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Posted Date: 17 May 2023

doi: 10.20944/preprints202302.0051.v13

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Article

Duality Principles and Numerical Procedures for a Large Class of Non-Convex Models in the Calculus of Variations

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Abstract: This article develops duality principles and numerical results for a large class of non-convex variational models. The main results are based on fundamental tools of convex analysis, duality theory and calculus of variations. More specifically the approach is established for a class of non-convex functionals similar as those found in some models in phase transition. Finally, in the last section we present a concerning numerical example and the respective software.

Keywords: duality theory; non-convex analysis; numerical method for a non-smooth model

MSC: 49N15

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1. Introduction

In this section we establish a dual formulation for a large class of models in non-convex optimization.

The main duality principle is applied to double well models similar as those found in the phase transition theory.

Such results are based on the works of J.J. Telega and W.R. Bielski [2,3,14,15] and on a D.C. optimization approach developed in Toland [16].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [5–7,9,13].

Finally, in this text we adopt the standard Einstein convention of summing up repeated indices, unless otherwise indicated.

In order to clarify the notation, here we introduce the definition of topological dual space.

Definition 1.1 (Topological dual spaces)**.** *Let U be a Banach space. We shall define its dual topological space, as the set of all linear continuous functionals defined on U. We suppose such a dual space of U, may be represented by another Banach space U^*, through a bilinear form $\langle \cdot, \cdot \rangle_U : U \times U^* \to \mathbb{R}$ (here we are referring to standard representations of dual spaces of Sobolev and Lebesgue spaces). Thus, given $f : U \to \mathbb{R}$ linear and continuous, we assume the existence of a unique $u^* \in U^*$ such that*

$$f(u) = \langle u, u^* \rangle_U, \forall u \in U. \tag{1}$$

The norm of f, denoted by $\|f\|_{U^*}$, is defined as

$$\|f\|_{U^*} = \sup_{u \in U}\{|\langle u, u^* \rangle_U| \; : \; \|u\|_U \leq 1\} \equiv \|u^*\|_{U^*}. \tag{2}$$

At this point we start to describe the primal and dual variational formulations.

2. A general duality principle non-convex optimization

In this section we present a duality principle applicable to a model in phase transition.

This case corresponds to the vectorial one in the calculus of variations.

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Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \rightarrow \mathbb{R}$ where

$$J(u) = F(\nabla u_1, \cdots, \nabla u_N) + G(u_1, \cdots, u_N) - \langle u_i, f_i \rangle_{L^2},$$

and where

$$V = \{u = (u_1, \cdots, u_N) \in W^{1,p}(\Omega; \mathbb{R}^N) \; : \; u = u_0 \text{ on } \partial\Omega\},$$

$f \in L^2(\Omega; \mathbb{R}^N)$, and $1 < p < +\infty$.

We assume there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = \inf_{u \in V} J(u).$$

Moreover, suppose F and G are Fréchet differentiable but not necessarily convex. A global optimum point may not be attained for J so that the problem of finding a global minimum for J may not be a solution.

Anyway, one question remains, how the minimizing sequences behave close the infimum of J.

We intend to use duality theory to approximately solve such a global optimization problem.

Denoting $V_0 = W_0^{1,p}(\Omega; \mathbb{R}^N)$, $Y_1 = Y_1^* = L^2(\Omega; \mathbb{R}^{N \times n})$, $Y_2 = Y_2^* = L^2(\Omega; \mathbb{R}^{N \times n})$, $Y_3 = Y_3^* = L^2(\Omega; \mathbb{R}^N)$, at this point we define, $F_1 : V \times V_0 \rightarrow \mathbb{R}$, $G_1 : V \rightarrow \mathbb{R}$, $G_2 : V \rightarrow \mathbb{R}$, $G_3 : V_0 \rightarrow \mathbb{R}$ and $G_4 : V \rightarrow \mathbb{R}$, by

$$\begin{aligned} F_1(\nabla u, \nabla \phi) &= F(\nabla u_1 + \nabla \phi_1, \cdots, \nabla u_N + \nabla \phi_N) + \frac{K}{2} \int_\Omega \nabla u_j \cdot \nabla u_j \, dx \\ &\quad + \frac{K_2}{2} \int_\Omega \nabla \phi_j \cdot \nabla \phi_j \, dx \end{aligned} \tag{3}$$

and

$$G_1(u_1, \cdots, u_n) = G(u_1, \cdots, u_N) + \frac{K_1}{2} \int_\Omega u_j \, u_j \, dx - \langle u_i, f_i \rangle_{L^2},$$

$$G_2(\nabla u_1, \cdots, \nabla u_N) = \frac{K_1}{2} \int_\Omega \nabla u_j \cdot \nabla u_j \, dx,$$

$$G_3(\nabla \phi_1, \cdots, \nabla \phi_N) = \frac{K_2}{2} \int_\Omega \nabla \phi_j \cdot \nabla \phi_j \, dx,$$

and

$$G_4(u_1, \cdots, u_N) = \frac{K_1}{2} \int_\Omega u_j \, u_j \, dx.$$

Define now $J_1 : V \times V_0 \rightarrow \mathbb{R}$,

$$J_1(u, \phi) = F(\nabla u + \nabla \phi) + G(u) - \langle u_i, f_i \rangle_{L^2}.$$

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Observe that

$$\begin{aligned}
J_1(u,\phi) &= F_1(\nabla u, \nabla \phi) + G_1(u) - G_2(\nabla u) - G_3(\nabla \phi) - G_4(u) \\
&\leq F_1(\nabla u, \nabla \phi) + G_1(u) - \langle \nabla u, z_1^* \rangle_{L^2} - \langle \nabla \phi, z_2^* \rangle_{L^2} - \langle u, z_3^* \rangle_{L^2} \\
&\quad + \sup_{v_1 \in Y_1} \{ \langle v_1, z_1^* \rangle_{L^2} - G_2(v_1) \} \\
&\quad + \sup_{v_2 \in Y_2} \{ \langle v_2, z_2^* \rangle_{L^2} - G_3(v_2) \} \\
&\quad + \sup_{u \in V} \{ \langle u, z_3^* \rangle_{L^2} - G_4(u) \} \\
&= F_1(\nabla u, \nabla \phi) + G_1(u) - \langle \nabla u, z_1^* \rangle_{L^2} - \langle \nabla \phi, z_2^* \rangle_{L^2} - \langle u, z_3^* \rangle_{L^2} \\
&\quad + G_2^*(z_1^*) + G_3^*(z_2^*) + G_4^*(z_3^*) \\
&= J_1^*(u, \phi, z^*), \qquad (4)
\end{aligned}$$

$\forall u \in V,\ \phi \in V_0,\ z^* = (z_1^*, z_2^*, z_3^*) \in Y^* = Y_1^* \times Y_2^* \times Y_3^*$.

Here we assume K, K_1, K_2 are large enough so that F_1 and G_1 are convex.

Hence, from the general results in [16], we may infer that

$$\inf_{(u,\phi) \in V \times V_0} J(u,\phi) = \inf_{(u,\phi,z^*) \in V \times V_0 \times Y^*} J_1^*(u,\phi,z^*). \qquad (5)$$

On the other hand

$$\inf_{u \in V} J(u) \geq \inf_{(u,\phi) \in V \times V_0} J_1(u,\phi) \geq \inf_{u \in V} Q_J(u) = \inf_{u \in V} J(u),$$

where $Q_J(u)$ refers to a standard quasi-convex regularization of J.

From these last two results we may obtain

$$\inf_{u \in V} J(u) = \inf_{(u,\phi,z^*) \in V \times V_0 \times Y^*} J_1^*(u,\phi,z^*).$$

Moreover, from standards results on convex analysis, we may have

$$\begin{aligned}
\inf_{u \in V} J_1^*(u,\phi,z^*) &= \inf_{u \in V} \{ F_1(\nabla u, \nabla \phi) + G_1(u) \\
&\quad - \langle \nabla u, z_1^* \rangle_{L^2} - \langle \nabla \phi, z_2^* \rangle_{L^2} - \langle u, z_3^* \rangle_{L^2} \\
&\quad + G_2^*(z_1^*) + G_3^*(z_2^*) + G_4^*(z_3^*) \} \\
&= \sup_{(v_1^*, v_2^*) \in C^*} \{ -F_1^*(v_1^* + z_1^*, \nabla \phi) - G_1^*(v_2^* + z_3^*) - \langle \nabla \phi, z_2^* \rangle_{L^2} \\
&\quad + G_2^*(z_1^*) + G_3^*(z_2^*) + G_4^*(z_3^*) \}, \qquad (6)
\end{aligned}$$

where

$$C^* = \{ v^* = (v_1^*, v_2^*) \in Y_1^* \times Y_3^* \ :\ -\operatorname{div}(v_1^*)_i + (v_2^*)_i = \mathbf{0}, \forall i \in \{1, \cdots, N\} \},$$

$$F_1^*(v_1^* + z_1^*, \nabla \phi) = \sup_{v_1 \in Y_1} \{ \langle v_1, z_1^* + v_1^* \rangle_{L^2} - F_1(v_1, \nabla \phi) \},$$

and

$$G_1^*(v_2^* + z_2^*) = \sup_{u \in V} \{ \langle u, v_2^* + z_2^* \rangle_{L^2} - G_1(u) \}.$$

Thus, defining

$$J_2^*(\phi, z^*, v^*) = F_1^*(v_1^* + z_1^*, \nabla \phi) - G_1^*(v_2^* + z_3^*) - \langle \nabla \phi, z_2^* \rangle_{L^2} + G_2^*(z_1^*) + G_3^*(z_2^*) + G_4^*(z_3^*),$$

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we have got

$$\begin{aligned}\inf_{u \in V} J(u) &= \inf_{(u,\phi) \in V \times V_0} J_1(u,\phi) \\ &= \inf_{(u,\phi,z^*) \in V \times V_0 \times Y^*} J_1^*(u,\phi,z^*) \\ &= \inf_{z^* \in Y^*} \left\{ \inf_{\phi \in V_0} \left\{ \sup_{v^* \in C^*} J_2^*(\phi,z^*,v^*) \right\} \right\}. \end{aligned} \tag{7}$$

Finally, observe that

$$\begin{aligned} &\inf_{u \in V} J(u) \\ = &\inf_{z^* \in Y^*} \left\{ \inf_{\phi \in V_0} \left\{ \sup_{v^* \in C^*} J_2^*(\phi,z^*,v^*) \right\} \right\} \\ \geq &\sup_{v^* \in C^*} \left\{ \inf_{(z^*,\phi) \in Y^* \times V_0} J_2^*(\phi,z^*,v^*) \right\}. \end{aligned} \tag{8}$$

This last variational formulation corresponds to a concave relaxed formulation in v^* concerning the original primal formulation.

3. Another duality principle for a simpler related model in phase transition with a respective numerical example

In this section we present another duality principle for a related model in phase transition.

Let $\Omega = [0,1] \subset \mathbb{R}$ and consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = \frac{1}{2}\int_\Omega ((u')^2 - 1)^2\, dx + \frac{1}{2}\int_\Omega u^2\, dx - \langle u, f \rangle_{L^2},$$

and where

$$V = \{u \in W^{1,4}(\Omega) \ : \ u(0) = 0 \text{ and } u(1) = 1/2\}$$

and $f \in L^2(\Omega)$.

A global optimum point is not attained for J so that the problem of finding a global minimum for J has no solution.

Anyway, one question remains, how the minimizing sequences behave close the infimum of J.

We intend to use duality theory to approximately solve such a global optimization problem.

Denoting $V_0 = W_0^{1,4}(\Omega)$, at this point we define, $F : V \to \mathbb{R}$ and $F_1 : V \times V_0 \to \mathbb{R}$ by

$$F(u) = \frac{1}{2}\int_\Omega ((u')^2 - 1)^2\, dx,$$

and

$$F_1(u,\phi) = \frac{1}{2}\int_\Omega ((u' + \phi')^2 - 1)^2\, dx.$$

Observe

$$F(u) \geq \inf_{\phi \in V_0} F_1(u,\phi) \geq Q_F(u),\ \forall u \in V,$$

where $Q_F(u)$ refers to a quasi-convex regularization of F.

We define also

$$F_2 : V \times V_0 \to \mathbb{R},$$

$$F_3 : V \times V_0 \to \mathbb{R}$$

and

$$G : V \times V_0 \to \mathbb{R}$$

by

$$F_2(u,\phi) = \frac{1}{2}\int_\Omega ((u' + \phi')^2 - 1)^2\,dx + \frac{1}{2}\int_\Omega u^2\,dx - \langle u, f\rangle_{L^2},$$

$$\begin{aligned} F_3(u,\phi) &= F_2(u,\phi) + \frac{K}{2}\int_\Omega (u')^2\,dx \\ &+ \frac{K_1}{2}\int_\Omega (\phi')^2\,dx \end{aligned} \tag{9}$$

and

$$\begin{aligned} G(u,\phi) &= \frac{K}{2}\int_\Omega (u')^2\,dx \\ &+ \frac{K_1}{2}\int_\Omega (\phi')^2\,dx \end{aligned} \tag{10}$$

Observe that if $K > 0, K_1 > 0$ is large enough, both F_3 and G are convex.
Denoting $Y = Y^* = L^2(\Omega)$ we also define the polar functional $G^* : Y^* \times Y^* \to \mathbb{R}$ by

$$G^*(v^*, v_0^*) = \sup_{(u,\phi)\in V\times V_0} \{\langle u, v^*\rangle_{L^2} + \langle \phi, v_0^*\rangle_{L^2} - G(u,\phi)\}.$$

Observe that

$$\inf_{u\in U} J(u) \geq \inf_{((u,\phi),(v^*,v_0^*))\in V\times V_0\times[Y^*]^2} \{G^*(v^*,v_0^*) - \langle u, v^*\rangle_{L^2} - \langle \phi, v_0^*\rangle_{L^2} + F_3(u,\phi)\}.$$

With such results in mind, we define a relaxed primal dual variational formulation for the primal problem, represented by $J_1^* : V \times V_0 \times [Y^*]^2 \to \mathbb{R}$, where

$$J_1^*(u,\phi,v^*,v_0^*) = G^*(v^*,v_0^*) - \langle u, v^*\rangle_{L^2} - \langle \phi, v_0^*\rangle_{L^2} + F_3(u,\phi).$$

Having defined such a functional, we may obtain numerical results by solving a sequence of convex auxiliary sub-problems, through the following algorithm.

1. Set $K \approx 150$ and $K_1 = K/20$ and $0 < \varepsilon \ll 1$.
2. Choose $(u_1, \phi_1) \in V \times V_0$, such that $\|u_1\|_{1,\infty} \ll K/4$ and $\|\phi_1\|_{1,\infty} \ll K/4$.
3. Set $n = 1$.
4. Calculate $(v_n^*, (v_0^*)_n)$ solution of the system of equations:

$$\frac{\partial J_1^*(u_n, \phi_n, v_n^*, (v_0^*)_n)}{\partial v^*} = \mathbf{0}$$

and

$$\frac{\partial J_1^*(u_n, \phi_n, v_n^*, (v_0^*)_n)}{\partial v_0^*} = \mathbf{0},$$

that is

$$\frac{\partial G^*(v_n^*, (v_0^*)_n)}{\partial v^*} - u_n = 0$$

and

$$\frac{\partial G^*(v_n^*, (v_0^*)_n)}{\partial v_0^*} - \phi_n = 0$$

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so that
$$v_n^* = \frac{\partial G(u_n, \phi_n)}{\partial u}$$
and
$$(v_0^*)_n^* = \frac{\partial G(u_n, \phi_n)}{\partial \phi}$$

5. Calculate (u_{n+1}, ϕ_{n+1}) by solving the system of equations:
$$\frac{\partial J_1^*(u_{n+1}, \phi_{n+1}, v_n^*, (v_0^*)_n)}{\partial u} = \mathbf{0}$$
and
$$\frac{\partial J_1^*(u_{n+1}, \phi_{n+1}, v_n^*, (v_0^*)_n)}{\partial \phi} = \mathbf{0}$$
that is
$$-v_n^* + \frac{\partial F_3(u_{n+1}, \phi_{n+1})}{\partial u} = \mathbf{0}$$
and
$$-(v_0^*)_n + \frac{\partial F_3(u_{n+1}, \phi_{n+1})}{\partial \phi} = \mathbf{0}$$
6. If $\max\{\|u_n - u_{n+1}\|_\infty, \|\phi_{n+1} - \phi_n\|_\infty\} \leq \varepsilon$, then stop, else set $n := n+1$ and go to item 4.

For the case in which $f(x) = 0$, we have obtained numerical results for $K = 1500$ and $K_1 = K/20$. For such a concerning solution u_0 obtained, please see Figure 1. For the case in which $f(x) = \sin(\pi x)/2$, we have obtained numerical results for $K = 100$ and $K_1 = K/20$. For such a concerning solution u_0 obtained, please see Figure 2.

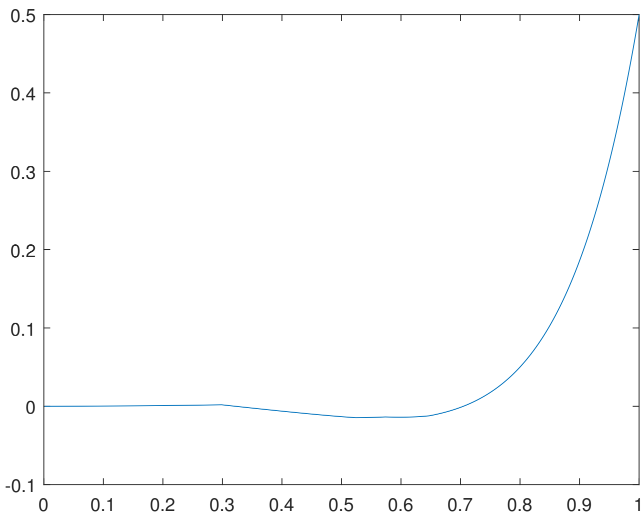

Figure 1. solution $u_0(x)$ for the case $f(x) = 0$.

Remark 3.1. *Observe that the solutions obtained are approximate critical points. They are not, in a classical sense, the global solutions for the related optimization problems. Indeed, such solutions reflect the average behavior of weak cluster points for concerning minimizing sequences.*

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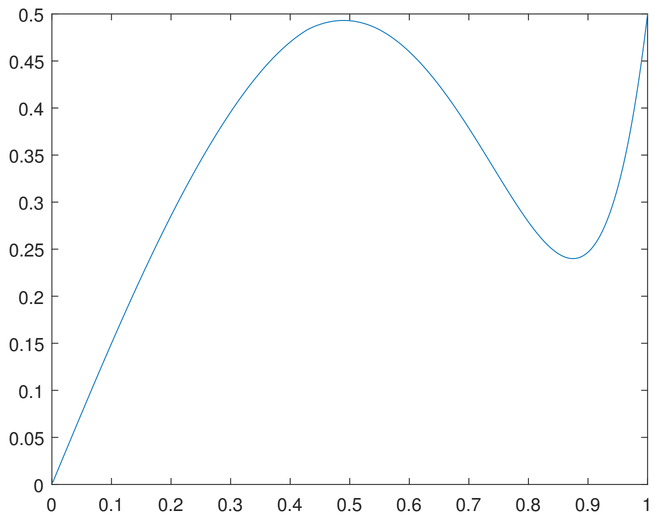

Figure 2. solution $u_0(x)$ for the case $f(x) = \sin(\pi x)/2$.

4. A convex dual variational formulation for a third similar model

In this section we present another duality principle for a third related model in phase transition. Let $\Omega = [0,1] \subset \mathbb{R}$ and consider a functional $J : V \rightarrow \mathbb{R}$ where

$$J(u) = \frac{1}{2}\int_\Omega \min\{(u'-1)^2, (u'+1)^2\}\,dx + \frac{1}{2}\int_\Omega u^2\,dx - \langle u, f\rangle_{L^2},$$

and where

$$V = \{u \in W^{1,2}(\Omega) \ : \ u(0) = 0 \text{ and } u(1) = 1/2\}$$

and $f \in L^2(\Omega)$.

A global optimum point is not attained for J so that the problem of finding a global minimum for J has no solution.

Anyway, one question remains, how the minimizing sequences behave close to the infimum of J.

We intend to use the duality theory to solve such a global optimization problem in an appropriate sense to be specified.

At this point we define, $F : V \rightarrow \mathbb{R}$ and $G : V \rightarrow \mathbb{R}$ by

$$\begin{aligned} F(u) &= \frac{1}{2}\int_\Omega \min\{(u'-1)^2, (u'+1)^2\}\,dx \\ &= \frac{1}{2}\int_\Omega (u')^2\,dx - \int_\Omega |u'|\,dx + 1/2 \\ &\equiv F_1(u'), \end{aligned} \tag{11}$$

and

$$G(u) = \frac{1}{2}\int_\Omega u^2\,dx - \langle u, f\rangle_{L^2}.$$

Denoting $Y = Y^* = L^2(\Omega)$ we also define the polar functional $F_1^* : Y^* \rightarrow \mathbb{R}$ and $G^* : Y^* \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_1^*(v^*) &= \sup_{v \in Y}\{\langle v, v^*\rangle_{L^2} - F_1(v)\} \\ &= \frac{1}{2}\int_\Omega (v^*)^2\,dx + \int_\Omega |v^*|\,dx, \end{aligned} \tag{12}$$

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and

$$\begin{aligned} G^*((v^*)') &= \sup_{u \in V}\{-\langle u', v^*\rangle_{L^2} - G(u)\} \\ &= \frac{1}{2}\int_\Omega ((v^*)' + f)^2\,dx - \frac{1}{2}v^*(1). \end{aligned} \tag{13}$$

Observe this is the scalar case of the calculus of variations, so that from the standard results on convex analysis, we have

$$\inf_{u \in V} J(u) = \max_{v^* \in Y^*}\{-F_1^*(v^*) - G^*(-(v^*)')\}.$$

Indeed, from the direct method of the calculus of variations, the maximum for the dual formulation is attained at some $\hat{v}^* \in Y^*$.

Moreover, the corresponding solution $u_0 \in V$ is obtained from the equation

$$u_0 = \frac{\partial G((\hat{v}^*)')}{\partial (v^*)'} = (\hat{v}^*)' + f.$$

Finally, the Euler-Lagrange equations for the dual problem stands for

$$\left\{ \begin{array}{ll} (v^*)'' + f' - v^* - \text{ sign}(v^*) = 0, & \text{in } \Omega, \\ (v^*)'(0) = 0,\ (v^*)'(1) = 1/2, & \end{array} \right. \tag{14}$$

where $\text{sign}(v^*(x)) = 1$ if $v^*(x) > 0$, $\text{sign}(v^*(x)) = -1$, if $v^*(x) < 0$ and

$$-1 \leq \text{ sign}(v^*(x)) \leq 1,$$

if $v^*(x) = 0$.

We have computed the solutions v^* and corresponding solutions $u_0 \in V$ for the cases in which $f(x) = 0$ and $f(x) = \sin(\pi x)/2$.

For the solution $u_0(x)$ for the case in which $f(x) = 0$, please see Figure 3.

For the solution $u_0(x)$ for the case in which $f(x) = \sin(\pi x)/2$, please see Figure 4.

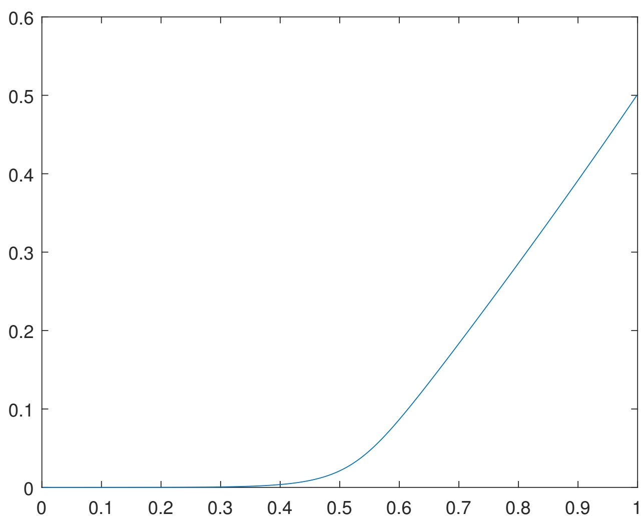

Figure 3. solution $u_0(x)$ for the case $f(x) = 0$.

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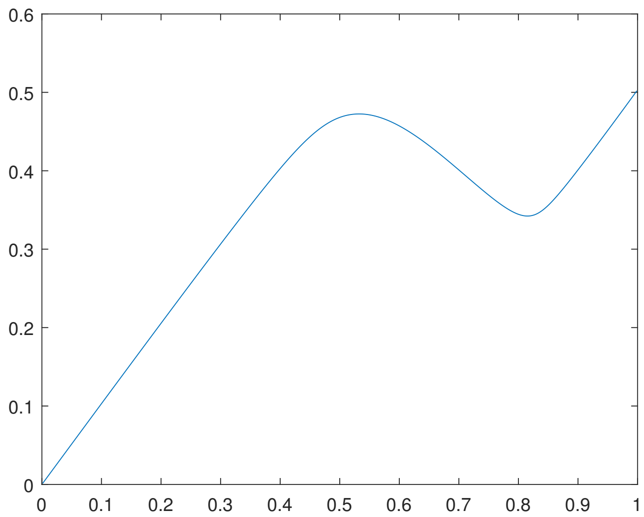

Figure 4. solution $u_0(x)$ for the case $f(x) = \sin(\pi x)/2$.

Remark 4.1. *Observe that such solutions u_0 obtained are not the global solutions for the related primal optimization problems. Indeed, such solutions reflect the average behavior of weak cluster points for concerning minimizing sequences.*

4.1. *The algorithm through which we have obtained the numerical results*

In this subsection we present the software in MATLAB through which we have obtained the last numerical results.

This algorithm is for solving the concerning Euler-Lagrange equations for the dual problem, that is, for solving the equation

$$\begin{cases} (v^*)'' + f' - v^* - \text{ sign}(v^*) = 0, & \text{in } \Omega, \\ (v^*)'(0) = 0, \ (v^*)'(1) = 1/2. \end{cases} \tag{15}$$

Here the concerning software in MATLAB. We emphasize to have used the smooth approximation

$$|v^*| \approx \sqrt{(v^*)^2 + e_1},$$

where a small value for e_1 is specified in the next lines.
****

1. clear all
2. $m_8 = 800$; (number of nodes)
3. $d = 1/m_8$;
4. $e_1 = 0.00001$;
5. $for\ i = 1 : m_8$

   $yo(i,1) = 0.01$;

   $y_1(i,1) = \sin(\pi * i/m_8)/2$;

   end;
6. $for\ i = 1 : m_8 - 1$

   $dy_1(i,1) = (y_1(i+1,1) - y_1(i,1))/d$;

   end;

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7. $for\ k = 1 : 3000$ (we have fixed the number of iterations)

   $i = 1;$

   $h_3 = 1/\sqrt{vo(i,1)^2 + e_1};$

   $m_{12} = 1 + d^2 * h_3 + d^2;$

   $m_{50}(i) = 1/m_{12};$

   $z(i) = m_{50}(i) * (dy_1(i,1) * d^2);$
8. $for\ i = 2 : m_8 - 1$

   $h_3 = 1/\sqrt{vo(i,1)^2 + e_1};$

   $m_{12} = 2 + h_3 * d^2 + d^2 - m50(i-1);$

   $m50(i) = 1/m_{12};$

   $z(i) = m_{50}(i) * (z(i-1) + dy_1(i,1) * d^2);$

   $end;$
9. $v(m_8, 1) = (d/2 + z(m_8 - 1))/(1 - m_{50}(m_8 - 1));$
10. $for\ i = 1 : m_8 - 1$

    $v(m_8 - i, 1) = m_{50}(m_8 - i) * v(m_8 - i + 1) + z(m_8 - i);$

    $end;$
11. $v(m_8/2, 1)$
12. $vo = v;$

    $end;$
13. $for\ i = 1 : m_8 - 1$

    $u(i,1) = (v(i+1,1) - v(i,1))/d + y_1(i,1);$

    $end;$
14. $for\ i = 1 : m_8 - 1$

    $x(i) = i * d;$

    $end;$

    $plot(x, u(:,1))$

****

5. An improvement of the convexity conditions for a non-convex related model through an approximate primal formulation

In this section we develop an approximate primal dual formulation suitable for a large class of variational models.

Here, the applications are for the Kirchhoff-Love plate model, which may be found in Ciarlet, [10].

At this point we start to describe the primal variational formulation.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set which represents the middle surface of a plate of thickness h. The boundary of Ω, which is assumed to be regular (Lipschitzian), is denoted by $\partial\Omega$. The vectorial basis related to the cartesian system $\{x_1, x_2, x_3\}$ is denoted by $(\mathbf{a}_\alpha, \mathbf{a}_3)$, where $\alpha = 1, 2$ (in general Greek indices stand for 1 or 2), and where $\mathbf{a}_3$ is the vector normal to Ω, whereas $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal vectors parallel to Ω. Also, $\mathbf{n}$ is the outward normal to the plate surface.

The displacements will be denoted by

$$\hat{\mathbf{u}} = \{\hat{u}_\alpha, \hat{u}_3\} = \hat{u}_\alpha \mathbf{a}_\alpha + \hat{u}_3 \mathbf{a}_3.$$

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The Kirchhoff-Love relations are

$$\hat{u}_\alpha(x_1,x_2,x_3) = u_\alpha(x_1,x_2) - x_3 w(x_1,x_2)_{,\alpha}$$
$$\text{and } \hat{u}_3(x_1,x_2,x_3) = w(x_1,x_2). \tag{16}$$

Here $-h/2 \leq x_3 \leq h/2$ so that we have $u = (u_\alpha, w) \in U$ where

$$\begin{aligned} U &= \Big\{ u = (u_\alpha, w) \in W^{1,2}(\Omega;\mathbb{R}^2) \times W^{2,2}(\Omega), \\ &\quad u_\alpha = w = \frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \Big\} \\ &= W_0^{1,2}(\Omega;\mathbb{R}^2) \times W_0^{2,2}(\Omega). \end{aligned}$$

It is worth emphasizing that the boundary conditions here specified refer to a clamped plate.

We also define the operator $\Lambda : U \rightarrow Y \times Y$, where $Y = Y^* = L^2(\Omega;\mathbb{R}^{2\times 2})$, by

$$\Lambda(u) = \{\gamma(u), \kappa(u)\},$$

$$\gamma_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{w_{,\alpha} w_{,\beta}}{2},$$

$$\kappa_{\alpha\beta}(u) = -w_{,\alpha\beta}.$$

The constitutive relations are given by

$$N_{\alpha\beta}(u) = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u), \tag{17}$$

$$M_{\alpha\beta}(u) = h_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}(u), \tag{18}$$

where: $\{H_{\alpha\beta\lambda\mu}\}$ and $\left\{h_{\alpha\beta\lambda\mu} = \frac{h^2}{12} H_{\alpha\beta\lambda\mu}\right\}$, are symmetric positive definite fourth order tensors. From now on, we denote $\{\overline{H}_{\alpha\beta\lambda\mu}\} = \{H_{\alpha\beta\lambda\mu}\}^{-1}$ and $\{\overline{h}_{\alpha\beta\lambda\mu}\} = \{h_{\alpha\beta\lambda\mu}\}^{-1}$.

Furthermore $\{N_{\alpha\beta}\}$ denote the membrane force tensor and $\{M_{\alpha\beta}\}$ the moment one. The plate stored energy, represented by $(G \circ \Lambda) : U \rightarrow \mathbb{R}$ is expressed by

$$(G \circ \Lambda)(u) = \frac{1}{2}\int_\Omega N_{\alpha\beta}(u)\gamma_{\alpha\beta}(u)\, dx + \frac{1}{2}\int_\Omega M_{\alpha\beta}(u)\kappa_{\alpha\beta}(u)\, dx \tag{19}$$

and the external work, represented by $F : U \rightarrow \mathbb{R}$, is given by

$$F(u) = \langle w, P \rangle_{L^2} + \langle u_\alpha, P_\alpha \rangle_{L^2}, \tag{20}$$

where $P, P_1, P_2 \in L^2(\Omega)$ are external loads in the directions $\mathbf{a}_3$, $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively. The potential energy, denoted by $J : U \rightarrow \mathbb{R}$ is expressed by:

$$J(u) = (G \circ \Lambda)(u) - F(u)$$

Define now $J_3 : \tilde{U} \rightarrow \mathbb{R}$ by

$$J_3(u) = J(u) + J_5(w).$$

where

$$J_5(w) = 10 \int_\Omega \frac{a^{K\,b\,w}}{\ln(a)\, K^{3/2}}\, dx + 10 \int_\Omega \frac{a^{-K(b\,w - 1/100)}}{\ln(a)\, K^{3/2}}\, dx.$$

In such a case for $a = 2.71$, $K = 185$, $b = P/|P|$ in Ω and

$$\tilde{U} = \{u \in U \;:\; \|w\|_\infty \leq 0.01 \text{ and } P\, w \geq 0 \text{ a.e. in } \Omega\},$$

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we get

$$\begin{aligned} \frac{\partial J_3(u)}{\partial w} &= \frac{\partial J(u)}{\partial w} + \frac{\partial J_5(u)}{\partial w} \\ &\approx \frac{\partial J(u)}{\partial w} + \mathcal{O}(\pm 3.0), \end{aligned} \tag{21}$$

and

$$\begin{aligned} \frac{\partial^2 J_3(u)}{\partial w^2} &= \frac{\partial^2 J(u)}{\partial w^2} + \frac{\partial^2 J_5(u)}{\partial w^2} \\ &\approx \frac{\partial^2 J(u)}{\partial w^2} + \mathcal{O}(850). \end{aligned} \tag{22}$$

This new functional J_3 has a relevant improvement in the convexity conditions concerning the previous functional J.

Indeed, we have obtained a gain in positiveness for the second variation $\frac{\partial^2 J(u)}{\partial w^2}$, which has increased of order $\mathcal{O}(700 - 1000)$.

Moreover the difference between the approximate and exact equation

$$\frac{\partial J(u)}{\partial w} = \mathbf{0}$$

is of order $\mathcal{O}(\pm 3.0)$ which corresponds to a small perturbation in the original equation for a load of $P = 1500\ N/m^2$, for example. Summarizing, the exact equation may be approximately solved in an appropriate sense.

6. An approximate convex variational formulation for another related model

In this section, we obtain an approximate convex variational formulation for a related model, more specifically, for a Ginzburg-Landau type equation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2}\int_\Omega (u^2 - \beta)^2 \, dx \\ &\quad - \langle u, f \rangle_{L^2}, \end{aligned} \tag{23}$$

where $\gamma > 0$, $\alpha > 0$, $\beta > 0$, $V = W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.

We define

$$A^+ = \{u \in V \ : \ u f \geq 0, \text{ a.e. in } \Omega\},$$

$$V_2 = \{u \in V \ : \ \|u\|_\infty \leq 1\},$$

and

$$V_1 = V_2 \cap A^+.$$

At this point we define $v = u/10$ so that

$$\begin{aligned} J(u) &= J_1(v) \\ &= \frac{10^2\gamma}{2}\int_\Omega \nabla v \cdot \nabla v \, dx + \frac{\alpha}{2}\int_\Omega ((10v)^2 - \beta) \, dx \\ &\quad - \langle 10v, f \rangle_{L^2}. \end{aligned} \tag{24}$$

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Moreover we define

$$\begin{aligned} J_2(v) &= \frac{1}{10} J_1(v) \\ &= \frac{10\gamma}{2}\int_\Omega \nabla v \cdot \nabla v \, dx + \frac{\alpha}{20}\int_\Omega ((10v)^2 - \beta)\, dx \\ & -\langle v, f \rangle_{L^2}, \end{aligned} \tag{25}$$

and $J_3 : U_3 \rightarrow \mathbb{R}$ where

$$J_3(v) = J_2(v) + J_5(v)$$

where

$$J_5(v) = K_1 \left(\int_\Omega \frac{a^{K^3\,(5\,b\,w)}}{\ln(a)\;K^4}\, dx + \int_\Omega \frac{a^{-K^3\,(5\,b\,w - 0.5)}}{\ln(a)\;K^4}\, dx \right).$$

Here $K_1 = 1/360$, $a = 2.71$, $K = 2$, $b = f/|f|$ in Ω and

$$U_2 = \{v \in V \;:\; f\,v \geq 0, \text{ a.e. in } \Omega\},$$

$$U_4 = \{v \in V \;:\; \|v\|_\infty \leq 1/10\},$$

and

$$U_3 = U_2 \cap U_4.$$

Thus, with such numerical values, we may obtain

$$\begin{aligned} \frac{\partial J_3(v)}{\partial v} &= \frac{\partial J_2(v)}{\partial v} + \frac{\partial J_5(v)}{\partial v} \\ &\approx \frac{\partial J_2(v)}{\partial v} + \mathcal{O}(\pm\, 0.3), \end{aligned} \tag{26}$$

and

$$\begin{aligned} \frac{\partial^2 J_3(v)}{\partial v^2} &= \frac{\partial^2 J_2(v)}{\partial v^2} + \frac{\partial^2 J_5(v)}{\partial v^2} \\ &\approx \frac{\partial^2 J_2(v)}{\partial v^2} + \mathcal{O}(7.0). \end{aligned} \tag{27}$$

Remark 6.1. *This new functional J_1 has a relevant improvement in the convexity conditions concerning the previous functional J.*

Indeed, we have obtained a gain in positiveness for the second variation $\frac{\partial^2 J_2(v)}{\partial v^2}$, which has increased of order $\mathcal{O}(5-14)$.

Moreover the difference between the approximate and exact equation

$$\frac{\partial J_2(v)}{\partial v} = \mathbf{0}$$

is of order $\mathcal{O}(\pm 0.3)$ which for appropriate parameters $\gamma > 0$, $\alpha > 0$ and $\beta > 0$, corresponds to a small perturbation in the original equation. Summarizing, the exact equation may be approximately solved in an appropriate sense.

Finally, for this last example, we highlight it is relatively easy to improve even more both such an approximation quality and the convexity conditions concerning the original variational model.

With such statements and results in mind, we may prove the following theorem.

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Theorem 6.2. *Suppose $\gamma > 0$, $\alpha > 0$ and $\beta > 0$ are such that*

$$\frac{\partial^2 J_3(v)}{\partial v^2} > \mathbf{0},$$

in U_3

Assume also, $v_0 \in U_3$ is such that

$$\delta J_3(v_0) = \mathbf{0}.$$

Under such hypotheses, J_3 is convex on U_3 so that

$$J_3(v_0) = \min_{v \in U_3} J_3(v).$$

Moreover,

$$\delta J(u_0) = \mathbf{0} + \mathcal{O}(\pm\, 0.3),$$

where $u_0 = 10 v_0 \in V_1$

Proof. From the hypotheses

$$\frac{\partial^2 J_3(v)}{\partial v^2} > \mathbf{0}$$

in U_3, so that J_3 is convex on the convex set U_3.

Consequently, since $\delta J_3(v_0) = \mathbf{0}$, we obtain

$$J_3(v_0) = \min_{v \in U_3} J_3(v).$$

Finally, from the approximation indicated in the last remark and $u_0 \in V_1$ we get

$$\delta J(u_0) = \mathbf{0} + \mathcal{O}(\pm\, 0.3).$$

The proof is complete.

□

7. An exact convex dual variational formulation for a non-convex primal one

In this section we develop a convex dual variational formulation suitable to compute a critical point for the corresponding primal one.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = F(u_x, u_y) - \langle u, f \rangle_{L^2},$$

$V = W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.

Here we denote $Y = Y^* = L^2(\Omega)$ and $Y_1 = Y_1^* = L^2(\Omega) \times L^2(\Omega)$.

Defining

$$V_1 = \{u \in V \ : \ \|u\|_{1,\infty} \leq K_1\}$$

for some appropriate $K_1 > 0$, suppose also F is twice Fréchet differentiable and

$$\det\left\{\frac{\partial^2 F(u_x, u_y)}{\partial v_1 \partial v_2}\right\} \neq 0,$$

$\forall u \in V_1$.

Define now $F_1 : V \to \mathbb{R}$ and $F_2 : V \to \mathbb{R}$ by

$$F_1(u_x, u_y) = F(u_x, u_y) + \frac{\varepsilon}{2}\int_\Omega u_x^2\,dx + \frac{\varepsilon}{2}\int_\Omega u_y^2\,dx,$$

and

$$F_2(u_x, u_y) = \frac{\varepsilon}{2}\int_\Omega u_x^2\,dx + \frac{\varepsilon}{2}\int_\Omega u_y^2\,dx,$$

where here we denote $dx = dx_1 dx_2$.

Moreover, we define the respective Legendre transform functionals F_1^* and F_2^* as

$$F_1^*(v^*) = \langle v_1, v_1^*\rangle_{L^2} + \langle v_2, v_2^*\rangle_{L^2} - F_1(v_1, v_2),$$

where $v_1, v_2 \in Y$ are such that

$$v_1^* = \frac{\partial F_1(v_1, v_2)}{\partial v_1},$$

$$v_2^* = \frac{\partial F_1(v_1, v_2)}{\partial v_2},$$

and

$$F_2^*(v^*) = \langle v_1, v_1^* + f_1\rangle_{L^2} + \langle v_2, v_2^*\rangle_{L^2} - F_2(v_1, v_2),$$

where $v_1, v_2 \in Y$ are such that

$$v_1^* + f_1 = \frac{\partial F_2(v_1, v_2)}{\partial v_1},$$

$$v_2^* = \frac{\partial F_2(v_1, v_2)}{\partial v_2}.$$

Here f_1 is any function such that

$$(f_1)_x = f, \text{ in } \Omega.$$

Furthermore, we define

$$\begin{aligned} J^*(v^*) &= -F_1^*(v^*) + F_2^*(v^*) \\ &= -F_1^*(v^*) + \frac{1}{2\varepsilon}\int_\Omega (v_1^* + f_1)^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (v_2^*)^2\,dx. \end{aligned} \quad (28)$$

Observe that through the target conditions

$$v_1^* + f_1 = \varepsilon u_x,$$

$$v_2^* = \varepsilon u_y,$$

we may obtain the compatibility condition

$$(v_1^* + f_1)_y - (v_2^*)_x = 0.$$

Define now

$$A^* = \{v^* = (v_1^*, v_2^*) \in B_r(0,0) \subset Y_1^* \;:\; (v_1^* + f_1)_y - (v_2^*)_x = 0, \text{ in } \Omega\},$$

for some appropriate $r > 0$ such that J^* is convex in $B_r(0,0)$.

Consider the problem of minimizing J^* subject to $v^* \in A^*$.

Assuming $r > 0$ is large enough so that the restriction in r is not active, at this point we define the associated Lagrangian

$$J_1^*(v^*, \varphi) = J^*(v^*) + \langle \varphi, (v_1^* + f)_y - (v_2^*)_x \rangle_{L^2},$$

where φ is an appropriate Lagrange multiplier.

Therefore

$$\begin{aligned} J_1^*(v^*) &= -F_1^*(v^*) + \frac{1}{2\varepsilon}\int_\Omega (v_1^* + f_1)^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (v_2^*)^2\,dx \\ &\quad + \langle \varphi, (v_1^* + f)_y - (v_2^*)_x \rangle_{L^2}. \end{aligned} \tag{29}$$

The optimal point in question will be a solution of the corresponding Euler-Lagrange equations for J_1^*.

From the variation of J_1^* in v_1^* we obtain

$$-\frac{\partial F_1^*(v^*)}{\partial v_1^*} + \frac{v_1^* + f}{\varepsilon} - \frac{\partial \varphi}{\partial y} = 0. \tag{30}$$

From the variation of J_1^* in v_2^* we obtain

$$-\frac{\partial F_1^*(v^*)}{\partial v_2^*} + \frac{v_2^*}{\varepsilon} + \frac{\partial \varphi}{\partial x} = 0. \tag{31}$$

From the variation of J_1^* in φ we have

$$(v_1^* + f)_y - (v_2^*)_x = 0.$$

From this last equation, we may obtain $u \in V$ such that

$$v_1^* + f = \varepsilon u_x,$$

and

$$v_2^* = \varepsilon u_y.$$

From this and the previous extremal equations indicated we have

$$-\frac{\partial F_1^*(v^*)}{\partial v_1^*} + u_x - \frac{\partial \varphi}{\partial y} = 0,$$

and

$$-\frac{\partial F_1^*(v^*)}{\partial v_2^*} + u_y + \frac{\partial \varphi}{\partial x} = 0.$$

so that

$$v_1^* + f = \frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_1},$$

and

$$v_2^* = \frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_2}.$$

From this and equation (36) and (37) we have

$$\begin{aligned} & -\varepsilon\left(\frac{\partial F_1^*(v^*)}{\partial v_1^*}\right)_x - \varepsilon\left(\frac{\partial F_1^*(v^*)}{\partial v_2^*}\right)_y \\ & + (v_1^* + f_1)_x + (v_2^*)_y \\ = & -\varepsilon u_{xx} - \varepsilon u_{yy} + (v_1^*)_x + (v_2^*)_y + f = 0. \end{aligned} \tag{32}$$

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Replacing the expressions of v_1^* and v_2^* into this last equation, we have

$$-\varepsilon u_{xx} - \varepsilon u_{yy} + \left(\frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_1}\right)_x + \left(\frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_2}\right)_y + f = 0,$$

so that

$$\left(\frac{\partial F(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_1}\right)_x + \left(\frac{\partial F(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_2}\right)_y + f = 0, \text{ in } \Omega. \quad (33)$$

Observe that if

$$\nabla^2 \varphi = 0$$

then there exists $\hat{u}$ such that u and φ are also such that

$$u_x - \varphi_y = \hat{u}_x$$

and

$$u_y + \varphi_x = \hat{u}_y.$$

The boundary conditions for φ must be such that $\hat{u} \in W_0^{1,2}$.
From this and equation (39) we obtain

$$\delta J(\hat{u}) = \mathbf{0}.$$

Summarizing, we may obtain a solution $\hat{u} \in W_0^{1,2}$ of equation $\delta J(\hat{u}) = \mathbf{0}$ by minimizing J^* on A^*.
Finally, observe that clearly J^* is convex in an appropriate large ball $B_r(0,0)$ for some appropriate $r > 0$

8. An exact convex dual variational formulation for a non-convex primal one

In this section we develop a convex dual variational formulation suitable to compute a critical point for the corresponding primal one.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = F(u_x, u_y) - \langle u, f \rangle_{L^2},$$

$V = W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.
Here we denote $Y = Y^* = L^2(\Omega)$ and $Y_1 = Y_1^* = L^2(\Omega) \times L^2(\Omega)$.
Defining

$$V_1 = \{u \in V \ : \ \|u\|_{1,\infty} \leq K_1\}$$

for some appropriate $K_1 > 0$, suppose also F is twice Fréchet differentiable and

$$\det\left\{\frac{\partial^2 F(u_x, u_y)}{\partial v_1 \partial v_2}\right\} \neq 0,$$

$\forall u \in V_1$.
Define now $F_1 : V \to \mathbb{R}$ and $F_2 : V \to \mathbb{R}$ by

$$F_1(u_x, u_y) = F(u_x, u_y) + \frac{\varepsilon}{2}\int_\Omega u_x^2 \, dx + \frac{\varepsilon}{2}\int_\Omega u_y^2 \, dx,$$

and

$$F_2(u_x, u_y) = \frac{\varepsilon}{2}\int_\Omega u_x^2 \, dx + \frac{\varepsilon}{2}\int_\Omega u_y^2 \, dx,$$

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where here we denote $dx = dx_1 dx_2$.

Moreover, we define the respective Legendre transform functionals F_1^* and F_2^* as

$$F_1^*(v^*) = \langle v_1, v_1^* \rangle_{L^2} + \langle v_2, v_2^* \rangle_{L^2} - F_1(v_1, v_2),$$

where $v_1, v_2 \in Y$ are such that

$$v_1^* = \frac{\partial F_1(v_1, v_2)}{\partial v_1},$$

$$v_2^* = \frac{\partial F_1(v_1, v_2)}{\partial v_2},$$

and

$$F_2^*(v^*) = \langle v_1, v_1^* + f_1 \rangle_{L^2} + \langle v_2, v_2^* \rangle_{L^2} - F_2(v_1, v_2),$$

where $v_1, v_2 \in Y$ are such that

$$v_1^* + f_1 = \frac{\partial F_2(v_1, v_2)}{\partial v_1},$$

$$v_2^* = \frac{\partial F_2(v_1, v_2)}{\partial v_2}.$$

Here f_1 is any function such that

$$(f_1)_x = f, \text{ in } \Omega.$$

Furthermore, we define

$$\begin{aligned} J^*(v^*) &= -F_1^*(v^*) + F_2^*(v^*) \\ &= -F_1^*(v^*) + \frac{1}{2\varepsilon}\int_\Omega (v_1^* + f_1)^2\, dx + \frac{1}{2\varepsilon}\int_\Omega (v_2^*)^2\, dx. \end{aligned} \tag{34}$$

Observe that through the target conditions

$$v_1^* + f_1 = \varepsilon u_x,$$

$$v_2^* = \varepsilon u_y,$$

we may obtain the compatibility condition

$$(v_1^* + f_1)_y - (v_2^*)_x = 0.$$

Define now

$$A^* = \{v^* = (v_1^*, v_2^*) \in B_r(0,0) \subset Y_1^* \;:\; (v_1^* + f_1)_y - (v_2^*)_x = 0, \text{ in } \Omega\},$$

for some appropriate $r > 0$ such that J^* is convex in $B_r(0,0)$.

Consider the problem of minimizing J^* subject to $v^* \in A^*$.

Assuming $r > 0$ is large enough so that the restriction in r is not active, at this point we define the associated Lagrangian

$$J_1^*(v^*, \varphi) = J^*(v^*) + \langle \varphi, (v_1^* + f)_y - (v_2^*)_x \rangle_{L^2},$$

where φ is an appropriate Lagrange multiplier.

Therefore

$$\begin{aligned} J_1^*(v^*) &= -F_1^*(v^*) + \frac{1}{2\varepsilon}\int_\Omega (v_1^* + f_1)^2\, dx + \frac{1}{2\varepsilon}\int_\Omega (v_2^*)^2\, dx \\ &\quad + \langle \varphi, (v_1^* + f)_y - (v_2^*)_x \rangle_{L^2}. \end{aligned} \tag{35}$$

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The optimal point in question will be a solution of the corresponding Euler-Lagrange equations for J_1^*.

From the variation of J_1^* in v_1^* we obtain

$$-\frac{\partial F_1^*(v^*)}{\partial v_1^*} + \frac{v_1^* + f}{\varepsilon} - \frac{\partial \varphi}{\partial y} = 0. \tag{36}$$

From the variation of J_1^* in v_2^* we obtain

$$-\frac{\partial F_1^*(v^*)}{\partial v_2^*} + \frac{v_2^*}{\varepsilon} + \frac{\partial \varphi}{\partial x} = 0. \tag{37}$$

From the variation of J_1^* in φ we have

$$(v_1^* + f)_y - (v_2^*)_x = 0.$$

From this last equation, we may obtain $u \in V$ such that

$$v_1^* + f = \varepsilon u_x,$$

and

$$v_2^* = \varepsilon u_y.$$

From this and the previous extremal equations indicated we have

$$-\frac{\partial F_1^*(v^*)}{\partial v_1^*} + u_x - \frac{\partial \varphi}{\partial y} = 0,$$

and

$$-\frac{\partial F_1^*(v^*)}{\partial v_2^*} + u_y + \frac{\partial \varphi}{\partial x} = 0.$$

so that

$$v_1^* + f = \frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_1},$$

and

$$v_2^* = \frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_2}.$$

From this and equation (36) and (37) we have

$$\begin{aligned} & -\varepsilon\left(\frac{\partial F_1^*(v^*)}{\partial v_1^*}\right)_x - \varepsilon\left(\frac{\partial F_1^*(v^*)}{\partial v_2^*}\right)_y \\ & +(v_1^* + f_1)_x + (v_2^*)_y \\ = \; & -\varepsilon u_{xx} - \varepsilon u_{yy} + (v_1^*)_x + (v_2^*)_y + f = 0. \end{aligned} \tag{38}$$

Replacing the expressions of v_1^* and v_2^* into this last equation, we have

$$-\varepsilon u_{xx} - \varepsilon u_{yy} + \left(\frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_1}\right)_x + \left(\frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_2}\right)_y + f = 0,$$

so that

$$\left(\frac{\partial F(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_1}\right)_x + \left(\frac{\partial F(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_2}\right)_y + f = 0, \text{ in } \Omega. \tag{39}$$

Observe that if

$$\nabla^2 \varphi = 0$$

then there exists $\hat{u}$ such that u and φ are also such that

$$u_x - \varphi_y = \hat{u}_x$$

and

$$u_y + \varphi_x = \hat{u}_y.$$

The boundary conditions for φ must be such that $\hat{u} \in W_0^{1,2}$.

From this and equation (39) we obtain

$$\delta J(\hat{u}) = \mathbf{0}.$$

Summarizing, we may obtain a solution $\hat{u} \in W_0^{1,2}$ of equation $\delta J(\hat{u}) = \mathbf{0}$ by minimizing J^* on A^*.

Finally, observe that clearly J^* is convex in an appropriate large ball $B_r(0,0)$ for some appropriate $r > 0$

9. Another primal dual formulation for a related model

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular boundary denoted by $\partial\Omega$.

Consider the functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2}\int_\Omega (u^2 - \beta)^2 \, dx \\ &\quad -\langle u, f\rangle_{L^2}, \end{aligned} \tag{40}$$

$\alpha > 0,\ \beta > 0,\ \gamma > 0$, $V = W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.

Denoting $Y = Y^* = L^2(\Omega)$, define now $J_1^* : V \times Y^* \to \mathbb{R}$ by

$$\begin{aligned} J_1^*(u, v_0^*) &= -\frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx - \langle u^2, v_0^*\rangle_{L^2} \\ &\quad + \frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2v_0^* u - f)^2 \, dx + \langle u, f\rangle_{L^2} \\ &\quad + \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx + \beta\int_\Omega v_0^* \, dx, \end{aligned} \tag{41}$$

Define also

$$A^+ = \{u \in V \ : \ u\, f \geq 0, \text{ a.e. in } \Omega\},$$

$$V_2 = \{u \in V \ : \ \|u\|_\infty \leq K_3\},$$

and

$$V_1 = V_2 \cap A^+$$

for some appropriate $K_3 > 0$ to be specified.

Moreover define

$$B^* = \{v_0^* \in Y^* \ : \ \|v_0^*\|_\infty \leq K\}$$

for some appropriate $K > 0$ to be specified.

Observe that, denoting

$$\varphi = -\gamma\nabla^2 u + 2v_0^* u - f$$

we have

$$\frac{\partial^2 J_1^*(u, v_0^*)}{\partial (v_0^*)^2} = \frac{1}{\alpha} + 4K_1 u^2$$

$$\frac{\partial^2 J_1^*(u, v_0^*)}{\partial u^2} = \gamma\nabla^2 - 2v_0^* + K_1(-\gamma\nabla^2 + 2v_0^*)^2$$

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and

$$\frac{\partial^2 J_1^*(u,v_0^*)}{\partial u \partial v_0^*} = K_1(2\varphi + 2(-\gamma\nabla^2 u + 2v_0^* u)) - 2u$$

so that

$$\begin{aligned} & \det\{\delta^2 J_1^*(u,v_0^*)\} \\ = & \frac{\partial^2 J_1^*(u,v_0^*)}{\partial (v_0^*)^2}\frac{\partial^2 J_1^*(u,v_0^*)}{\partial u^2} - \left(\frac{\partial^2 J_1^*(u,v_0^*)}{\partial u \partial v_0^*}\right)^2 \\ = & \frac{K_1(-\gamma\nabla^2 + 2v_0^*)^2}{\alpha} - \frac{\gamma\nabla^2 + 2v_0^* + 4\alpha u^2}{\alpha} \\ & -4K_1^2\varphi^2 - 8K_1\varphi(-\gamma\nabla^2 + 2v_0^*)u + 8K_1\varphi u \\ & +4K_1(-\gamma\nabla^2 u + 2v_0^* u)u. \end{aligned} \tag{42}$$

Observe now that a critical point $\varphi = 0$ and $(-\gamma\nabla^2 u + 2v_0^* u)u = fu \geq 0$ in Ω. Therefore, for an appropriate large $K_1 > 0$, also at a critical point, we have

$$\begin{aligned} & \det\{\delta^2 J_1^*(u,v_0^*)\} \\ = & 4K_1 fu - \frac{\delta^2 J(u)}{\alpha} + K_1\frac{(-\gamma\nabla^2 + 2v_0^*)^2}{\alpha} > \mathbf{0}. \end{aligned} \tag{43}$$

Remark 9.1. *From this last equation we may observe that J_1^* has a large region of convexity about any critical point $(u_0, \hat{v}_0^*)$, that is, there exists a large $r > 0$ such that J_1^* is convex on $B_r(u_0, \hat{v}_0^*)$.*

With such results in mind, we may easily prove the following theorem.

Theorem 9.2. *Assume $K_1 \gg \max\{1, K, K_3\}$ and suppose $(u_0, \hat{v}_0^*) \in V_1 \times B^*$ is such that*

$$\delta J_1^*(u_0, \hat{v}_0^*) = \mathbf{0}.$$

Under such hypotheses, there exists $r > 0$ such that J_1^* is convex in $E^* = B_r(u_0, \hat{v}_0^*) \cap (V_1 \times B^*)$,

$$\delta J(u_0) = \mathbf{0},$$

and

$$-J(u_0) = J_1(u_0, \hat{v}_0^*) = \inf_{(u,v_0^*) \in E^*} J_1^*(u, v_0^*).$$

10. A third primal dual formulation for a related model

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular boundary denoted by $\partial\Omega$. Consider the functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) = & \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2}\int_\Omega (u^2 - \beta)^2 \, dx \\ & -\langle u, f\rangle_{L^2}, \end{aligned} \tag{44}$$

$\alpha > 0,\ \beta > 0,\ \gamma > 0$, $V = W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.

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Denoting $Y = Y^* = L^2(\Omega)$, define now $J_1^* : V \times Y^* \times Y^* \to \mathbb{R}$ by

$$\begin{aligned} J_1^*(u, v_0^*, v_1^*) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{1}{2}\int_\Omega K\, u^2 \, dx \\ &\quad -\langle u, v_1^* \rangle_{L^2} + \frac{1}{2}\int_\Omega \frac{(v_1^*)^2}{(-2v_0^* + K)} \, dx \\ &\quad + \frac{1}{2(\alpha+\varepsilon)}\int_\Omega (v_0^* - \alpha(u^2 - \beta))^2 \, dx + \langle u, f \rangle_{L^2} \\ &\quad -\frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx - \beta \int_\Omega v_0^* \, dx, \end{aligned} \tag{45}$$

where $\varepsilon > 0$ is a small real constant.

Define also

$$A^+ = \{u \in V \;:\; u\, f \geq 0, \text{ a.e. in } \Omega\},$$

$$V_2 = \{u \in V \;:\; \|u\|_\infty \leq K_3\},$$

and

$$V_1 = V_2 \cap A^+$$

for some appropriate $K_3 > 0$ to be specified.

Moreover define

$$B^* = \{v_0^* \in Y^* \;:\; \|v_0^*\|_\infty \leq K_4\}$$

and

$$D^* = \{v_1^* \in Y^* \;:\; \|v_1^*\| \leq K_5\},$$

for some appropriate real constants $K_4, K_5 > 0$ to be specified.

Remark 10.1. *Define now*

$$H_1(u, v_0^*) = -\gamma\nabla^2 + 2v_0^* + 4\alpha u^2$$

and

$$\hat{E}_{v_0^*} = \{u \in V \;:\; H_1(u, v_0^*) \geq \mathbf{0}\}.$$

For a fixed $v_0^* \in B^*$, we are going to prove that $C^* = \hat{E}_{v_0^*} \cap V_1$ is a convex set.

Assume, for a finite dimensional problem version, in a finite differences or finite element context, that

$$-\gamma\nabla^2 - 2\alpha\beta \leq \mathbf{0},$$

so that for $K_1 > 0$ be sufficiently large, we have

$$-\gamma\nabla^2 + 2v_0^* - K_1 u^2 \leq \mathbf{0}.$$

Observe now that

$$H_1(u, v_0^*) = -\gamma\nabla^2 + 2v_0^* - K_1 u^2 + 4\alpha u^2 + K_1 u^2.$$

Let $u_1, u_2 \in C^*$ and $\lambda \in [0,1]$.

Thus

$$\text{sign } (u_1) = \text{ sign } (u_2) \text{ in } \Omega$$

so that

$$\lambda|u_1| + (1-\lambda)|u_2| = |\lambda u_1 + (1-\lambda)u_2| \text{ in } \Omega.$$

Observe now that

$$H_1(u_1, v_0^*) \geq \mathbf{0}$$

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and

$$H_1(u_2, v_0^*) \geq \mathbf{0}$$

so that

$$4\alpha u_1^2 + K_1 u_1^2 \geq \gamma\nabla^2 - 2v_0^* + K_1 u_1^2 \geq \mathbf{0},$$

and

$$4\alpha u_2^2 + K_1 u_1^2 \geq \gamma\nabla^2 - 2v_0^* + K_1 u_2^2 \geq \mathbf{0},$$

so that

$$\sqrt{4\alpha + K_1}|u_1| \geq \sqrt{\gamma\nabla^2 - 2v_0^* + K_1 u_1^2}$$

and

$$\sqrt{4\alpha + K_1}|u_2| \geq \sqrt{\gamma\nabla^2 - 2v_0^* + K_1 u_2^2}.$$

From such results we obtain

$$\begin{aligned} \sqrt{4\alpha + K_1}|\lambda u_1 + (1-\lambda)u_2| &= \sqrt{4\alpha + K_1}(\lambda|u_1| + (1-\lambda|u_2|) \\ &\geq \lambda\sqrt{\gamma\nabla^2 - 2v_0^* + K_1 u_1^2} + (1-\lambda)\sqrt{\gamma\nabla^2 - 2v_0^* + K_1 u_2^2} \\ &\geq \sqrt{\gamma\nabla^2 - 2v_0^* + K_1(\lambda u_1 + (1-\lambda)u_2)^2}. \end{aligned} \tag{46}$$

From this we obtain

$$(4\alpha + K_1)(\lambda u_1 + (1-\lambda)u_2)^2 \geq \gamma\nabla^2 - 2v_0^* + K_1(\lambda u_1 + (1-\lambda)u_2)^2,$$

so that

$$H_1(\lambda u_1 + (1-\lambda)u_2, v_0^*) \geq \mathbf{0}.$$

Hence $\hat{E}_{v_0^*}$ is convex. Since V_1 is also clearly convex, we have obtained that $C^* = \hat{E}_{v_0^*} \cap V_1$ is convex. Such a result we will be used many times in the next sections.

Observe that, defining

$$\varphi = v_0^* - \alpha(u^2 - \beta)$$

we may obtain

$$\frac{\partial^2 J_1^*(u, v_0^*, v_1^*)}{\partial u^2} = -\gamma\nabla^2 + K + \frac{\alpha}{\alpha+\varepsilon}4u^2 - 2\varphi\frac{\alpha}{\alpha+\varepsilon}$$

$$\frac{\partial^2 J_1^*(u, v_0^*, v_1^*)}{\partial (v_1^*)^2} = \frac{1}{-2v_0^* + K}$$

and

$$\frac{\partial^2 J_1^*(u, v_0^*, v_1^*)}{\partial u \partial v_1^*} = -1$$

so that

$$\begin{aligned} &\det\left\{\frac{\partial^2 J_1^*(u, v_0^*, v_1^*)}{\partial u \partial v_1^*}\right\} \\ =\ & \frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial (v_1^*)^2}\frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial u^2} - \left(\frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial u \partial v_1^*}\right)^2 \\ =\ & \frac{-\gamma\nabla^2 + 2v_0^* + 4\frac{\alpha^2}{\alpha+\varepsilon}u^2 - 2\frac{\alpha}{\alpha+\varepsilon}\varphi}{-2v_0^* + K} \\ \equiv\ & H(u, v_0^*). \end{aligned} \tag{47}$$

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However, at a critical point, we have $\varphi = \mathbf{0}$ so that, we define

$$C^*_{v_0^*} = \{u \in V \ : \ \varphi \leq \mathbf{0}\}.$$

From such results, assuming $K \gg \max\{K_3, K_4, K_5\}$, define now

$$E_{v_0^*} = \{u \in V \ : \ H(u, v_0^*) > \mathbf{0}\}.$$

Observe that similarly as it was develop in remark 10.1, we may prove that $E_{v_0^*}$ is a convex set. With such results in mind, we may easily prove the following theorem.

Theorem 10.2. *Suppose* $(u_0, \hat{v}_1^*, \hat{v}_0^*) \in E^* = (V_1 \cap C_{\hat{v}_0^*} \cap E_{\hat{v}_0^*}) \times D^* \times B^*$ *is such that*

$$\delta J_1^*(u_0, \hat{v}_0^*, \hat{v}_1^*) = \mathbf{0}.$$

Under such hypotheses, we have that

$$\delta J(u_0) = \mathbf{0}$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} J(u) \\ &= J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) \\ &= \inf_{(u, v_1^*) \in V_1 \times D^*} \left\{ \sup_{v_0^* \in B^*} J_1^*(u, v_1^*, v_0^*) \right\} \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{(u, v_1^*) \in V_1 \times D^*} J_1^*(u, v_1^*, v_0^*) \right\}. \end{aligned} \tag{48}$$

Proof. The proof that

$$\delta J(u_0) = \mathbf{0}$$

and

$$J(u_0) = J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*)$$

may be easily made similarly as in the previous sections.

Moreover, from the hypotheses, we have

$$J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) = \inf_{(u, v_1^*) \in V_1 \times D^*} J_1^*(u, v_1^*, \hat{v}_0^*)$$

and

$$J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) = \sup_{v_0^* \in B^*} J_1^*(u_0, \hat{v}_1^*, v_0^*).$$

From this, from a standard saddle point theorem and the remaining hypotheses, we may infer that

$$\begin{aligned} J(u_0) &= J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) \\ &= \inf_{(u, v_1^*) \in V_1 \times D^*} \left\{ \sup_{v_0^* \in B^*} J_1^*(u, v_1^*, v_0^*) \right\} \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{(u, v_1^*) \in V_1 \times D^*} J_1^*(u, v_1^*, v_0^*) \right\}. \end{aligned} \tag{49}$$

Moreover, observe that

$$\begin{aligned} J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*) &= \inf_{(u,v_1^*)\in V_1\times D^*} J_1^*(u,v_1^*,\hat{v}_0^*) \\ &\leq \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\frac{K}{2}\int_\Omega u^2\,dx \\ &\quad +\langle u^2,\hat{v}_0^*\rangle_{L^2}-\frac{K}{2}\int_\Omega u^2\,dx \\ &\quad -\frac{1}{2\alpha}\int_\Omega (\hat{v}_0^*)^2\,dx-\beta\int_\Omega \hat{v}_0^*\,dx \\ &\quad +\frac{1}{2(\alpha+\varepsilon)}\int_\Omega (\hat{v}_0^*-\alpha(u^2-\beta))^2\,dx-\langle u,f\rangle_{L^2} \\ &\leq \sup_{v_0^*\in Y^*}\left\{\frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\langle u^2,v_0^*\rangle\right. \\ &\quad -\frac{1}{2\alpha}\int_\Omega (v_0^*)^2\,dx-\beta\int_\Omega v_0^*\,dx \\ &\quad \left.+\frac{1}{2(\alpha+\varepsilon)}\int_\Omega (v_0^*-\alpha(u^2-\beta))^2\,dx-\langle u,f\rangle_{L^2}\right\} \\ &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\,dx \\ &\quad -\langle u,f\rangle_{L^2},\ \forall u\in V_1. \end{aligned} \tag{50}$$

Summarizing, we have got

$$J(u_0)=J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*)\leq \inf_{u\in V_1} J(u).$$

From such results, we may infer that

$$\begin{aligned} J(u_0) &= \inf_{u\in V_1} J(u) \\ &= J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*) \\ &= \inf_{(u,v_1^*)\in V_1\times D^*}\left\{\sup_{v_0^*\in B^*} J_1^*(u,v_1^*,v_0^*)\right\} \\ &= \sup_{v_0^*\in B^*}\left\{\inf_{(u,v_1^*)\in V_1\times D^*} J_1^*(u,v_1^*,v_0^*)\right\}. \end{aligned} \tag{51}$$

The proof is complete. $\square$

11. A fourth primal dual formulation for a related model

Let $\Omega\subset\mathbb{R}^3$ be an open, bounded and connected set with a regular boundary denoted by $\partial\Omega$. Consider the functional $J:V\to\mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\,dx \\ &\quad -\langle u,f\rangle_{L^2}, \end{aligned} \tag{52}$$

$\alpha>0,\ \beta>0,\ \gamma>0,\ V=W_0^{1,2}(\Omega)$ and $f\in L^2(\Omega)$.

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Denoting $Y = Y^* = L^2(\Omega)$, define now $J_1^* : V \times Y^* \to \mathbb{R}$ by

$$\begin{aligned} J_1^*(u, v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx - \langle u^2, v_0^* \rangle_{L^2} \\ &+ \frac{1}{2(\alpha+\varepsilon)}\int_\Omega (v_0^* - \alpha(u^2-\beta))^2 \, dx - \langle u, f \rangle_{L^2} \\ &- \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx - \beta \int_\Omega v_0^* \, dx, \end{aligned} \tag{53}$$

where $\varepsilon > 0$ is a small real constant.

Define also

$$A^+ = \{u \in V \; : \; u\, f \geq 0, \text{ a.e. in } \Omega\},$$

$$V_2 = \{u \in V \; : \; \|u\|_\infty \leq K_3\},$$

and

$$V_1 = V_2 \cap A^+$$

for some appropriate real constant $K_3 > 0$.

Moreover define

$$B^* = \{v_0^* \in Y^* \; : \; \|v_0^*\|_\infty \leq K_4\}$$

for some appropriate real constant $K_4 > 0$.

Observe that, denoting $\varphi = v_0^* - \alpha(u^2 - \beta)$, we may obtain

$$\begin{aligned} \frac{\partial^2 J_1^*(u, v_0^*)}{\partial u^2} &= -\gamma\nabla^2 + 2v_0 \\ &+ \frac{\alpha^2}{\alpha+\varepsilon}4u^2 - 2\frac{\varphi}{\alpha+\varepsilon}\alpha \\ &\equiv H(u, v_0^*), \end{aligned} \tag{54}$$

and

$$\frac{\partial^2 J_1^*(u, v_0^*)}{\partial (v_0^*)^2} = -\frac{1}{\alpha} + \frac{1}{\alpha+\varepsilon} < \mathbf{0}$$

However, at a critical point, we have $\varphi = \mathbf{0}$ so that, we define

$$C_{v_0^*}^* = \{u \in V \; : \; \varphi \leq \mathbf{0}\}.$$

Define also,

$$E_{v_0^*} = \{u \in V \; : \; H(u, v_0^*) > \mathbf{0}\}.$$

Remark 11.1. *Similarly as it was developed in remark 10.1 we may prove that such a $E_{v_0^*}$ is a convex set.*

With such results in mind, we may easily prove the following theorem.

Theorem 11.2. *Suppose $(u_0, \hat{v}_0^*) \in E^* = (V_1 \cap C_{\hat{v}_0^*} \cap E_{\hat{v}_0^*}) \times B^*$ is such that*

$$\delta J_1^*(u_0, \hat{v}_0^*) = \mathbf{0}.$$

Under such hypotheses, we have that

$$\delta J(u_0) = \mathbf{0}$$

and

 doi:10.20944/preprints202302.0051.v13

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} J(u) \\ &= J_1^*(u_0, \hat{v}_0^*) \\ &= \inf_{u \in V_1} \left\{ \sup_{v_0^* \in B^*} J_1^*(u, v_0^*) \right\} \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{u \in V_1} J_1^*(u, v_0^*) \right\}. \end{aligned} \tag{55}$$

Proof. The proof that

$$\delta J(u_0) = \mathbf{0}$$

and

$$J(u_0) = J_1^*(u_0, \hat{v}_0^*)$$

may be easily made similarly as in the previous sections.

Moreover, from the hypotheses, we have

$$J_1^*(u_0, \hat{v}_0^*) = \inf_{u \in V_1} J_1^*(u, \hat{v}_0^*)$$

and

$$J_1^*(u_0, \hat{v}_0^*) = \sup_{v_0^* \in B^*} J_1^*(u_0, v_0^*).$$

From this, from a standard saddle point theorem and the remaining hypotheses, we may infer that

$$\begin{aligned} J(u_0) &= J_1^*(u_0, \hat{v}_0^*) \\ &= \inf_{u \in V_1} \left\{ \sup_{v_0^* \in B^*} J_1^*(u, v_0^*) \right\} \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{u \in V_1} J_1^*(u, v_0^*) \right\}. \end{aligned} \tag{56}$$

Moreover, observe that

$$\begin{aligned} J_1^*(u_0, \hat{v}_0^*) &= \inf_{u \in V_1} J_1^*(u, \hat{v}_0^*) \\ &\leq \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} \\ &\quad - \frac{1}{2\alpha} \int_\Omega (\hat{v}_0^*)^2 \, dx - \beta \int_\Omega \hat{v}_0^* \, dx \\ &\quad + \frac{1}{2(\alpha + \varepsilon)} \int_\Omega (\hat{v}_0^* - \alpha(u^2 - \beta))^2 \, dx - \langle u, f \rangle_{L^2} \\ &\leq \sup_{v_0^* \in Y^*} \left\{ \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^* \rangle \right. \\ &\quad - \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx - \beta \int_\Omega v_0^* \, dx \\ &\quad \left. + \frac{1}{2(\alpha + \varepsilon)} \int_\Omega (v_0^* - \alpha(u^2 - \beta))^2 \, dx - \langle u, f \rangle_{L^2} \right\} \\ &= \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx \\ &\quad - \langle u, f \rangle_{L^2}, \ \forall u \in V_1. \end{aligned} \tag{57}$$

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Summarizing, we have got

$$J(u_0) = J_1^*(u_0, \hat{v}_0^*) \leq \inf_{u \in V_1} J(u).$$

From such results, we may infer that

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} J(u) \\ &= J_1^*(u_0, \hat{v}_0^*) \\ &= \inf_{u \in V_1} \left\{ \sup_{v_0^* \in B^*} J_1^*(u, v_0^*) \right\} \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{u \in V_1} J_1^*(u, v_0^*) \right\}. \end{aligned} \tag{58}$$

The proof is complete. □

12. One more primal dual formulation for a related model

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular boundary denoted by $\partial\Omega$. Consider the functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2} \int_\Omega (u^2 - \beta)^2 \, dx \\ &\quad - \langle u, f \rangle_{L^2}, \end{aligned} \tag{59}$$

$\alpha > 0,\ \beta > 0,\ \gamma > 0,\ V = W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.
Denoting $Y = Y^* = L^2(\Omega)$, define now $J_1^* : V \times Y^* \times Y^* \to \mathbb{R}$ by

$$\begin{aligned} J_1^*(u, v_1^*, v_0^*) &= \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{K}{2} \int_\Omega u^2 \, dx - \langle u, v_1^* \rangle_{L^2} \\ &\quad + \frac{1}{2} \int_\Omega \frac{(v_1^*)^2}{-2v_0^* + K} dx - \langle u, f \rangle_{L^2} \\ &\quad + \frac{K^2}{2} \int_\Omega \left(\frac{v_1^* + f}{-\gamma \nabla^2 + K} - \frac{v_1^*}{-2v_0^* + K} \right)^2 dx \\ &\quad - \frac{1}{2\alpha} \int_\Omega (v_0^*)^2 \, dx - \beta \int_\Omega v_0^* \, dx, \end{aligned} \tag{60}$$

Define also

$$A^+ = \{u \in V \ : \ u f \geq 0, \text{ a.e. in } \Omega\},$$

$$V_2 = \{u \in V \ : \ \|u\|_\infty \leq K_3\},$$

and

$$V_1 = V_2 \cap A^+$$

specifically for a constant $K_3 = \sqrt{\frac{1}{5\alpha}}$.
Moreover define

$$B^* = \{v_0^* \in Y^* \ : \ \|v_0^*\|_\infty \leq K_4\}$$

and

$$D^* = \{v_1^* \in Y^* \ : \ \|v_1^*\|_\infty \leq K_5\}$$

for some appropriate real constants $K_4 > 0$ and $K_5 > 0$.

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Observe that

$$\frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial u^2} = -\gamma \nabla^2 + K,$$

$$\frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial (v_1^*)^2} = \frac{1}{-2v_0^* + K} + \frac{K^2(-\gamma\nabla^2 + 2v_0^*)^2}{[(-\gamma\nabla^2 + K)(-2v_0^* + K)]^2},$$

$$\frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial u \, \partial v_1^*} = -1,$$

so that

$$\begin{aligned} \det\left(\frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial u \, \partial v_1^*}\right) &= \frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial (v_1^*)^2} \frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial u^2} - \left(\frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial u \, \partial v_1^*}\right)^2 \\ &= \mathcal{O}\left(\frac{K^2(2(-\gamma\nabla^2 + 2v_0^*) + 2(-\gamma\nabla^2 + 2v_0^*)^2)}{(-\gamma\nabla^2 + K)(-2v_0^* + K)^2}\right) \\ &\equiv H(v_0^*). \end{aligned} \quad (61)$$

With such results in mind, we may easily prove the following theorem.

Theorem 12.1. *Assume $K \gg \max\{K_3, K_4, K_5, 1\}$ and suppose $(u_0, \hat{v}_1^*, \hat{v}_0^*) \in V_1 \times D^* \times B^*$ is such that*

$$\delta J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}.$$

Suppose also $H(\hat{v}_0^*) > 0$.

Under such hypotheses, we have that

$$\delta J(u_0) = \mathbf{0}$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1}\left\{ J(u) + \frac{K^2}{2}\int_\Omega \left(\frac{(-\gamma\nabla^2 u + 2\hat{v}_0^* u - f)}{-\gamma\nabla^2 + K}\right)^2 dx \right\} \\ &= J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) \\ &= \inf_{(u, v_1^*) \in V_1 \times D^*}\left\{ \sup_{v_0^* \in B^*} J_1^*(u,, v_1^*, v_0^*) \right\} \\ &= \sup_{v_0^* \in B^*}\left\{ \inf_{(u, v_1^*) \in V_1 \times D^*} J_1^*(u, v_1^*, v_0^*) \right\}. \end{aligned} \quad (62)$$

Proof. The proof that

$$\delta J(u_0) = -\gamma\nabla^2 u_0 + 2\alpha(u^2 - \beta)u_0 - f = \mathbf{0},$$

$$\hat{v}_0^* = \alpha(u_0^2 - \beta)$$

and

$$J(u_0) = J(u_0) + \frac{K^2}{2}\int_\Omega \left(\frac{(-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 - f)}{-\gamma\nabla^2 + K}\right)^2 dx = J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*)$$

may be easily made similarly as in the previous sections.

Moreover, from the hypotheses, we have

$$J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) = \inf_{(u, v_1^*) \in V_1 \times D^*} J_1^*(u, v_1^*, \hat{v}_0^*)$$

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and

$$J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*)=\sup_{v_0^*\in B^*} J_1^*(u_0,\hat{v}_1^*,v_0^*).$$

From this, from a standard saddle point theorem and the remaining hypotheses, we may infer that

$$\begin{aligned} J(u_0) &= J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*) \\ &= \inf_{(u,v_1^*)\in V_1\times D^*}\left\{\sup_{v_0^*\in B^*} J_1^*(u,v_1^*,v_0^*)\right\} \\ &= \sup_{v_0^*\in B^*}\left\{\inf_{(u,v_1^*)\in V_1\times D^*} J_1^*(u,v_1^*,v_0^*)\right\}. \end{aligned} \tag{63}$$

Moreover, observe that

$$\begin{aligned} J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*) &= \inf_{(u,v_1^*)\in V_1\times D^*} J_1^*(u,v_1^*,\hat{v}_0^*) \\ &\leq \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\langle u^2,\hat{v}_0^*\rangle_{L^2} \\ &\quad -\frac{1}{2\alpha}\int_\Omega(\hat{v}_0^*)^2\,dx-\beta\int_\Omega \hat{v}_0^*\,dx-\langle u,f\rangle_{L^2} \\ &\quad +\frac{K^2}{2}\int_\Omega\left(\frac{(-\gamma\nabla^2 u+2\hat{v}_0^*u-f)}{-\gamma\nabla^2+K}\right)^2\,dx \\ &\leq \sup_{v_0^*\in Y^*}\left\{\frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\langle u^2,v_0^*\rangle\right. \\ &\quad -\frac{1}{2\alpha}\int_\Omega(v_0^*)^2\,dx-\beta\int_\Omega v_0^*\,dx-\langle u,f\rangle_{L^2} \\ &\quad \left.+\frac{K^2}{2}\int_\Omega\left(\frac{(-\gamma\nabla^2 u+2\hat{v}_0^*u-f)}{-\gamma\nabla^2+K}\right)^2\,dx\right\} \\ &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\frac{\alpha}{2}\int_\Omega(u^2-\beta)^2\,dx \\ &\quad -\langle u,f\rangle_{L^2} \\ &\quad +\frac{K^2}{2}\int_\Omega\left(\frac{(-\gamma\nabla^2 u+2\hat{v}_0^*u-f)}{-\gamma\nabla^2+K}\right)^2\,dx,\ \forall u\in V_1. \end{aligned} \tag{64}$$

From this we have got

$$\begin{aligned} J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*) &\leq \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\frac{\alpha}{2}\int_\Omega(u^2-\beta)^2\,dx-\langle u,f\rangle_{L^2} \\ &\quad +\frac{K^2}{2}\int_\Omega\left(\frac{(-\gamma\nabla^2 u+2\hat{v}_0^*u-f)}{-\gamma\nabla^2+K}\right)^2\,dx,\ \forall u\in V_1. \end{aligned} \tag{65}$$

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Therefore, from such results we may obtain

$$
\begin{aligned}
J(u_0) &= \inf_{u\in V_1}\left\{J(u)+\frac{K^2}{2}\int_\Omega\left(\frac{(-\gamma\nabla^2 u+2\hat{v}_0^* u-f)}{-\gamma\nabla^2+K}\right)^2\,dx\right\}\\
&= J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*)\\
&= \inf_{(u,v_1^*)\in V_1\times D^*}\left\{\sup_{v_0^*\in B^*} J_1^*(u,,v_1^*,v_0^*)\right\}\\
&= \sup_{v_0^*\in B^*}\left\{\inf_{(u,v_1^*)\in V_1\times D^*} J_1^*(u,v_1^*,v_0^*)\right\}. \qquad (66)
\end{aligned}
$$

The proof is complete. $\square$

13. Another primal dual formulation for a related model

In this section we present another primal dual formulation.
Let $\Omega\subset\mathbb{R}^3$ be an open, bounded and connected set with a regular boundary denoted by $\partial\Omega$.
Consider the functional $J: V\to\mathbb{R}$ where

$$
\begin{aligned}
J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\,dx\\
&\quad -\langle u,f\rangle_{L^2}, \qquad (67)
\end{aligned}
$$

$\alpha>0,\ \beta>0,\ \gamma>0,\ V=W_0^{1,2}(\Omega)$ and $f\in L^2(\Omega)$.
Denoting $Y=Y^*=L^2(\Omega)$, define now $J_1^*: V\times Y^*\to\mathbb{R}$ by

$$
\begin{aligned}
J_1^*(u,v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\langle u^2,v_0^*\rangle_{L^2}\\
&\quad +\frac{\alpha-\varepsilon}{2}\int_\Omega u^4\,dx-\langle u,f\rangle_{L^2}\\
&\quad -\frac{1}{2\varepsilon}\int_\Omega (v_0^*+\alpha\beta)^2\,dx, \qquad (68)
\end{aligned}
$$

and $J_2^*: V\times Y^*\to\mathbb{R}$, by

$$
\begin{aligned}
J_2^*(u,v_0^*) &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\langle u^2,v_0^*\rangle_{L^2}\\
&\quad +\frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u+2v_0^* u-2(\alpha-\varepsilon)u^3-f)^2\,dx\\
&\quad +\frac{\alpha-\varepsilon}{2}\int_\Omega u^4\,dx-\langle u,f\rangle_{L^2}\\
&\quad -\frac{1}{2\varepsilon}\int_\Omega (v_0^*+\alpha\beta)^2\,dx, \qquad (69)
\end{aligned}
$$

Define also

$$A^+=\{u\in V\ :\ u\,f\geq 0,\ \text{a.e. in }\Omega\},$$

$$V_2=\{u\in V\ :\ \|u\|_\infty\leq K_3\},$$

and

$$V_1=V_2\cap A^+.$$

Moreover define

$$B^*=\{v_0^*\in Y^*\ :\ \|v_0^*\|_\infty\leq K_4\}$$

for some appropriate constants $K_3>0$ and $K_4>0$.

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Observe that, for $K_1 = 1/\sqrt{\varepsilon}$, we have

$$\begin{aligned} \frac{\partial^2 J_2^*(u, v_0^*)}{\partial u^2} &= (-\gamma\nabla^2 + 2v_0^* + 6(\alpha-\varepsilon)u^2) + K_1(-\gamma\nabla^2 + 2v_0^* + 6(\alpha-\varepsilon)u^2)^2 \\ &+ K_1(-\gamma\nabla^2 u + 2v_0^* u + 2(\alpha-\varepsilon)u^3 - f)12(\alpha-\varepsilon)u\,dx, \end{aligned} \tag{70}$$

$$\begin{aligned} \frac{\partial^2 J_2^*(u, v_0^*)}{\partial (v_0^*)^2} &= K_1 4u^2 - \frac{1}{\varepsilon} \\ &< \mathbf{0}, \ \forall u \in V_1, \ v_0^* \in B^*. \end{aligned} \tag{71}$$

Define now

$$A_2(u, v_0^*) = (-\gamma\nabla^2 u + 2v_0^* u + 2(\alpha-\varepsilon)u^3 - f)12(\alpha-\varepsilon)u,$$

$$C^* = \{(u, v_0^*) \in V \times B^* \ : \ \|A_2(u, v_0^*)\|_\infty \leq \varepsilon_1$$

for a small real parameter $\varepsilon_1 > 0$.

Finally, define

$$E_{v_0^*} = \left\{ u \in V \ : \ \frac{\partial^2 A_2(u, v_0^*)}{\partial u^2} > \mathbf{0} \right\}.$$

Remark 13.1. *Similarly as it was developed in remark 10.1 we may prove that such a $E_{v_0^*}$ is a convex set.*

Thus,

$$E_{v_0^*} \cap V_1$$

is a convex set, $\forall v_0^* \in B^*$ (for the proof of a similar result please see Theorem 8.7.1 at pages 297, 298 and 299 in [5]).)

With such results in mind, we may easily prove the following theorem.

Theorem 13.2. *Assume $K_1 \gg 1 \gg \varepsilon_1$ and suppose $(u_0, \hat{v}_0^*) \in V_1 \times B^*$ is such that*

$$\delta J_2^*(u_0, \hat{v}_0^*) = \mathbf{0}$$

and $u_0 \in E_{\hat{v}_0^*}$.

Under such hypotheses, we have that

$$\delta J(u_0) = \mathbf{0}$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (-\gamma\nabla^2 u + 2\hat{v}_0^* u + 2(\alpha-\varepsilon)u^3 - f)^2\,dx \right\} \\ &= J_2^*(u_0, \hat{v}_0^*) \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{u \in V_1} J_2^*(u, v_0^*) \right\}. \end{aligned} \tag{72}$$

Proof. The proof that

$$\delta J(u_0) = -\gamma\nabla^2 u_0 + 2\alpha(u^2 - \beta)u_0 - f = \mathbf{0},$$

and

$$J(u_0) = J(u_0) + \frac{K_1}{2} \int_\Omega (-\gamma\nabla^2 u_0 + 2\hat{v}_0^* u_0 + 2(\alpha-\varepsilon)u_0^3 - f)^2\,dx = J_2^*(u_0, \hat{v}_0^*)$$

may be easily made similarly as in the previous sections.

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Moreover, from the hypotheses and from the above lines, since J_2^* is concave in v_0^* on $V_1 \times B^*$ and $u_0 \in E_{\hat{v}_0^*}$, we have that

$$J_2^*(u_0, \hat{v}_0^*) = \inf_{u \in V_1} J_2^*(u, \hat{v}_0^*)$$

and

$$J_2^*(u_0, \hat{v}_0^*) = \sup_{v_0^* \in B^*} J_2^*(u_0, v_0^*).$$

From this, from the standard Saddle Point Theorem and the remaining hypotheses, we may infer that

$$\begin{aligned} J(u_0) &= J_2^*(u_0, \hat{v}_0^*) \\ &= \inf_{u \in V_1} \left\{ \sup_{v_0^* \in B^*} J_2^*(u, v_1^*, v_0^*) \right\} \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{u \in V_1} J_2^*(u, v_0^*) \right\}. \end{aligned} \quad (73)$$

Moreover, observe that

$$\begin{aligned} J_2^*(u_0, \hat{v}_0^*) &= \inf_{u \in V_1} J_2^*(u, \hat{v}_0^*) \\ &\leq \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, \hat{v}_0^* \rangle_{L^2} + \frac{\alpha - \varepsilon}{2} \int_\Omega u^4 \, dx \\ &\quad - \frac{1}{2\varepsilon} \int_\Omega (\hat{v}_0^* + \alpha\beta)^2 \, dx - \langle u, f \rangle_{L^2} \\ &\quad + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u + 2(\alpha - \varepsilon) u^3 - f)^2 \, dx \\ &\leq \sup_{v_0^* \in Y^*} \left\{ \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \langle u^2, v_0^* \rangle_{L^2} + \frac{\alpha - \varepsilon}{2} \int_\Omega u^4 \, dx \right. \\ &\quad - \frac{1}{2\varepsilon} \int_\Omega (v_0^* + \alpha\beta)^2 \, dx - \langle u, f \rangle_{L^2} \\ &\quad \left. + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u + 2(\alpha - \varepsilon) u^3 - f)^2 \, dx \right\} \\ &= J(u) + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u + 2(\alpha - \varepsilon) u^3 - f)^2 \, dx, \ \forall u \in V_1. \end{aligned} \quad (74)$$

From this we have got

$$J_2^*(u_0, \hat{v}_0^*) \leq J(u) + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u + 2(\alpha - \varepsilon) u^3 - f)^2 \, dx, \ \forall u \in V_1. \quad (75)$$

Therefore, from such results we may obtain

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1} \left\{ J(u) + \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + 2\hat{v}_0^* u + 2(\alpha - \varepsilon) u^3 - f)^2 \, dx \right\} \\ &= J_2^*(u_0, \hat{v}_0^*) \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{u \in V_1} J_2^*(u, v_0^*) \right\}. \end{aligned} \quad (76)$$

The proof is complete. □

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14. A convex (in fact concave) dual formulation for a related model

In this section we present a convex dual formulation for the model in question.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular boundary denoted by $\partial\Omega$.

Consider the functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2}\int_\Omega (u^2-\beta)^2 \, dx \\ &\quad -\langle u, f\rangle_{L^2}, \end{aligned} \tag{77}$$

$\alpha > 0,\ \beta > 0,\ \gamma > 0, V = W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.

Denoting $Y = Y^* = L^2(\Omega)$, define now $J_1^* : [Y^*]^6 \to \mathbb{R}$ (with exact penalization) by

$$\begin{aligned} J_1^*(v_1^*, v_2^*, v_3^*, v_0^*, z_1^*, z_2^*) &= -\int_\Omega \frac{(v_1^*+v_3^*-f)^2}{-\gamma\nabla^2}\, dx - \int_\Omega \frac{(v_2^*-v_3^*)^2}{-\gamma\nabla^2}\, dx \\ &\quad -\frac{1}{2}\int_\Omega \frac{(-v_1^*+z_1^*)^2}{2v_0^*+K(A)}\, dx - \frac{1}{2}\int_\Omega \frac{(-v_2^*+z_2^*)^2}{K(1-A)}\, dx \\ &\quad +\int_\Omega \frac{(z_1^*)^2}{K}\, dx + \int_\Omega \frac{(z_2^*)^2}{K}\, dx \\ &\quad -\frac{1}{2\alpha}\int_\Omega (v_0^*)^2\, dx - \beta\int_\Omega v_0^*\, dx. \end{aligned} \tag{78}$$

Define also

$$B^+ = \{u \in V \ : \ u\, f \geq 0, \text{ a.e. in } \Omega\},$$

$$V_2 = \{u \in V \ : \ \|u\|_\infty \leq K_3\},$$

and

$$V_1 = V_2 \cap B^+.$$

Moreover define

$$B^* = \{v_0^* \in Y^* \ : \ \|v_0^*\|_\infty \leq K_4\}$$

for some appropriate constants $K_3 > 0$ and $K_4 > 0$.

Define also

$$D^* = \{(v_1^*, v_2^*, v_3^*) = w^* \in [Y^*]^2 \ : \ \|w^*\|_\infty \leq K_5\},$$

for an appropriate $K_5 > 0$ to be specified.

Observe that, for appropriate $0 < A < 1$, J_1^* is concave in $v^* = (v_1^*, v_2^*, v_3^*, v_0^*)$ and convex in $z^* = (z_1^*, z_2^*)$ on $D^* \times B^* \times [Y^*]^2$. With such results in mind, we may easily prove the following theorem.

Theorem 14.1. *Assume an appropriate* $0 < A < 1$ *and* $K \gg \max\{1, K_3, K_4, K_5\}$ *and suppose* $(\hat{v}^*, \hat{z}^*) \in D^* \times B^* \times [Y^*]^2$ *is such that*

$$\delta J_2^*(\hat{v}^*, \hat{z}^*) = \mathbf{0}.$$

Suppose also

$$u_0 = 2\left(\frac{\hat{v}_2^* - \hat{v}_3^*}{-\gamma\nabla^2}\right) \in V_1.$$

Under such hypotheses, we have that

$$\delta J(u_0) = \mathbf{0}$$

and

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$$
\begin{aligned}
J(u_0) &= J_1^*(\hat{v}^*, \hat{z}^*) \\
&= \sup_{v^* \in D^* \times B^*} \left\{ \inf_{z^* \in [Y^*]^2} J_1^*(v^*, z^*) \right\}.
\end{aligned} \tag{79}
$$

Proof. The proof that

$$\delta J(u_0) = -\gamma \nabla^2 u_0 + 2\alpha(u^2 - \beta)u_0 - f = \mathbf{0},$$

and

$$J(u_0) = J_1^*(\hat{v}^*, \hat{z}^*)$$

may be easily made similarly as in the previous sections.

Moreover, from the hypotheses and from the above lines, since J_1^* is concave in v^* and convex in z^* on $D^* \times B^* \times [Y^*]^2$, we have

$$J_1^*(\hat{v}^*, \hat{z}^*) = \sup_{v^* \in D^* \times B^*} J_2^*(v^*, \hat{z})$$

and

$$J_1^*(\hat{v}^*, \hat{z}^*) = \inf_{z^* \in [Y^*]^2} J_1^*(\hat{v}^*, z^*).$$

From this, from the standard Min-Max Theorem and the remaining hypotheses, we may infer that

$$
\begin{aligned}
J(u_0) &= J_1^*(\hat{v}^*, \hat{z}^*) \\
&= \sup_{v^* \in D^* \times B^*} \left\{ \inf_{z^* \in [Y^*]^2} J_1^*(v^*, z^*) \right\}.
\end{aligned} \tag{80}
$$

The proof is complete. □

Remark 14.2. *The functional*

$$J_3^*(v^*) = \inf_{z^* \in [Y^*]^2} J_1^*(v^*, z^*)$$

is indeed a concave dual variational formulation for a critical point of the primal model in question.

15. An algorithm for a related model in shape optimization

The next two subsections have been previously published by Fabio Silva Botelho and Alexandre Molter in [5], Chapter 21.

15.1. Introduction

Consider an elastic solid which the volume corresponds to an open, bounded, connected set, denoted by $\Omega \subset \mathbb{R}^3$ with a regular (Lipschitzian) boundary denoted by $\partial\Omega = \Gamma_0 \cup \Gamma_t$ where $\Gamma_0 \cap \Gamma_t = \emptyset$. Consider also the problem of minimizing the functional $\hat{J} : U \times B \to \mathbb{R}$ where

$$\hat{J}(u, t) = \frac{1}{2}\langle u_i, f_i \rangle_{L^2(\Omega)} + \frac{1}{2}\langle u_i, \hat{f}_i \rangle_{L^2(\Gamma_t)},$$

subject to

$$
\begin{cases}
(H_{ijkl}(t)e_{kl}(u))_{,j} + f_i = 0 \text{ in } \Omega, \\
H_{ijkl}(t)e_{kl}(u)n_j - \hat{f}_i = 0, \text{ on } \Gamma_t, \ \forall i \in \{1, 2, 3\}.
\end{cases} \tag{81}
$$

Here $\mathbf{n} = (n_1, n_2, n_3)$ denotes the outward normal to $\partial\Omega$ and

$$U = \{u = (u_1, u_2, u_3) \in W^{1,2}(\Omega; \mathbb{R}^3) \ : \ u = (0,0,0) = \mathbf{0} \text{ on } \Gamma_0\},$$

$$B = \left\{t : \Omega \to [0,1] \text{ measurable } \ : \ \int_\Omega t(x)\,dx = t_1|\Omega|\right\},$$

where

$$0 < t_1 < 1$$

and $|\Omega|$ denotes the Lebesgue measure of Ω.

Moreover $u = (u_1, u_2, u_3) \in W^{1,2}(\Omega; \mathbb{R}^3)$ is the field of displacements relating the cartesian system $(0, x_1, x_2, x_3)$, resulting from the action of the external loads $f \in L^2(\Omega; \mathbb{R}^3)$ and $\hat{f} \in L^2(\Gamma_t; \mathbb{R}^3)$.

We also define the stress tensor $\{\sigma_{ij}\} \in Y^* = Y = L^2(\Omega; \mathbb{R}^{3\times 3})$, by

$$\sigma_{ij}(u) = H_{ijkl}(t)e_{kl}(u),$$

and the strain tensor $e : U \to L^2(\Omega; \mathbb{R}^{3\times 3})$ by

$$e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ \forall i,j \in \{1,2,3\}.$$

Finally,

$$\{H_{ijkl}(t)\} = \{tH^0_{ijkl} + (1-t)H^1_{ijkl}\},$$

where H^0 corresponds to a strong material and H^1 to a very soft material, intending to simulate voids along the solid structure.

The variable t is the design one, which the optimal distribution values along the structure are intended to minimize its inner work with a volume restriction indicated through the set B.

The duality principle obtained is developed inspired by the works in [2,3]. Similar theoretical results have been developed in [9], however we believe the proof here presented, which is based on the min-max theorem is easier to follow (indeed we thank an anonymous referee for his suggestion about applying the min-max theorem to complete the proof). We highlight throughout this text we have used the standard Einstein sum convention of repeated indices.

Moreover, details on the Sobolev spaces addressed may be found in [1]. In addition, the primal variational development of the topology optimization problem has been described in [9].

The main contributions of this work are to present the detailed development, through duality theory, for such a kind of optimization problems. We emphasize that to avoid the check-board standard and obtain appropriate robust optimized structures without the use of filters, it is necessary to discretize more in the load direction, in which the displacements are much larger.

15.2. Mathematical formulation of the topology optimization problem

Our mathematical topology optimization problem is summarized by the following theorem.

Theorem 15.1. *Consider the statements and assumptions indicated in the last section, in particular those refereing to Ω and the functional $\hat{J} : U \times B \to \mathbb{R}$.*

Define $J_1 : U \times B \to \mathbb{R}$ by

$$J_1(u,t) = -G(e(u),t) + \langle u_i, f_i\rangle_{L^2(\Omega)} + \langle u_i, \hat{f}_i\rangle_{L^2(\Gamma_t)},$$

where

$$G(e(u),t) = \frac{1}{2}\int_\Omega H_{ijkl}(t)e_{ij}(u)e_{kl}(u)\,dx,$$

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and where

$$dx = dx_1 dx_2 dx_3.$$

Define also $J^* : U \to \mathbb{R}$ *by*

$$\begin{aligned} J^*(u) &= \inf_{t \in B}\{J_1(u,t)\} \\ &= \inf_{t \in B}\{-G(e(u),t) + \langle u_i, f_i\rangle_{L^2(\Omega)} + \langle u_i, \hat{f}_i\rangle_{L^2(\Gamma_t)}\}. \end{aligned} \tag{82}$$

Assume there exists $c_0, c_1 > 0$ *such that*

$$H^0_{ijkl} z_{ij} z_{kl} > c_0 z_{ij} z_{ij}$$

and

$$H^1_{ijkl} z_{ij} z_{kl} > c_1 z_{ij} z_{ij},\ \forall z = \{z_{ij}\} \in \mathbb{R}^{3\times 3},\ \text{ such that } z \neq \mathbf{0}.$$

Finally, define $J : U \times B \to \mathbb{R} \cup \{+\infty\}$ *by*

$$J(u,t) = \hat{J}(u,t) + Ind(u,t),$$

where

$$Ind(u,t) = \begin{cases} 0, & \text{if } (u,t) \in A^*, \\ +\infty, & \text{otherwise }, \end{cases} \tag{83}$$

where $A^* = A_1 \cap A_2$,

$$A_1 = \{(u,t) \in U \times B \ : \ (\sigma_{ij}(u))_{,j} + f_i = 0, \text{ in } \Omega,\ \forall i \in \{1,2,3\}\}$$

and

$$A_2 = \{(u,t) \in U \times B \ : \ \sigma_{ij}(u) n_j - \hat{f}_i = 0, \text{ on } \Gamma_t,\ \forall i \in \{1,2,3\}\}.$$

Under such hypotheses, there exists $(u_0, t_0) \in U \times B$ *such that*

$$\begin{aligned} J(u_0,t_0) &= \inf_{(u,t) \in U \times B} J(u,t) \\ &= \sup_{\hat{u} \in U} J^*(\hat{u}) \\ &= J^*(u_0) \\ &= \hat{J}(u_0,t_0) \\ &= \inf_{(t,\sigma) \in B \times C^*} G^*(\sigma,t) \\ &= G^*(\sigma(u_0),t_0), \end{aligned} \tag{84}$$

where

$$\begin{aligned} G^*(\sigma,t) &= \sup_{v \in Y}\{\langle v_{ij}, \sigma_{ij}\rangle_{L^2(\Omega)} - G(v,t)\} \\ &= \frac{1}{2}\int_\Omega \overline{H}_{ijkl}(t)\sigma_{ij}\sigma_{kl}\, dx, \end{aligned} \tag{85}$$

$$\{\overline{H}_{ijkl}(t)\} = \{H_{ijkl}(t)\}^{-1}$$

and $C^* = C_1 \cap C_2$, *where*

$$C_1 = \{\sigma \in Y^* \ : \ \sigma_{ij,j} + f_i = 0, \text{ in } \Omega,\ \forall i \in \{1,2,3\}\}$$

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and

$$C_2 = \{\sigma \in Y^* \;:\; \sigma_{ij}n_j - \hat{f}_i = 0, \; on\; \Gamma_t, \; \forall i \in \{1,2,3\}\}.$$

Proof. Observe that

$$\begin{aligned}
\inf_{(u,t)\in U\times B} J(u,t) &= \inf_{t\in B}\left\{\inf_{u\in U} J(u,t)\right\} \\
&= \inf_{t\in B}\left\{\sup_{\hat{u}\in U}\left\{\inf_{u\in U}\left\{\frac{1}{2}\int_\Omega H_{ijkl}(t)e_{ij}(u)e_{kl}(u)\,dx\right.\right.\right. \\
&\quad +\langle \hat{u}_i, (H_{ijkl}(t)e_{kl}(u))_{,j} + f_i\rangle_{L^2(\Omega)} \\
&\quad \left.\left.\left. -\langle \hat{u}_i, H_{ijkl}(t)e_{kl}(u)n_j - \hat{f}_i\rangle_{L^2(\Gamma_t)}\right\}\right\}\right\} \\
&= \inf_{t\in B}\left\{\sup_{\hat{u}\in U}\left\{\inf_{u\in U}\left\{\frac{1}{2}\int_\Omega H_{ijkl}(t)e_{ij}(u)e_{kl}(u)\,dx\right.\right.\right. \\
&\quad -\int_\Omega H_{ijkl}(t)e_{ij}(\hat{u})e_{kl}(u)\,dx \\
&\quad \left.\left.\left. +\langle \hat{u}_i, f_i\rangle_{L^2(\Omega)} + \langle \hat{u}_i, \hat{f}_i\rangle_{L^2(\Gamma_t)}\right\}\right\}\right\} \\
&= \inf_{t\in B}\left\{\sup_{\hat{u}\in U}\left\{-\int_\Omega H_{ijkl}(t)e_{ij}(\hat{u})e_{kl}(\hat{u})\,dx\right.\right. \\
&\quad \left.\left. \langle \hat{u}_i, f_i\rangle_{L^2(\Omega)} + \langle \hat{u}_i, \hat{f}_i\rangle_{L^2(\Gamma_t)}\right\}\right\} \\
&= \inf_{t\in B}\left\{\inf_{\sigma\in C^*} G^*(\sigma,t)\right\}. \qquad (86)
\end{aligned}$$

Also, from this and the min-max theorem, there exist $(u_0, t_0) \in U \times B$ such that

$$\begin{aligned}
\inf_{(u,t)\in U\times B} J(u,t) &= \inf_{t\in B}\left\{\sup_{\hat{u}\in U} J_1(u,t)\right\} \\
&= \sup_{u\in U}\left\{\inf_{t\in B} J_1(u,t)\right\} \\
&= J_1(u_0,t_0) \\
&= \inf_{t\in B} J_1(u_0,t) \\
&= J^*(u_0). \qquad (87)
\end{aligned}$$

Finally, from the extremal necessary condition

$$\frac{\partial J_1(u_0,t_0)}{\partial u} = \mathbf{0}$$

we obtain

$$(H_{ijkl}(t_0)e_{kl}(u_0))_{,j} + f_i = 0 \text{ in } \Omega,$$

and

$$H_{ijkl}(t_0)e_{kl}(u_0)n_j - \hat{f}_i = 0 \text{ on } \Gamma_t, \; \forall i \in \{1,2,3\},$$

so that

$$G(e(u_0)) = \frac{1}{2}\langle (u_0)_i, f_i\rangle_{L^2(\Omega)} + \frac{1}{2}\langle (u_0)_i, \hat{f}_i\rangle_{L^2(\Gamma_t)}.$$

Hence $(u_0, t_0) \in A^*$ so that $Ind(u_0, t_0) = 0$ and $\sigma(u_0) \in C^*$.

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Moreover

$$\begin{aligned} J^*(u_0) &= -G(e(u_0)) + \langle (u_0)_i, f_i \rangle_{L^2(\Omega)} + \langle (u_0)_i, \hat{f}_i \rangle_{L^2(\Gamma_t)} \\ &= G(e(u_0)) \\ &= G(e(u_0)) + Ind(u_0, t_0) \\ &= J(u_0, t_0) \\ &= G^*(\sigma(u_0), t_0). \end{aligned} \tag{88}$$

This completes the proof. □

15.3. About a concerning algorithm and related numerical method

For numerically solve this optimization problem in question, we present the following algorithm

1. Set $t_1 = 0.5$ in Ω and $n = 1$.
2. Calculate $u_n \in U$ such that
$$J_1(u_n, t_n) = \sup_{u \in U} J_1(u, t_n).$$
3. Calculate $t_{n+1} \in B$ such that
$$J_1(u_n, t_{n+1}) = \inf_{t \in B} J_1(u_n, t).$$
4. If $\|t_{n+1} - t_n\|_\infty < 10^{-4}$ or $n > 100$ then stop, else set $n := n + 1$ and go to item 2.

We have developed a software in finite differences for solving such a problem.

For a two dimensional beam of dimensions $1m \times 0.5m$ and $t_1 = 0.63$ we have obtained the following results:

1. Case A: For the optimal shape for a clamped beam at left (cantilever) and load $P = -4 \cdot 10^6 N\mathbf{j}$ at $(x, y) = (1, 0.25)$, please Figure 5.

   In this case the mesh was 28×24.

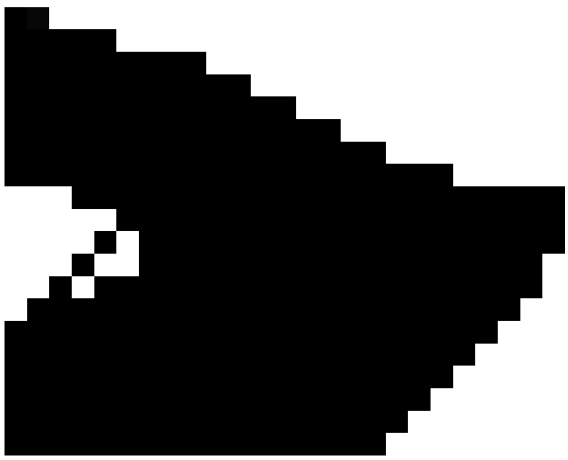

Figure 5. Density $t(x, y)$ for the Case A.

 doi:10.20944/preprints202302.0051.v13

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