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Solve the 3x+1 problem by the multiplication and division of binary numbers

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Abstract

The $3x+1$ problem asks the following: Suppose we start with a positive integer, and if it is odd then multiply it by 3 and add 1, and if it is even, divide it by 2. Then repeat this process as long as you can. Do you eventually reach the integer 1, no matter what you started with? Collatz conjecture (or $3n+1$ problem) has been explored for about 85 years. In this paper, we convert an integer number from decimal to binary number, and convert the Collatz function to binary function, which is multiplication and division of two binary numbers. Finally the iternation of the Collatz function, eventually reach the integer 1, thus we solve the $3x+1$ problem completely.

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Key words: $3x+1$ problem, Collatz conjecture, Sharkovsky ordering path

1 Introduction

The $3x+1$ problem is one of the unsolved problems in mathematics. It is also known as the Collatz conjecture, $3x+1$ mapping, Ulam conjecture, Kakutani's problem, Thwaites conjecture, Hasse's algorithm, or Syracuse problem [1]. Paul Erdos (1913-1996) commented on the intractability of problem $3n+1$ [2]: "Mathematics is not ready for those problems yet".

The $2x+1$ problem is that, take any positive integer x, If x is even, divide x by 2. If x is odd, multiply x by 3 and add 1. Repeat this process continuously. The conjecture states that no matter which number you start with, you will always reach 1 eventually.

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2 Terminology and notations

We will use the notations as in [4,7]. we describe a Collatz function as

$$T(n) = \begin{cases} 3n+1, & \text{if } n \text{ is odd number,} \\ \frac{n}{2} & \text{if } n \text{ is even number.} \end{cases} \tag{1}$$

Let N denote the set of positive integers. For $n \in N$, and $k = 0, 1, 2, 3, \cdots$, $T^0(n)$ and $T^{k+1}(n)$ denote n and $T(T^k(n)$, respectively. The $3x + 1$ problem concerns the behavior of the iterates of the Collatz function, for any integer n, there must exist an integer r, so that

$$T^r(n) = 1.$$

2.1 The modified Sarkovskii ordering and integer lattice

We remove the last row number to the first column, get an integer lattice[6] of the modified Sarkovskii ordering as

$$\begin{array}{lllllllllll}
1, & 3, & 5, & 7, & 9, & 11, & 13, & 15, & 17, & 19, & \cdots \\
2, & 2\cdot 3, & 2\cdot 5, & 2\cdot 7, & 2\cdot 9, & 2\cdot 11, & 2\cdot 13, & 2\cdot 15, & 2\cdot 17, & 2\cdot 19, & \cdots \\
2^2, & 2^2\cdot 3, & 2^2\cdot 5, & 2^2\cdot 7, & 2^2\cdot 9, & 2^2\cdot 11, & 2^2\cdot 13, & 2^2\cdot 15, & 2^2\cdot 17, & 2^2\cdot 19, & \cdots \\
2^3, & 2^3\cdot 3, & 2^3\cdot 5, & 2^3\cdot 7, & 2^3\cdot 9, & 2^3\cdot 11, & 2^3\cdot 13, & 2^3\cdot 15, & 2^3\cdot 17, & 2^3\cdot 19, & \cdots \\
2^4, & 2^4\cdot 3, & 2^4\cdot 5, & 2^4\cdot 7, & 2^4\cdot 9, & 2^4\cdot 11, & 2^4\cdot 13, & 2^4\cdot 15, & 2^4\cdot 17, & 2^4\cdot 19, & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

In the first row, its are odd number from left to right, that are $1, 3, 5, 7, 9, 11, 13, \cdots$, from the second row, each number is multiplying each number in its previous row by 2, and so on.

2.2 The algebraic formula and Collatz graph

If we draw a line segment of arrow between two digits in the lattice of integer in the modified Sarkovskii ordering, those are the original value x, and its value of Collatz function $T(x)$, and connect $T(x)$ to $T^2(x)$, and so on $T^2(x)$ to $T^3(x), \cdots$, thus we get a graph, which can be called as *Collatz graph*. For different integer m, n, besides 1, 4, 2, there is not other common vertices in

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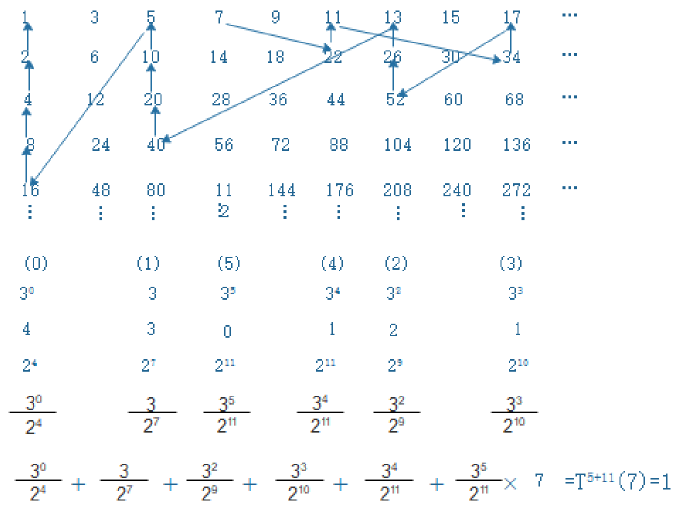

Fig. 1. The Collatz graph of $T^{16}(7) = T(5, 11, 7) = 1$ in the lattice of integers in the modified Sarkovskii ordering and the algebraic formula.

their Collatz graphs. Using the Collatz function $T(x)$, We obtain an algebraic formula of $\frac{1}{2^4}, \frac{3}{2^7}, \frac{3^2}{2^9}, \cdots, \frac{3^m}{2^r} \cdot x$. Here r is the number of perpendicular segments, m is the oblique segments in the Collatz graph,

$$T^{m+r}(n) = T(m, r, n) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \cdots + \frac{3^m}{2^r} \cdot x = 1,$$

For example, $n = 7$, $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1 \to \cdots$, the algebraic formula is

$$T^{16}(7) = T(5, 11, 7) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{11}} \cdot 7 = 1,$$

and the Collatz graph is Fig. 1.

And $n = 36$, the algebraic formula is

$$T^{21}(36) = T(6, 15, 36) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{15}} \cdot 36 = 1$$

and the Collatz graph is Fig. 2.

3 Numerical example

We propose the following algebraic formulas,

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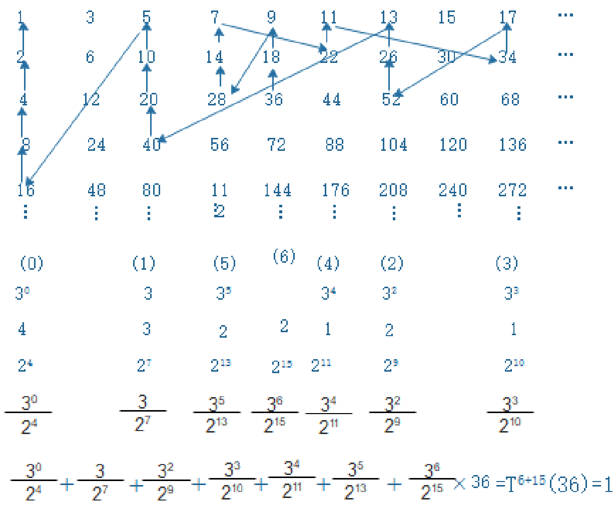

Fig. 2. The Collatz graph of $T^{21}(36) = T(6, 15, 36) = 1$ in the lattice of integers in the modified Sarkovskii ordering and the algebraic formula.

$$T^{20}(18) = T(6, 14, 18) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{13}} + \frac{3^6}{2^{14}} \cdot 18 = 1,$$

$$T^{15}(23) = T(4, 11, 23) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{11}} \cdot 23 = 1,$$
$$T^{17}(15) = T(5, 12, 15) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{12}} + \frac{3^5}{2^{12}} \cdot 15 = 1,$$
$$T^{12}(17) = T(3, 9, 17) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^9} \cdot 17 = 1,$$
$$T^{16}(397) = T(5, 11, 397) = \frac{1}{2^4} + \frac{3}{2^7} + \frac{3^2}{2^9} + \frac{3^3}{2^{10}} + \frac{3^4}{2^{11}} + \frac{3^5}{2^{17}} + \frac{3^6}{2^{20}} + \frac{3^7}{2^{20}} \cdot 397 = 1,$$
$${}^{19}(61) = T(5, 14, 61) = \frac{1}{2^4} + \frac{3}{2^9} + \frac{3^2}{2^{10}} + \frac{3^3}{2^{11}} + \frac{3^4}{2^{14}} + \frac{3^5}{2^{14}} \cdot 61 = 1.$$

4 Convert the integer number from decimal to binary

Be inspired by the above, we use binary to describe the Collatz function (1) an the follows. We denote binary number which is a string of 0s and 1s, $n = (1 \cdots \times)_2$, $\times$ is 1 or 0, $3 = (11)_2$,

$$T(n) = T((1 \cdots \times)_2) = \begin{cases} (11)_2 \cdot (1 \cdots 1)_2 + 1, & \text{if } n \text{ is odd number,} \\ \frac{(1 \cdots 0)_2}{(10)_2}, & \text{if } n \text{ is even number.} \end{cases} \tag{2}$$

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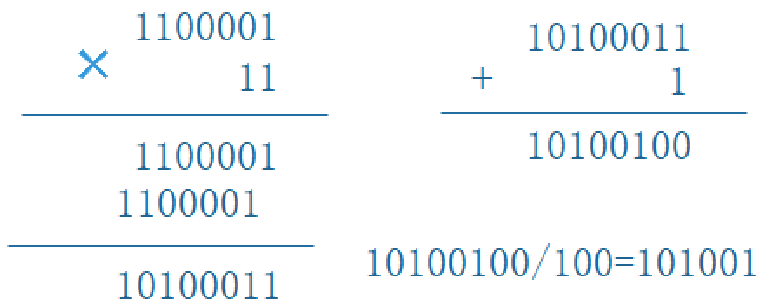

Fig. 3. For the Collatz function $T(97)$ in binary, the first step is the multiplication in left, the second step is division in right bottom.

$$T(n) = T((1 \cdots \times)_2) = \begin{cases} (1 \times \times \times \times 0 \cdots 0)_2, & \text{if } n \text{ is odd number,} \\ (1 \times \times \times 1)_2, & \text{if } n \text{ is even number.} \end{cases} \tag{3}$$

Namely, when n is odd number, we multipe it with $(11)_2$ and add 1 to the end of the binary number, we give an example, $T(97) = T(1100001)$ in Fig. 3 in the following. When n is even number, the division is equal to delete zeros at the end in binary number. We give the iteration of the Collatz function for 7, 97 in binary as the following tables.

Example 1 *For 7=(111)$_2$, we calculate the iteration of the Collatz function*

ith	*0*	*1*	*2*	*3*	*4*	*5*	*7*	*8*	*11*	*12*	*16*
decimal	*7*	*22*	*11*	*34*	*17*	*52*	*13*	*40*	*5*	*16*	*1*
binary	*111*	*10110*	*1011*	*100010*	*10001*	*110100*	*1101*	*101000*	*101*	*10000*	*1*

Example 2 *For 97=(100100)$_2$, we calculate the iteration of the Collatz func-*

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tion as the following,

0	**1100001**	30	11001110	60	**110101001**	90	1101100010
1	100100100	31	**1100111**	61	10011111100	91	**110110001**
2	10010010	32	100110110	62	1001111110	92	10100010100
3	**1001001**	33	**10011011**	63	**100111111**	93	1010001010
4	11011100	34	111010010	64	1110111110	94	**101000101**
5	1101110	35	**11101001**	65	**111011111**	95	1111010000
6	**110111**	36	1010111100	66	10110011110	96	111101000
7	11111010	37	101011110	67	**1011001111**	97	11110100
8	**1111101**	38	**10101111**	68	100001101110	98	1111010
9	10100110	39	1000001110	69	**10000110111**	99	**111101**
10	**1010011**	40	**100000111**	70	110010100110	100	10111000
11	11111010	41	11000010110	71	**11001010011**	101	1011100
12	**1111101**	42	**1100001011**	72	1001011111010	102	101110
13	101111000	43	10010100010	73	**100101111101**	103	**10111**
14	10111100	44	**1001010001**	74	1110001111000	104	1000110
15	1011110	45	11011110100	75	111000111100	105	**100011**
16	**101111**	46	1101111010	76	11100011110	106	1101010
17	10001110	47	**110111101**	77	**1110001111**	107	**110101**
18	**1101011**	48	10100111000	78	101010101110	108	10100000
19	101000010	49	1010011100	79	**10101010111**	109	1010000
20	**10100001**	50	101001110	80	1000000000110	110	101000
21	111100100	51	**10100111**	81	**100000000011**	111	10100
22	11110010	52	111110110	82	1100000001010	112	1010
23	**1111001**	53	**11111011**	83	**110000000101**	113	**101**
24	101101100	54	1011110010	84	10010000010000	114	10000
25	10110110	55	**101111001**	85	1001000001000	115	1000
26	**1011011**	56	10001101100	86	100100000100	116	100
27	100010010	57	1000110110	87	10010000010	117	10
28	**10001001**	58	**100011011**	88	**1001000001**	118	1
29	110011100	59	1101010010	89	11011000100		

We can rewrite the Collatz conjecture in binary as the following, it become an easy problem.

Fact 3 *For any positive integer, under the Collatz function, the sequence of integer number in binary eventually must reach the integer 1.*

PROOF. *For an binary number of odd integer, when we add 1 to the end, and add the shifted binary number one place to the left, finally, the result number must with zeros at the end, we remove this zeros. Thus repeat this process as long as we can, eventually we must reach 1.*

Remark 4 *We can say that* $3x + 1$ *problem is an converse proposition of period three implies chaos [4] and an example.*

5 Conclusion

In the integer lattice in the modifying the Sarkovskii ordering, denote the composition of the Collatz function as a algebraic formula about the $\frac{3^m}{2^r}$, we give a bridge of algebraic formula with graphs. We completely solve the $3x+1$ problem.

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