$

and define G_1, G_2 similarly [thus the matrices A_1, A defined after (4.11) now have determinants $\det A_1 = \bar x_1 J(F,G) \neq 0$, $\det A = \bar x_2 J(F,G) \neq 0$, and again by replacing (F_i, G_i) by $(F_i, G_i)A_1^{-1}$ for $i = 0, 1$, we can assume $A_1 = I_2$]. Similar to (4.12), we can write [from now on, we only need the linear parts of F_i, G_i],

$$F_1 \equiv x, \qquad F_2 \equiv -a_{\boldsymbol{k}} x + b_{\boldsymbol{k}} \frac{\bar x_2 x + y}{\bar x_2 + \bar y_2}, \tag{4.31}$$

$$G_1 \equiv y, \qquad G_2 \equiv cx + dy, \tag{4.32}$$

where we have written the linear part of F_2 as above simply for later convenience, we emphasis that $a_{\boldsymbol{k}}, b_{\boldsymbol{k}}$ depend on $\boldsymbol{k}$ (of course other coefficients also depend on $\boldsymbol{k}$) and are in fact non-negative as shown in the next lemma. We define q_1, q_2 accordingly [similar to, but a slightly different from, (4.13), simply due to the different definitions in (4.30) and (4.10); we emphasis that the choice of $\mathcal{E}$ depend on $\boldsymbol{k}$: in general the larger $\boldsymbol{k}$ is, the smaller $\mathcal{E}$; but in any case once $\boldsymbol{k}$ is chosen we can always choose sufficiently small $\mathcal{E}$, cf. also Remark 4.5],

$$q_1 := (\dot x_1, \dot y_1) = \big(\bar x_1(1 + s\mathcal{E}), \bar y_1 + t\mathcal{E}\big), \qquad q_2 := (\dot x_2, \dot y_2) = \big(\bar x_2(1 + u\mathcal{E}), \bar y_2 + v\mathcal{E}\big). \tag{4.33}$$

In particular, by (4.31) we have as in (4.15),

$$s = -a_{\boldsymbol{k}} u + b_{\boldsymbol{k}} \frac{\bar x_2 u + v}{\bar x_2 + \bar y_2} + O(\mathcal{E})^1. \tag{4.34}$$

Lemma 4.10. *Assume we have* (4.1) *and Assumption* 4.2. *We have* $a_{\boldsymbol{k}} \geq 0$, $b_{\boldsymbol{k}} \geq 0$.

Proof. Assume $a_{\boldsymbol{k}\,\mathrm{im}} \neq 0$ or $a_{\boldsymbol{k}\,\mathrm{re}} < 0$ or $b_{\boldsymbol{k}\,\mathrm{im}} \neq 0$ or $b_{\boldsymbol{k}\,\mathrm{re}} < 0$ [cf. Convention 2.1 (1)]. Then from (4.34) one can easily choose u, v [with $u_{\mathrm{im}} \neq 0$, $u_{\mathrm{re}} < 0$, $\big(\frac{\bar x_2 u + v}{\bar x_2 + \bar y_2}\big)_{\mathrm{im}} \neq 0$, $\big(\frac{\bar x_2 u + v}{\bar x_2 + \bar y_2}\big)_{\mathrm{re}} > 0$ such that either $(a_{\boldsymbol{k}} u)_{\mathrm{re}} > 0$ or $\big(b_{\boldsymbol{k}}\big(\frac{\bar x_2 u + v}{\bar x_2 + \bar y_2}\big)\big)_{\mathrm{re}} < 0$, and so $s_{\mathrm{re}} < 0$] satisfying [cf. (4.33) and (4.34)],

$$\begin{aligned} &0 < k_1 := |\dot x_1| = \boldsymbol{k}|1 + s\mathcal{E}| < \boldsymbol{k}, \qquad 0 < k_2 := |\dot x_2| = \boldsymbol{k}|1 + u\mathcal{E}| < \boldsymbol{k}, \\ &|\dot x_2 + \dot y_2| = \gamma_{\boldsymbol{k},\boldsymbol{k}} \Big| 1 + \Big(\frac{\bar x_2 u + v}{\bar x_2 + \bar y_2}\Big)\mathcal{E}\Big| > \gamma_{\boldsymbol{k},\boldsymbol{k}}, \end{aligned} \tag{4.35}$$

i.e., $0 < k_1 < \boldsymbol{k}$ and $0 < k_2 < \boldsymbol{k}$ with $\gamma_{k_1,k_2} \geq |\dot x_2 + \dot y_2| > \gamma_{\boldsymbol{k},\boldsymbol{k}}$, a contradiction with Lemmas 4.4 and 4.6. Thus $a_{\boldsymbol{k}} \geq 0$ and $b_{\boldsymbol{k}} \geq 0$. □

Lemma 4.11. *Assume we have* (4.1) *and Assumption* 4.2. *For any fixed* $N \in \mathbb{R}_{>0}$, *let* $\delta' \in \mathbb{R}_{>0}$ *be such that* $\delta' > \ln(\boldsymbol{k})^{-N}$ *(where* $\ln(\cdot)$ *is the natural logarithmic function), we have* $\boldsymbol{k} < \gamma_{\boldsymbol{k},\boldsymbol{k}} < (1 + \delta')\boldsymbol{k}$.

Proof. By (3.25), we have $h_{\bar p_1, \bar p_2} \sim \boldsymbol{k}$. If $\gamma_{\boldsymbol{k},\boldsymbol{k}} \geq (1 + \delta')\boldsymbol{k}$, then,

$$|\bar y_2| \geq |\bar x_2 + \bar y_2| - |\bar x_2| \geq \delta' \boldsymbol{k} \geq \ln(\boldsymbol{k})^{-N} \boldsymbol{k} \succ \boldsymbol{k}^{\frac{m}{m+1}} \sim h_{\bar p_1, \bar p_2}^{\frac{m}{m+1}}, \tag{4.36}$$

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 December 2022 doi:10.20944/preprints202212.0171.v1

a contradiction with (3.25). □

Lemma 4.12. *Assume we have* (4.1) *and Assumption* 4.2*. For any fixed* $\delta \in \mathbb{R}_{>0}$ *with* $\delta < \frac{1}{m}$*, we have* $b_{\boldsymbol{k}} \geq 1 + \delta + a_{\boldsymbol{k}}$ *for all* $\boldsymbol{k} > 0$.

Proof. Assume the lemma does not hold, then we can choose sufficiently small $\delta_1 > 0$ (which can depend on $\boldsymbol{k}$) such that

$$(1+\delta_1)b_{\boldsymbol{k}} < 1 + \delta - \delta_1 + a_{\boldsymbol{k}}. \tag{4.37}$$

Let $\ell \gg \boldsymbol{k}$. We define V_0 to be some connected component of the subset of V consisting of elements $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big)$ satisfying ,

$$\begin{aligned}&\text{(i) } 1 \leq \big(\boldsymbol{k}^{-1}|x_2|\big)^{1+\delta-\delta_1-\ell^{-3}} \leq \boldsymbol{k}^{-1}|x_1| \leq \big(\boldsymbol{k}^{-1}|x_2|\big)^{1+\delta} \leq \ell^{1+\delta}, \quad \text{(ii) } \ell^{-4} \leq |x_1| \leq \ell^4,\\ &\text{(iii) } \ell^{-4} \leq |x_2| \leq \ell^4, \quad \text{(iv) } \frac{\gamma_{\boldsymbol{k},\boldsymbol{k}}^{-1}|x_2+y_2|}{\big(\boldsymbol{k}^{-1}|x_2|\big)^{1+\delta_1}} + (|x_2| + |x_2+y_2|)\mathcal{E}^3 \geq 1 + \mathcal{E}^2.\end{aligned} \tag{4.38}$$

Then we have (4.2) and (4.3).

Remark 4.13. Recall from statements inside the bracket before (4.33) and Remark 4.5 that when $\boldsymbol{k}$ is fixed, $\mathcal{E}$ can be fixed, and we can assume $\mathcal{E} < \ell^{-\ell}$. We emphasis that $\mathcal{E}$ used in (4.38) is the same as that used in the local bijectivity of Keller maps in (4.33). There is no problem in doing so since we only use the local bijectivity of Keller maps to show that V_1 is nonempty [in (4.38), $\ell, \boldsymbol{k}, \mathcal{E}$ are simply some chosen (and fixed) positive real numbers, cf. Remark 4.7].

Let $(p_1, p_2) \in V_0$. In (4.38) (i), if the equality occurs in the first inequality, or two equalities simultaneously occur in the second and third inequalities, then $|x_2| = |x_1| = \boldsymbol{k}$, but by (4.38) (iv), $|x_2 + y_2| > \gamma_{\boldsymbol{k},\boldsymbol{k}}$, a contradiction with the definition of $\gamma_{\boldsymbol{k},\boldsymbol{k}}$. If the equality occurs in the last inequality of (4.38) (i), then we obtain that $|x_2| \sim \ell$, $|x_1| \preceq \ell^{1+\delta}$, but by (4.38) (iv), $|x_2 + y_2| \succeq \ell^{1+\delta_1} \succ |x_2|$. Thus $|y_2| \sim |x_2 + y_2| \succ \ell^{1+\delta_1}$. Again by (3.25), we have $h_{p_1,p_2} \sim \max\{|x_1|, |x_2|\} \preceq \ell^{1+\delta} \prec \ell^{1+\frac{1}{m}} \prec |y_2|^{\frac{m+1}{m}}$, a contradiction with (3.25). By (4.38) (i), we easily see that no equality can hold in any inequality of (4.38) (ii), (iii). This shows that Proposition 4.1 (ii) holds.

Next, we choose suitable u, v such that (4.38) (i), (iv) hold for (q_1, q_2) [defined in (4.33)], i.e.,

$$\begin{aligned}&\text{(i) } 1 \leq |1 + u\mathcal{E}|^{1+\delta-\delta_1-\ell^{-3}} \leq |1 + s\mathcal{E}| \leq |1 + u\mathcal{E}|^{1+\delta} \leq \ell^{1+\delta},\\ &\text{(ii) } \frac{\left|1 + \frac{\bar{x}_2 u + v}{\bar{x}_2 + \bar{y}_2}\mathcal{E}\right|}{|1 + u\mathcal{E}|^{1+\delta_1}} + O(\mathcal{E})^3 \geq 1 + \mathcal{E}^2.\end{aligned} \tag{4.39}$$

We take $u, v \in \mathbb{C}$ such that

$$u = 1, \quad \frac{\bar{x}_2 u + v}{\bar{x}_2 + \bar{y}_2} = \frac{a_{\boldsymbol{k}} + 1 + \delta - \delta_1}{b_{\boldsymbol{k}}}, \qquad \text{then} \quad s = 1 + \delta - \delta_1 + O(\mathcal{E})^1, \tag{4.40}$$

where the last equation is obtained from (4.34). Then by comparing the coefficients of $\mathcal{E}^1$, one can easily see that all inequalities in (4.39) (i) are strict inequalities. Further, the coefficient of $\mathcal{E}^1$ in the left hand-side of (4.39) (ii) is $\frac{a_{\boldsymbol{k}}+1+\delta-\delta_1}{b_{\boldsymbol{k}}} - (1+\delta_1) > 0$ by (4.37). Thus we can choose V_0 such that $(q_1, q_2) \in V_0$, i.e., $V_0 \neq \emptyset$. This shows that Proposition 4.1 holds, a contradiction with Assumption 4.2. This proves the lemma. □

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 December 2022 doi:10.20944/preprints202212.0171.v1

Lemma 4.14. *Assume we have* (4.1) *and Assumption* 4.2. *For any fixed* $\delta \in \mathbb{R}_{>0}$, *we have* $(1-\delta^5)b_{\boldsymbol{k}} \le 1 + a_{\boldsymbol{k}}$ *for all* $\boldsymbol{k} \gg 1$.

Proof. Define V_0 to be some connected component of the subset of V consisting of elements $(p_1,p_2) = \big((x_1,y_1),(x_2,y_2)\big)$ satisfying,

$$\text{(i)}\ (1-\delta^5)^{1+\boldsymbol{k}^{-3}} \le \big(\boldsymbol{k}^{-1}|x_2|\big)^{1+\boldsymbol{k}^{-3}} \le \boldsymbol{k}^{-1}|x_1| \le \big(\boldsymbol{k}^{-1}|x_2|\big)^{1-\boldsymbol{k}^{-3}} \le 1, \quad \text{(ii)}\ \boldsymbol{k}^{-2} \le |x_1| \le \boldsymbol{k}^2,$$
$$\text{(iii)}\ \boldsymbol{k}^{-2} \le |x_2| \le \boldsymbol{k}^2, \quad \text{(iv)}\ \frac{\gamma_{\boldsymbol{k},\boldsymbol{k}}^{-1}|x_2+y_2|}{\big(\boldsymbol{k}^{-1}|x_2|\big)^{1-\delta^5}} + (|x_2|+|x_2+y_2|)\mathcal{E}^3 \ge 1+\mathcal{E}^2. \tag{4.41}$$

We have (4.2) and (4.3). Let $(p_1,p_2) \in V_0$. If the equality occurs in the first inequality of (4.41) (i), then we obtain that $|x_2| = (1-\delta^5)\boldsymbol{k}$, $|x_1| \le \boldsymbol{k}$, and by (4.41) (iv), Lemma 4.11, we have

$$|x_2+y_2| > (1-\delta^5)^{1-\delta^5}\gamma_{\boldsymbol{k},\boldsymbol{k}} > \Big(1-\delta^5+\delta^{10}+O(\delta)^{15}\Big)\boldsymbol{k} > |x_2|. \tag{4.42}$$

As before, we obtain that $|y_2| \ge |x_2+y_2| - |x_2| \sim |x_2+y_2| \succeq \boldsymbol{k} \succ \boldsymbol{k}^{\frac{m}{m+1}} \sim h_{p_1,p_2}^{\frac{m}{m+1}}$, a contradiction with (3.25). If two equalities simultaneously occur in the second and third inequalities of (4.41) (i), or the equality occurs in the last inequality, then $|x_2| = |x_1| = \boldsymbol{k}$, but by (4.41) (iv), $|x_2+y_2| > \gamma_{\boldsymbol{k},\boldsymbol{k}}$, a contradiction with definition (4.6). By (4.41) (i), we easily see that no equality can hold in any inequality of (4.41) (ii), (iii). Hence Proposition 4.1 (ii) holds.

Next, we choose suitable u, v such that (4.41) (i), (iv) hold for (q_1,q_2), i.e.,

$$\text{(i)}\ (1-\delta^5)^{1+\boldsymbol{k}^{-3}} \le |1+u\mathcal{E}|^{1+\boldsymbol{k}^{-3}} \le |1+s\mathcal{E}| \le |1+u\mathcal{E}|^{1-\boldsymbol{k}^{-3}} \le 1,$$
$$\text{(ii)}\ \frac{\Big|1+\frac{\bar{x}_2u+v}{\bar{x}_2+\bar{y}_2}\mathcal{E}\Big|}{|1+u\mathcal{E}|^{1-\delta^5}} + O(\mathcal{E})^3 \ge 1+\mathcal{E}^2. \tag{4.43}$$

Take $u, v \in \mathbb{C}$ such that [the last equation is obtained from (4.34)],

$$u = -1, \quad \frac{\bar{x}_2u+v}{\bar{x}_2+\bar{y}_2} = -\frac{1+a_{\boldsymbol{k}}}{b_{\boldsymbol{k}}}, \qquad \text{and} \quad s = -1+O(\mathcal{E})^1. \tag{4.44}$$

By comparing the coefficients of $\mathcal{E}^1$, we see that all inequalities in (4.43) (i) are strict inequalities. Further, the coefficient of $\mathcal{E}^1$ in the left hand-side of (4.43) (ii) is $1-\delta^5-\frac{1+a_{\boldsymbol{k}}}{b_{\boldsymbol{k}}}$, which is positive if the assertion of the lemma is not true; in this case, we can choose V_0 such that $(q_1,q_2) \in V_0$, i.e., $V_0 \ne \emptyset$, and we obtain a contradiction with Assumption 4.2. This proves the lemma. □

Assume we have (4.1) and Assumption 4.2. The above two lemmas show that $a_{\boldsymbol{k}} \ge \frac{1-\delta^4(1+\delta)}{\delta^4}$. Since δ is arbitrarily sufficiently small number, we see that $a_{\boldsymbol{k}}$ (thus also $b_{\boldsymbol{k}}$) is unbounded, i.e.,

$$\lim_{\boldsymbol{k}\to\infty} a_{\boldsymbol{k}} = \lim_{\boldsymbol{k}\to\infty} b_{\boldsymbol{k}} = \infty, \qquad \text{and} \qquad \lim_{\boldsymbol{k}\to\infty} \frac{a_{\boldsymbol{k}}}{b_{\boldsymbol{k}}} = 1. \tag{4.45}$$

Proof of Proposition 4.1. Assume conversely we have Assumption 4.2. Then we have (4.1) and Assumption 4.2. We first fix some choices of positive numbers satisfying,

$$1 \ll \ell_0 := \delta_0^{-1} \ll \ell_1 := \delta_1^{-1} \ll \ell_2 := \delta_2^{-1} \ll \ell := \delta^{-1} \ll \boldsymbol{k} \ll \mathcal{E}^{-1}. \tag{4.46}$$

For instance, it is enough to take $\ell_0 = 10^{10}$, $\ell_1 = \ell_0^{\ell_0}$, $\ell_2 = \ell_1^{\ell_1}$, $\ell = \ell_2^{\ell_2}$, $\boldsymbol{k} = (\ell \boldsymbol{s}_5)^\ell$ (where $\boldsymbol{s}_5$ satisfies Proposition 3.5) and $\mathcal{E} < \boldsymbol{k}^{-\boldsymbol{k}}$. For any $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in V$, we denote [recall from (4.29), (4.45) that $|\bar{x}_1| = |\bar{x}_2| = \boldsymbol{k}$, $|\bar{x}_2 + \bar{y}_2| = \gamma_{\boldsymbol{k},\boldsymbol{k}}$, and $\alpha_0 > 0$],

$$X_1 = \frac{x_1}{\bar{x}_1}, \quad X_2 = \frac{x_2}{\bar{x}_2}, \quad Z = \frac{x_2 + y_2}{\bar{x}_2 + \bar{y}_2}, \quad \widetilde{X}_1 = \big((1 + \alpha_0 \mathcal{E}) X_1\big)^\ell, \quad \alpha_0 = \ell_0 b_{\boldsymbol{k}} - \ell_0^4 \delta > 0. \tag{4.47}$$

We now define V_1 to be some connected component of the subset of V consisting of elements $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big)$ satisfying the following [we suggest that readers do not need to check details at this moment — we will explain everything when our arguments are carried on step by step so that all will become clear; throughout the section, some multi-valued functions may appear; for instance, B_2 defined in (4.48) (v) is a multi-valued function on $\widetilde{X}_1$, X_2, Z; from our arguments below one can see that locally there always exists a unique choice of each multi-valued function satisfying (4.48), therefore globally there exists a unique choice of each multi-valued function by the fact that all B_2, B_2, B_3 are locally holomorphic functions on $\widetilde{X}_1$, X_2, Z],

$$\begin{aligned}
&\text{(i) } 1 \le |B_1|^{\frac{5\delta_0}{8} - 3\delta_0^2} \le |B_2| \le |B_1|^{\frac{5\delta_0}{8}} \le \ell_1^{\frac{5\delta_0}{8}}, \quad \text{(ii) } (1-\delta)|B_1|^{-\ell_0^2} \le |\widetilde{X}_1| \le \ell_2,\\
&\text{(iii) } (1-\delta)|B_1|^{\ell_0 - 3} \le |X_2| \le \ell_2, \quad \text{(iv) } B_3' := |B_3 B_1^{-\frac{\ell_0^2}{2} + \frac{5}{8} + \frac{3\delta_0}{8}}| + (|x_2| + |x_2 + y_2|)\mathcal{E}^3 \ge 1 + \mathcal{E}^2,\\
&\text{(v) } B_1 = \frac{Z\Big(\frac12\big(1 + \frac{\delta_0^2}{2}\big) + \frac12\big(1 - \frac{\delta_0^2}{2}\big)\widetilde{X}_1^2 B_1^{2\ell_0^2}\Big)}{X_2^{1+\delta_0^2+\delta_0^3}\widetilde{X}_1 B_3^2},\\
&\text{(vi) } B_2 = \frac{1}{2 - \widetilde{X}_1 X_2^{\ell_0^2 + 2 + 2\delta_0 - \delta_0^4} Z^{-\ell_0^2} B_1^{\frac{\ell_0^2}{2} - \ell_0 + \frac38}}, \qquad \text{(vii) } B_3 = \frac{X_2^{\ell_0^2}}{\widetilde{X}_1 Z^{\ell_0^2 + 1}}.
\end{aligned} \tag{4.48}$$

Then (4.48) can be rewritten as the form in (4.2). Now we divide the proof of Proposition 4.1 into three lemmas.

Lemma 4.15. *When conditions* (4.48) *hold, we have the following* [*in particular, we have* (4.3)],

$$\begin{aligned}
&(1)\ |X_1| < |\widetilde{X}_1|^\delta = 1 + O(\delta)^1, \qquad (2)\ Z = X_2 + O(\delta)^2, \qquad (3)\ B_3'' := |B_3 B_1^{-\frac{\ell_0^2}{2} + \frac58 + \frac{3\delta_0}{8}}| > 1,\\
&(4)\ B_1 = \widetilde{X}_1 X_2^{2 - \delta_0^2 - \delta_0^3}\Big(\frac12\big(1 + \frac{\delta_0^2}{2}\big) + \frac12\big(1 - \frac{\delta_0^2}{2}\big)\widetilde{X}_1^2 B_1^{2\ell_0^2}\Big) + O(\delta)^2,\\
&(5)\ B_2 = \frac{1}{2 - \widetilde{X}_1 X_2^{2 + 2\delta_0 - \delta_0^4} B_1^{\frac{\ell_0^2}{2} - \ell_0 + \frac38}} + O(\delta)^2, \qquad (6)\ B_3 = \frac{1}{\widetilde{X}_1 X_2} + O(\delta)^2,\\
&(7)\ (1-\delta)|B_1|^{-\ell_0^2} < |\widetilde{X}_1| < \ell_2, \qquad (8)\ (1-\delta)|B_1|^{\ell_0 - 3} < |X_2| < \ell_2,\\
&(9)\ \widetilde{X}_1 = \frac{A_1\Big(1 - \sqrt{1 - \big(1 - \frac{\delta_0^4}{4}\big)A_2}\Big)}{1 - \frac{\delta_0^2}{2}}, \quad \textit{where}\\
&(10)\ A_1 = Y^{-1} X_2^{\delta_0^2 + \delta_0^3} B_3^2 B_1^{-2\ell_0^2 + 1}, \quad (11)\ A_2 = Y^2 X_2^{-2(\delta_0^2 + \delta_0^3)} B_3^{-4} B_1^{2\ell_0^2 - 2}, \quad (12)\ Y = \frac{Z}{X_2}.
\end{aligned} \tag{4.49}$$

Proof. By (4.47), (4.48) (i), (ii), we obtain (4.49) (1). To prove (4.49) (2), for the sake of convenience to state our arguments, we regard $\boldsymbol{k}$ as a variable and take $\boldsymbol{k} \gg \ell$ [which means that other elements

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 December 2022 doi:10.20944/preprints202212.0171.v1

in (4.46) are regarded as fixed and we choose $\boldsymbol{k}$ to be sufficiently larger than ℓ, cf. Remark 4.13; in this sense, $1 \sim_{\boldsymbol{k}} \ell_0 \sim_{\boldsymbol{k}} \ell_1 \sim_{\boldsymbol{k}} \ell_2 \sim_{\boldsymbol{k}} \ell \prec_{\boldsymbol{k}} \boldsymbol{k} \prec_{\boldsymbol{k}} \mathcal{E}^{-1}$; here to avoid confusion, we use the subscript "${}_{\boldsymbol{k}}$" to indicate that $\boldsymbol{k}$ is regarded as a variable]. Then $|B_2| \sim_{\boldsymbol{k}} |B_2| \sim_{\boldsymbol{k}} 1$ and $|X_1| \preceq_{\boldsymbol{k}} 1 \sim_{\boldsymbol{k}} |X_2|$ by (4.48) (i), (iii), (4.49) (1). By (4.29), (4.47), we have $|x_1| = \boldsymbol{k}|X_1| \preceq_{\boldsymbol{k}} \boldsymbol{k}|X_2| = |x_2| \sim_{\boldsymbol{k}} \boldsymbol{k}$. Thus by (3.25), we must have

$$h_{p_1,p_2} \sim_{\boldsymbol{k}} \max\{|x_1|,|x_2|\} \sim_{\boldsymbol{k}} \boldsymbol{k}, \quad |y_2| \prec_{\boldsymbol{k}} h_{p_1,p_2}, \quad |x_2+y_2| \sim_{\boldsymbol{k}} |x_2| \sim_{\boldsymbol{k}} \boldsymbol{k}. \tag{4.50}$$

Write $x_2 = (x_2+y_2)(1+\mu_2)$ for some $\mu_2 \in \mathbb{C}$, then by (3.25),

$$|(x_2+y_2)\mu_2| = |x_2-(x_2+y_2)| = |y_2| < h_{p_1,p_2}^{\frac{m}{m+1}} \sim_{\boldsymbol{k}} |x_2+y_2|^{\frac{m}{m+1}}, \tag{4.51}$$

i.e., $\mu_2 \preceq_{\boldsymbol{k}} |x_2+y_2|^{-\frac{1}{m+1}} \sim_{\boldsymbol{k}} \boldsymbol{k}^{-\frac{1}{m+1}} \ll \delta^2$. Thus $|\mu_2| = O(\delta)^2$. Similarly, we can write $\bar{x}_2 = (\bar{x}_2+\bar{y}_2)(1+\bar{\mu}_2)$ with $|\bar{\mu}_2| = O(\delta)^2$ (cf. Lemma 4.11). Hence

$$X_2 = \frac{(\bar{x}_2^{-1}x_2)Z}{(\bar{x}_2+\bar{y}_2)^{-1}(x_2+y_2)} = \frac{(1+\mu_2)Z}{1+\bar{\mu}_2} = Z\Big(1+O(\delta)^2\Big),$$

which with (4.48) (iii) gives (4.49) (2). By (4.29), (4.47), and (4.48) (iii), (4.49) (4), we see that $(|x_2|+|x_2+y_2|)\mathcal{E}^3 < \mathcal{E}^2$, which with (4.48) (iv) implies (4.49) (3). By (4.48) (i)–(iii), (4.49) (2), we have

$$a^{0+O(\delta)^1} = 1+O(\delta)^1, \quad a\Big(1+O(\delta)^1\Big) = a+O(\delta)^1 \ \text{ for all } \ a \in A_0 \ \text{ or } \ a^{-1} \in A_0, \ \text{ where}$$
$$A_0 := \Big\{\widetilde{X}_1, X_2, Z, B_1, B_2, B_3, \Big(\frac{1}{2}(1+\frac{\delta_0^2}{2})+\frac{1}{2}(1-\frac{\delta_0^2}{2})\widetilde{X}_1^2 B_1^{2\ell_0^2}\Big), 2-\widetilde{X}_1 X_2^{2+2\delta_0-\delta_0^4} B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}}\Big\}.$$

Thus we have (4.49) (4)–(6) by (4.48) (v)–(vii). To prove (4.49) (7), (8), this time we regard $\ell_2 = \delta_2^{-1}$ as a variable (then $1 \sim_{\ell_2} \ell_0 \sim_{\ell_2} \ell_1 \prec_{\ell_2} \ell_2 \prec_{\ell_2} \ell \prec_{\ell_2} \boldsymbol{k} \prec_{\ell_2} \mathcal{E}^{-1}$). We have $|B_1| \sim_{\ell_2} |B_2| \sim_{\ell_2} 1$ by (4.48) (i). Then $|\widetilde{X}_1|, |X_2|, |B_3| \succeq_{\ell_2} 1$ by the first inequalities of (4.48) (ii), (iii) and (4.49) (3). Thus $|\widetilde{X}_1| \sim_{\ell_2} 1 \sim_{\ell_2,} |X_2|$ by (4.49) (6). In particular we have the last inequalities of (4.49) (7), (8).

Now assume $(1-\delta)|B_1|^{-\ell_0^2} \geq |\widetilde{X}_1|$. Then by (4.48) (ii), $|\widetilde{X}_1| = (1-\delta)|B_1|^{-\ell_0^2}$. By (4.49) (3), (6), and the fact that $|B_1| \geq 1$, we can obtain the following from (4.49) (4),

$$\begin{aligned}
1 &\leq \Big|B_1^{-1}\widetilde{X}_1^{-1+\delta_0^2+\delta_0^3}B_3^{-2+\delta_0^2+\delta_0^3}\Big|\Big(\frac{1}{2}\Big(1+\frac{\delta_0^2}{2}\Big)+\frac{1}{2}\Big(1-\frac{\delta_0^2}{2}\Big)|\widetilde{X}_1 B_1^{\ell_0^2}|^2\Big)+O(\delta)^2 \\
&\leq (1-\delta)^{-1+\delta_0^2+\delta_0^3}|B_1|^{-1-\ell_0^2(-1+\delta_0^2+\delta_0^3)+(2-\delta_0^2-\delta_0^3)(-\frac{\ell_0^2}{2}+\frac{5}{8}+\frac{3\delta_0}{8})}\Big(1+\Big(-1+\frac{\delta_0^2}{2}\Big)\delta+O(\delta)^2\Big)+O(\delta)^2 \\
&\leq \Big(1+\Big(-\frac{\delta_0^2}{2}+O(\delta_0)^3\Big)\delta+O(\delta)^2\Big)|B_1|^{-\frac{1}{4}+O(\delta_0)^1}+O(\delta)^2 \\
&\leq 1+\Big(-\frac{\delta_0^2}{2}+O(\delta_0)^3\Big)\delta+O(\delta)^2 < 1,
\end{aligned} \tag{4.52}$$

a contradiction. This proves (4.49) (7). Next assume $(1-\delta)|B_1|^{\ell_0-3} \ge |X_2|$. Then $|X_2| = (1-\delta)|B_1|^{\ell_0-3}$ by (4.48) (iii). By (4.48) (i), (4.49) (3), (5), (6), we obtain

$$\begin{aligned}
2 &\le |B_2|^{-1} + |\widetilde{X}_1 X_2^{2+2\delta_0-\delta_0^4} B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}}| + O(\delta)^2 \le |B_1|^{-\frac{5\delta_0}{8}+3\delta_0^2} + |B_3^{-1} X_2^{1+2\delta_0-\delta_0^4} B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}}| + O(\delta)^2 \\
&\le 1 + (1-\delta)^{1+2\delta_0-\delta_0^4} |B_1|^{-\frac{\ell_0^2}{2}+\frac{5}{8}+\frac{3\delta_0}{8}+(1+2\delta_0-\delta_0^4)(\ell_0-3)+\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}} + O(\delta)^2 \\
&\le 1 + \Big(1 + \big(-1 + O(\delta_0)^1\big)\delta + O(\delta)^2\Big) |B_1|^{-\frac{45}{8}\delta_0 + O(\delta_0)^2} + O(\delta)^2 \\
&\le 2 + \Big(-1 + O(\delta_0)^1\Big)\delta + O(\delta)^2 < 2,
\end{aligned} \tag{4.53}$$

which is again a contradiction. This proves (4.49) (8). Finally, if we regard (4.48) (v) as an equation on $\widetilde{X}_1$, then there are two solutions for $\widetilde{X}_1$, one is stated as in (4.49) (9). We prove as follows that the other solution does not satisfy our requirement: note that locally there is only one choice of $\widetilde{X}_1$, thus globally there is only one choice of $\widetilde{X}_1$ since V_0 is connected; in Lemma 4.17, we will show that we can choose $(q_1, q_2) \in V_0$ such that (4.49) (9) holds; thus it holds globally. □

Lemma 4.16. *Proposition* 4.1 (ii) *holds.*

Proof. Now let $(p_1, p_2) \in V_1$. First assume in (4.48) (i), the equality occurs in the first inequality, or two equalities simultaneously occur in the second and third inequalities. Then $|B_1| = |B_2| = 1$ and $|B_3| > 1$ by (4.49) (3). We have $|\widetilde{X}_1| \ge 1 + O(\delta)^1$, $|X_2| \ge 1 + O(\delta)^1$ by (4.49) (7), (8). Thus by (4.49) (3), (6), we obtain

$$|a| = 1 + O(\delta)^1 \quad \text{for} \quad a \in A_1 := \big\{\widetilde{X}_1, X_2, \widetilde{Z} := Z X_2^{-(1+\delta_0^2+\delta_0^3)}, B_3\big\}. \tag{4.54}$$

By (4.48) (vi), (vii), we obtain [where (4.55) (ii) is obtained from (4.55) (iii)],

$$\text{(i) } |\widetilde{X}_1| = |B_3^{-1} X_2^{\ell_0^2} Z^{-\ell_0^2-1}| < |X_2^{\ell_0^2} Z^{-\ell_0^2-1}|, \qquad \text{(ii) } |Z| < |X_2|^{1+\frac{\delta_0^2}{2}+O(\delta_0)^3},$$

$$\text{(iii) } 1 = 2 - |B_2|^{-1} \le |2 - B_2^{-1}| = |\widetilde{X}_1 X_2^{\ell_0^2+2+2\delta_0-\delta_0^4} Z^{-\ell_0^2} B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac{3}{8}}| < |X_2^{2\ell_0^2+2+2\delta_0-\delta_0^4} Z^{-2\ell_0^2-1}|. \tag{4.55}$$

By (4.48) (v), we obtain (for convenience, we denote $\boldsymbol{x}_1 = |\widetilde{X}_1|$, $\boldsymbol{x}_2 = |X_2|$, $\boldsymbol{z} = |\widetilde{Z}|$, $\boldsymbol{b} = |B_3|$; since $\boldsymbol{b} > 1$ we have the strict inequality),

$$1 \le \boldsymbol{z}\boldsymbol{x}_1^{-1}\boldsymbol{b}^{-2}\Big(\frac{1}{2}\big(1+\frac{\delta_0^2}{2}\big) + \frac{1}{2}\big(1-\frac{\delta_0^2}{2}\big)\boldsymbol{x}_1^2\Big) < \beta_1 := \boldsymbol{z}\boldsymbol{x}_1^{-1}\Big(\frac{1}{2}\big(1+\frac{\delta_0^2}{2}\big) + \frac{1}{2}\big(1-\frac{\delta_0^2}{2}\big)\boldsymbol{x}_1^2\Big). \tag{4.56}$$

We claim

$$\text{(i) } \boldsymbol{x}_1 < \boldsymbol{z}^{2\ell_0^2+\delta_0^5} \ \text{ if } \ \boldsymbol{z} \ge 1, \quad \text{or} \quad \text{(ii) } \boldsymbol{x}_1 < \boldsymbol{z}^{2\ell_0^2-\delta_0^5} \ \text{ if } \ \boldsymbol{z} < 1. \tag{4.57}$$

Say $\boldsymbol{z} \ge 1$ and $\boldsymbol{x}_1 \ge \boldsymbol{z}^{2\ell_0+\delta_0^5}$. Noting that β_1 is a strictly decreasing function on $\boldsymbol{x}_1$ when other variables are fixed and when all variables satisfy (4.54) (since $\frac{\partial\beta_1}{\partial\boldsymbol{x}_1}|_{(\boldsymbol{x}_1,\boldsymbol{z})=(1,1)} = -\frac{\delta_0^2}{2} < 0$ and $0 < \delta \ll \delta_0$), we obtain from (4.56),

$$1 < \beta_1 \le \beta_1\Big|_{\boldsymbol{x}_1=\boldsymbol{z}^{2\ell_0^2+\delta_0^5}} = \beta_2 := \boldsymbol{z}^{-2\ell_0^2+1-\delta_0^5}\Big(\frac{1}{2}\big(1+\frac{\delta_0^2}{2}\big) + \frac{1}{2}\big(1-\frac{\delta_0^2}{2}\big)\boldsymbol{z}^{4\ell_0^2+2\delta_0^5}\Big). \tag{4.58}$$

Noting that β_2 is a strictly decreasing on $\boldsymbol{z}$ (as $\frac{d\beta_2}{dz}|_{\boldsymbol{z}=1} = -\frac{\delta_0^7}{2} < 0$), we obtain that $\boldsymbol{z} < 1$, a contradiction with the assumption. This proves (4.57) (i). Similarly, we have (4.57) (ii) if $\boldsymbol{z} < 1$.

This shows that in general we have $\boldsymbol{x}_1 < \boldsymbol{z}^{2\ell_0^2+O(\delta_0)^5}$, i.e., $|\widetilde X_1| < |Z^{2\ell_0^2+O(\delta_0)^5}X_2^{-2\ell_0^2(1+\delta_0^2+\delta_0^3)+O(\delta_0)^5}|$. Using this in the first inequality of (4.55) (iii) gives that $1 < |Z^{\ell_0^2+O(\delta_0)^5}X_2^{-\ell_0^2-\delta_0^4+O(\delta_0)^5}|$. Thus $|X_2| < |Z|^{1+O(\delta_0)^5}$. Then (4.55) (iii) gives that $1 < |Z|^{1+O(\delta_0)^1}$, i.e., $k := |Z| > 1$ and $k_2 := |X_2| < |Z|^{\frac{\ell_0^2+O(\delta_0)^5}{\ell_0^2+\delta_0^4+O(\delta_0)^5}} < |Z| = k$. Further $k_1 := |X_1| < |\widetilde X_1|^\delta < |X_2^{\ell_0^2}Z^{-\ell_0^2-1}|^\delta < |Z^{-1}|^\delta < 1$ by (4.49) (1), (4.55) (i). By (4.29), (4.47), we see that $|x_2| = k_2\boldsymbol{k} < k\boldsymbol{k}$, $|x_1| = k_1\boldsymbol{k} < \boldsymbol{k} < k\boldsymbol{k}$, $|x_2+y_2| = k\gamma_{\boldsymbol{k},\boldsymbol{k}}$. We obtain [where the first two inequalities follow from Lemmas 4.4 and 4.6, while the second from definition (4.6)],

$$\gamma_{k\boldsymbol{k},k\boldsymbol{k}} > \gamma_{|x_1|,k\boldsymbol{k}} \ge \gamma_{|x_1|,|x_2|} \ge |x_2+y_2| = k\gamma_{\boldsymbol{k},\boldsymbol{k}}, \tag{4.59}$$

which is a contradiction with Lemma 4.8.

Next assume in (4.48) (i), the equality occurs in the last inequality, i.e., $|B_1| = \ell_1$. By the second and third inequalities of (4.48) (i), we have $|B_2| = \ell_1^{\frac{5\delta_0}{8}+O(\delta_0)^2} \succ_{\ell_1} 1$, which with (4.49) (5) implies

$$|\widetilde X_1 X_2^{2+O(\delta_0)^1}B_1^{\frac{\ell_0^2}{2}(1+O(\delta_0)^1)}| = |\widetilde X_1 X_2^{2+2\delta_0-\delta_0^4}B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac38}| \sim_{\ell_1} 1. \tag{4.60}$$

The above together with (4.49) (3), (6) shows

$$\text{(i) } |X_2|^{1+2\delta_0-\delta_0^4} = |\widetilde X_1 X_2^{2+2\delta_0-\delta_0^4}B_1^{\frac{\ell_0^2}{2}-\ell_0+\frac38}|\cdot|\widetilde X_1X_2|^{-1}\cdot|B_1|^{-(\frac{\ell_0^2}{2}-\ell_0+\frac38)} \sim_{\ell_1} |B_3B_1^{-(\frac{\ell_0^2}{2}-\ell_0+\frac38)}|,$$

$$\begin{aligned}\text{(ii) } \alpha_1 &:= |B_1^{-1}\widetilde X_1X_2^{2-\delta_0^2-\delta_0^3}| = |B_1^{-1}(\tilde X_1X_2)|\cdot|X_2^{1-\delta_0^2-\delta_0^3}| \sim_{\ell_1} |B_1^{-1}B_3^{-1}|\cdot|B_3B_1^{-(\frac{\ell_0^2}{2}-\ell_0+\frac38)}|^{\frac{1-\delta_0^2-\delta_0^3}{1+2\delta_0-\delta_0^4}} \\ &= |B_3|^{-1+\frac{1-\delta_0^2-\delta_0^3}{1+2\delta_0-\delta_0^4}}\cdot|B_1|^{-1-(\frac{\ell_0^2}{2}-\ell_0+\frac38)\frac{1-\delta_0^2-\delta_0^3}{1+2\delta_0-\delta_0^4}} \prec_{\ell_1} 1.\end{aligned} \tag{4.61}$$

Equ. (4.61) (ii) with (4.49) (4) implies that we must have $\alpha_2 := |\widetilde X_1^2B_1^{2\ell_0^2}| \succ_{\ell_1} 1$ and $\alpha_1\alpha_2 \sim_{\ell_1} 1$. Thus

$$|\widetilde X_1^3X_2^{2+O(\delta_0)^1}B_1^{2\ell_0^2(1+O(\delta_0)^1)}| = |\widetilde X_1^3X_2^{2-\delta_0^2-\delta_0^3}B_1^{2\ell_0^2-1}| = \alpha_1\alpha_2 \sim_{\ell_1} 1. \tag{4.62}$$

Thus we can easily obtain from (4.60), (4.62),

$$\text{(i) } |\widetilde X_1| \sim_{\ell_1} |B_1|^{-\frac{3\ell_0^2}{4}(1+O(\delta_0)^1)}, \qquad \text{(ii) } |X_2| \sim_{\ell_1} |B_1|^{\frac{\ell_0^2}{8}(1+O(\delta_0)^1)}. \tag{4.63}$$

From this and (4.49) (2), (6), (11), (12), we have

$$|A_2| \sim_{\ell_1} |\widetilde X_1^4X_2^{4+O(\delta_0)^1}B_1^{2\ell_0^2(1+O(\delta_0)^1)}| \sim_{\ell_1} |B_1|^{-\frac{\ell_0^2}{2}(1+O(\delta_0)^1)} \prec_{\ell_1} 1. \tag{4.64}$$

Thus we can expand $\alpha_3 := 1-\sqrt{1-(1-\frac{\delta_0^4}{4})A_2}$ as a power series of A_2 to obtain

$$\alpha_3 = \frac12\Big(1-\frac{\delta_0^4}{4}\Big)A_2 + O(A_2)^2 \sim_{\ell_1} A_2. \tag{4.65}$$

Then by (4.49) (6), (9)–(12), (4.63), we have

$$|\widetilde X_1| \sim_{\ell_1} |A_1A_2| \sim_{\ell_1} |X_2^{0+O(\delta_0)^1}B_3^{-2}B_1^{-1}| \sim_{\ell_1} |X_2^{2+O(\delta_0)^1}\widetilde X_1^2B_1^{-1}| \sim_{\ell_1} |B_1|^{-\frac{5\ell_0^2}{4}(1+O(\delta_0)^1)}, \tag{4.66}$$

which is a contradiction with (4.63) (i).

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 9 December 2022 doi:10.20944/preprints202212.0171.v1

Finally by (4.49) (7), (8), we see that no equality can occur in any inequality of (4.48) (ii), (iii). This proves that Proposition 4.1 (ii) holds. □

Lemma 4.17. $V_0 \neq \emptyset$.

Proof. We choose suitable u, v such that (4.48) holds for (q_1, q_2). Note from (4.29), (4.47) that setting (p_1, p_2) to (q_1, q_2) implies that X_1, X_2, y_2 are set to $1 + s\mathcal{E}$, $1 + u\mathcal{E}$, $\bar{y}_2 + v\mathcal{E}$ respectively. We take $u, v \in \mathbb{C}$ such that,

$$\text{(i) } u = 0, \qquad \text{(ii) } \bar{v} := \frac{\bar{x}_2 u + v}{\bar{x}_2 + \bar{y}_2} = -\ell_0, \qquad \text{(iii) } s = s_1 + O(\mathcal{E})^1 \ \text{ with } \ s_1 = -\ell_0 b_{\boldsymbol{k}}, \tag{4.67}$$

where (4.67) (iii) is obtained from (4.34). Thus we obtain from (4.47) that $\widetilde{X}_1 = 1 + \tilde{s}\mathcal{E} + O(\mathcal{E})^2$ with $\tilde{s} = \ell(\alpha_0 + s_1) = -\ell_0^4$ and $Z = \frac{\bar{x}_2(1+u\mathcal{E})+\bar{y}_2+v\mathcal{E}}{\bar{x}_2+\bar{y}_2} = 1 + \bar{v}\mathcal{E}$. Then we see from (4.48) (vii) that $B_3 = 1 + c_3\mathcal{E} + O(\mathcal{E})^2$ with $c_3 = \ell_0^2 u - \tilde{s} - (\ell_0^2 + 1)\bar{v} = \ell_0^4 + \ell_0^3 + \ell_0$. We can uniquely choose B_1 of the form $B_1 = 1 + c_1\mathcal{E} + O(\mathcal{E})^2$ such that (4.48) (v) holds, where c_1 is determined from (4.48) (v) as follows,

$$c_1 = \bar{v} - (1 + \delta_0^2 + \delta_0^3)u - \tilde{s} - 2c_3 + \frac{1}{2}\Big(1 - \frac{\delta_0^2}{2}\Big)(2\tilde{s} + 2\ell_0^2 c_1), \tag{4.68}$$

and we solve that $c_1 = \frac{\ell_0^2(4+4\delta_0-\delta_0^2+6\delta_0^3)}{2-3\delta_0^2}$. We see from (4.48) (vi) that $B_2 = 1 + c_2\mathcal{E} + O(\mathcal{E})^2$ with

$$c_2 = \tilde{s} + (\ell_0^2 + 2 + 2\delta_0 - \delta_0^4)u - \ell_0^2\bar{v} + \Big(\frac{\ell_0^2}{2} - \ell_0 + \frac{3}{8}\Big)c_1 = \frac{5\ell_0}{4} - \frac{51}{16} + O(\delta_0)^1. \tag{4.69}$$

Then one can easily observe that both sides of (4.49) (9) are elements of the form $1 + O(\mathcal{E})^1$ [thus (4.49) (9) holds for (q_1, q_2)]. Now we obtain (note that $c_1 = 2\ell_0^2 + 2\ell_0 + \frac{5}{2} + 6\delta_0 + \frac{15\delta_0^2}{4} + O(\delta_0)^3$),

$$0 \ < \ \Big(\frac{5\delta_0}{8} - 3\delta_0^2\Big)c_1 = \frac{5\ell_0}{4} - \frac{19}{4} + O(\delta_0)^1 \ < \ c_2 \ < \ \frac{5\delta_0 c_1}{8} = \frac{5\ell_0}{4} + \frac{5}{4} + O(\delta_0)^1. \tag{4.70}$$

We see that all inequalities in (4.48) (i) are strict inequalities. Obviously, all inequalities in (4.48) (ii), (iii) are strict inequalities. Further, the coefficient of $\mathcal{E}^1$ in B_3' is

$$c_3' = c_3 + \Big(-\frac{\ell_0^2}{2} + \frac{5}{8} + \frac{3\delta_0}{8}\Big)c_1 = \frac{7}{16} + O(\delta_0)^1 > 0, \tag{4.71}$$

i.e., the inequality in (4.48) (iv) is a strict inequality. Hence $(q_1, q_2) \in V_0$, i.e., $V_0 \neq \emptyset$. □

Now Proposition 4.1 follows from Lemmas 4.16 and 4.17. □

Proposition 4.18. *Assume we have* (4.1)*. For any* $(p_1, p_2) \in V_0$*, there exists* $(q_1, q_2) = \big((\dot{x}_1, \dot{y}_1), (\dot{x}_2, \dot{y}_2)\big) \in V_0$ *such that*

$$\ell_{q_1,q_2} > \ell_{p_1,p_2}. \tag{4.72}$$

Proof. Let $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in V_0$, i.e., (4.2) holds. Note that (4.3) implies that $x_1, x_2, x_2 + y_2 \neq 0$. Similar to (4.10) and (4.30), we define

$$F_1 = F\big(x_1(1 + x), y_1 + y\big), \quad F_2 = F\big(x_2(1 + x), y_2 + y\big), \tag{4.73}$$

and define G_1, G_2 similarly. Define q_1, q_2 accordingly [similar to (4.13) and (4.33)],

$$q_1 := (\dot{x}_1, \dot{y}_1) = \big(x_1(1 + s\mathcal{E}), y_1 + t\mathcal{E}\big), \quad q_2 := (\dot{x}_2, \dot{y}_2) = \big(x_2(1 + u\mathcal{E}), y_2 + v\mathcal{E}\big). \tag{4.74}$$

As in (4.15) and (4.34), we have [here we write the linear part of F_2 as $ax+b\frac{x_2x+y}{x_2+y_2}$, cf. (4.31)],

$$s=s_1+O(\mathcal{E})^1,\quad s_1=au+b\frac{x_2u+v}{x_2+y_2}. \tag{4.75}$$

If no equality occurs in any inequality of (4.2) (a), then we only need to consider (4.72), which can be regarded as a special case below. Thus by Proposition 4.1 (ii), we may assume that the equality occurs in the second or third inequality of (4.2) (a). In any case, the two inequations we need to consider can be always stated as the following, for some $\tilde\alpha_i,\tilde\beta_i\in\mathbb{C}$, $\kappa'_1,\kappa'_2\in\mathbb{R}_{>0}$, where $\bar v=\frac{x_2u+v}{x_2+y_2}$ is regarded as a new variable,

$$\begin{aligned}
&\text{(i) } C_1:=\big|1+(\tilde\alpha_1u+\tilde\alpha_2\bar v)\mathcal{E}+(\tilde\beta_1u^2+\tilde\beta_2u\bar v+\tilde\beta_3\bar v^2)\mathcal{E}^2\big|-1+O(\mathcal{E})^3\ge 0,\\
&\text{(ii) } C_2:=\kappa'_1\big|1+(\tilde\alpha_3u+\tilde\alpha_4\bar v)\mathcal{E}+(\tilde\beta_4u^2+\tilde\beta_5u\bar v+\tilde\beta_6\bar v^2)\mathcal{E}^2\big|+|1+u\mathcal{E}|\\
&\qquad +\kappa'_2|1+\bar v\mathcal{E}|-(\kappa'_1+1+\kappa'_2)+O(\mathcal{E})^3>0.
\end{aligned} \tag{4.76}$$

First assume $\tilde\alpha_1\ne 0$. We can take $u=-\tilde\alpha_1^{-1}\tilde\alpha_2\bar v+(\beta_1\bar v^2+\beta_2w)\mathcal{E}$ for some $\beta_i,w\in\mathbb{C}$ with $w_{\rm re}>0$ so that C_1 has the form,

$$C_1=|1+w\mathcal{E}^2|-1+O(\mathcal{E})^3=w_{\rm re}\mathcal{E}^2+O(\mathcal{E})^3>0, \tag{4.77}$$

i.e., (4.76) (i) holds. Then (4.76) (ii) becomes the following, for some $\tilde\alpha_i\in\mathbb{C}$,

$$\begin{aligned}
C_2=\;&\kappa'_1\big|1+\tilde\alpha_5\bar v\mathcal{E}+(\tilde\alpha_6\bar v^2+\tilde\alpha_7w)\mathcal{E}^2\big|+\big|1+\tilde\alpha_8\bar v\mathcal{E}+(\tilde\alpha_9\bar v^2+\tilde\alpha_{10}w)\mathcal{E}^2\big|\\
&+\kappa'_2|1+\bar v\mathcal{E}|-(\kappa'_1+1+\kappa'_2)+O(\mathcal{E})^3>0.
\end{aligned} \tag{4.78}$$

By comparing the coefficients of $\mathcal{E}^1$, we immediately obtain that if $c_0:=\kappa'_1\tilde\alpha_5+\tilde\alpha_8+\kappa'_2\ne 0$, we can always choose $\bar v\in\mathbb{C}$ with $(c_0\bar v)_{\rm re}>0$ to satisfy (4.78). Thus assume $c_0=0$. Then C_2 in (4.78) becomes an $O(\mathcal{E})^2$ element. One can compute [observe that $\tilde\alpha_6$, $\tilde\alpha_7$, $\tilde\alpha_9$, $\tilde\alpha_{10}$ do not contribute to the left-hand side of (4.79)],

$$\begin{aligned}
\tilde\beta:=\;&\mathrm{C_{oeff}}(C_2,\bar v_{\rm re}^2\mathcal{E}^2)+\mathrm{C_{oeff}}(C_2,\bar v_{\rm im}^2\mathcal{E}^2)\\
=\;&\frac12\Big(\kappa'_1(\kappa'_1+1)\tilde\alpha_{5\,\rm im}^2+\kappa'_1(\kappa'_1+1)\Big(\tilde\alpha_{5\,\rm re}+\frac{\kappa'_2}{\kappa'_1+1}\Big)^2+\frac{\kappa'_2(\kappa'_1+1+\kappa'_2)}{\kappa'_1+1}\Big)>0.
\end{aligned} \tag{4.79}$$

Thus we can choose $\bar v$ with $\bar v_{\rm re}^2$ being sufficiently larger than $\bar v_{\rm im}^2$ if $\mathrm{C_{oeff}}(C'_2,\bar v_{\rm re}^2\mathcal{E}^2)>0$ or with $\bar v_{\rm im}^2$ being sufficiently larger than $\bar v_{\rm re}^2$ if $\mathrm{C_{oeff}}(C'_2,v_{\rm im}^2\mathcal{E}^2)>0$, to guarantee that (4.78) holds (when w is fixed). This proves Theorem 4.18 for the case that $\tilde\alpha_1\ne 0$.

Assume $\tilde\alpha_1=0$. By symmetry, we may also assume $\tilde\alpha_2=0$. Then we have one of the following,

$$\text{(i) } C_1=0,\qquad\text{or}\qquad\text{(ii) } C_1=|1+g(u,\bar v)\mathcal{E}^k|-1+O(\mathcal{E})^{k+1}, \tag{4.80}$$

for some nonzero homogeneous polynomial $g(u,\bar v)$ of $u,\bar v$ of degree $k\in\mathbb{Z}_{\ge2}$ [assume we have (4.80) (ii) as (4.76) (i) holds trivially for case (4.80) (i)]. In case $c_1:=\kappa'_1\tilde\alpha_3+1\ne0$ [see (4.76) (ii)], we can solve the problem as follows: First take $\bar v=\alpha u$ for some $\alpha\in\mathbb{C}$ with $g(u,\alpha u)\ne0$ [in this case $g(u,\alpha u)=b'u^k$ for some $b'\in\mathbb{C}_{\ne0}$] and with $|\alpha|$ being sufficiently small, then we choose $u\in\mathbb{C}$ with $(c_1u)_{\rm re}>0$ so that (4.76) (ii) holds [since $|\alpha|$ is sufficiently small (say we choose α with $0<|\alpha|\ll|c_1|$), our choice of u with $(c_1u)_{\rm re}>0$ can guarantee that (4.76) (ii) holds], and further $(b'u^k)_{\rm re}>0$ (this can be done since $k\ge2$). If $c_2:=\kappa'_1\tilde\alpha_4+\kappa'_2\ne0$, we can solve the problem symmetrically.

Assume $c_1 = c_2 = 0$. One can compute,

$$\mathrm{C}_{\mathrm{oeff}}(C_2, u_{\mathrm{re}}^2\mathcal{E}^2)+\mathrm{C}_{\mathrm{oeff}}(C_2, u_{\mathrm{im}}^2\mathcal{E}^2)=\frac{\kappa_1'+1}{2\kappa_1'}>0,$$

$$\mathrm{C}_{\mathrm{oeff}}(C_2, \bar v_{\mathrm{re}}^2\mathcal{E}^2)+\mathrm{C}_{\mathrm{oeff}}(C_2, \bar v_{\mathrm{im}}^2\mathcal{E}^2)=\frac{(\kappa_1'+\kappa_2')\kappa_2'}{2\kappa_1'}>0. \tag{4.81}$$

If $g(u,\bar v)$ does not depend on $\bar v$ [i.e., $g(u,\bar v) = b'u^k$ for some $b' \in \mathbb{C}_{\neq 0}$], then we can first choose $u \in \mathbb{C}$ to satisfy that $g(u,\bar v)_{\mathrm{re}} > 0$ then choose $\bar v \in \mathbb{C}$ with $\bar v_{\mathrm{re}}^2$ being sufficiently larger than $\bar v_{\mathrm{im}}^2$ if $\mathrm{C}_{\mathrm{oeff}}(C_2, \bar v_{\mathrm{re}}^2\mathcal{E}^2) > 0$ or with $\bar v_{\mathrm{im}}^2$ being sufficiently larger than $\bar v_{\mathrm{re}}^2$ if $\mathrm{C}_{\mathrm{oeff}}(C_2, \bar v_{\mathrm{im}}^2\mathcal{E}^2) > 0$, to guarantee that $\mathrm{C}_{\mathrm{oeff}}(C_2, \mathcal{E}^2) > 0$, i.e., (4.76) (ii) holds. Thus assume $g(u,\bar v)$ depends on $\bar v$. We set $\bar v = \alpha u$ with $\alpha, u \in \mathbb{C}$ being determined later such that $|\alpha|$ is sufficiently small. Then (4.80) (ii) and (4.76) (ii) become the following, for some $\tilde\alpha_{11} \in \mathbb{C}$, and some non-constant polynomial $g_0(\alpha)$ of α [where the term $-\kappa_1'^{-1}(1+\alpha)u\mathcal{E}$ in C_2 is obtained by the assumption that $c_1 = c_2 = 0$, i.e., $\tilde\alpha_3 = -\kappa_1'^{-1}$, $\tilde\alpha_4 = -\kappa_2'\kappa_1'^{-1}$],

(i) $C_1 = |1 + g_0(\alpha)u^k\mathcal{E}^k| - 1 + O(\mathcal{E})^{k+1} \ge 0,$

(ii) $C_2 = \kappa_1'\big|1-\kappa_1'^{-1}(1+\kappa_2'\alpha)u\mathcal{E}+\tilde\alpha_{11}u^2\mathcal{E}\big|+|1+u\mathcal{E}|+\kappa_2'|1+\alpha u\mathcal{E}|-(\kappa_1'+1+\kappa_2')+O(\mathcal{E})^3>0.$ (4.82)

One can compute,

$$\beta := \mathrm{C}_{\mathrm{oeff}}(C_2, u_{\mathrm{re}}^2\mathcal{E}^2) + \mathrm{C}_{\mathrm{oeff}}(C_2, u_{\mathrm{im}}^2\mathcal{E}^2)$$
$$= \frac{1}{2\kappa_1'}\Big((\kappa_2'\alpha_{\mathrm{re}}+1)^2 + \kappa_2'\alpha_{\mathrm{im}}^2(\kappa_1'+\kappa_2') + \alpha_{\mathrm{re}}^2\kappa_1'\kappa_2' + \kappa_1'\Big) > 0. \tag{4.83}$$

We can always choose $u \in \mathbb{C}$ with u_{re}^2 being sufficiently larger than u_{im}^2 if $\mathrm{C}_{\mathrm{oeff}}(C_2, u_{\mathrm{re}}^2\mathcal{E}^2) > 0$ or with u_{im}^2 being sufficiently larger than u_{re}^2 if $\mathrm{C}_{\mathrm{oeff}}(C_2, u_{\mathrm{im}}^2\mathcal{E}^2) > 0$, to guarantee that $\mathrm{C}_{\mathrm{oeff}}(C_2, \mathcal{E}^2) > 0$, i.e., (4.82) (ii) holds; and further $(g_0(\alpha)u^k)_{\mathrm{re}} > 0$ by some suitable choose of $\alpha \in \mathbb{C}$ [when $|\alpha|$ is sufficiently small, one can guarantee that the choice of α does not affect the inequality in (4.82) (ii) by noting that when $|\alpha|$ is sufficiently small, β defined in (4.83) is bigger than a positive number which is independent of α], i.e., (4.82) (i) holds. This proves Proposition 4.18. □

As in the proof of Lemma 4.3, V_0 is a compact subset of $\mathbb{C}^4$. Then Proposition 4.18 gives a contradiction, which shows that (4.1) does not hold, i.e., we have Theorem 1.3.

5. Proofs of Theorems 1.4 and 1.1

To proof Theorem 1.4, as in (4.9) and (4.30), take $(p_1, p_2) = \big((x_1, y_1), (x_2, y_2)\big) \in V$ and set (and define G_1, G_2 similarly)

$$F_1 = F(x_1 + \alpha_1 x, y_1 + y), \qquad F_2 = F(x_2 + \alpha_2 x, y_2 + y), \quad \text{where} \tag{5.1}$$

$$\alpha_1 = \begin{cases} 1 & \text{if } x_1 = \xi_1, \\ x_1 - \xi_1 & \text{else,} \end{cases} \qquad \alpha_2 = \begin{cases} 1 & \text{if } x_2 = \xi_2, \\ x_2 - \xi_2 & \text{else.} \end{cases} \tag{5.2}$$

Define q_1, q_2 accordingly [cf. (4.13) and (4.33)],

$$q_1 := (\dot x_1, \dot y_1) = (x_1 + \alpha_1 s\mathcal{E}, y_1 + t\mathcal{E}), \qquad q_2 := (\dot x_2, \dot y_2) = (x_2 + \alpha_2 u\mathcal{E}, y_2 + v\mathcal{E}). \tag{5.3}$$

 doi:10.20944/preprints202212.0171.v1

Then we have as in (4.15) and (4.75),

$$s = au + bv + O(\mathcal{E})^1 \text{ for some } a, b \in \mathbb{C} \text{ with } (a, b) \neq (0, 0). \tag{5.4}$$

Note that $(x_1, x_2) \neq (\xi_1, \xi_2)$. First suppose $x_1 \neq \xi_1$, $x_2 \neq \xi_2$ (then $\alpha_1 = x_1 - \xi_1$, $\alpha_2 = x_2 - \xi_2$). In this case, by (1.5), (1.6), we need to choose u, v such that,

$$C_1 := \beta_1 |1 + s\mathcal{E}|^2 + \beta_2 |1 + u\mathcal{E}|^2 - (\beta_1 + \beta_2) < 0, \tag{5.5}$$

where $\beta_1 = |x_1 - \xi_1|^2 > 0$, $\beta_2 = |x_2 - \xi_2|^2 > 0$. Using (5.4) in (5.5), we immediately see (by comparing the coefficients of $\mathcal{E}^1$) that if $b \neq 0$ or $a \neq -\beta_1\beta_2^{-1}$, then we have a solution for (5.5). Thus assume

$$b = 0, \qquad a = -\beta_2\beta_1^{-1} \in \mathbb{R}_{\neq 0}. \tag{5.6}$$

Similar to (4.12), we can write (note that $d \neq 0$),

$$\begin{aligned} &\text{(i) } F_1 = \sum_{i\ge 2} a_i y^i + x\Big(1 + \sum_{i\ge 1} \hat{a}_i y^i\Big) + \cdots, \ \ F_2 = \sum_{i\ge 2} b_i z^i + ax\Big(1 + \sum_{i\ge 1} \hat{b}_i z^i\Big) + \cdots, \\ &\text{(ii) } G_1 = y + \sum_{i\ge 2} c_i y^i + x \sum_{i\ge 1} c'_i y^i + \cdots, \ \ G_2 = z + \sum_{i\ge 2} d_i z^i + x \sum_{i\ge 1} d'_i z^i + \cdots, \ \ z = cx + dy, \end{aligned} \tag{5.7}$$

for some $a_i, \hat{a}_i, b_i, \hat{b}_i, c_i, c'_i, d_i, d'_i \in \mathbb{C}$, and where we regard F_2, G_2 as polynomials on x, z and we omit terms with x-degree ≥ 2.

Lemma 5.1. *There exists some $i \geq 2$ such that $(a_i, c_i) \neq (b_i, d_i)$.*

Proof. Otherwise by (5.1), (5.7), we obtain (and the like for G),

$$\begin{aligned} &F(x_1, y_1 + \boldsymbol{k}) = F_1\big|_{(x,y)=(0,\boldsymbol{k})} = F_2\big|_{(x,z)=(0,\boldsymbol{k})} = F(x_2, y_2 + d^{-1}\boldsymbol{k}), \quad \text{i.e.,} \\ &\sigma(\hat{p}_1) = \sigma(\hat{p}_2) \ \text{ with } \hat{p}_1 := (\hat{x}_1, \hat{y}_1) = (x_1, y_1 + \boldsymbol{k}), \ \hat{p}_2 := (\hat{x}_2, \hat{y}_2) = (x_2, y_2 + d^{-1}\boldsymbol{k}), \end{aligned} \tag{5.8}$$

for all $\boldsymbol{k} \gg 1$. Since $p_1 \neq p_2$, we have $\hat{p}_1 \neq \hat{p}_2$ when $\boldsymbol{k} \gg 1$, i.e., $(\hat{p}_1, \hat{p}_2) \in V$. Then $h_{\hat{p}_1,\hat{p}_2} \sim \boldsymbol{k}$ and $|\hat{y}_2| \sim \boldsymbol{k} \succ h_{\hat{p}_1,\hat{p}_2}^{\frac{m}{m+1}}$, a contradiction with (3.25). ☐

Lemma 5.2. *Let $i_0 \geq 2$ be the minimal number satisfying Lemma 5.1. We can assume*

$$\text{(i) } a_i = 0,\ i = 2, ..., 2i_0, \ \text{(ii) } b_i = 0,\ i = 2, ..., 2i_0, \ \text{(iii) } c_i = d_i,\ i = 2, ..., i_0 - 1, \ c_{i_0} \neq d_{i_0}. \tag{5.9}$$

Proof. By replacing F_j by $F_j + \sum_{i=2}^{2i_0} \beta_i G_j^i$ for some $\beta_i \in \mathbb{C}$ and $j = 1, 2$ (observe that i_0 is still the minimal number satisfying Lemma 5.1 after the replacement by considering either $c_{i_0} \neq d_{i_0}$ or $c_{i_0} = d_{i_0}$, $a_{i_0} \neq b_{i_0}$), thanks to the term y in G_1, we can then suppose (5.9) (i) holds. Assume $b_k \neq 0$ for some $k \leq 2i_0$. Take minimal such $k \geq 2$. Setting [noting from (5.7) that this amounts to setting $x = u\mathcal{E} = \check{u}\mathcal{E}^k$, $z = w\mathcal{E}$ in F_2, G_2, and setting $x = s\mathcal{E}$, $y = t\mathcal{E}$ in F_1, G_1, and letting $F_1 = F_2$, $G_1 = G_2$ to solve s, t],

$$u = \check{u}\mathcal{E}^{k-1}, \qquad v = d^{-1}(w - cu), \tag{5.10}$$

and regarding $\check{u}, w$ as new variables, by (5.7) (i), we have

$$F_2|_{(x,z)=(\check{u}\mathcal{E}^k, w\mathcal{E})} = (b_k w^k + a\check{u})\mathcal{E}^k + O(\mathcal{E})^{k+1} = F_1(s\varepsilon, t\varepsilon) = O(\mathcal{E})^{k+1} + s\mathcal{E}\Big(1 + O(t\mathcal{E})^1\Big) + O(s\mathcal{E})^2. \tag{5.11}$$

This shows that we have $s\mathcal{E} = O(\mathcal{E})^k$ and the right-hand side of (5.11) becomes $s\mathcal{E} + O(\mathcal{E})^{k+1}$. Hence,

$$s = (b_k w^k + a\check{u})\mathcal{E}^{k-1} + O(\mathcal{E})^k. \tag{5.12}$$

Using this, (5.6) and the first equation of (5.10) in (5.5), we obtain by choosing $\check{u} = 0$ and $w \in \mathbb{C}$ with $(b_k w^k)_{\rm re} < 0$,

$$\begin{aligned} C_0 &:= \beta_1\Big|1 + (-\beta_1^{-1}\beta_2\check{u} + b_k w^k)\mathcal{E}^k + O(\mathcal{E})^{k+1}\Big|^2 + \beta_2|1 + \check{u}\mathcal{E}^k|^2 - (\beta_1 + \beta_2) \\ &= 2\beta_1(b_k w^k)_{\rm re}\mathcal{E}^k + O(\mathcal{E})^{k+1} < 0. \end{aligned}$$

This proves Theorem 1.4. Therefore, we can assume (5.9) (ii) holds. Then we have (5.9) (iii) by Lemma 5.1. □

Lemma 5.3. *We have $\hat{a}_i = \hat{b}_i$ for $1 \le i \le i_0 - 2$ and $\kappa_0 := \hat{b}_{i_0-1} - \hat{a}_{i_0-1} = i_0(c_{i_0} - d_{i_0}) \neq 0$.*

Proof. For $1 \le i \le i_0 - 1$, by (5.7) we have,

$$\begin{aligned} &\hat{a}_i + \sum_{1\le j<i}(j+1)\hat{a}_{j-i}c_{j+1} + (i+1)c_{i+1} = \mathrm{C_{oeff}}\big(J(F_1, G_1), x^0y^i\big) = 0, \\ &\hat{b}_i + \sum_{1\le j<i}(j+1)\hat{b}_{j-i}d_{j+1} + (i+1)d_{i+1} = \frac{\mathrm{C_{oeff}}\big(J(F_2, G_2), x^0y^i\big)}{ad} = 0. \end{aligned} \tag{5.13}$$

Induction on i for $1 \le i \le i_0$, we obtain the lemma by (5.9) (iii). □

Now we set,

$$u = u_1\boldsymbol{i}\mathcal{E}^{i_0-1}, \qquad v = d^{-1}(w - cu), \tag{5.14}$$

for $u_1 \in \mathbb{R}_{\neq 0}$. As in (5.11), one can see that $s\mathcal{E} = O(\mathcal{E})^{i_0}$. Thus by (5.7) (ii), we have

$$G_1(s\mathcal{E}, t\mathcal{E}) = t\mathcal{E} + \sum_{i=2}^{i_0}c_i(t\mathcal{E})^i + O(\mathcal{E})^{i_0+1} = G_2|_{(x,z)=(u_1\boldsymbol{i}\mathcal{E}^{i_0}, w\mathcal{E})} = w\mathcal{E} + \sum_{i=2}^{i_0}d_i(w\mathcal{E})^i + O(\mathcal{E})^{i_0+1}. \tag{5.15}$$

Hence $t\mathcal{E} = w\mathcal{E} + O(\mathcal{E})^{i_0}$. Using this and (5.7) (i), we obtain,

$$\begin{aligned} F_1(s\mathcal{E}, t\mathcal{E}) &= s\mathcal{E}\Big(1 + \sum_{i=1}^{i_0-1}\hat{a}_i(t\mathcal{E})^i\Big) + O(\mathcal{E})^{2i_0} \\ &= F_2|_{(x,z)=(u_1\boldsymbol{i}\mathcal{E}^{i_0}, w\mathcal{E})} = au_1\boldsymbol{i}\mathcal{E}^{i_0}\Big(1 + \sum_{i=1}^{i_0-1}\hat{b}_i(w\mathcal{E})^i\Big) + O(\mathcal{E})^{2i_0}, \end{aligned} \tag{5.16}$$

which with Lemma 5.3 gives that $s\mathcal{E} = au_1\boldsymbol{i}\mathcal{E}^{i_0}(1 + \kappa_1 w^{i_0-1}\mathcal{E}^{i_0-1}) + O(\mathcal{E})^{2i_0}$. Thus C_0 defined in (5.5) becomes,

$$\begin{aligned} C_0 &= \beta_1\Big|1 - \beta_2\beta_1^{-1}u_1\boldsymbol{i}\mathcal{E}^{i_0}(1 + \kappa_1 w^{i_0-1}\mathcal{E}^{i_0-1}) + O(\mathcal{E})^{2i_0}\Big|^2 + \beta_2|1 + u_1\boldsymbol{i}\mathcal{E}^{i_0}|^2 - \beta_1 - \beta_2 \\ &= 2\beta_2 u_1(\kappa_1 w^{i_0-1})_{\rm im}\mathcal{E}^{2i_0-1} + O(\mathcal{E})^{2i_0}, \end{aligned} \tag{5.17}$$

which is negative if we choose $u_1 = 1$ and $w \in \mathbb{C}$ with $(\kappa_1 w^{i_0-1})_{\rm im} < 0$. This proves Theorem 1.4 in case $\beta_1 > 0$, $\beta_2 > 0$.

Now if $x_1 = \xi_1$ (then $x_2 \neq \xi_2$), then the first term of C_0 becomes $|s\mathcal{E}|^2 = O(\mathcal{E})^2$ and we can easily choose any u with $u_{\rm re} < 0$ to satisfy that $C_0 < 0$. Similarly, if $x_2 = \xi_2$ (then $x_1 \neq \xi_1$), then

 doi:10.20944/preprints202212.0171.v1

the second term of C_0 becomes $|u\mathcal{E}|^2 = O(\mathcal{E})^2$ and we can easily choose u with $(au)_{\rm re} < 0$ (in case $a \neq 0$) or v with $(bv)_{\rm re} < 0$ (in case $b \neq 0$) to satisfy that $C_0 < 0$. This proves Theorem 1.4.

Now we can prove Theorem 1.1 as follows. The second assertion of Theorem 1.1 follows from [8, 24]. To prove the first statement, assume conversely that there exists a Jacobian pair $(F, G) \in \mathbb{C}[x, y]^2$ satisfying (3.2) such that (1.1) holds. Then we have Theorems 1.2–1.4. Fix any $(\bar{p}_1, \bar{p}_2) \in V$ and define $V_1 = \{(p_1, p_2) \in V \mid \ell_{p_1,p_2} \le \ell_{\bar{p}_1,\bar{p}_2}\}$. By (1.5), as in the proof of Lemma 4.3, V_1 is a compact nonempty subset of $\mathbb{C}^4$. Then (1.6) gives a contradiction. This proves Theorem 1.1.

References

[1] A. Abdesselam, The Jacobian conjecture as a problem of perturbative quantum field theory, *Ann. Henri Poincaré* **4** (2003), 199–215.

[2] S. Abhyankar, Some thoughts on the Jacobian conjecture. I., *J. Algebra* **319** (2008), 493–548.

[3] H. Appelgate, H. Onishi, The Jacobian conjecture in two variables, *J. Pure Appl. Algebra* **37** (1985), 215–227.

[4] P.K. Adjamagbo, A. van den Essen, A proof of the equivalence of the Dixmier, Jacobian and Poisson conjectures, *Acta Math. Vietnam.* **32** (2007), 205–214.

[5] H. Bass, The Jacobian conjecture, *Algebra and its Applications* (New Delhi, 1981), 1–8, Lecture Notes in Pure and Appl. Math. **91**, Dekker, New York, 1984.

[6] H. Bass, E.H. Connell, D. Wright, The Jacobian conjecture: reduction of degree and formal expansion of the inverse, *Bull. Amer. Math. Soc.* **7** (1982), 287–330.

[7] A. Belov-Kanel, M. Kontsevich, The Jacobian Conjecture is stably equivalent to the Dixmier Conjecture, *Mosc. Math. J.* **7** (2007), 209–218.

[8] A. Bialynicki-Birula, M. Rosenlicht, Injective morphisms of real algebraic varieties, *Proc. Amer. Math. Soc.* **13** (1962), 200–203.

[9] Z. Charzyński, J. Chadzyński, P. Skibiński, A contribution to Keller's Jacobian conjecture. IV. *Bull. Soc. Sci. Lett. Łódź* **39** (1989), no. 11, 6 pp.

[10] M. Chamberland, G. Meisters, A mountain pass to the Jacobian conjecture, *Canad. Math. Bull.* **41** (1998), 442–451.

[11] J. Dixmier, Sur les algebres de Weyl, *Bull. Soc. Math. France* **96** (1968), 209–242.

[12] L.M. Drużkowski, The Jacobian conjecture: survey of some results, *Topics in Complex Analysis* (Warsaw, 1992), 163–171, Banach Center Publ., 31, Polish Acad. Sci., Warsaw, 1995.

[13] M. de Bondt, A. van den Essen, A reduction of the Jacobian conjecture to the symmetric case, *Proc. Amer. Math. Soc.* **133** (2005), 2201–2205.

[14] G.P. Egorychev, V.A. Stepanenko, The combinatorial identity on the Jacobian conjecture, *Acta Appl. Math.* **85** (2005), 111–120.

[15] E. Hamann, Algebraic observations on the Jacobian conjecture, *J. Algebra* **265** (2003), 539–561.

[16] Z. Jelonek, The Jacobian conjecture and the extensions of polynomial embeddings, *Math. Ann.* **294** (1992), 289–293.

[17] S. Kaliman, On the Jacobian conjecture, *Proc. Amer. Math. Soc.* **117** (1993), 45–51.

[18] T. Kambayashi, M. Miyanishi, On two recent views of the Jacobian conjecture, *Affine Algebraic Geometry*, 113–138, Contemp. Math. **369**, Amer. Math. Soc., Providence, RI, 2005.

[19] M. Kirezci, The Jacobian conjecture. I, II. *İstanbul Tek. Üniv. Bül.* **43** (1990), 421–436, 451–457.

[20] L. Makar-Limanov, U. Turusbekova, U. Umirbaev, Automorphisms and derivations of free Poisson algebras in two variables, *J. Algebra* **322** (2009), 3318–3330.

[21] C.B. Miranda-Neto, An ideal-theoretic approach to Keller maps, *Proc. Edinburgh Math. Soc.*, doi:10.1017/S0013091519000099

[22] T.T. Moh, On the Jacobian conjecture and the configurations of roots, *J. Reine Angew. Math.* **340** (1983), 140–212.

[23] M. Nagata, Some remarks on the two-dimensional Jacobian conjecture, *Chinese J. Math.* **17** (1989), 1–7.

[24] D.J. Newman, One-one polynomial maps, *Proc. Amer. Math. Soc.* **11** (1960), 867–870.

[25] A. Nowicki, On the Jacobian conjecture in two variables. *J. Pure Appl. Algebra* **50** (1988), 195–207.

[26] K. Rusek, A geometric approach to Keller's Jacobian conjecture, *Math. Ann.* **264** (1983), 315–320.

[27] S. Smale, Mathematical problems for the next century, *Math. Intelligencer* **20** (1998), 7-15.

[28] M.H. Shih, J.W. Wu, On a discrete version of the Jacobian conjecture of dynamical systems, *Nonlinear Anal.* **34** (1998), 779–789.

[29] V. Shpilrain, J.T. Yu, Polynomial retracts and the Jacobian conjecture, *Trans. Amer. Math. Soc.* **352** (2000), 477–484.

[30] Y. Su, Poisson algebras, Weyl algebras and Jacobi pairs, arXiv:1107.1115v9.

[31] Y. Su, X. Xu, Central simple Poisson algebras, *Science in China A* **47** (2004), 245–263.

[32] A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture,* Progress in Mathematics **190**, Birkhäuser Verlag, Basel, 2000.

[33] A. van den Essen, The sixtieth anniversary of the Jacobian conjecture: a new approach, *Polynomial automorphisms and related topics,* Ann. Polon. Math. **76** (2001), 77–87.

[34] D. Wright, The Jacobian conjecture: ideal membership questions and recent advances, *Affine Algebraic Geometry*, 261–276, Contemp. Math. **369**, Amer. Math. Soc., Providence, RI, 2005.

[35] J.T. Yu, Remarks on the Jacobian conjecture, *J. Algebra* **188** (1997), 90–96.