Article

A Solution of The Collatz Conjecture Problem

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Abstract: Research Collatz odd sequence, change $(\times 3 + 1) \div 2^k$ operation in Collatz Conjecture to $(\times 3 + 2^m - 1) \div 2^k$ operation. Build $(\times 3 + 2^m - 1) \div 2^k$ odd tree model and transform position model for odds in tree. Via comparing actual and virtual positions, prove if a $(\times 3 + 2^m - 1) \div 2^k$ odd sequence can not converge after infinite steps of $(\times 3 + 2^m - 1) \div 2^k$ operation, the sequence must walk out of boundary of the tree.

Keywords: Collatz Conjecture; Collatz odd sequence; $(\times 3 + 2^m - 1) \div 2^k$ odd sequence; $(\times 3 + 2^m - 1) \div 2^k$ odd tree

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Terms:

$(\times 3 + 1) \div 2^k$: $\forall b \in O\{1, 3, 5, 7, 9...\}$, $O\{...\}$ is odd set, do $b \times 3 + 1$, then do k times $\div 2$ repeatedly until get an odd. At this point, these odds are called Collatz odds.

$(\times 3 + 2^m - 1) \div 2^k$: $\forall a \in O\{3, 5, 7, 9...\}$, highest binary bit is 2^{m-1}, do $a \times 3 + 2^m - 1$, then do k times $\div 2$ repeatedly until get an odd.

1. Introduction About The Collatz Conjecture

The Collatz Conjecture is a famous math conjecture, named after mathematician Lothar Collatz, who introduced the idea in 1937. It is also known as the 3x + 1 conjecture, the Ulam conjecture etc. Many mathematicians have tried to prove it true or false and have expanded it to more digits scale. But until today, it has not yet been proved.

The Collatz Conjecture concerns sequences of positive integers in which each term is obtained from the previous one as follows: if the previous integer is even, the next integer is the previous integer divided by 2. If the previous integer is odd, the next term is the previous integer multiply 3 and plus 1. The conjecture is that these sequences always reach 1, no matter which positive integer is chosen to start the sequence.

Here is an example for a typical integer $x = 27$, takes up to 111 steps, increasing or decreasing step by step, climbing as high as 9232 before descending to 1.

27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1.

If the conjecture is false, there should exist some starting number which gives rise to a sequence that does not contain 1. Such a sequence would either enter a repeating cycle that excludes 1, or increase without bound. No such sequence has been found by human or computer after verified a lot of numbers can reach to 1. It is very difficult to prove these two cases exist or not.

This paper will try to prove the conjecture true from a special view. Because any even can become odd through $\div 2^k$ operation, this paper will research only odd characters in the conjecture sequence. The equivalence conjecture becomes: with random starting odd x, do $(\times 3 + 1) \div 2^k$ operation repeatedly, it always converges to 1. The above sequence can be written as follows, in which numbers on arrows are k in $\div 2^k$ in each step:

$27 \xrightarrow{1} 41 \xrightarrow{2} 31 \xrightarrow{1} 47 \xrightarrow{1} 71 \xrightarrow{1} 107 \xrightarrow{1} 161 \xrightarrow{2} 121 \xrightarrow{2} 91 \xrightarrow{1} 137 \xrightarrow{2} 103 \xrightarrow{1} 155 \xrightarrow{1} 233 \xrightarrow{2} 175 \xrightarrow{1} 263 \xrightarrow{1} 395 \xrightarrow{1} 593 \xrightarrow{2} 445 \xrightarrow{3} 167 \xrightarrow{1} 251 \xrightarrow{1} 377 \xrightarrow{2} 283 \xrightarrow{1} 425 \xrightarrow{2} 319 \xrightarrow{1} 479 \xrightarrow{1} 719 \xrightarrow{1} 1079 \xrightarrow{1} 1619 \xrightarrow{1} 2429 \xrightarrow{3} 911 \xrightarrow{1} 1367 \xrightarrow{1} 2051 \xrightarrow{1} 3077 \xrightarrow{4} 577 \xrightarrow{2} 433 \xrightarrow{2} 325 \xrightarrow{4} 61 \xrightarrow{3} 23 \xrightarrow{1} 35 \xrightarrow{1} 53 \xrightarrow{5} 5 \xrightarrow{4} 1$

2. Equivalence of $(\times 3 + 1) \div 2^k$ And $(\times 3 + 2^m - 1) \div 2^k$ Operation

Lemma 2.1: $\forall a \in O\{3, 5, 7, 9...\}$, highest binary bit is $2^{m-1}(m \geq 2)$, do $(\times 3 + 2^m - 1) \div 2^k$ operation get odd b, with complement odd $2^m - a$ do $(\times 3 + 1) \div 2^k$ operation get odd c, then their times of $\div 2$ operation is same, $b + c = 2^j$, and $b > c$.

Prove: First, $2^m > a > 2^{m-1} > 2^m - a$

$3 \times a + 2^m - 1 = b \times 2^{p_1}$

$3 \times (2^m - a) + 1 = c \times 2^{p_2}$

$3 \times (2^m - a) + 1 + 3 \times a + 2^m - 1 = 2^{m+2} = b \times 2^{p_1} + c \times 2^{p_2}$

Because both b and c are odds, then

$2^{p_1} = 2^{p_2}$, and $b + c = 2^{m+2-p_1} = 2^j$

$3 \times a + 2^m - 1 - (3 \times (2^m - a) + 1) = 6 \times a - 2^{m+1} - 4 > 6 \times 2^{m-1} - 2^{m+1} - 4 = 2^m - 4 \geq 0$

Then $b > c$.

For example, $(9 \times 3 + 16 - 1) \div 2 = 21, (7 \times 3 + 1) \div 2 = 11, 21 + 11 = 2^5, 21 > 11$

This states, with any Collatz odd $2^m - a(a > 2^m - a)$ do $(\times 3 + 1) \div 2^k$ operation, we can do $(\times 3 + 2^m - 1) \div 2^k$ operation with its complement odd a instead, they are equivalent.

For example, use 7 as start Collatz odd, Collatz sequence is: $7, 11, 17, 13, 5, 1$. Use 9 as corresponding start complement odd, $(\times 3 + 2^m - 1) \div 2^k$ sequence is: $9, 21, 47, 51, 27, 7$.

Lemma 2.2: $\forall a = 2^m - 1(m \geq 2)$, do $(\times 3 + 2^m - 1) \div 2^k$ operation get odd b, $b = a$.

Prove: $a \times 3 + 2^m - 1 = (2^m - 1) \times 3 + 2^m - 1 = (2^m - 1) \times 2^2$

$b = (2^m - 1) \times 2^2 \div 2^2 = 2^m - 1 = a$

We call odds of form $2^m - 1$ is convergence state for any odds(> 3), if they can reach to after doing $(\times 3 + 2^m - 1) \div 2^k$ operation repeatedly.

Lemma 2.3: $\forall a \in O\{3, 5, 7, 9...\}$, highest binary bit is $2^{m-1}(m \geq 2)$, do $\times 3 + 2^m - 1$ operation get an even, result must grow 2 binary bits, with odd part of result do $\times 3 + 2^m - 1$ operation repeatedly, m is highest binary bit number plus 1 of the odd part produced in each step, then count of succession binary bit 1 in head part of odd produced in each step must remain unchanged or increased, and must increase 1 within 3 steps, until reach to convergence state.

Prove: $a = 2^{m-1} + a_1 \times 2^{m-2} + ...a_{m-2} \times 2 + 1, (a_1...a_{m-2} = 0 or 1)$

$a \times 3 + 2^m - 1 = 2^{m+1} + 2^{m-1} + 3 \times (a_1 \times 2^{m-2} + ...a_{m-2} \times 2) + 2$, tail part $3 \times (a_1 \times 2^{m-2} + ...a_{m-2} \times 2) + 2$ is possible to carry 0, 1 or 2 bits to head part. If carries 0 bit, head part changes nothing; If carries 1 bit, head part $+2^{m-1}$; If carries 2 bits, head part $+2^m + 0 \times 2^{m-1}$. All cases highest binary bit is 2^{m+1}, grows 2 binary bits.

If head of first odd or odd part of result has only one succession binary bit 1, has following 4 cases:

Case 1: odd part is with binary form 1000*...1, head part becomes 101000 after doing $\times 3 + 2^m$, tail part *...1 is possible to carry 0, 1 or 2 bits to head part after doing $\times 3 - 1$. If carries 0 bit, head part changes nothing; If carries 1 bit, head part +1, becomes 101001; If carries 2 bits, head part +10, becomes 101010. All cases form of head part is same with case 3, count of succession binary bit 1 remains unchanged.

Case 2: odd part is with binary form 1001*...1, head part becomes 101011 after doing $\times 3 + 2^m$, tail part *...1 is possible to carry 0, 1 or 2 bits to head part after doing $\times 3 - 1$. If carries 0 bit, head part changes nothing, its form is same with case 3; If carries 1 bit, head part +1, becomes 101100, its form is same with case 4; If carries 2 bits, head part +10, becomes 101101, its form is same with case 4. All cases count of succession binary bit 1 remains unchanged.

Case 3: odd part is with binary form 1010*...1, head part becomes 101110 after doing $\times 3 + 2^m$, tail part *...1 is possible to carry 0, 1 or 2 bits to head part after doing $\times 3 - 1$. If carries 0 bit, head part changes nothing, its form is same with case 4, count of succession binary bit 1 remains unchanged; If carries 1 bit, head part +1, becomes 101111, its form is same with case 4, count of succession binary bit 1 remains unchanged; If carries 2 bits, head part +10, becomes 110000, count of succession binary bit 1 increases 1.

Case 4: odd part is with binary form 1011*...1, head part becomes 110001 after doing $\times 3 + 2^m$, tail part *...1 is possible to carry 0, 1 or 2 bits to head part after doing $\times 3 - 1$. All cases count of succession binary bit 1 increases 1.

Hence, within 3 steps count of succession binary bit 1 in head part of odd increases 1, until reach to convergence state. In convergence state, count of succession binary bit 1 always remains unchanged.

Similar to cases of head of first odd or odd part of result produced in each step have more than one succession binary bit 1.

For example, $1110001 + 11100010 + 2^7 - 1 = 111010010$, $11101001 + 111010010 + 2^8 - 1 = 1110111010$, $111011101 + 1110111010 + 2^9 - 1 = 11110010110$.

Corollary: If exists loop sequence $b_1, b_2, ...b_i, b_1, b_2, ...b_i (i > 3)$ in Collatz odd sequence, expand in corresponding $(\times 3 + 2^m - 1) \div 2^k$ odd sequence, get odd sequence $a_1, a_2, ...a_i, a_{i+1}, a_{i+2}, ...a_{2i}$, then $a_{i+1} \neq a_1, a_{i+2} \neq a_2, ...a_{2i} \neq a_i$.

3. $(\times 3 + 2^m - 1) \div 2^k$ **Odd Tree**

Use all odds in order in set $O\{3, 5, 7, 9...\}$ to build a "tree":

...

L6: 129(321.1) 131(81.3) 133(327.1) 135(165.2) 137(333.1) 139(21.5) 141(339.1) 143(171.2) 145(345.1) 147(87.3) 149(351.1) 151(177.2) 153(357.1) 155(45.4) 157(363.1) 159(183.2) 161(369.1) 163(93.3) 165(375.1) 167(189.2) 169(381.1) 171(3.8) 173(387.1) 175(195.2) 177(393.1) 179(99.3) 181(399.1) 183(201.2) 185(405.1) 187(51.4) 189(411.1) 191(207.2) 193(417.1) 195(105.3) 197(423.1) 199(213.2) 201(429.1) 203(27.5) 205(435.1) 207(219.2) 209(441.1) 211(111.3) 213(447.1) 215(225.2) 217(453.1) 219(57.4) 221(459.1) 223(231.2) 225(465.1) 227(117.3) 229(471.1) 231(237.2) 233(477.1) 235(15.6) 237(483.1) 239(243.2) 241(489.1) 243(123.3) 245(495.1) 247(249.2) 249(501.1) 251(63.4) 253(507.1) 255

L5: 65(161.1) 67(41.3) 69(167.1) 71(85.2) 73(173.1) 75(11.5) 77(179.1) 79(91.2) 81(185.1) 83(47.3) 85(191.1) 87(97.2) 89(197.1) 91(25.4) 93(203.1) 95(103.2) 97(209.1) 99(53.3) 101(215.1) 103(109.2) 105(221.1) 107(7.6) 109(227.1) 111(115.2) 113(233.1) 115(59.3) 117(239.1) 119(121.2) 121(245.1) 123(31.4) 125(251.1) 127

L4: 33(81.1) 35(21.3) 37(87.1) 39(45.2) 41(93.1) 43(3.6) 45(99.1) 47(51.2) 49(105.1) 51(27.3) 53(111.1) 55(57.2) 57(117.1) 59(15.4) 61(123.1) 63

L3: 17(41.1) 19(11.3) 21(47.1) 23(25.2) 25(53.1) 27(7.4) 29(59.1) 31

L2: 9(21.1) 11(3.4) 13(27.1) 15

L1: 5(11.1) 7

L0: 3

Graph 3.1 $(\times 3 + 2^m - 1) \div 2^k$ odd tree

All odds in the tree do $(\times 3 + 2^m - 1) \div 2^k$ operation, $a.b$ in () means result is $a \times 2^b$ after front odd doing $(\times 3 + 2^m - 1) \div 2^k$ operation, value b is called step property of the front odd. Layer i has 2^i elements, first element is $2^{i+1} + 1$, last element is $2^{i+2} - 1$, which is convergence state.

In this tree, because element count of each layer is 2 times of which of downward layer, we can transform all element positions to one specific layer, layer $i - 2$ transform to layer $i - 1$ do $\times 2$, layer i transform to layer $i - 1$ do $\div 2$..., then all transformed positions(to layer $i - 1$) can not exceed 2^{i-1}.

Lemma 3.1: Suppose a is any odd before convergence state in layer $m - 1$, $a \times 3 + 2^{m+1} - 1 = b \times 2^{p_1}$, b is odd, then b is in layer $m + 1 - p_1$.

Prove: According to Lemma 2.3, highest binary bit of $b \times 2^{p_1}$ is 2^{m+2}, then highest binary bit of b is 2^{m+2-p_1}, is in layer $m + 1 - p_1$.

Lemma 3.2: Suppose a is any odd before convergence state in layer $m - 1$, highest binary bit is 2^m, $2^{m+1} - a > 3$, do $(\times 3 + 2^m - 1) \div 2^k$ get odd b, b do $(\times 3 + 2^m - 1) \div 2^k$ get odd c. Then transform position(to layer $m - 1$) of b is bigger than which of a. If step property of a is $p_1 = 2$, step property of b is $p_2 = 2$, transform position increment ratio from a to c is $\frac{3}{4}$.

Prove: Position of a in layer $m - 1$ is: $\frac{a - 2^m + 1}{2}$,

$3 \times a + 2^{m+1} - 1 = b \times 2^{p_1}$, b is in layer $m - p_1 + 1$

Position of b in layer $m - p_1 + 1$ is: $\frac{b - 2^{m-p_1+2}+1}{2}$,

Position of b in layer $m - 1$ is: $\frac{b - 2^{m-p_1+2}+1}{2^{3-p_1}}$

$3^2 \times a + 3 \times 2^{m+1} - 3 + 2^{m+3} - 2^{p_1} = c \times 2^{p_1+p_2}$, c is in layer $m + 3 - p_1 - p_2$

Position of c in layer $m + 3 - p_1 - p_2$ is: $\frac{c - 2^{m+4-p_1-p_2}+1}{2}$

Position of c in layer $m - 1$ is: $\frac{c - 2^{m+4-p_1-p_2}+1}{2^{5-p_1-p_2}}$

Transform position increment from odd a to b is:

$\delta = \frac{b - 2^{m-p_1+2}+1}{2^{3-p_1}} - \frac{a - 2^{m}+1}{2} = \frac{b + 1 - 2^{2-p_1} \times a - 2^{2-p_1}}{2^{3-p_1}}$
$= \frac{b \times 2^{p_1} + 2^{p_1} - 2^2 \times a - 2^2}{2^3} = \frac{3 \times a + 2^{m+1} - 1 + 2^{p_1} - 2^2 \times a - 2^2}{2^3}$
$= \frac{2^{m+1} - a + 2^{p_1} - 5}{2^3}$

Only when $2^{m+1} - a = 1$ and $p_1 = 2$ or $2^{m+1} - a = 3$ and $p_1 = 1$, $\delta = 0$, these two cases are convergence state or quasi convergence state of Collatz odd sequence. Other cases $\delta > 0$, even in expanding loop sequence. When $2^{m+1} - a > 3$, the bigger p_1 is, the bigger δ will be.

$\frac{b - 2^{m-p_1+2}+1}{2^{3-p_1}} - \frac{a - 2^{m}+1}{2} = \frac{b + 1 - 2^{2-p_1} \times a - 2^{2-p_1}}{2^{3-p_1}}$

$\frac{c - 2^{m+4-p_1-p_2}+1}{2^{5-p_1-p_2}} - \frac{b - 2^{m-p_1+2}+1}{2^{3-p_1}} = \frac{c + 1 - 2^{2-p_2} \times b - 2^{2-p_2}}{2^{5-p_1-p_2}}$, transform position increment ratio from a to c

is:

$r = \frac{c + 1 - 2^{2-p_2} \times b - 2^{2-p_2}}{2^{5-p_1-p_2}} \times \frac{2^{3-p_1}}{b + 1 - 2^{2-p_1} \times a - 2^{2-p_1}}$
$= \frac{c \times 2^{p_2} + 2^{p_2} - 2^2 \times b - 2^2}{2^2} \times \frac{2^{p_1}}{b \times 2^{p_1} + 2^{p_1} - 2^2 \times a - 2^2}$
$= \frac{c \times 2^{p_1+p_2} + 2^{p_1+p_2} - 2^{2+p_1} \times b - 2^{2+p_1}}{2^2} \times \frac{1}{3 \times a + 2^{m+1} - 1 + 2^{p_1} - 2^2 \times a - 2^2}$
$= \frac{3^2 \times a + 3 \times 2^{m+1} - 3 + 2^{m+3} - 2^{p_1} + 2^{p_1+p_2} - 2^{2+p_1} \times b - 2^{2+p_1}}{2^2} \times \frac{1}{2^{m+1} + 2^{p_1} - a - 5}$
$= \frac{3^2 \times a + 3 \times 2^{m+1} - 3 + 2^{m+3} - 2^{p_1} + 2^{p_1+p_2} - 2^2 \times (3 \times a + 2^{m+1} - 1) - 2^{2+p_1}}{2^2} \times \frac{1}{2^{m+1} + 2^{p_1} - a - 5}$
$= \frac{3 \times 2^{m+1} - 3 \times a - 5 \times 2^{p_1} + 2^{p_1+p_2} + 1}{2^2} \times \frac{1}{2^{m+1} + 2^{p_1} - a - 5}$
$= \frac{3 \times (2^{m+1} + 2^{p_1} - a - 5) + 2^{p_1+p_2} - 8 \times 2^{p_1} + 16}{2^2 \times (2^{m+1} + 2^{p_1} - a - 5)}$
$= \frac{3}{4} + \frac{2^{p_1+p_2} - 8 \times 2^{p_1} + 16}{2^2 \times (2^{m+1} + 2^{p_1} - a - 5)}$
$= \frac{3}{4} + \frac{2^{p_1} \times (2^{p_2} - 8) + 16}{2^2 \times (2^{m+1} + 2^{p_1} - a - 5)}$

Because $2^{m+1} - a > 3$, $2^{m+1} + 2^{p_1} - a - 5 > 0$. Then if $p_2 = 1$ and $p_1 \geq 2$ or $p2 = 2$ and $p1 \geq 3$, $r < \frac{3}{4}$; if $p_1 = p_2 = 2$, $r = \frac{3}{4}$; other cases $r > \frac{3}{4}$.

Obviously, the transform position increment regularity is also suitable for all following steps, as long as in each step corresponding Collatz odd is bigger than 3.

4. Convergence of Transform Position For Odds in $(\times 3 + 2^m - 1) \div 2^k$ Odd Tree

Suppose start odd a is any odd before convergence state in layer $m - 1$, s_0 is its position in layer $m - 1$, $s_i (i \geq 1)$ is transform position(to layer $m - 1$) of odd produced in step i, deduce common transform position s_i:

$s_0 = \frac{a - 2^m + 1}{2} = 2^{m-1} + \frac{1 - (2^{m+1} - a)}{2}$

$s_1 = \frac{3 \times a + 2^{m+1} - 1 - 2^{m+2} + 2^{p_1}}{2^3} = 2^{m-1} - \frac{3^1 \times (2^{m+1} - a) + 1 - 2^{p_1}}{2^{2 \times 1+1}} = 2^{m-1} + \frac{2^{p_1} - 3^1 \times (2^{m+1} - a) - 1}{2^{2 \times 1+1}}$

$s_2 = \frac{3^2 \times a + 3 \times 2^{m+1} - 3 + 2^{m+3} - 2^{p_1} - 2^{m+4} + 2^{p_1+p_2}}{2^5} = 2^{m-1} + \frac{2^{p_1+p_2} - 3^2 \times (2^{m+1} - a) - 3 - 2^{p_1}}{2^{2 \times 2+1}}$

...

Use mathematical induction get

$s_i = 2^{m-1} + \frac{2^{p_1+p_2+\cdots p_i} - 3^i \times (2^{m+1} - a) - 3^{i-1} - 3^{i-2} \times 2^{p_1} - \ldots 2^{p_1+p_2+\cdots p_{i-1}}}{2^{2i+1}}$.

Furthmore, use $s_{(p_1, p_2 \ldots p_i)}$ to represent transform position(to layer $m - 1$) of odd produced in step i from start odd a in layer $m - 1$. We can change the value of step property $p_k (1 \leq k \leq i)$ to different positive integer or delete middle steps in order to compare two transform positions. At this point, the modified transform position is called virtual transform position, use same common formula to calculate two kinds of transform position.

Lemma 4.1: a is positive rational number, $\forall a, 2^{m+1} > a > 2^{m+1} - a > 1(m > 1)$, highest binary bit of its integer part is 2^m, do operation similar with $(\times 3 + 2^m - 1) \div 2^k$: $a \times 3 + 2^{m+1} - 1 = b \times 2^2$, $3 \times b \times 2^2 + 2^{m+3} - 2^2 = c \times 2^4$. Then $2^{m+1} > b > 2^{m+1} - b > 1, 2^{m+1} > c > 2^{m+1} - c > 1$, virtual transform position(to layer $m - 1$) of b is bigger than which of a, virtual transform position increment ratio from a to c is $\frac{3}{4}$.

Prove: $b = \frac{3}{4} \times a + 2^{m-1} - \frac{1}{4} < \frac{3}{4} \times 2^{m+1} + 2^{m-1} - \frac{1}{4} = 2^{m+1} - \frac{1}{4} < 2^{m+1}$

Because $a > 2^{m+1} - a, a > 2^m$

$b = \frac{3}{4} \times a + 2^{m-1} - \frac{1}{4} > \frac{3}{4} \times 2^m + 2^{m-1} - \frac{1}{4} = 2^m + 2^{m-2} - \frac{1}{4} > 2^m$, then $b > 2^{m+1} - b$

Because $2^{m+1} - a > 1, a < 2^{m+1} - 1$

$2^{m+1} - b = 2^{m+1} - \frac{3}{4} \times a - 2^{m-1} + \frac{1}{4} > 2^{m+1} - \frac{3}{4} \times (2^{m+1} - 1) - 2^{m-1} + \frac{1}{4} = 1$

Then $2^{m+1} > b > 2^{m+1} - b > 1$

Because $c = \frac{3}{4} \times b + 2^{m-1} - \frac{1}{4}$, use same way can prove $2^{m+1} > c > 2^{m+1} - c > 1$.

Because two kinds of transform position formula are same, virtual transform position increment from odd a to b is:

$\delta = s_1 - s_0 = \frac{2^{m+1} - a + 2^2 - 5}{2^3}$, when $2^{m+1} - a > 1, \delta > 0$

Virtual transform position increment ratio from a to c is:

$r = \frac{3}{4} + \frac{2^2 \times (2^2 - 8) + 16}{2^2 \times (2^{m+1} + 2^2 - a - 5)} = \frac{3}{4}$

Obviously, the virtual transform position increment regularity is also suitable for all following steps, as long as step property of each step is 2.

Lemma 4.2: For any non-convergence start odd a in layer $m - 1$, if $2^{m+1} - a > 3$, $s_{(1,2,2,2...)} > s_{(2,2,2...)}$.

Prove:

$s_{(1,2)} - s_{(2)} = \frac{2^3 - 3^2 \times (2^{m+1} - a) - 3 - 2}{2^5} - \frac{2^2 - 3 \times (2^{m+1} - a) - 1}{2^3} = \frac{3}{4} \times \frac{2^{m+1} - a - 3}{2^3} > 0$

$s_{(1,2,2)} - s_{(2,2)} = \frac{2^5 - 3^3 \times (2^{m+1} - a) - 3^2 - 3^1 \times 2 - 2^3}{2^7} - \frac{2^4 - 3^2 \times (2^{m+1} - a) - 3 - 2^2}{2^5}$

$= \frac{2^5 - (3^3 \times (2^{m+1} - a) + 3^2 + 3^1 \times 2 + 2^3) - 4 \times 2^4 + 4 \times (3^2 \times (2^{m+1} - a) + 3 + 2^2)}{2^7}$

$= (\frac{3}{4})^2 \times \frac{2^{m+1} - a - 3}{2^3} > 0$

$s_{(1,2,2,2)} - s_{(2,2,2)} = \frac{2^{5+2} - 3 \times (3^3 \times (2^{m+1} - a) + 3^2 + 3^1 \times 2 + 2^3) - 2^5}{2^{7+2}} - \frac{2^{4+2} - 3 \times (3^2 \times (2^{m+1} - a) + 3 + 2^2) - 2^4}{2^{5+2}}$

$= \frac{3 \times 2^5 - 3 \times (3^3 \times (2^{m+1} - a) + 3^2 + 3^1 \times 2 + 2^3)}{2^{7+2}} - \frac{3 \times 2^4 - 3 \times (3^2 \times (2^{m+1} - a) + 3 + 2^2)}{2^{5+2}}$

$= (\frac{3}{4})^3 \times \frac{2^{m+1} - a - 3}{2^3} > 0$

If all $p_i = 2(1 \le i \le \infty)$, use mathematical induction get

$s_{(1,p_1,p_2...p_i)} - s_{(p_1,p_2...p_i)} = (\frac{3}{4})^i \times \frac{2^{m+1} - a - 3}{2^3} > 0$, then $s_{(1,2,2,2...)} > s_{(2,2,2...)}$.

Lemma 4.3: For any non-convergence start odd a in layer $m - 1$, if $2^{m+1} - a > 3$, $s_{(1,1,2,2,2...)} > s_{(2,2,2...)}$.

Prove:

$s_{(1,1,2)} - s_{(2)} = \frac{2^4 - 3^3 \times (2^{m+1} - a) - 3^2 - 3^1 \times 2 - 2^2}{2^7} - \frac{2^2 - 3 \times (2^{m+1} - a) - 1}{2^3} = \frac{3}{4} \times \frac{7 \times (2^{m+1} - a) - 17}{2^5} > 0$

$s_{(1,1,2,2)} - s_{(2,2)} = \frac{2^6 - 3^4 \times (2^{m+1} - a) - 3^3 - 3^2 \times 2 - 3 \times 2^2 - 2^4}{2^9} - \frac{2^4 - 3^2 \times (2^{m+1} - a) - 3 - 2^2}{2^5}$

$= (\frac{3}{4})^2 \times \frac{7 \times (2^{m+1} - a) - 17}{2^5} > 0$

$s_{(1,1,2,2,2)} - s_{(2,2,2)} = (\frac{3}{4})^3 \times \frac{7 \times (2^{m+1} - a) - 17}{2^5} > 0$

If all $p_i = 2(1 \le i \le \infty)$, use mathematical induction get

$s_{(1,1,p_1,p_2...p_i)} - s_{(p_1,p_2...p_i)} = (\frac{3}{4})^i \times \frac{7 \times (2^{m+1} - a) - 17}{2^5} > 0$, then $s_{(1,1,2,2,2...)} > s_{(2,2,2...)}$.

Lemma 4.4: For any non-convergence start odd a in layer $m - 1$, if after doing $(\times 3 + 2^m - 1) \div 2^k$ operation, corresponding Collatz odd in each step is bigger than 3, then $s_{(p_1,p_2...,p_i,1...,1,2^+,1...,1,2^+)} > s_{(p_1,p_2...,p_i,2,2)}$, where $2^+ \ge 2$.

Prove: $s_{(2^+,2)} - s_{(2,2)} = \frac{2^{2^+ + 2} - 3^2 \times (2^{m+1} - a) - 3 - 2^{2^+}}{2^5} - \frac{2^{2+2} - 3^2 \times (2^{m+1} - a) - 3 - 2^2}{2^5} = \frac{3 \times 2^{2^+} - 3 \times 2^2}{2^5} \ge 0$

According to Lemma 4.2 and Lemma 4.3

$s_{(p_1,p_2...,p_i,1...,1,2^+)} \ge s_{(p_1,p_2...,p_i,1...,1,2)} > s_{(p_1,p_2...,p_i,2)}$

$s_{(p_1,p_2...,p_i,1...,1,2^+,1...,1,2^+)} > s_{(p_1,p_2...,p_i,1...,1,2^+,2)} \ge s_{(p_1,p_2...,p_i,1...1,2,2)} > s_{(p_1,p_2...,p_i,2,2)}$.

The result can be extended to more cases such as $s_{(p_1,p_2...,p_i,1...,1,2^+,1...,1,2^+,1...,1,2^+)}\cdots$

For example, with start Collatz odd 27, Select 37 as corresponding start odd in $(\times 3 + 2^m - 1) \div 2^k$ odd tree, $s_0 = 3$, original transform positions(to layer 4) in following steps are:

$s_{(1,2)} = 8.5, s_{(1,2,1,1,1,2)} = 14.125, s_{(1,2,1,1,1,2,2)} = 14.59375, s_{(1,2,1,1,1,2,2,1,2)} = 15.203125, s_{(1,2,1,1,1,2,2,1,2,1,1,2)} = 15.66015625...$

Corresponding virtual transform positions(to layer 4) are:

$s_{(2)} = 6.25, s_{(2,2)} = 8.6875, s_{(2,2,2)} = 10.515625, s_{(2,2,2,2)} = 11.88671875, s_{(2,2,2,2,2)} = 12.9150390625...$

This is to say, if delete all (1) steps in long sequence and change all (2^+) steps to (2) steps, final virtual transform position is smaller than original, if corresponding original Collatz odd in each step is bigger than 3.

Lemma 4.5: If exists loop Collatz odd sequence, step count must be bigger than 2.

Prove: For any Collatz odd $a > 1$, suppose $3 \times a + 1 = a \times 2^{p_1}$.

Then $(2^{p_1} - 3) \times a = 1$, there exists no odd solution.

Suppose $3 \times a + 1 = b \times 2^{p_1}, 3 \times b + 1 = a \times 2^{p_2}$, where odd $b > 1$.

Then $9 \times a + 3 = 3 \times b \times 2^{p_1}, 3 \times b \times 2^{p_1} + 2^{p_1} = a \times 2^{p_1+p_2}$.

Get $2^{p_1} + 3 = (2^{p_1+p_2} - 9) \times a > 7 \times a$, and $2^{p_1} = \frac{3 \times a + 1}{b} < 3 \times a + 1$, then $3 \times a + 1 + 3 > 2^{p_1} + 3 > 7 \times a, a < 1$, it is contradictory.

Hence, if exists loop Collatz odd sequence, step count must be bigger than 2. This way, according to Lemma 2.3, we can expand loop Collatz odd sequence(if exists), get a ∞ steps $(\times 3 + 2^m - 1) \div 2^k$ odd sequence, and it is no longer loop sequence.

Lemma 4.6: If exists ∞ steps non-convergence Collatz odd sequence, step property of any tail part of the sequence is not possible to always be 1.

Lemma 4.7: It is not possible to exist loop Collatz odd sequence or ∞ steps non-convergence Collatz odd sequence.

Prove: If exists, odds in the sequence must be bigger than 3, change to $(\times 3 + 2^m - 1) \div 2^k$ odd sequence, expand loop Collatz odd sequence, then both cases we get ∞ steps $(\times 3 + 2^m - 1) \div 2^k$ odd sequence, there are many (1) and (2^+) steps.

Select a part sequence from original $(\times 3 + 2^m - 1) \div 2^k$ odd sequence, odd a in layer $m - 1$ as start odd, last step is (2^+), exists many (1) and (2^+) steps in the middle. Use common transform position formula to produce a transform position (to layer $m - 1$) sequence. Delete all (1) steps before last step and change all (2^+) steps to (2) steps, use common transform position formula to produce a new z steps virtual transform position sequence.

According to Lemma 4.1, transform position increment ratio of new transform position sequence is always $\frac{3}{4}$. According to Lemma 4.4, final transform position of original sequence is(can also be gotten from common transform position formula):

$s_{original} > s_{new} = \frac{a - 2^m + 1}{2} + \left(\frac{2^{m+1} - a + 2^2 - 5}{2^3}\right) \times (1 + \frac{3}{4} + (\frac{3}{4})^2 + ...(\frac{3}{4})^{z-1})$

Original Collatz sequence must appear (2^+) steps continuously after some (1) steps each time, count of (2^+) steps must be infinite.

When $z \to \infty$,

$s_{original} > s_{new} = \frac{a - 2^m + 1}{2} + \left(\frac{2^{m+1} - a - 1}{2^3}\right) \times 4 = 2^{m-1}$

Walk out of boundary of $(\times 3 + 2^m - 1) \div 2^k$ odd tree, it is not possible in real world.

5. Conclusion

This way, we have proved that the Collatz Conjecture is true. If exists loop or ∞ steps non-convergence Collatz odd sequence, change them to $(\times 3 + 2^m - 1) \div 2^k$ odd sequence, both cases exist ∞ steps, and will finally walk out of boundary of $(\times 3 + 2^m - 1) \div 2^k$ odd tree.

Reference

1. Wikipedia. *Collatz Conjecture*, TheFreeDictionary.com mirror. Ia.thefreedictionary.com

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