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Adaptive Parametric Identification of Systems with Multiple Nonlinearities

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Abstract. The system identification problem with multiple nonlinearities is relevant. Its decision depends on many factors. These include: feedbacks, the method of connecting nonlinear links, signal properties. They affect the identifiability of the system parameters. We introduced a condition for the excitation constancy for state variables, which considers the S-identifiability of the system. We propose system decomposition by measuring input to identify parameters. Each subsystem has an implicit identification representation. It guarantees obtaining estimates of subsystem parameters based on experimental data. The trajectories boundedness of adaptive system proved in parametric and coordinate spaces. Conditions guaranteeing exponential stability of the system obtained. Systems of self-oscillation generation and nonlinear correction of a nonlinear system consider. Conditions for the trajectories boundedness of the adaptive system obtained for these cases. The influence of nonlinearity and feedback on the system performance estimated.

Keywords: adaptation; identification; identifiability; stability; excitation constancy; Lyapunov vector function; self-oscillation

1. Introduction

Identification of systems with multiple nonlinearities studied in several papers. Parametric identification of nonlinearities [1] based on the use of frequency response inversion based on insufficient data. Sinusoidal tests used to detect nonlinearity. The type nonlinearity determines based on the analysis of the restoring force function. The curve fitting method is the basis for estimating nonlinearity parameters. The function description method [2] used for the parametric identification of a system with two nonlinear elements. An approach to parameters estimating of the transfer function of a second-order system suggests in [3]. The system contains dry and quadratic friction. Harmonic linearization of nonlinearities performed beforehand. Parameter estimates define as the solution of an equations system.

Identification of nonlinear systems, especially with multiple local nonlinearities [4] exhibiting disproportionate ratios of the degree of nonlinearity and present at a single or multiple spatial location, is a challenging inverse problem. Identification of such complex nonlinear systems cannot handle by the existing conventional restoring force or by describing function methods. Meta support vector machine approach [4] uses for the identification of a spring mass system with several degrees of freedom. Systems identification with multiple nonlinearities considers in [5]. The approach is based on the Hilbert–Huang transform. Application stages of this approach and the amplitude-frequency modulation procedure considered. In [6], a nonlinear vibration system study with a known structure. Optimization method and recursive algorithm used to nonlinearity form select from a given class.

The second- and third-order system identification with dry friction nonlinearities connected in parallel studies in [7]. The identification procedure base on application of the weighted least squares method. Identifiable conditions obtained and the parameters finding algorithm for a discrete nonlinear system [8] with feedback proposed. Nonlinear mechanical vibrations study in [9]. A gray

box model or a model based on semi-physical principles applied. It assumed that observed nonlinearities localized in the physical space.

The system with two nonlinearities considers in [10]. The structure one nonlinearity is a priori known. The second nonlinearity approximates by a time series. In [11], the method presented to design mathematical models for the dynamic system on a set of local nonlinear attachments. Devices installed in the same primary construction. The identification method uses on the measurements processing, and the synthesis of the model based on physical laws. The closed system identification considered in [12]. Frequency methods applied for estimation system. Convolution schemes use and the learning theory apply in [13]. Voltaire and Taylor series use to design models. State space models and learning algorithms based on variational auto-coders used for nonlinear systems (NS) parameters identification [14].

The review [15] contains the methods analysis applied for the nonlinear process's identification in the structure's dynamics. Disadvantages of approaches based on linearization, harmonic balance, and the surface restoring force method note. Identification algorithms based on the Hilbert frequency transformation proposed in [16]. The chaos theory used to identify bifurcation processes in [17]. The systems Identification with feedback considered in [18,19]. The proposed approach is based on the transfer function analysis of the system. The approaches analysis of feedback systems (FS) identification consider in [20]. The algorithm determining (AD) parameter estimates asymptotic bias in a nonlinear FS proposed in [21]. AD based on the correlation analysis between the system input and the noise. Very often, the FS identification replaces by the open system parameters assessment [22,23]. A complex structure is the identification problem of nonlinear FS. A recursive identification procedure proposes in [24]. The forward channel describes by an autoregressive model, and the reverse channel describes by a nonlinear model.

So, the review shows that most of the studies based on NS frequency identification methods. Estimating approaches to the nonlinearity structure proposing sometimes. Different procedures apply to linearization of the nonlinearity from a specified class. Approaches to NS linearization with feedback considered. The identification problem of systems with multiple nonlinearities is less studied. The complexity of such systems is the main difficulty for the identification. Identification approaches and methods of such systems base on localization of nonlinearity.

We propose an approach to adaptive identification of a system with multiple nonlinearities. This is a complex problem, and its solution boils down to: (i) the requirements formation on state variables for system parameters evaluation; (ii) conditions analysis guaranteeing the consideration system nonlinear properties (S-synchronizability); (iii) synthesis of an adaptive system. We give a solution to these problems.

2. Problem statement

Consider the system S_F

$$\dot{X}(t) = AX(t) + DF_1(X,t) + BU(t), \tag{1}$$

$$\mathcal{L}Y(t) = CX(t) + F_2(X,t), \tag{2}$$

where $X \in \mathbb{R}^m$ is the state vector, $A \in \mathbb{R}^{m \times m}$ is the state matrix, $D \in \mathbb{R}^{m \times q}$, $F_1(X,t): \mathbb{R}^m \to \mathbb{R}^q$ is nonlinear vector-function, $U \in \mathbb{R}^k$ is input (control) vector, $B \in \mathbb{R}^{m \times k}$, $Y \in \mathbb{R}^n$ is output vector, $C \in \mathbb{R}^{n \times m}$, $F_2(X,t): \mathbb{R}^m \to \mathbb{R}^n$ is perturbation vector (measurement errors), $\mathcal{L}$ is operator to obtain the vector Y. $\mathcal{L}$ can be a differential operator.

Information set for S_F

$$\mathbb{I}_o = \left\{ Y(t), U(t), t \in [t_0, t_N] \right\} \tag{3}$$

where $t_N < \infty$.

Assumption 1. Elements $\varphi_{1,i}(x_j) \in F_1(X,t)$, $\varphi_{2,i}(x_j) \in F_2(X,t)$ are smooth, one-valued functions and belong to the class

$$\mathcal{F}_\varphi = \left\{ \gamma_1 \xi^2 \le \varphi(\xi)\xi \le \gamma_2 \xi^2, \ \xi \ne 0, \ \varphi(0) = 0, \ \gamma_1 \ge 0, \gamma_2 < \infty \right\} \tag{4}$$

where $\xi \in \mathbb{R}$ is input of a nonlinear element.

The condition $\varphi_i^1(x_j) = \varphi_i^1\left(\varphi_k^1(x_j)\right)$ $i \ne k$ may be met in some cases. We apply the model

$$M_F : \begin{cases} \dot{\hat{X}}(t) = \hat{A}(t)\hat{X}(t) + \hat{D}(t)F_1(X,t) + \hat{B}(t)U(t), \\ \mathcal{L}\hat{Y}(t) = CX(t) + F_2(X,t) \end{cases} \tag{5}$$

for the parameters estimation of matrices A, D, B, where $\hat{A}(t)$, $\hat{D}(t)$, $\hat{B}(t)$ are matrices with tuning parameters.

Problem: construct the model (5) for the system (1) based on the analysis I_o and assumption 1, and find the tuning laws $\hat{A}(t)$, $\hat{D}(t)$ и $\hat{B}(t)$ to

$$\lim_{t\to\infty}\left\|\hat{Y}(t) - Y(t)\right\| \le \delta_y, \quad \delta_y \ge 0, \tag{6}$$

where $\|\cdot\|$ is the Euclidean norm.

We use the S-synchronizability concept of the system (1).

3. S-synchronizability of system S_F

S-synchronizability (SS) guarantees the structural identifiability for the system S_F. SS bases on the property's analysis of geometric frameworks S_{ey} [26]. S_{ey} reflects system S_F nonlinear properties, and properties S_{ey} depend on the parameters I_o. An unsuccessful choice of input can give to a so-called "insignificant" framework NS_{ey} [25]. Therefore, the input (control) choice is the important problem in the identification systems design.

Let the framework S_{ey} closed, and its area is not zero. Denote the S_{ey} height as $h(S_{ey})$, where height is the distance between two points on framework opposite sides. Then the framework S_{ey}, and consequently, the system is identifiable or h-identifiable [25] if:

(i) the input $U(t)$ bounded, piecewise, continuous and constantly excited (see below);

(ii) $\delta_S > 0$ exists such that $h(S_{ey}) \ge \delta_S$.

Structural identifiability conditions of the system. Let $\mathcal{D}_y = \mathrm{dom}(S_{ey})$ is the domain S_{ey}, $D_y = D_y(\mathcal{D}_y) = \max_t y(t) - \min_t y(t)$ is diameter $\mathcal{D}_y$; $U(t) \in \mathbb{U}$, where $\mathbb{U}$ is acceptable inputs set for the system. The set $\mathbb{U}$ contains representative inputs.

Definition 1. The input $U(t) \in \mathbb{U}_S \subseteq \mathbb{U}$ is S-*synchronizing* the system S_F if the domain $\mathcal{D}_y$ of the framework S_{ey} has a maximum diameter D_y on the set $\{Y(t), t \in \mathbb{J}\}$.

Remark 1. We understand the system (1), (2) synchronization as the choice of such the input $U_h(t) \in \mathbb{U}$, which reflects all the features of the framework S_{ey}.

Consider a reference framework S_{ey}^{ref}. S_{ey}^{ref} is the structure S_{ey} that reflects all the properties of the nonlinear function $\varphi(y)$. Let $D_y^{ref} = D_y(S_{ey}^{ref})$ be the diameter of the framework S_{ey}^{ref}. D_y^{ref} exists for a system (1), (2) with an S-synchronizing input. If $S_{ey} \cong S_{ey}^{ref}$, then

$$\left| D_y - D_y^{ref} \right| \le \varepsilon_y, \tag{7}$$

where $\varepsilon_y \geq 0$, $\cong$ is a sign of closeness. Subset $\mathbb{U}_S$ elements have the property

$$\left| D_y \left(S_{ey} \left(U(t) \big|_{U \in \mathbb{U}_S} \right) \right) - D_y^{ref} \right| \leq \varepsilon_y. \tag{8}$$

Consider the framework S_{ey}. Denote the framework S_{ey} center on the set $\mathbb{J}_y = \{ y(t) \}$ as c_S, and the area $\mathcal{D}_y$ center denote as c_{D_y}. Let $S_{ey} = \mathcal{F}_{S_{ey}}^l \cup \mathcal{F}_{S_{ey}}^r$. Determine secants $\gamma_S^\upsilon = a^\upsilon y + b^\upsilon$ for fragments $\mathcal{F}_{S_{ey}}^l, \mathcal{F}_{S_{ey}}^r$, where $\upsilon = r, l$; a^υ, b^υ, a^υ, and b^υ are parameters γ_S^υ.

Theorem 1 [25]. Let the set $\mathbb{U}_S$ of synchronizing inputs for the system S_F obtain and (i) $\varepsilon \geq 0$ exists such that $\left| c_S - c_{D_y} \right| \leq \varepsilon$; (ii) the condition $\left\| a^l \right| - \left| a^r \right\| \leq \delta_h$ is hold, where a^l, a^r are secant coefficients for fragments $\mathcal{F}_{S_{ey}}^l, \mathcal{F}_{S_{ey}}^r$, $\delta_h \geq 0$. Then the system S_F is h_{δ_h}-identifiable or structurally identifiable, and $\left(U_h(t) \in \mathbb{U}_S \right) \in S$.

4. On excitation constancy condition

The condition excitation constant (EC) is important in parametric identification problems. If the system is nonlinear, then the EC condition may not be sufficient. The system must be S-synchronized as we consider the nonlinear properties of the system. The EC condition has the form.

Definition 2. Vector $P \in \mathbb{R}^m$ constantly excited with level α or has property $\mathcal{PE}_\alpha$ if

$$\mathcal{PE}_\alpha: \ P(t)P^T(t) \geq \alpha I_l \tag{9}$$

for $\alpha > 0$ and $\forall t \geq t_0$ at some interval $T > 0$, where $I_l \in \mathbb{R}^{m \times m}$ is unity matrix.

The condition (9) has a different form

$$\frac{1}{T} \int_{t_0}^{t_0 + T} P(t)P^T(t) \geq \alpha I_l. \tag{10}$$

We use condition (9) next.

Assumption 2. The frequencies set $\Omega_P(\omega)$ correspond to $P(t)$ when $P(t)$ have expansion into the Fourier series.

Assumption 3. $\Omega_S(\omega)$ is the set of allowable input frequencies that ensure S-synchronizability of the system.

If $\Omega_P(\omega)$ guarantees the system S-synchronizability by the variable $P \in \mathbb{R}^m \Rightarrow P(t) \in S$, then a vector $P(t)$ is representative and applicable for the adaptive identification. Assuming the $P(t)$ boundedness, we write the condition EC for $B_P(t) = P(t)P^T(t)$ as

$$\mathcal{PE}_{\alpha,\bar{\alpha}}^S: \ \left(\alpha I_l \leq B_P(t) \leq \bar{\alpha} I_l \right) \& \left(\Omega_P(\omega) \subseteq \Omega_S(\omega) \right) \quad \forall t \geq t_0, \tag{11}$$

where $\bar{\alpha} > 0$ is some number.

5. Structural-parametric approach to system S_F identification

We propose a method for the system S_F identification based on the structural-parametric approach (SPA) [27]. The SPA realization depends on a priori information. The system S_F has a complex form. Therefore, the adaptive algorithms synthesis is based on a priori information about the system structure.

Let the composition of subsystems included in S_F be known. Knowing the output vector dimension the matrix A can decompose into n blocks (subsystem $S_{F,n} \subseteq S_F, \{ s_{F,j} \} \in S_{F,n}, j < n$). Perform the subsystems analysis and select subsystems that contain nonlinearities $S_{F,nel} \subseteq S_{F,n}$.

Apply SPA to each element $s_{F,nel,k} \in S_{F,nel}$. If the $\{s_{F,j}\} \in S_{F,n} \setminus S_{F,nel}$ subsystem does not contain nonlinearities, then apply the adaptive identification procedure.

Remark 2. The SPA based on the system S-synchronizability and the condition (9) fulfillment.

The SPA contains two procedures under uncertainty: (i) SF1 structural analysis, (ii) parametric (adaptive) identification. These stages describe in [27].

Remark 3. The system's structural identifiability (S-synchronizability) affects the connection of subsystems and the correlation of variables. The relationships analysis excludes influencing relationships and gives a solution to the system (nonlinearity) structural identifiability problem. The mutual influence diagram construction is possible only if the condition $\mathcal{PE}_{\alpha,\tilde{\alpha}}^{S}$ is true.

Remark 4. The system S-synchronizability evaluation is based on the framework S_{ey} construction and analysis. S_{ey} [25,26] reflects the nonlinearity structure of the corresponding subsystem $s_{F,nel,k}$.

Remark 5. Obtained estimates of nonlinearities structure in S_F are the basis for the adaptive parametric identification implementation.

Remark 6. We have information about nonlinearity. Therefore, the structural analysis stage for S_F was not performed.

Remark 7. Structural identification and structural identifiability based on a single approach. Structural identifiability guarantees the solution of the structural identification problem for nonlinear systems.

6. Adaptive identification of system S_F

Consider the subsystem $s_{F,i} \in S_F$, $\dim s_{F,i} = n_i$, $\mathcal{L}$ is a linear operator in (2). Let the information set $\mathbb{I}_{o,i} \subset \mathbb{I}_o$ be known for $s_{F,i}$. n_i-th order differential equation

$$W_{s_{F,i}}(\upsilon) y_{s_{F,i}} = \sum_{k=1}^{n_i} \sum_{j=1}^{n_i} \left(d_{1,s_{F,i}}^{k,h} f_{s_{F,i}}^{h,j} + b_{s_{F,i}}^{k,j} u_{s_{F,i}}^{j} \right) \tag{12}$$

corresponds to the equations system describing the subsystem $s_{F,i}$

$$\begin{cases} \dot{X}_{s_{F,i}} = A_{s_{F,i}} X_{s_{F,i}} + D_{1,s_{F,i}} F_{1,s_{F,i}}(X) + B_{s_{F,i}} U_{s_{F,i}}, \\ y_{s_{F,i}} = x_{s_{F,i}}^{1}, \end{cases} \tag{13}$$

where $X_{s_{F,i}} \in \mathbb{R}^{n_i}$ is state vector, $x_{s_{F,i}}^{1}$ is the first element $X_{s_{F,i}}$; $D_{1,s_{F,i}}$, $F_{1,s_{F,i}}$ and $B_{s_{F,i}}$ are matrices of corresponding dimensions; $d_{1,s_{F,i}}^{k,h} \in D_{1,s_{F,i}}, f_{s_{F,i}}^{j} \in F_{1,s_{F,i}}, b_{s_{F,i}}^{k,j} \in B_{s_{F,i}}, u_{s_{F,i}}^{j} \in U_{s_{F,i}}$; $\upsilon = d/dt$; $W_{s_{F,i}}(\upsilon)$ is degree n_i polynomial. $A_{s_{F,i}} \in \mathbb{R}^{n_i \times n_i}$ 2 is a Hurwitz matrix.

Divide the left and right parts of (12) into a degree $n_i - 1$ polynomial

$$H_{s_{F,i}}(\upsilon) = \prod_{k=1}^{n_i - 1} (\upsilon + \mu_k), \tag{14}$$

where $\mu_k > 0$ does not coincide with roots of the polynomial $W_{s_{F,i}}(\upsilon)$. Then (12)

$$\dot{y}_{s_{F,i}} = -\eta_{s_{F,i}} y_{s_{F,i}} + \sum_{k=1}^{n_i - 1} \sum_{j=1}^{n_i - 1} \left(\tilde{d}_{s_{F,i}}^{k,h} p_{f_{s_{F,i}}^{h,j}} + \tilde{b}_{s_{F,i}}^{k,j} p_{u_{s_{F,i}}^{j}} \right), \tag{15}$$

where $\eta_{s_{F,i}} > 0$, $\tilde{d}_{s_{F,i}}^{k,h}$ and $\tilde{b}_{s_{F,i}}^{k,j}$ depend on subsystem parameters $s_{F,i}$ and μ_k, variables $p_{z_{s_{F,i}}^{j}}$ $(z = f, u)$ satisfy the equation

$$\dot{p}_{z^j_{s_{F,i}}} = -\mu_j p_{z^j_{s_{F,i}}} + z^j_{s_{F,i}}. \tag{16}$$

Remark 8. The specificity and structure of the right part (15) define by the matrix $A_{s_{F,i}}$ and the polynomial $H_{s_{F,i}}(\upsilon)$.

The representation (15) allows us to estimate the parameters of $s_{F,i}$ on a given set of $\mathbb{I}_{o,i}$. Apply the adaptive model

$$\dot{\hat{y}}_{s_{F,i}} = -k_{s_{F,i}} \left(\hat{y}_{s_{F,i}} - y_{s_{F,i}} \right) + \hat{\kappa}_{s_{F,i}} y_{s_{F,i}} + \sum_{k=1}^{n_i-1} \sum_{j=1}^{n_i-1} \left(\hat{\tilde{d}}^{k,h}_{s_{F,i}} p_{f^{h,j}_{s_{F,i}}} + \hat{\tilde{b}}^{k,j}_{s_{F,i}} p_{u^j_{s_{F,i}}} \right), \tag{17}$$

where $k_{s_{F,i}} > 0$; $\hat{\kappa}_{s_{F,i}}$, $\hat{\tilde{d}}^{k,h}_{s_{F,i}}$, $\hat{\tilde{b}}^{k,j}_{s_{F,i}}$ are tuned parameters. The equation for identification error

$$\dot{e}_{s_{F,i}} = -k_{s_{F,i}} e_{s_{F,i}} + \Delta\kappa_{s_{F,i}} y_{s_{F,i}} + \sum_{k=1}^{n_i-1} \sum_{j=1}^{n_i-1} \left(\Delta\tilde{d}^{k,h}_{s_{F,i}} p_{f^{h,j}_{s_{F,i}}} + \Delta\tilde{b}^{k,j}_{s_{F,i}} p_{u^j_{s_{F,i}}} \right), \tag{18}$$

where $e_{s_{F,i}} = \hat{y}_{s_{F,i}} - y_{s_{F,i}}$, $\Delta\kappa_{s_{F,i}} = \hat{\kappa}_{s_{F,i}} - \eta_{s_{F,i}}$, $\Delta\tilde{d}^{k,h}_{s_{F,i}} = \hat{\tilde{d}}^{k,h}_{s_{F,i}} - \tilde{d}^{k,h}_{s_{F,i}}$, $\Delta\tilde{b}^{k,j}_{s_{F,i}} = \hat{\tilde{b}}^{k,j}_{s_{F,i}} - \tilde{b}^{k,j}_{s_{F,i}}$.

Lyapunov function (LF) for the identification algorithms synthesis has the form $V_{e,i}\left(e_{s_{F,i}}\right) = 0.5 e^2_{s_{F,i}}$. Then

$$\dot{V}_{e,i}\left(e_{s_{F,i}}\right) = e_{s_{F,i}} \left[-k_{s_{F,i}} e_{s_{F,i}} + \Delta\kappa_{s_{F,i}} y_{s_{F,i}} + \sum_{k=1}^{n_i-1} \sum_{j=1}^{n_i-1} \left(\Delta\tilde{d}^{k,h}_{s_{F,i}} p_{f^{h,j}_{s_{F,i}}} + \Delta\tilde{b}^{k,j}_{s_{F,i}} p_{u^j_{s_{F,i}}} \right) \right]. \tag{19}$$

If $p_{f^{h,j}_{s_{F,i}}}$, $p_{u^j_{s_{F,i}}}$ have the property $PE^S_{\alpha,\bar{\alpha}}$, then obtain from $\dot{V}_i\left(e_{s_{F,i}}\right) < 0$

$$\Delta\dot{\kappa}_{s_{F,i}} = -\gamma_{\kappa,s_{F,i}} e_{s_{F,i}} y_{s_{F,i}}, \tag{20}$$

$$\Delta\dot{\tilde{d}}^{k,h}_{1,s_{F,i}} = -\gamma_{k,h,s_{F,i}} e_{s_{F,i}} p_{f^{h,j}_{s_{F,i}}}, \tag{21}$$

$$\Delta\dot{\tilde{b}}^{k,j}_{s_{F,i}} p_{u^j_{s_{F,i}}} = -\gamma_{k,j,s_{F,i}} e_{s_{F,i}} p_{u^j_{s_{F,i}}}, \tag{22}$$

where $\gamma_{\kappa,s_{F,i}}$, $\gamma_{k,h,s_{F,i}}$, $\gamma_{k,j,s_{F,i}}$ are positive numbers ensuring algorithms convergence. Get algorithms to adjust model (17) parameters from (20)-(22)

$$\dot{\hat{\kappa}}_{s_{F,i}} = -\gamma_{\kappa,s_{F,i}} e_{s_{F,i}} y_{s_{F,i}}, \tag{20a}$$

$$\dot{\hat{\tilde{d}}}^{k,h}_{1,s_{F,i}} = -\gamma_{k,h,s_{F,i}} e_{s_{F,i}} p_{f^{h,j}_{s_{F,i}}}, \tag{21a}$$

$$\dot{\hat{\tilde{b}}}^{k,j}_{s_{F,i}} p_{u^j_{s_{F,i}}} = -\gamma_{k,j,s_{F,i}} e_{s_{F,i}} p_{u^j_{s_{F,i}}}, \tag{22a}$$

So, the adaptive system for identification of subsystem $s_{F,i}$ parameters describe by the equations (18), (20) – (22). Denote it as $AS_{F,i}$.

Consider LF

$$V_{\Delta,i} = 0.5\gamma_{\kappa,s_{F,i}}^{-1}\left(\Delta\kappa_{s_{F,i}}\right)^2 + 0.5\mathrm{Sp}\left(\Delta D_{1,s_{F,i}}\Gamma_{k,h,s_{F,i}}^{-1}\Delta D_{1,s_{F,i}}^T\right) + 0.5\mathrm{Sp}\left(\Delta B_{s_{F,i}}\Gamma_{k,j,s_{F,i}}^{-1}\Delta B_{s_{F,i}}^T\right), \tag{23}$$

where $\mathrm{Sp}(\cdot)$ is spur of matrix, $\Gamma_{k,h,s_{F,i}} = \mathrm{diag}(\gamma_{k,h,s_{F,i}})$, $\Gamma_{k,j,s_{F,i}}^{-1} = \mathrm{diag}\left(\gamma_{k,j,s_{F,i}}\right)$, $\Delta\tilde{d}_{s_{F,i}}^{k,h} \in \Delta D_{1,s_{F,i}}$ and $\Delta\tilde{b}_{s_{F,i}}^{k,j} \in \Delta B_{s_{F,i}}$.

Let $\Delta K_{s_{F,i}} \triangleq \left[\Delta D_{1,s_{F,i}}, \Delta B_{s_{F,i}}\right]$ and

$$V_{s_{F,i}}(t) = V_e(t) + V_{\Delta,i}(t). \tag{24}$$

Theorem 2. Let (i) functions $V_{e,i}(t)$, $V_{\Delta,i}(t)$ are positive definite and satisfy conditions

$$\inf_{|e|\to\infty} V_{e,i}(e_{s_{F,i}}) \to \infty, \quad \inf_{\left(\|\Delta K_{s_{F,i}}\|,\|\Delta\kappa_{s_{F,i}}\|\right)\to\infty} V_{\Delta,i}\left(\Delta\kappa_{s_{F,i}}, \Delta K_{s_{F,i}}\right) \to \infty;$$

(ii) $A_{s_{F,i}} \in \mathbb{R}^{n_i \times n_i}$ is Hurwitz matrix; (iii) variables $p_{f_{s_{F,i}}^{h,j}}$, $p_{u_{s_{F,i}}^j}$ have the property $\mathcal{PE}_{\alpha,\bar{\alpha}}^S$. The all trajectories of the system $AS_{F,i}$ bounded belong to the domain $G_t = \left\{\left(e_{s_{F,i}}, \Delta\kappa_{s_{F,i}}, \Delta K_{s_{F,i}}\right) : V_{s_{F,i}}(t) \leq V_{s_{F,i}}(t_0)\right\}$, and the estimate

$$2k_{s_{F,i}}\int_{t_0}^{t} V_{e,i}(\tau)\,d\tau \leq V_{s_{F,i}}(t_0) - V_{s_{F,i}}(t). \tag{25}$$

is valid.

The Theorem 2 proof gives in Appendix A.

Let

$$0.5\,\underline{g}\left\{\mathrm{Sp}\left(\Delta K_{s_{F,i}}(t)\Delta K_{s_{F,i}}^T(t)\right) + \left(\Delta\kappa_{s_{F,i}}\right)^2\right\} \leq V_{\Delta,i}(t) \leq 0.5\,\bar{g}\left\{\mathrm{Sp}\left(\Delta K_{s_{F,i}}(t)\Delta K_{s_{F,i}}^T(t)\right) + \left(\Delta\kappa_{s_{F,i}}\right)^2\right\}, \tag{26}$$

where $\Gamma_{s_{F,i}} = \Gamma_{k,h,s_{F,i}} \dot{+} \Gamma_{k,j,s_{F,i}}$, $\dot{+}$ is the sign of the matrices direct sum, $\beta_1\left(\Gamma_{s_{F,i}}\right)$, $\beta_l\left(\Gamma_{s_{F,i}}\right)$ are minimum and maximum matrix $\Gamma_{s_{F,i}}$ eigenvalues, $\underline{g} = \min\left(\beta_l^{-1}\left(\Gamma_{s_{F,i}}\right), \gamma_{\kappa,s_{F,i}}^{-1}\right)$ $\bar{g} = \max\left(\beta_1^{-1}\left(\Gamma_{s_{F,i}}\right), \gamma_{\kappa,s_{F,i}}^{-1}\right)$.

The exponential stability proof is based on ensuring the $\mathcal{M}^+$-property for functions $\dot{V}_{e,i}(t)$, $\dot{V}_{\Delta,i}(t)$ [29].

Definition 3 [29]. A non-positive quadratic form $W(Y,X)$ has the $\mathcal{M}^+$-property or $W(Y,X) \in \mathcal{M}^+$ if fair

$$W(Y,X) = -c_y\|Y\|^2 + c_x\|X\|^2, \tag{27}$$

for any $Y \in R^m$, $X \in R^n$ in some bounded domain

$$\Omega_D = \left\{Y \in R^m, X \in R^n : \|Y\|^2 + \|X\|^2 \leq \chi, \ \chi \geq 0\right\}, \tag{28}$$

where $\|Y\|$ is the Euclidean norm of the vector Y, $c_y > 0$, $c_x \geq 0$.

$\mathcal{M}^+$-property is the constructive completeness sign the quadratic form $W(Y,X)$. It reduces the $W(Y,X)$ properties analysis to the characteristic's evaluation of the M-matrix [30].

Lemma 1. $\dot{V}_{e,i}(t)$ has the $\mathcal{M}^+$ -property

$$\dot{V}_{e,i} \leq -k_{s_{F,i}} V_{e,i} + \frac{\bar{\alpha}\bar{\vartheta}}{2k_{s_{F,i}}} V_{\Delta,i} \tag{29}$$

if matrix $P^T_{f^{h,j}_{s_{F,i}},u^j_{s_{F,i}}}(t) P_{f^{h,j}_{s_{F,i}},u^j_{s_{F,i}}}(t) \in \mathcal{PE}^S_{\alpha_P,\bar{\alpha}_P}$ and $y_{s_{F,i}}(t) \in \mathcal{PE}^S_{\alpha_{y_i},\bar{\alpha}_{y_i}}$, where $\alpha_{P_i}, \bar{\alpha}_{P_i}, \alpha_{y_i}, \bar{\alpha}_{y_i}$ are positive numbers, $\bar{\alpha} = \max\left(\bar{\alpha}_{P_i}, \bar{\alpha}_{y_i}\right)$.

The Lemma 1 proof gives in Appendix B.

Lemma 2. If Lemma 1 conditions satisfy, then $\dot{V}_{\Delta,i}(t)$ has the $\mathcal{M}^+$ -property

$$\dot{V}_{\Delta,i} \leq -\frac{3}{8}\pi\alpha_i V_{\Delta,i} + \frac{8}{3}\pi V_{e,i}, \tag{30}$$

where $\pi \geq 0$.

The Lemma 1 proof gives in Appendix C.

Consider the Lyapunov vector function $V_{e_i,\Delta_i}(t) = \left[V_{e,i}(t) \, V_{\Delta,i}(t)\right]^T$. Let functions $s_\rho(t) \geq 0$ exist such that

$$V_\rho(t) \leq s_\rho(t) \quad \forall(t \geq t_0) \& \left(V_\rho(t_0) \leq s_\rho(t_0)\right), \quad \rho = e,i;\Delta,i. \tag{31}$$

Analysis of adaptive system properties (18), (20) – (22) based on the study of the inequalities system

$$\begin{bmatrix} \dot{V}_{e,i} \\ \dot{V}_{\Delta,i} \end{bmatrix} \leq \begin{bmatrix} -k_{s_{F,i}} & \dfrac{\bar{\alpha}\bar{\vartheta}}{2k_{s_{F,i}}} \\ \dfrac{8}{3}\pi & -\dfrac{3}{8}\pi\alpha_i \end{bmatrix} \begin{bmatrix} V_{e,i} \\ V_{\Delta,i} \end{bmatrix}, \tag{32}$$

Apply (31) and obtain a vector comparison system for (32)

$$\dot{S} = A_V S, \tag{33}$$

where $S = \left[s_\varepsilon \, s_K\right]^T$, $A_V \in R^{2\times2}$ is M-matrix

$$A_V = \begin{bmatrix} -k_{s_{F,i}} & \dfrac{\bar{\alpha}\bar{\vartheta}}{2k_{s_{F,i}}} \\ \dfrac{8}{3}\pi & -\dfrac{3}{8}\pi\alpha_i \end{bmatrix}, \quad S = \begin{bmatrix} s_e \\ s_N \end{bmatrix}.$$

The stability conditions for the M-matrix A_V have the form [30]

$$-m_1(A_V) > 0, \quad m_2(A_V) > 0,$$

where m_1, m_2 are diagonal minors of the matrix A_V.

These conditions have the form for A_V

$$k_{s_{F,i}} > 0, \quad k_{s_{F,i}} \geq \frac{4}{3}\sqrt{\frac{2\bar{\alpha}\bar{\vartheta}}{\alpha_i}}. \tag{34}$$

So, the following statement is true.

Theorem 3. Let (i) positive-definite Lyapunov functions

$$V_{e,i}(t) = 0.5 e_{s_{F,i}}(t), \quad V_{\Delta,i}(t) = 0.5 \gamma_{\kappa,s_{F,i}}^{-1} \left(\Delta \kappa_{s_{F,i}} \right)^2 + 0.5 \mathrm{Sp}\left(\Delta D_{1,s_{F,i}} \Gamma_{k,h,s_{F,i}}^{-1} \Delta D_{1,s_{F,i}}^T \right) + 0.5 \mathrm{Sp}\left(\Delta B_{s_{F,i}} \Gamma_{k,j,s_{F,i}}^{-1} \Delta B_{s_{F,i}}^T \right)$$

have the infinitesimal higher limit for $\left| e_{s_{F,i}}(t) \right| \to 0$, $\left| \Delta \kappa_{s_{F,i}} \right| \to 0, \left\| \Delta K_{s_{F,i}} \right\| \to 0$; (ii) matrix $P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}^T (t) P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}(t) \in \mathcal{PE}_{\alpha_P,\bar{\alpha}_P}^S$ and $y_{s_{F,i}}(t) \in \mathcal{PE}_{\alpha_{y_i},\bar{\alpha}_{y_i}}^S$ are piecewise continuous bounded and $W_{s_{F,i}}(t) \in \mathcal{PE}_{\alpha,\bar{\alpha}}^S$, $y_{s_{F,i}}(t) \in \mathcal{PE}_{\alpha_{y_i},\bar{\alpha}_{y_i}}^S$; (iii) the equality

$$e_{s_{F,i}} \sum_{k=1}^{n_i-1} \sum_{j=1}^{n_i-1} \left(\Delta \tilde{d}_{s_{F,i}}^{k,h} p_{f_{s_{F,i}}^{h,j}} + \Delta \tilde{b}_{s_{F,i}}^{k,j} p_{u_{s_{F,i}}^{j}} \right) = \pi \left\{ \mathrm{Sp}\left(\Delta K_{s_{F,i}}(t) P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}^T P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}} \Delta K_{s_{F,i}}^T(t) \right) + \left(\Delta \kappa_{s_{F,i}} y_{s_{F,i}} \right)^2 + e_{s_{F,i}}^2 \right\}$$

is fair in area $O_v(O)$, where $\pi > 0$, $O = \{0, 0^{n_i \times n_i}\} \subset \mathbb{R} \times \mathbb{R}^{n_i \times n_i} \times \mathbb{J}_{0,\infty}$, $0^{n_i \times n_i} \in \mathbb{R}^{n_i \times n_i}$ is null matrix, O_v is some neighborhood of the point O, $t \in [0,\infty] = \mathbb{J}_{0,\infty}$; (iv) Lemma 1, 2 conditions satisfied for $\dot{V}_{e,i}$ and $\dot{V}_{\Delta,i}$; (v) the estimate (26) fair for function $V_{\Delta,i}(t)$; (vi) the inequalities system (32) is valid for $\dot{V}_{e,i}$, $\dot{V}_{\Delta,i}$; (vii) the upper solution for $V_{e_i,\Delta_i}(t) = \left[V_{e,i}(t) V_{\Delta,i}(t) \right]^T$ satisfies equation (33) if inequality (31) is true for elements $V_{e,i}, V_{\Delta,i}$. Then the adaptive system (18), (20) – (22) is exponentially stable with the estimate

$$V_{e_i,\Delta_i}(t) \le e^{A_V(t-t_0)} S(t_0), \tag{35}$$

if conditions (34) held.

Theorem 3 shows if the information matrix $W_{s_{F,i}}(t) = P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}^T (t) P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}(t)$ is constantly excited, then the adaptive system (18), (20) – (22) guarantees consistent parameters estimates for system $s_{F,i}$. This statement is valid if (34) is fulfilled.

Remark 9. The EC condition ($\mathcal{PE}_{\alpha,\bar{\alpha}}^S$) is the base for the stability proof of the adaptive system. EC indirectly consider when applying the Lyapunov function. The approach proposed above ensures the $\mathcal{M}^+$-property for the Lyapunov function. It bases on consideration the condition $\mathcal{PE}_{\alpha,\bar{\alpha}}^S$. The Lyapunov function analysis considers features of adaptive algorithms when the $\mathcal{PE}_{\alpha,\bar{\alpha}}^S$ condition applies.

7. Nonlinear correction of nonlinear system

The system contains an amplifier with an electric motor and a relay control described by a function $f_1(u)$. The correcting device is a nonlinear velocity feedback with a parabolic characteristic $f_2(x_2)$

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -a_1 x_2 - a_2 x_3 + b f_1(u), \\ y = x_1, \end{cases} \tag{36}$$

$$f_1(u) = \begin{cases} c, & \text{if } u \ge d, \\ 0, & \text{if } -d \le u \le d, \\ -c, & \text{if } u < -d, \end{cases} \tag{37}$$

where $u = \omega - x_1 - f_2(x_2)$ is control, $f_2(x_2) = k_{os} x_2^2 sign(x_2)$, ω is input, $c > 0, d > 0$.

Equation (13) and polynomials in (13), (14) have the form

$$W(\upsilon)y = bf_1(u),\tag{38}$$

$$W(\upsilon) = \upsilon^3 + a_2\upsilon^2 + a_1\upsilon,\tag{39}$$

$$H(\upsilon) = (\upsilon + \mu_1)(\upsilon + \mu_2)\tag{40}$$

where μ_1, μ_2 are positive numbers that do not coincide with the roots of the equation $W(\upsilon) = 0$.

Adaptive model and algorithms for estimating system parameters (42), (43)

$$\dot{\hat{y}} = -k_y e - \hat{\eta} y + \hat{v}_1 p_{y,\mu_1} + \hat{v}_2 p_{y,\mu_2} + \hat{b}_1 p_{f_1,\mu_1} + \hat{b}_2 p_{f_1,\mu_2}\tag{44}$$

$$\begin{aligned}
\dot{\hat{\eta}} &= -\gamma_\eta ey,\\
\dot{\hat{v}}_1 &= -\gamma_{v_1} ep_{y,\mu_1},\\
\dot{\hat{v}}_2 &= -\gamma_{v_2} ep_{y,\mu_2},\\
\dot{\hat{b}}_1 &= -\gamma_{b_1} p_{f_1,\mu_1},\\
\dot{\hat{b}}_2 &= -\gamma_{b_2} p_{f_1,\mu_2},
\end{aligned}\tag{45}$$

where $e = \hat{y} - y$, $i = \eta, v_1, v_2, b_1, b_2$. Equation for error $e(t)$

$$\dot{e} = -ke - \Delta\eta y + \Delta v_1 p_{y,\mu_1} + \Delta v_2 p_{y,\mu_2} + \Delta b_1 p_{f_1,\mu_1} + \Delta b_2 p_{f_1,\mu_2},\tag{46}$$

where $k > 0$, $\Delta\sigma = \hat{\sigma} - \sigma$, $\sigma = \eta, v_1, v_2, b_2, b_2$.

Denote $\Delta K \triangleq [\Delta\eta, \Delta v_1, \Delta v_2, \Delta b_1, \Delta b_2]^T$. The vector tuning law follows from (35)

$$\Delta\dot{K} = -\Gamma_K eP_y,\tag{47}$$

where $\Gamma_K = \mathrm{diag}(\gamma_\eta, \gamma_{v_1}, \gamma_{v_2}, \gamma_{b_1}, \gamma_{b_2})$, $P_y = [y, p_{y,\mu_1}, p_{y,\mu_2}, p_{f_1,\mu_1}, p_{f_1,\mu_2}]^T$.

The system (44), and (46) trajectories boundedness follows from Theorem 2. The simulation results show in Figures 1-4. Present the model (44) parameter tuning in Figures 1 and 2. Figure 3 reflects the change in the estimation error. The output form determines the error change. Figure 3 shows also the change in $e(t)$ depending on the dynamics of the system output (36). The model adequacy in the output space demonstrates in Figure 4.
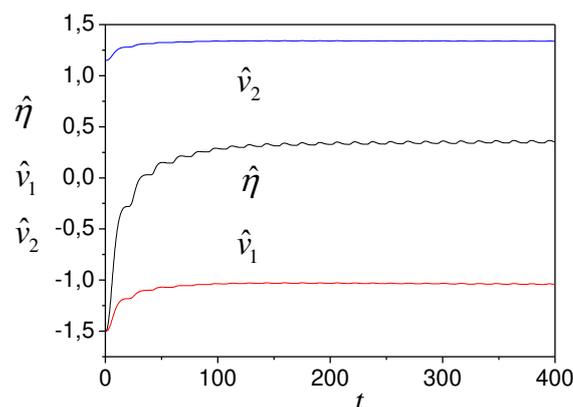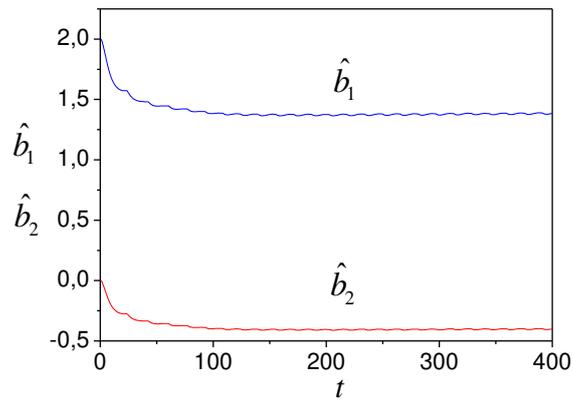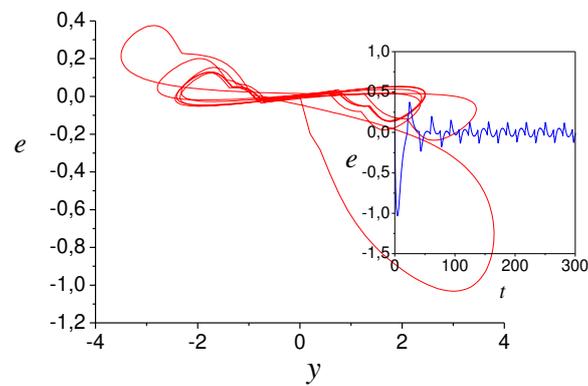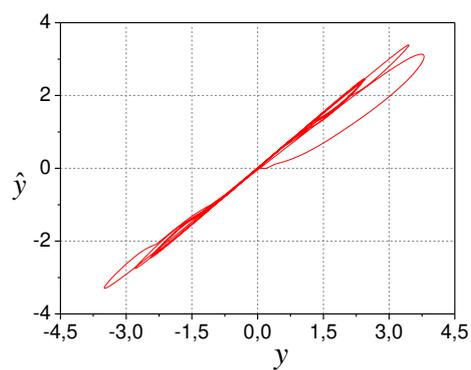
Figure 1. Model (44) parameters tuning.

Figure 2. Model (44) parameters tuning.

Figure 3. Changing the estimation error.

Figure 4. Model adequacy in output space.

Note [32] that the system (36), (37) is unidentifiable by the function F2 on the measurements set. In this case, we can use indirect information about the dependence of $u_1 = \omega - x_1$ on x_2. It is true since the relationship between u_1 and u.

Consider the case when the structure of nonlinearity is a priori unknown. Use the relationship diagram (RD) [28] for system (36). The relationship between u_1 and u is significant. Therefore,

consider the framework $S_{x_3 u_1}$ described by the function $f_{x_3 u_1} : u_1 \to x_3$ (Figure 5). $f_1(u)$ belongs to the relay functions class with a dead zone. Replacing u with u_1 changes the boundaries of the dead zone. This problem solves in the identification process. Construct the framework $S_{u_1 x_3}$ described by the mapping $\chi_{u_1 x_3} : x_3 \to u_1$ (the structure $S_{u_1 x_3}$ coincides with $S_{x_3 u_1}$), and the secant

$$\hat{u}_1 = \pi_0 + \pi_1 x_3 ,$$

where $\pi_0 = 23.63, \pi_1 = 0.129$. Coefficient of determination is $r_{u_1, x_3} = 0.66$. It indicates the presence of nonlinear properties.
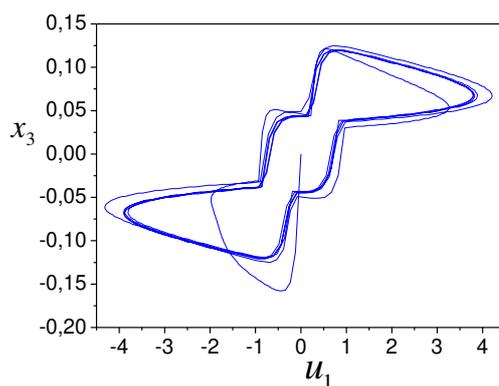

Figure 5. Framework $S_{x_3 u_1}$.

Determine the discrepancy $\varepsilon_{u_1} \triangleq \hat{u}_1 - u_1$. It contains information about the function f_2. Construction $\varepsilon_{u_1} = \varepsilon_{u_1}(x_2)$ and approximate the curve on the interval $\mathbb{J}_{x_2} = [-0.64; 1]$. Obtain a model with a quadratic structure. The model structure coincides with f_2. So, we have performed a structural identification of the system (36).

8. Self-oscillation generation system

The self-oscillation system contains (i) a control object (variables y_1, y_2), (ii) nonlinear (variable y_3) and linear (variable y_4) converters, (iii) an amplifier-converter with a nonlinear actuator (variable y_5)

$$\dot{Y} = AY + DF(Y)$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & k_0 \\ 0 & 0 & -\dfrac{1}{T_1} & 0 & 0 \\ 0 & \dfrac{k_2}{T_2} & 0 & -\dfrac{1}{T_2} & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{1}{T_3} \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \dfrac{1}{T_1} & 0 \\ 0 & 0 \\ 0 & -\dfrac{1}{T_3} \end{bmatrix}, F(Y) = \begin{bmatrix} f_1(y_1) \\ f_3(y_3 + y_4) \end{bmatrix}, \qquad (48)$$

where $T_i > 0$ is time constant, $g \geq 0$, y_5 is input. $f_i(x)$ ($i = 1, 3$) is a saturation function with a dead zone

$$f_i(x) = \begin{cases} c, & \text{if } x \geq d_{2,i}, \\ 2(x - d_{1,i}), & \text{if } d_{1,i} < x < d_{2,i}, \\ 0, & \text{if } -d_{1,i} \leq x \leq d_{1,i}, \\ 2(x + d_{1,i}), & \text{if } -d_{1,i} < x, \\ -c, & \text{if } x < -d_{2,i}, \end{cases} \tag{49}$$

The object phase portrait shows in Figure 6. It shows self-oscillations in the system. We transform the first two equations of the system (48) using the approach from Section 6. Present these equations as (38) and divide (38) by $\upsilon + \mu$. Transform variables y_1, y_5
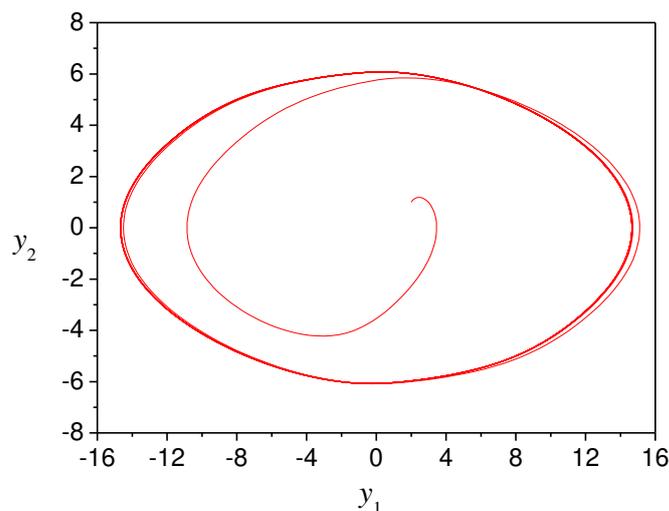

Figure 6. Object phase portrait.

$$\dot{p}_{y_1,\mu} = -\mu p_{y_1,\mu} + y_1,$$
$$\dot{p}_{y_5,\mu} = -\mu p_{y_5,\mu} + y_5, \tag{50}$$

and obtain the identification form

$$\dot{y}_1 = a_{11} y_1 + a_{12} p_{y_1} + a_{15} p_{y_5},$$
$$\dot{y}_3 = -a_3 y_3 + a_{31} f_1(y_1),$$
$$\dot{y}_4 = -a_4 y_4 + a_{42} y_2,$$
$$\dot{y}_5 = -a_5 y_5 + a_{53} f_3(y_3 + y_4), \tag{51}$$

The adaptive system for evaluating system (51) parameters has the form

$$\dot{\hat{y}}_1 = -k_1 e_1 + \hat{a}_{11} y_1 + \hat{a}_{12} p_{y_1} + \hat{a}_{15} p_{y_5},$$
$$\dot{\hat{y}}_2 = +\hat{a}_{11} y_1 + \hat{a}_{12} p_{y_1} + \hat{a}_{15} p_{y_5}$$
$$\dot{\hat{y}}_3 = -k_3 e_3 + \hat{a}_{31}(f_1(y_1) - y_3),$$
$$\dot{\hat{y}}_4 = -k_4 e_4 + \hat{a}_{41} y_4 + \hat{a}_{42} \hat{y}_2,$$
$$\dot{\hat{y}}_5 = -k_5 e_5 + \hat{a}_{51}(y_5 + f_3(y_3 + y_4)), \tag{52}$$

where $e_i = \hat{y}_i - y_i$, $i = 1; 3; 4; 5$.

Consider the Lyapunov functions $V_i(e_i) = 0.5e_i^2$ and get adaptive algorithms for tuning system parameters (52)

$$
\begin{aligned}
&\dot{\hat{a}}_{11} = -\gamma_{11}e_1 y_1, \quad \dot{\hat{a}}_{12} = -\gamma_{12}e_1 p_{y_1}, \\
&\dot{\hat{a}}_{15} = -\gamma_{15}e_1 p_{y_5}, \quad \dot{\hat{a}}_{31} = -\gamma_{31}e_3\left(f_1(y_1) - y_3\right), \\
&\dot{\hat{a}}_{41} = -\gamma_{41}e_4 y_4, \quad \dot{\hat{a}}_{42} = -\gamma_{42}e_4 \hat{y}_2, \\
&\dot{\hat{a}}_{51} = -\gamma_{51}e_5\left(y_5 + f_3(y_3 + y_4)\right),
\end{aligned}
\tag{53}
$$

where $\gamma_{ij} > 0$.

Apply Theorem 2 and get the boundedness of trajectories in an adaptive system

$$
\begin{aligned}
&\dot{e}_1 = -k_1 e_1 + \Delta a_{11} y_1 + \Delta a_{12} p_{y_1} + \Delta a_{15} p_{y_5}, \\
&e_2 = +\Delta a_{11} y_1 + \Delta a_{12} p_{y_1} + \Delta a_{15} p_{y_5}, \\
&\dot{e}_3 = -k_3 e_3 + \Delta a_{31}\left(f_1(y_1) - y_3\right), \\
&\dot{e}_4 = -k_4 e_4 - \Delta a_{41} y_4 + \Delta a_{42} \hat{y}_2, \\
&\dot{e}_5 = -k_5 e_5 - \Delta a_{51}\left(y_5 + f_3(y_3 + y_4)\right),
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
&\Delta \dot{a}_{11} = -\gamma_{11}e_1 y_1, \quad \Delta \dot{a}_{12} = -\gamma_{12}e_1 p_{y_1}, \\
&\Delta \dot{a}_{15} = -\gamma_{15}e_1 p_{y_5}, \quad \Delta \dot{a}_{31} = -\gamma_{31}e_3\left(f_1(y_1) - y_3\right), \\
&\Delta \dot{a}_{41} = \gamma_{41}e_4 y_4, \quad \Delta \dot{a}_{42} = -\gamma_{42}e_4 \hat{y}_2, \\
&\Delta \dot{a}_{51} = \gamma_{51}e_5\left(y_5 + f_3(y_3 + y_4)\right),
\end{aligned}
\tag{55}
$$

where $\Delta a_{ij}(t) = \hat{a}_{ij}(t) - a_{ij}$.

Show the simulation results in Figures 7-10. Figures 7 and 8 represent the system (52) parameters tuning, and Figure 9 shows the change e_i. Results confirm the adaptive system trajectories boundedness.
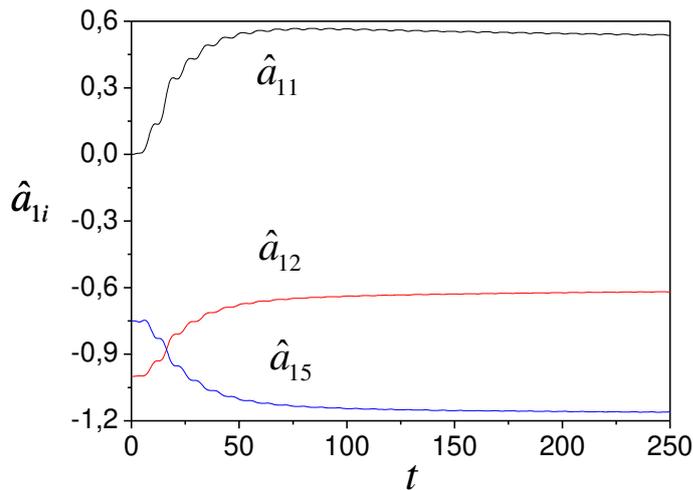

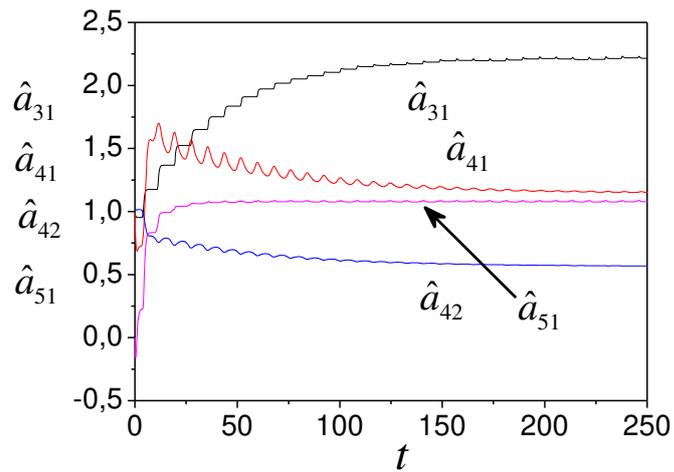**Figure 7.** Tuning model parameters for estimation y_1.

Figure 8. Tuning model parameters for estimation y_3, y_4, y_5.

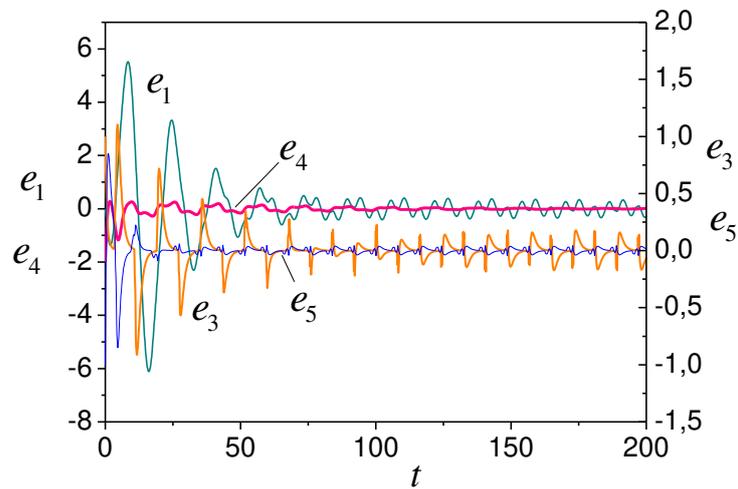**Figure 9.** System errors (52) for various variables (green is e_1, red is e_4, orange is e_3, blue is e_5).

Tuning parameters in the space $\left(e_i, \hat{a}_{ij}\right)$ shows in Figure 10. We see that $y_5 \in \mathcal{PE}_{\alpha,\bar{a}}^S$ and S-synchronizability execution does not guarantee the asymptotic stability of the system. Nonlinearity is the reason for such properties of an adaptive system.
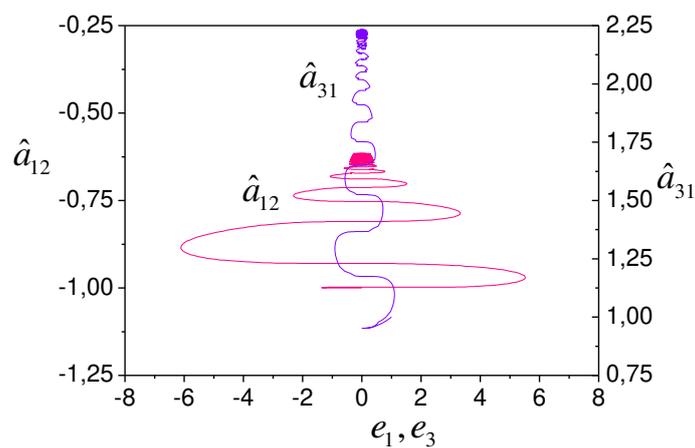
Figure 10. Tuning parameters in space $\left(e_i, \hat{a}_{ij} \right)$.

Consider Lyapunov functions $V_{e,1-5}(t) = 0.5 \sum\limits_{i=1, i \neq 2}^{5} e_i^2(t)$,

$$V_{\Delta,1-5}(t) = \underbrace{0.5 \left(\gamma_{11}^{-1} \Delta a_{11}^2(t) + \gamma_{12}^{-1} \Delta a_{12}^2(t) + \gamma_{15}^{-1} \Delta a_{15}^2(t) \right)}_{V_{\Delta,1}} + \underbrace{0.5 \gamma_{31}^{-1} \Delta a_{31}^2(t)}_{V_{\Delta,3}} +$$

$$+ \underbrace{0.5 \left(\gamma_{41}^{-1} \Delta a_{41}^2(t) + \gamma_{42}^{-1} \Delta a_{42}^2(t) \right)}_{V_{\Delta,4}} + \underbrace{0.5 \gamma_{51}^{-1} \Delta a_{51}^2(t)}_{V_{\Delta,5}},$$

$$V_{e,\Delta} = \left[V_{e,1}, V_{\Delta,1}, V_{e,3}, V_{\Delta,3}, V_{e,4}, V_{\Delta,4}, V_{e,5}, V_{\Delta,5} \right]^T. \tag{56}$$

Theorem 4. Let (i) positive-definite Lyapunov functions

$$V_{e,i}(t) = 0.5 e_i(t), \quad V_{\Delta,1}(t) = 0.5 \left(\gamma_{11}^{-1} \Delta a_{11}^2(t) + \gamma_{12}^{-1} \Delta a_{12}^2(t) + \gamma_{15}^{-1} \Delta a_{15}^2(t) \right), \quad V_{\Delta,i}(t) = 0.5 \gamma_{ij}^{-1} \left(\Delta a_{ij} \right)^2, i \geq 3$$

have an infinitesimal upper limit at $\left| e_i(t) \right| \to 0$, $\left| \Delta a_{ij} \right| \to 0$; (ii) $y_i(t) \in \mathcal{P}E_{\alpha_{y_i}, \bar{\alpha}_{y_i}}^S$, $p_{y_1} \in \mathcal{P}E_{\alpha_{p_{y_1}}, \bar{\alpha}_{p_{y_1}}}^S$, $p_{y_5} \in \mathcal{P}E_{\alpha_{p_{y_5}}, \bar{\alpha}_{p_{y_5}}}^S$, $\alpha = \max \left(\bar{\alpha}_{y_1}, \bar{\alpha}_{p_{y_1}}, \bar{\alpha}_{p_{y_5}} \right)$, $i = \overline{1,5}$, $i \neq 2$; (iii) the inequality is fulfilled for $V_{\Delta,1}(t) = 0.5 \Delta G_1^T(t) \Gamma_1^{-1} \Delta G_1(t)$

$$0.5 \underline{\vartheta}_1 \Delta G_1^T(t) \Delta G_1(t) \leq V_{\Delta,1}(t) \leq 0.5 \bar{\vartheta}_1 \Delta G_1^T(t) \Delta G_1(t), \tag{57}$$

where $\Delta G_1(t) = \left[\Delta a_{11}(t), \Delta a_{12}(t), \Delta a_{15}(t) \right]^T$, $\Gamma_1 = \mathrm{diag} \left(\gamma_{11}, \gamma_{11}, \gamma_{15} \right)$, $\underline{\vartheta}_1 = \beta_l^{-1}(\Gamma_1)$, $\bar{\vartheta}_1 = \beta_l^{-1}(\Gamma_1)$, $\beta_l(\Gamma_1)$ and $\beta_l(\Gamma_1)$ are minimum and maximum eigenvalues of the matrix Γ_1; (iv) $\alpha_{f_1, y_3} - \underline{\upsilon}_3 \leq \left(f_1(y_1) - y_3 \right)^2 \leq \bar{\alpha}_{f_1, y_3} + \upsilon_3$, where $\underline{\upsilon}_3 \geq 0$, $\upsilon_3 \geq 0$; (v) $\gamma_4 = \max \left(\gamma_{41}, \gamma_{42} \right)$; (vi) the equality $e_i \Delta a_{ij} \omega_i = \pi_i \left(e_i^2 + \left(\Delta a_{ij} \right)^2 \omega_i^2 \right)$ fulfilled in the domain $O_v(O)$, where $\pi > 0$, $O = \{ 0^4, 0^n \} \subset \mathbb{R}^4 \times \mathbb{R}^n \times \mathbb{J}_{0,\infty}$, 0^4 and 0^n are zero vectors, n is number of adjustable parameters, O_v is some neighborhood of a point O, $t \in [0, \infty] = \mathbb{J}_{0,\infty}$; (vii) the matrix system of inequalities

$$\dot{V}_{e,\Delta} \leq A_{e,\Delta} V_{e,\Delta} + B_{e,\Delta}, \tag{58}$$

is valid for $V_{e,\Delta}(t)$, where $A_{e,\Delta}$ block-diagonal matrix

$$A_{e,\Delta} = \mathrm{diag} \left(A_{e,\Delta,1}, A_{e,\Delta,3} A_{e,\Delta,4}, A_{e,\Delta,5} \right), \quad B_{e,\Delta} = \begin{bmatrix} 0 & 0 & \tilde{\upsilon}_3 \vartheta_3 & \pi_3 \chi & 0 & 0 & 0 & 0 \end{bmatrix}^T,$$

$$A_{e,\Delta,1} = \begin{bmatrix} -k_1 & \dfrac{\alpha\beta_l(\Gamma_1)}{k_1} \\ \dfrac{\pi_1}{3} & -\dfrac{3}{4}\pi_1\alpha_{P_1}\beta_1(\Gamma_1) \end{bmatrix}, A_{e,\Delta,3} = \begin{bmatrix} -k_3 & \dfrac{\alpha_{f_1,y_3}\gamma_{31}}{k_3} \\ \dfrac{\pi_3\overline{\alpha}_{f_1,y_3}}{3(\alpha_{f_1,y_3}-\underline{\upsilon}_3)} & -\dfrac{3}{4}\pi_3\alpha_{f_1,y_3}\gamma_{31} \end{bmatrix},$$

$$A_{e,\Delta,4} = \begin{bmatrix} -k_4 & \dfrac{\alpha_{y_1,\hat{y}_2}}{k_4}\gamma_4 \\ \dfrac{2\pi_{41}}{3} & -\dfrac{3}{4}\pi_{41}\tilde{\alpha}_{y_1,\hat{y}_2}\gamma_4 \end{bmatrix}, A_{e,\Delta,5} = \begin{bmatrix} -k_5 & \dfrac{\overline{\alpha}_{y_5}+c^2}{k_5}\gamma_{51} \\ \dfrac{2\pi_5}{3} & -\dfrac{3\pi_5}{4}\alpha_{y_5.f_3}\gamma_{51} \end{bmatrix};$$

(viii) the upper solution for $V_{e,\Delta}(t)$ satisfies the equation $\dot{S}_{e,\Delta} = A_{e,\Delta}S_{e,\Delta} + B_{e,\Delta}$ if functions $s_i(t) \geq 0$ exist such that $V_{e,\Delta,i}(t) \leq s_i(t)$ $\forall(t \geq t_0)\&(V_{e,\Delta,i}(t_0) \leq s_i(t_0))$, $i = 1,3;4;5$, where $S_{e,\Delta} \in \mathbb{R}^6$, s_i is vector $S_{e,\Delta}$ element. Then the adaptive system (54), (55) is exponentially dissipative with the estimate

$$V_{e,\Delta}(t) \leq e^{A_{e,\Delta}(t-t_0)}S(t_0) + \int_{t_0}^{t} e^{A_{e,\Delta}(t-\tau)}B_{e,\Delta}d\tau \tag{59}$$

if

$$k_1 \geq \frac{2}{3}\sqrt{\frac{\alpha\beta_l(\Gamma_1)}{\alpha_{P_1}k_1\beta_1(\Gamma_1)}}, \quad k_3 \geq \frac{2}{3}\sqrt{\frac{\overline{\alpha}_{f_1,y_3}}{(\alpha_{f_1,y_3}-\underline{\upsilon}_3)}}, \quad k_4 \geq \frac{2}{3}\sqrt{2\frac{\alpha_{y_1,\hat{y}_2}}{\tilde{\alpha}_{y_1,\hat{y}_2}}}, \quad k_5 \geq \frac{2}{3}\sqrt{2\frac{\overline{\alpha}_{y_5}+c^2}{\alpha_{y_5.f_3}}} \tag{60}$$

The Theorem 4 proof gives in Appendix D.

As follows from Theorem 4, system (48), (49) limiting properties depend on nonlinearity, feedback and constant excitation. In particular, this is true for block 3. The Theorem 4 statement confirms by Figure 11. It reflects the change of parametric error norm for channels 3-5

$$N_\Delta(t) = \sqrt{\Delta a_{31}^2(t) + \Delta a_{41}^2(t) + \Delta a_{42}^2(t) + \Delta a_{51}^2(t)}. \tag{61}$$
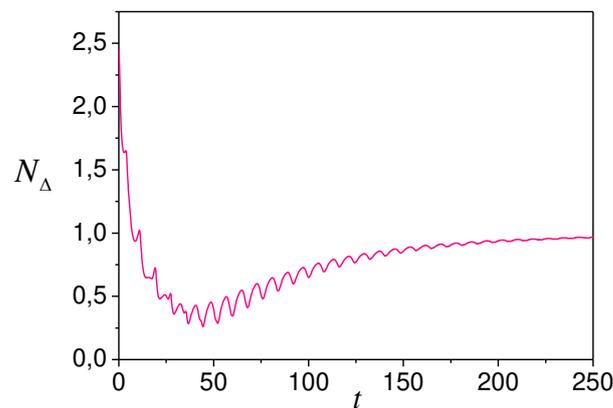

Figure 11. Parametric residual norm.

So, the nonlinearities and feedback give biased estimates for the system parameters (48).

9. Conclusion

We propose the approach to adaptive identification for systems with multiple nonlinearities. It uses the transformation of equations based on the experimental data available set. Adaptive identification system designed. System decomposition into several subsystems is proposed for the adaptation process simplify. The trajectories boundedness in the adaptive system proved. The S-synchronizability problem of the system considers, and the excitation constancy condition modification proposes. The modification considers the requirement for nonlinear system structural identifiability. The Lyapunov vector function method applies to prove exponential stability.

Nonlinear correction of a nonlinear system considers. Adaptive algorithms obtain for estimating system parameters. The system trajectories limitation proved. A nonlinear system considers for self-oscillation generation with nonlinear feedback. The adaptive parametric identification system develops. We study the effect of feedback and nonlinearity. Simulation results confirm theoretical results.

Appendix A. Proof of Theorem 2

Consider the Lyapunov function $V_{s_{F,i}}(t)$ (24). $\dot{V}_{s_{F,i}}(t)$ has the form of subsystem $s_{F,i}$ motions

$$\dot{V}_{s_{F,i}} = -k_{s_{F,i}}e^2 + \dot{V}_{i,\Delta} - \dot{V}_{i,\Delta} \leq -2k_{s_{F,i}}V_{e,i}. \tag{A.1}$$

Apply Theorem 2 condition (i). As $\dot{V}_{s_{F,i}}(t) < 0$, the $s_{F,i}$-subsystem is stable. Integrate $\dot{V}(t)$

$$V_{s_{F,i}}(t_0) - 2k_{s_{F,i}}\int_{t_0}^{t}V_e(\tau)d\tau \geq V_{s_{F,i}}(t). \tag{A.2}$$

$V_{s_{F,i}}\left(e, \Delta\kappa_{s_{F,i}}, \Delta K_{s_{F,i}}\right)$ satisfies the Theorem 2 condition (1). Therefore, all trajectories of the $AS_{F,i}$ -subsystem belong to the domain

$$\mathbf{G}_t = \left\{\left(e, \Delta\kappa_{s_{F,i}}, \Delta K_{s_{F,i}}\right) : V_{s_{F,i}}(t) \leq V_{s_{F,i}}(t_0)\right\}. \tag{A.3}$$

The desired estimate follows from (A.2). ∎

Appendix B. Proof of Lemma 1

Consider $\dot{V}_{e,i}(t)$

$$\dot{V}_{e,i}\left(e_{s_{F,i}}\right) = e_{s_{F,i}}\left[-k_{s_{F,i}}e_{s_{F,i}} + \Delta\kappa_{s_{F,i}}y_{s_{F,i}} + \sum_{k=1}^{n_i-1}\sum_{j=1}^{n_i-1}\left(\Delta\tilde{d}_{s_{F,i}}^{k,h}p_{f_{s_{F,i}}^{h,j}} + \Delta\tilde{b}_{s_{F,i}}^{k,j}p_{u_{s_{F,i}}^{j}}\right)\right] \leq$$
$$-k_{s_{F,i}}e_{s_{F,i}}^2 + \left|e_{s_{F,i}}\right|\left|\Delta\kappa_{s_{F,i}}y_{s_{F,i}}\right| + \left|e_{s_{F,i}}\right|\left|\sum_{k=1}^{n_i-1}\sum_{j=1}^{n_i-1}\left(\Delta\tilde{d}_{s_{F,i}}^{k,h}p_{f_{s_{F,i}}^{h,j}} + \Delta\tilde{b}_{s_{F,i}}^{k,j}p_{u_{s_{F,i}}^{j}}\right)\right|. \tag{B.1}$$

Apply the inequality [30]

$$-az^2 + bz \leq \frac{-az^2}{2} + \frac{b^2}{2a}, \quad a > 0, \; b \geq 0, \; z \geq 0 \tag{B.2}$$

and (26). Then

$$\dot{V}_{e,i} \leq -k_{s_{F,i}}e_{s_{F,i}}^2 + \left|e_{s_{F,i}}\right|\left|\Delta\kappa_{s_{F,i}}y_{s_{F,i}}\right| + \left|e_{s_{F,i}}\right|\left|\sum_{k=1}^{n_i-1}\sum_{j=1}^{n_i-1}\left(\Delta\tilde{d}_{s_{F,i}}^{k,h}p_{f_{s_{F,i}}^{h,j}} + \Delta\tilde{b}_{s_{F,i}}^{k,j}p_{u_{s_{F,i}}^{j}}\right)\right| \leq$$
$$-\frac{k_{s_{F,i}}}{2}e_{s_{F,i}}^2 + \frac{1}{2k_{s_{F,i}}}\left[\mathrm{Sp}\left(\Delta K_{s_{F,i}}(t)P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}^{T}P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}\Delta K_{s_{F,i}}^{T}(t)\right) + \left(\Delta\kappa_{s_{F,i}}y_{s_{F,i}}\right)^2\right]' \tag{B.3}$$

where $P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}$ is matrix with elements $p_{f_{s_{F,i}}^{h,j}}$ and $p_{u_{s_{F,i}}^{j}}$.

Let $P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}^{T}(t) P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}(t) \in \mathcal{PE}_{\alpha_P,\bar{\alpha}_P}^{S}$ and $y_{s_{F,i}}(t) \in \mathcal{PE}_{\alpha_{y_i},\bar{\alpha}_{y_i}}^{S}$, where $\alpha_{P_i}, \bar{\alpha}_{P_i}, \alpha_{y_i}, \bar{\alpha}_{y_i}$ are positive numbers. $\mathcal{M}^+$-property for $\dot{V}_{e,i}$ obtains from (B.3)

$$\dot{V}_{e,i} \le -k_{s_{F,i}} V_{e,i} + \frac{\bar{\alpha}\bar{\vartheta}}{2 k_{s_{F,i}}} V_{\Delta,i}, \tag{B.4}$$

where $\bar{\alpha} = \max\left(\bar{\alpha}_{P_i}, \bar{\alpha}_{y_i}\right)$. ∎

Appendix C. Proof of Lemma 2

Consider $\dot{V}_{\Delta,i}(t)$

$$\dot{V}_{\Delta,i} = -e_{s_{F,i}} \Delta\kappa_{s_{F,i}} y_{s_{F,i}} - e_{s_{F,i}} \sum_{k=1}^{n_i-1}\sum_{j=1}^{n_i-1}\left(\Delta\tilde{d}_{s_{F,i}}^{k,h} p_{f_{s_{F,i}}^{h,j}} + \Delta\tilde{b}_{s_{F,i}}^{k,j} p_{u_{s_{F,i}}^{j}}\right). \tag{C.1}$$

Let for $\left(\forall t \ge t^* > t_0\right) \& \left(\forall\left(e, \Delta K_{s_{F,i}}, \Delta\kappa_{s_{F,i}}\right)\right) \in O_\nu(O))$

$$e_{s_{F,i}} \sum_{k=1}^{n_i-1}\sum_{j=1}^{n_i-1}\left(\Delta\tilde{d}_{s_{F,i}}^{k,h} p_{f_{s_{F,i}}^{h,j}} + \Delta\tilde{b}_{s_{F,i}}^{k,j} p_{u_{s_{F,i}}^{j}}\right) = \pi\left\{\mathrm{Sp}\left(\Delta K_{s_{F,i}}(t) P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}^{T} P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}} \Delta K_{s_{F,i}}^{T}(t)\right) + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2 + e_{s_{F,i}}^2\right\}, \tag{C.2}$$

where $O = \{0,\ 0^{n_i\times n_i}\} \subset \mathbb{R}\times\mathbb{R}^{n_i\times n_i}\times\mathbb{J}_{0,\infty}$ is system equilibrium position, $O_\nu(O)$ is some neighborhood of a point O, $0^{n_i\times n_i} \in \mathbb{R}^{n_i\times n_i}$ is null matrix, $t \in [0,\infty] = \mathbb{J}_{0,\infty}$, $\pi > 0$ is some number.

Let $W_{s_{F,i}} \triangleq P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}^{T} P_{f_{s_{F,i}}^{h,j},u_{s_{F,i}}^{j}}$, $\sigma_{s_{F,i}} \triangleq \mathrm{Sp}\left(\Delta K_{s_{F,i}}(t) W_{s_{F,i}} \Delta K_{s_{F,i}}^{T}(t)\right)$. Apply (C.2) and present (C.1) as

$$\dot{V}_{\Delta,i} = -\pi e_{s_{F,i}}^2 - \pi\left\{\mathrm{Sp}\left(\Delta K_{s_{F,i}}(t) W_{s_{F,i}} \Delta K_{s_{F,i}}^{T}(t)\right) + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2\right\} \le$$

$$-\frac{3}{4}\pi\left(\sigma_{s_{F,i}} + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2\right) - \pi\left(\frac{1}{4}\left(\sigma_{s_{F,i}} + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2\right) + e_{s_{F,i}}^2 \pm 2\pi\frac{1}{2}\left|e_{s_{F,i}}\right|\sqrt{\sigma_{s_{F,i}} + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2}\right) \le$$

$$-\frac{3}{4}\pi\left(\sigma_{s_{F,i}} + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2\right) - \pi\left(\left|e_{s_{F,i}}\right| + \sqrt{\sigma_{s_{F,i}} + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2}\right)^2 +$$

$$+\pi\left|e_{s_{F,i}}\right|\sqrt{\sigma_{s_{F,i}} + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2} \le -\frac{3}{4}\pi\left(\sigma_{s_{F,i}} + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2\right) + \pi\left|e_{s_{F,i}}\right|\sqrt{\sigma_{s_{F,i}} + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2}. \tag{C.3}$$

Apply to (C.3) inequalities (B.2), (26) and obtain

$$\dot{V}_{\Delta,i} \le -\frac{3}{8}\pi\left(\sigma_{s_{F,i}} + \left(\Delta\kappa_{s_{F,i}} y_{s_{F,i}}\right)^2\right) + \frac{4}{3}\pi e_{s_{F,i}}^2 \le$$

$$\le -\frac{3}{8}\pi\alpha_i\left(\sigma_{s_{F,i}} + \left(\Delta\kappa_{s_{F,i}}\right)^2\right) + \frac{8}{3}\pi V_{e,i} \le -\frac{3}{8}\pi\alpha_i V_{\Delta,i} + \frac{8}{3}\pi V_{e,i} \tag{C.4}$$

where $\alpha_i = \min\left(\alpha_{P_i}, \alpha_{y_i}\right)$.

So, $\mathcal{M}^+$-property for $\dot{V}_{\Delta,i}$

$$\dot{V}_{\Delta,i} \le -\frac{3}{8}\pi\alpha_i V_{\Delta,i} + \frac{8}{3}\pi V_{e,i}. \blacksquare \tag{C.5}$$

Appendix D. Proof of Theorem 4

Consider the Lyapunov vector function (56). Elements $V_{e,\Delta}$ satisfy Theorem 4 condition (i). Ensure the $\mathcal{M}^+$ -property for each component $\dot{V}_{e,\Delta,i}$. Apply the approach used in proving Lemmas 1 and 2. Consider the proof procedure with the example $V_{e,1}$. $\dot{V}_{e,1}$ has form

$$\begin{aligned}
\dot{V}_{e,1} &= -k_1 e_1^2 + e_1 \left(\Delta a_{11} y_1 + \Delta a_{12} p_{y_1} + \Delta a_{15} p_{y_5} \right) \leq \\
&\leq k_1 e_1^2 + |e_1| \left| \Delta a_{11} y_1 + \Delta a_{12} p_{y_1} + \Delta a_{15} p_{y_5} \right|
\end{aligned} \tag{D.1}$$

Apply to (D.1) inequalities (B.2) and obtain

$$\dot{V}_{e,1} \leq -\frac{k_1}{2} e_1^2 + \frac{1}{2k_1} \left(\left(\Delta a_{11} \right)^2 y_1^2 + \left(\Delta a_{12} \right)^2 p_{y_1}^2 + \left(\Delta a_{15} \right)^2 p_{y_5}^2 \right). \tag{D.2}$$

Let $V_{\Delta,1}(t) = 0.5 \Delta G_1^T(t) \Gamma_1^{-1} \Delta G_1(t)$, $\Delta G_1(t) = \left[\Delta a_{11}(t), \Delta a_{12}(t), \Delta a_{15}(t) \right]^T$, $\Gamma_1 = \text{diag}\left(\gamma_{11}, \gamma_{11}, \gamma_{15} \right)$, $\underline{\vartheta}_1 = \beta_l^{-1}(\Gamma_1)$, $\overline{\vartheta}_1 = \beta_l^{-1}(\Gamma_1)$, $\beta_l(\Gamma_1)$, $\beta_l(\Gamma_1)$ are minimum and maximum eigenvalues of the matrix Γ_1. Theorem 4 condition (iii) holds for $V_{\Delta,1}(t)$. Then (D.2)

$$\dot{V}_{e,1} \leq -\frac{k_1}{2} e_1^2 + \frac{1}{2k_1} \left(\left(\Delta a_{11} \right)^2 y_1^2 + \left(\Delta a_{12} \right)^2 p_{y_1}^2 + \left(\Delta a_{15} \right)^2 p_{y_5}^2 \right) \leq -k_1 V_{e,1} + \frac{\alpha \beta_l(\Gamma_1)}{k_1} V_{\Delta,1}. \tag{D.3}$$

So, the $\mathcal{M}^+$ -property obtain for $\dot{V}_{e,3}$. Consider $\dot{V}_{e,3}$

$$\begin{aligned}
\dot{V}_{e,3} &\leq -\frac{k_3}{2} e_3^2 + \frac{1}{2k_3} \left(\Delta a_{31} \right)^2 \left(f_1(y_1) - y_3 \right)^2 \leq \\
&\quad -\frac{k_3}{2} e_3^2 + \frac{\overline{\alpha}_{f_1,y_3} + \upsilon_3}{2k_3} \left(\Delta a_{31} \right)^2 \leq -k_3 V_{e,3} + \frac{\left(\overline{\alpha}_{f_1,y_3} + \upsilon_3 \right) \gamma_{31}}{k_3} V_{\Delta,3},
\end{aligned} \tag{D.4}$$

where $\alpha_{f_1,y_3} - \underline{\upsilon}_3 \leq \left(f_1(y_1) - y_3 \right)^2 \leq \overline{\alpha}_{f_1,y_3} + \upsilon_3$, $\underline{\upsilon}_3 \geq 0$, $\upsilon_3 \geq 0$. υ_3 levels the EC property, as the nonlinear properties of the system effect.

Apply Theorem 2 and get the adaptive system trajectories boundedness. Then (D.4)

$$\dot{V}_{e,3} \leq -k_3 V_{e,3} + \frac{\overline{\alpha}_{f_1,y_3} \gamma_{31}}{k_3} V_{\Delta,3} + \tilde{\upsilon}_3 \vartheta_3, \tag{D.5}$$

where $\vartheta_3(t) \geq 0$ is uncertainty appearing at $\left(f_1(y_1) - y_3 \right)^2 \notin \mathcal{PE}^S_{\alpha_{f_1,y_3}, \overline{\alpha}_{f_1,y_3}}$, $\tilde{\upsilon}_3 = \upsilon_3 \gamma_{31} / k_3$.

Apply the above approach and get the $\mathcal{M}^+$ - property for the remnants $\dot{V}_{e,i}$

$$\begin{aligned}
\dot{V}_{e,1} &\leq -\frac{k_1}{2} e_1^2 + \frac{\alpha \beta_l(\Gamma_1)}{k_1} V_{\Delta,1} = -k_1 V_{e,1} + \frac{\alpha \beta_l(\Gamma_1)}{k_1} V_{\Delta,1}, \\
\dot{V}_{e,3} &\leq -\frac{k_3}{2} e_3^2 + \frac{\overline{\alpha}_{f_1,y_3}}{2k_3} \left(\Delta a_{31} \right)^2 \leq -k_3 V_{e,3} + \frac{\overline{\alpha}_{f_1,y_3} \gamma_{31}}{k_3} V_{\Delta,3} + \tilde{\upsilon}_3 \vartheta_3, \\
\dot{V}_{e,4} &\leq -\frac{k_4}{2} e_4^2 + \frac{\alpha_{y_1,\hat{y}_2}}{2k_4} \left(\left(\Delta a_{41} \right)^2 + \left(\Delta a_{42} \right)^2 \right) \leq -k_4 V_{e,4} + \frac{\overline{\alpha}_{y_1,\hat{y}_2}}{k_4} \gamma_4 V_{\Delta,4}, \\
\dot{V}_{e,5} &\leq -\frac{k_5}{2} e_5^2 + \frac{\overline{\alpha}_{y_5} + c^2}{2k_5} \left(\Delta a_{51} \right)^2 \leq -k_5 V_{e,5} + \frac{\overline{\alpha}_{y_5} + c^2}{k_5} \gamma_{51} V_{\Delta,5},
\end{aligned} \tag{D.6}$$

where $\gamma_4 = \max\left(\gamma_{41}, \gamma_{42} \right)$.

Consider components $\dot{V}_{\Delta,i}$ and use the approach used for Lemma 2 proving. Let the condition $\Delta a_{ij} e_i \omega_i = \pi_i \left(e_i^2 + \Delta a_{ij}^2 \omega_i^2 \right)$ for $\left(\forall t \geq t^* > t_0 \right) \& \left(\forall \left(e, \Delta a_{ij} \right) \right) \in O_\nu(O)$, where $O = \{0, 0\} \subset \mathbb{R} \times \mathbb{R} \times \mathbb{J}_{0,\infty}$ is system equilibrium position, $O_\nu(O)$ is some neighborhood of a point O, $\pi_i \geq 0$, ω_i are corresponding variables in (48). Then

$$\dot{V}_{\Delta,1} = -e_1 \left(\Delta a_{11} y + \Delta a_{12} p_{y_1} + \Delta a_{15} p_{y_5} \right) = -\pi_1 e_1^2 - \pi_1 \Delta G_1^T P_1 P_1^T \Delta G_1 =$$

$$\frac{3}{4} \pi_1 \alpha_{P_1} \Delta G_1^T \Delta G_1 - \pi_1 \left(\frac{1}{4} \alpha_{P_1} \Delta G_1^T \Delta G_1 + e_1^2 \pm 2 \frac{1}{2} |e_1| \sqrt{\alpha_{P_1} \Delta G_1^T \Delta G_1} \right) =$$

$$-\frac{3}{4} \pi_1 \alpha_{P_1} \Delta G_1^T \Delta G_1 - \pi_1 \left(|e_1| + \frac{1}{2} \sqrt{\alpha_{P_1} \Delta G_1^T \Delta G_1} \right)^2 + \pi_1 \frac{1}{2} |e_1| \sqrt{\alpha_{P_1} \Delta G_1^T \Delta G_1} \leq \qquad (D.7)$$

$$\leq -\frac{3}{4} \pi_1 \alpha_{P_1} \Delta G_1^T \Delta G_1 + \pi_1 \frac{1}{2} |e_1| \sqrt{\alpha_{P_1} \Delta G_1^T \Delta G_1},$$

where $P_1(t) = \left[y_1(t), p_{y_1}(t), p_{y_5}(t) \right]^T$, $P_1 P_1^T \in \mathcal{PE}_{\alpha_{P_1}, \bar{\alpha}_{P_1}}^S$. Apply (B.2) and get the $\mathcal{M}^+$-property for $\dot{V}_{\Delta,1}$

$$\dot{V}_{\Delta,1} - \frac{3}{4} \pi_1 \alpha_{P_1} \beta_1 \left(\Gamma_1 \right) V_{\Delta,1} + \frac{\pi_1}{3} V_{e,1}. \qquad (D.8)$$

Similarly, obtain the $\mathcal{M}^+$-property for elements $\dot{V}_{\Delta,i}$

$$\begin{cases} \dot{V}_{\Delta,1} \leq -\frac{3}{4} \pi_1 \alpha_{P_1} \beta_1 \left(\Gamma_1 \right) V_{\Delta,1} + \frac{\pi_1}{3} V_{e,1}, \\[2mm] \dot{V}_{\Delta,3} \leq -\frac{3\pi_3}{4} \alpha_{f_1, y_3} \gamma_{31} V_{\Delta,3} + \frac{\pi_3 \bar{\alpha}_{f_1, y_3}}{3 \left(\alpha_{f_1, y_3} - \upsilon_3 \right)} V_{e,3} + \pi_3 \chi, \\[2mm] \dot{V}_{\Delta,4} \leq -\frac{3}{4} \pi_{41} \tilde{\alpha}_{y_1, \hat{y}_2} \gamma_4 V_{\Delta,4} + \frac{2\pi_{41}}{3} V_{e,4}, \\[2mm] \dot{V}_{\Delta,5} \leq -\frac{3\pi_5}{4} \alpha_{y_5 . f_3} \gamma_{51} V_{\Delta,5} + \frac{2\pi_5}{3} V_{e,5}, \end{cases} \qquad (D.9)$$

where χ is the result of the violation of the EC condition. The χ boundedness follows from Theorem 2.

The derivative $V_{e,\Delta}$ satisfies the matrix system of inequalities

$$\dot{V}_{e,\Delta} \leq A_{e,\Delta} V_{e,\Delta} + B_{e,\Delta}, \qquad (D.10)$$

where $A_{e,\Delta} = \mathrm{diag} \left(A_{e,\Delta,1}, A_{e,\Delta,3} A_{e,\Delta,4}, A_{e,\Delta,5} \right)$, $B_{e,\Delta} = \begin{bmatrix} 0 & 0 & \tilde{\upsilon}_3 \vartheta_3 & \pi_3 \chi & 0 & 0 & 0 & 0 \end{bmatrix}^T$,

$$A_{e,\Delta,1} = \begin{bmatrix} -k_1 & \dfrac{\alpha \beta_l \left(\Gamma_1 \right)}{k_1} \\[3mm] \dfrac{\pi_1}{3} & -\dfrac{3}{4} \pi_1 \alpha_{P_1} \beta_1 \left(\Gamma_1 \right) \end{bmatrix}, A_{e,\Delta,3} = \begin{bmatrix} -k_3 & \dfrac{\alpha_{f_1, y_3} \gamma_{31}}{k_3} \\[3mm] \dfrac{\pi_3 \bar{\alpha}_{f_1, y_3}}{3 \left(\alpha_{f_1, y_3} - \upsilon_3 \right)} & -\dfrac{3}{4} \pi_3 \alpha_{f_1, y_3} \gamma_{31} \end{bmatrix},$$

$$A_{e,\Delta,4} = \begin{bmatrix} -k_4 & \dfrac{\alpha_{y_1, \hat{y}_2}}{k_4} \gamma_4 \\[3mm] \dfrac{2\pi_{41}}{3} & -\dfrac{3}{4} \pi_{41} \tilde{\alpha}_{y_1, \hat{y}_2} \gamma_4 \end{bmatrix}, A_{e,\Delta,5} = \begin{bmatrix} -k_5 & \dfrac{\bar{\alpha}_{y_5} + c^2}{k_5} \gamma_{51} \\[3mm] \dfrac{2\pi_5}{3} & -\dfrac{3\pi_5}{4} \alpha_{y_5 . f_3} \gamma_{51} \end{bmatrix};$$

The comparison system is fair for (D.10)

$$\dot{S}_{e,\Delta} = A_{e,\Delta} S_{e,\Delta} + B_{e,\Delta} , \tag{D.11}$$

if functions $s_i(t) \geq 0$ exist such that the comparison system is fair for (D.10)

$$V_{e,\Delta,i}(t) \leq s_i(t) \qquad \forall \left(t \geq t_0\right) \& \left(V_{e,\Delta,i}\left(t_0\right) \leq s_i\left(t_0\right)\right), \quad i = 1,3;4;5 , \tag{D.12}$$

where $S_{e,\Delta} \in \mathbb{R}^6$, s_i is elements of the vector $S_{e,\Delta}$.

Form $A_{e,\Delta}$ shows that system (48) channels are isolated at the level of stability research. Therefore, perform the analysis $A_{e,\Delta}$ for each channel. $A_{e,\Delta}$ is an *M*-matrix [30]. The major minors $A_{e,\Delta}$ positivity is the condition for the asymptotic stability of the adaptive system. Have

$$k_1 \geq \frac{2}{3}\sqrt{\frac{\alpha\beta_l\left(\Gamma_1\right)}{\alpha_{P_1}k_1\beta_1\left(\Gamma_1\right)}} , \quad k_3 \geq \frac{2}{3}\sqrt{\frac{\overline{\alpha}_{f_1,y_3}}{\left(\alpha_{f_1,y_3} - \underline{\upsilon_3}\right)}} , \quad k_4 \geq \frac{2}{3}\sqrt{2\frac{\alpha_{y_1,\hat{y}_2}}{\tilde{\alpha}_{y_1,\hat{y}_2}}} , \quad k_5 \geq \frac{2}{3}\sqrt{2\frac{\overline{\alpha}_{y_5} + c^2}{\alpha_{y_5 \cdot f_3}}}$$

We see that the limiting properties (48) depend on the nonlinear properties of the system, feedback, and constant excitation. In particular, this is true for channel 3 in (D.10). ∎

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