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Fabio Botelho *

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Keywords: Duality theory; non-convex analysis; numerical method for a non-smooth model

Article

Duality Principles and Numerical Procedures for a Large Class of Non-Convex Models in the Calculus of Variations

Fabio Silva Botelho

Department of Mathematics, Federal University of Santa Catarina, Florianópolis - SC, Brazil

Abstract: This article develops duality principles and numerical results for a large class of non-convex variational models. The main results are based on fundamental tools of convex analysis, duality theory and calculus of variations. More specifically the approach is established for a class of non-convex functionals similar as those found in some models in phase transition. Finally, in the last section we present a concerning numerical example and the respective software.

Keywords: Duality theory; non-convex analysis; numerical method for a non-smooth model

MSC: 49N15

1. Introduction

In this section we establish a dual formulation for a large class of models in non-convex optimization.

The main duality principle is applied to double well models similar as those found in the phase transition theory.

Such results are based on the works of J.J. Telega and W.R. Bielski [2,3,15,16] and on a D.C. optimization approach developed in Toland [17].

About the other references, details on the Sobolev spaces involved are found in [1]. Related results on convex analysis and duality theory are addressed in [5,7,8,10,14].

Finally, in this text we adopt the standard Einstein convention of summing up repeated indices, unless otherwise indicated.

In order to clarify the notation, here we introduce the definition of topological dual space.

Definition 1.1 (Topological dual spaces)**.** *Let U be a Banach space. We shall define its dual topological space, as the set of all linear continuous functionals defined on U. We suppose such a dual space of U, may be represented by another Banach space U^*, through a bilinear form $\langle \cdot, \cdot \rangle_U : U \times U^* \to \mathbb{R}$ (here we are referring to standard representations of dual spaces of Sobolev and Lebesgue spaces). Thus, given $f : U \to \mathbb{R}$ linear and continuous, we assume the existence of a unique $u^* \in U^*$ such that*

$$f(u) = \langle u, u^* \rangle_U, \forall u \in U. \tag{1}$$

The norm of f, denoted by $\|f\|_{U^*}$, is defined as

$$\|f\|_{U^*} = \sup_{u \in U} \{|\langle u, u^* \rangle_U| \; : \; \|u\|_U \leq 1\} \equiv \|u^*\|_{U^*}. \tag{2}$$

At this point we start to describe the primal and dual variational formulations.

2. A general duality principle non-convex optimization

In this section we present a duality principle applicable to a model in phase transition.

This case corresponds to the vectorial one in the calculus of variations.

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Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = F(\nabla u_1, \cdots, \nabla u_N) + G(u_1, \cdots, u_N) - \langle u_i, f_i \rangle_{L^2},$$

and where

$$V = \{u = (u_1, \cdots, u_N) \in W^{1,p}(\Omega; \mathbb{R}^N) \; : \; u = u_0 \text{ on } \partial\Omega\},$$

$f \in L^2(\Omega; \mathbb{R}^N)$, and $1 < p < +\infty$.

We assume there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = \inf_{u \in V} J(u).$$

Moreover, suppose F and G are Fréchet differentiable but not necessarily convex. A global optimum point may not be attained for J so that the problem of finding a global minimum for J may not be a solution.

Anyway, one question remains, how the minimizing sequences behave close the infimum of J.

We intend to use duality theory to approximately solve such a global optimization problem.

Denoting $V_0 = W_0^{1,p}(\Omega; \mathbb{R}^N)$, $Y_1 = Y_1^* = L^2(\Omega; \mathbb{R}^{N \times n})$, $Y_2 = Y_2^* = L^2(\Omega; \mathbb{R}^{N \times n})$, $Y_3 = Y_3^* = L^2(\Omega; \mathbb{R}^N)$, at this point we define, $F_1 : V \times V_0 \to \mathbb{R}$, $G_1 : V \to \mathbb{R}$, $G_2 : V \to \mathbb{R}$, $G_3 : V_0 \to \mathbb{R}$ and $G_4 : V \to \mathbb{R}$, by

$$\begin{aligned} F_1(\nabla u, \nabla \phi) &= F(\nabla u_1 + \nabla \phi_1, \cdots, \nabla u_N + \nabla \phi_N) + \frac{K}{2} \int_\Omega \nabla u_j \cdot \nabla u_j \, dx \\ &\quad + \frac{K_2}{2} \int_\Omega \nabla \phi_j \cdot \nabla \phi_j \, dx \end{aligned} \quad (3)$$

and

$$G_1(u_1, \cdots, u_n) = G(u_1, \cdots, u_N) + \frac{K_1}{2} \int_\Omega u_j \, u_j \, dx - \langle u_i, f_i \rangle_{L^2},$$

$$G_2(\nabla u_1, \cdots, \nabla u_N) = \frac{K_1}{2} \int_\Omega \nabla u_j \cdot \nabla u_j \, dx,$$

$$G_3(\nabla \phi_1, \cdots, \nabla \phi_N) = \frac{K_2}{2} \int_\Omega \nabla \phi_j \cdot \nabla \phi_j \, dx,$$

and

$$G_4(u_1, \cdots, u_N) = \frac{K_1}{2} \int_\Omega u_j \, u_j \, dx.$$

Define now $J_1 : V \times V_0 \to \mathbb{R}$,

$$J_1(u, \phi) = F(\nabla u + \nabla \phi) + G(u) - \langle u_i, f_i \rangle_{L^2}.$$

Observe that

$$\begin{aligned}
J_1(u,\phi) &= F_1(\nabla u,\nabla\phi)+G_1(u)-G_2(\nabla u)-G_3(\nabla\phi)-G_4(u) \\
&\leq F_1(\nabla u,\nabla\phi)+G_1(u)-\langle\nabla u,z_1^*\rangle_{L^2}-\langle\nabla\phi,z_2^*\rangle_{L^2}-\langle u,z_3^*\rangle_{L^2} \\
&\quad+\sup_{v_1\in Y_1}\{\langle v_1,z_1^*\rangle_{L^2}-G_2(v_1)\} \\
&\quad+\sup_{v_2\in Y_2}\{\langle v_2,z_2^*\rangle_{L^2}-G_3(v_2)\} \\
&\quad+\sup_{u\in V}\{\langle u,z_3^*\rangle_{L^2}-G_4(u)\} \\
&= F_1(\nabla u,\nabla\phi)+G_1(u)-\langle\nabla u,z_1^*\rangle_{L^2}-\langle\nabla\phi,z_2^*\rangle_{L^2}-\langle u,z_3^*\rangle_{L^2} \\
&\quad+G_2^*(z_1^*)+G_3^*(z_2^*)+G_4^*(z_3^*) \\
&= J_1^*(u,\phi,z^*), \qquad (4)
\end{aligned}$$

$\forall u\in V,\ \phi\in V_0,\ z^*=(z_1^*,z_2^*,z_3^*)\in Y^*=Y_1^*\times Y_2^*\times Y_3^*.$

Here we assume K,K_1,K_2 are large enough so that F_1 and G_1 are convex.

Hence, from the general results in [17], we may infer that

$$\inf_{(u,\phi)\in V\times V_0} J(u,\phi) = \inf_{(u,\phi,z^*)\in V\times V_0\times Y^*} J_1^*(u,\phi,z^*). \qquad (5)$$

On the other hand

$$\inf_{u\in V} J(u)\geq \inf_{(u,\phi)\in V\times V_0} J_1(u,\phi)\geq \inf_{u\in V} Q_J(u)=\inf_{u\in V} J(u),$$

where $Q_J(u)$ refers to a standard quasi-convex regularization of J.

From these last two results we may obtain

$$\inf_{u\in V} J(u)=\inf_{(u,\phi,z^*)\in V\times V_0\times Y^*} J_1^*(u,\phi,z^*).$$

Moreover, from standards results on convex analysis, we may have

$$\begin{aligned}
\inf_{u\in V} J_1^*(u,\phi,z^*) &= \inf_{u\in V}\{F_1(\nabla u,\nabla\phi)+G_1(u) \\
&\quad-\langle\nabla u,z_1^*\rangle_{L^2}-\langle\nabla\phi,z_2^*\rangle_{L^2}-\langle u,z_3^*\rangle_{L^2} \\
&\quad+G_2^*(z_1^*)+G_3^*(z_2^*)+G_4^*(z_3^*)\} \\
&= \sup_{(v_1^*,v_2^*)\in C^*}\{-F_1^*(v_1^*+z_1^*,\nabla\phi)-G_1^*(v_2^*+z_3^*)-\langle\nabla\phi,z_2^*\rangle_{L^2} \\
&\quad+G_2^*(z_1^*)+G_3^*(z_2^*)+G_4^*(z_3^*)\}, \qquad (6)
\end{aligned}$$

where

$$C^*=\{v^*=(v_1^*,v_2^*)\in Y_1^*\times Y_3^*\ :\ -\operatorname{div}(v_1^*)_i+(v_2^*)_i=\mathbf{0},\forall i\in\{1,\cdots,N\}\},$$

$$F_1^*(v_1^*+z_1^*,\nabla\phi)=\sup_{v_1\in Y_1}\{\langle v_1,z_1^*+v_1^*\rangle_{L^2}-F_1(v_1,\nabla\phi)\},$$

and

$$G_1^*(v_2^*+z_2^*)=\sup_{u\in V}\{\langle u,v_2^*+z_2^*\rangle_{L^2}-G_1(u)\}.$$

Thus, defining

$$J_2^*(\phi,z^*,v^*)=F_1^*(v_1^*+z_1^*,\nabla\phi)-G_1^*(v_2^*+z_3^*)-\langle\nabla\phi,z_2^*\rangle_{L^2}+G_2^*(z_1^*)+G_3^*(z_2^*)+G_4^*(z_3^*),$$

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we have got

$$\begin{aligned}
\inf_{u \in V} J(u) &= \inf_{(u,\phi) \in V \times V_0} J_1(u,\phi) \\
&= \inf_{(u,\phi,z^*) \in V \times V_0 \times Y^*} J_1^*(u,\phi,z^*) \\
&= \inf_{z^* \in Y^*} \left\{ \inf_{\phi \in V_0} \left\{ \sup_{v^* \in C^*} J_2^*(\phi, z^*, v^*) \right\} \right\}. \qquad (7)
\end{aligned}$$

Finally, observe that

$$\begin{aligned}
& \inf_{u \in V} J(u) \\
= & \inf_{z^* \in Y^*} \left\{ \inf_{\phi \in V_0} \left\{ \sup_{v^* \in C^*} J_2^*(\phi, z^*, v^*) \right\} \right\} \\
\geq & \sup_{v^* \in C^*} \left\{ \inf_{(z^*,\phi) \in Y^* \times V_0} J_2^*(\phi, z^*, v^*) \right\}. \qquad (8)
\end{aligned}$$

This last variational formulation corresponds to a concave relaxed formulation in v^* concerning the original primal formulation.

3. Another duality principle for a simpler related model in phase transition with a respective numerical example

In this section we present another duality principle for a related model in phase transition.

Let $\Omega = [0,1] \subset \mathbb{R}$ and consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = \frac{1}{2} \int_\Omega ((u')^2 - 1)^2 \, dx + \frac{1}{2} \int_\Omega u^2 \, dx - \langle u, f \rangle_{L^2},$$

and where

$$V = \{u \in W^{1,4}(\Omega) \; : \; u(0) = 0 \text{ and } u(1) = 1/2\}$$

and $f \in L^2(\Omega)$.

A global optimum point is not attained for J so that the problem of finding a global minimum for J has no solution.

Anyway, one question remains, how the minimizing sequences behave close the infimum of J.

We intend to use duality theory to approximately solve such a global optimization problem.

Denoting $V_0 = W_0^{1,4}(\Omega)$, at this point we define, $F : V \to \mathbb{R}$ and $F_1 : V \times V_0 \to \mathbb{R}$ by

$$F(u) = \frac{1}{2} \int_\Omega ((u')^2 - 1)^2 \, dx,$$

and

$$F_1(u,\phi) = \frac{1}{2} \int_\Omega ((u' + \phi')^2 - 1)^2 \, dx.$$

Observe

$$F(u) \geq \inf_{\phi \in V_0} F_1(u,\phi) \geq Q_F(u), \; \forall u \in V,$$

where $Q_F(u)$ refers to a quasi-convex regularization of F.

We define also

$$F_2 : V \times V_0 \to \mathbb{R},$$

$$F_3 : V \times V_0 \to \mathbb{R}$$

and

$$G : V \times V_0 \to \mathbb{R}$$

by

$$F_2(u, \phi) = \frac{1}{2}\int_\Omega ((u' + \phi')^2 - 1)^2\, dx + \frac{1}{2}\int_\Omega u^2\, dx - \langle u, f\rangle_{L^2},$$

$$\begin{aligned} F_3(u,\phi) &= F_2(u,\phi) + \frac{K}{2}\int_\Omega (u')^2\, dx \\ &+ \frac{K_1}{2}\int_\Omega (\phi')^2\, dx \end{aligned} \tag{9}$$

and

$$\begin{aligned} G(u,\phi) &= \frac{K}{2}\int_\Omega (u')^2\, dx \\ &+ \frac{K_1}{2}\int_\Omega (\phi')^2\, dx \end{aligned} \tag{10}$$

Observe that if $K > 0, K_1 > 0$ is large enough, both F_3 and G are convex.
Denoting $Y = Y^* = L^2(\Omega)$ we also define the polar functional $G^* : Y^* \times Y^* \to \mathbb{R}$ by

$$G^*(v^*, v_0^*) = \sup_{(u,\phi)\in V\times V_0} \{\langle u, v^*\rangle_{L^2} + \langle \phi, v_0^*\rangle_{L^2} - G(u,\phi)\}.$$

Observe that

$$\inf_{u\in U} J(u) \geq \inf_{((u,\phi),(v^*,v_0^*))\in V\times V_0\times [Y^*]^2} \{G^*(v^*, v_0^*) - \langle u, v^*\rangle_{L^2} - \langle \phi, v_0^*\rangle_{L^2} + F_3(u,\phi)\}.$$

With such results in mind, we define a relaxed primal dual variational formulation for the primal problem, represented by $J_1^* : V \times V_0 \times [Y^*]^2 \to \mathbb{R}$, where

$$J_1^*(u, \phi, v^*, v_0^*) = G^*(v^*, v_0^*) - \langle u, v^*\rangle_{L^2} - \langle \phi, v_0^*\rangle_{L^2} + F_3(u,\phi).$$

Having defined such a functional, we may obtain numerical results by solving a sequence of convex auxiliary sub-problems, through the following algorithm.

1. Set $K \approx 0.1$ and $K_1 = 120.0$ and $0 < \varepsilon \ll 1$.
2. Choose $(u_1, \phi_1) \in V \times V_0$, such that $\|u_1\|_{1,\infty} < 1$ and $\|\phi_1\|_{1,\infty} < 1$.
3. Set $n = 1$.
4. Calculate $(v_n^*, (v_0^*)_n)$ solution of the system of equations:

$$\frac{\partial J_1^*(u_n, \phi_n, v_n^*, (v_0^*)_n)}{\partial v^*} = \mathbf{0}$$

and

$$\frac{\partial J_1^*(u_n, \phi_n, v_n^*, (v_0^*)_n)}{\partial v_0^*} = \mathbf{0},$$

that is

$$\frac{\partial G^*(v_n^*, (v_0^*)_n)}{\partial v^*} - u_n = 0$$

and

$$\frac{\partial G^*(v_n^*, (v_0^*)_n)}{\partial v_0^*} - \phi_n = 0$$

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so that

$$v_n^* = \frac{\partial G(u_n, \phi_n)}{\partial u}$$

and

$$(v_0^*)_n^* = \frac{\partial G(u_n, \phi_n)}{\partial \phi}$$

5. Calculate (u_{n+1}, ϕ_{n+1}) by solving the system of equations:

$$\frac{\partial J_1^*(u_{n+1}, \phi_{n+1}, v_n^*, (v_0^*)_n)}{\partial u} = \mathbf{0}$$

and

$$\frac{\partial J_1^*(u_{n+1}, \phi_{n+1}, v_n^*, (v_0^*)_n)}{\partial \phi} = \mathbf{0}$$

that is

$$-v_n^* + \frac{\partial F_3(u_{n+1}, \phi_{n+1})}{\partial u} = \mathbf{0}$$

and

$$-(v_0^*)_n + \frac{\partial F_3(u_{n+1}, \phi_{n+1})}{\partial \phi} = \mathbf{0}$$

6. If $\max\{\|u_n - u_{n+1}\|_\infty, \|\phi_{n+1} - \phi_n\|_\infty\} \leq \varepsilon$, then stop, else set $n := n + 1$ and go to item 4.

At this point, we present the corresponding software in MAT-LAB, in finite differences and based on the one-dimensional version of the generalized method of lines.

Here the software.
****

1. clear all

```
m8=300;
d=1/m8;
K=0.1;
K1=120;
for i=1:m8
uo(i,1) = i^2 * d/2;
vo(i,1)=i*d/10;
yo(i,1)=sin(i*d*pi)/2;
end;
k=1;
b12=1.0;
while (b12 > 10^-4.3) and (k < 230000)
k=k+1;
for i=1:m8-1
duo(i,1)=(uo(i+1,1)-uo(i,1))/d;
dvo(i,1)=(vo(i+1,1)-vo(i,1))/d;
end;
m9=zeros(2,2);
m9(1,1)=1;
```

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i=1;

$f1 = 6 * (duo(i,1) + dvo(i,1))^2 - 2;$

m80(1,1,i)=-f1-K;

m80(1,2,i)=-f1;

m80(2,1,i)=-f1;

m80(2,2,i)=-f1-K1;

$y11(1,i) = K * (uo(i+1,1) - 2 * uo(i,1))/d^2 - yo(i,1);$

$y11(2,i) = K1 * (vo(i+1,1) - 2 * vo(i,1))/d^2;$

$m12 = 2 * m80(:,:,i) - m9 * d^2;$

m50(:,:,i)=m80(:,:,i)*inv(m12);

z(:,i)=inv(m12)*y11(:,i)*d^2;

for i=2:m8-1

$f1 = 6 * (duo(i,1) + dvo(i,1))^2 - 2;$

m80(1,1,i)=-f1-K;

m80(1,2,i)=-f1;

m80(2,1,i)=-f1;

m80(2,2,i)=-f1-K1;

$y11(1,i) = K * (uo(i+1,1) - 2 * uo(i,1) + uo(i-1,1))/d^2 - yo(i,1);$

$y11(2,i) = K1 * (vo(i+1,1) - 2 * vo(i,1) + vo(i-1,1))/d^2;$

$m12 = 2 * m80(:,:,i) - m9 * d^2 - m80(:,:,i) * m50(:,:,i-1);$

m50(:,:,i)=inv(m12)*m80(:,:,i);

$z(:,i) = inv(m12) * (y11(:,i) * d^2 + m80(:,:,i) * z(:,i-1));$

end;

U(1,m8)=1/2;

U(2,m8)=0.0;

for i=1:m8-1

U(:,m8-i)=m50(:,:,m8-i)*U(:,m8-i+1)+z(:,m8-i);

end;

for i=1:m8

u(i,1)=U(1,i);

v(i,1)=U(2,i);

end;

b12=max(abs(u-uo))

uo=u;

vo=v;

u(m8/2,1)

end;

for i=1:m8

y(i)=i*d;

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```
end;
plot(y,uo)
************************************
```

For the case in which $f(x) = 0$, we have obtained numerical results for $K = 0.1$ and $K_1 = 120$. For such a concerning solution u_0 obtained, please see Figure 1. For the case in which $f(x) = \sin(\pi x)/2$, we have obtained numerical results also for $K = 0.1$ and $K_1 = 120$. For such a concerning solution u_0 obtained, please see Figure 2.

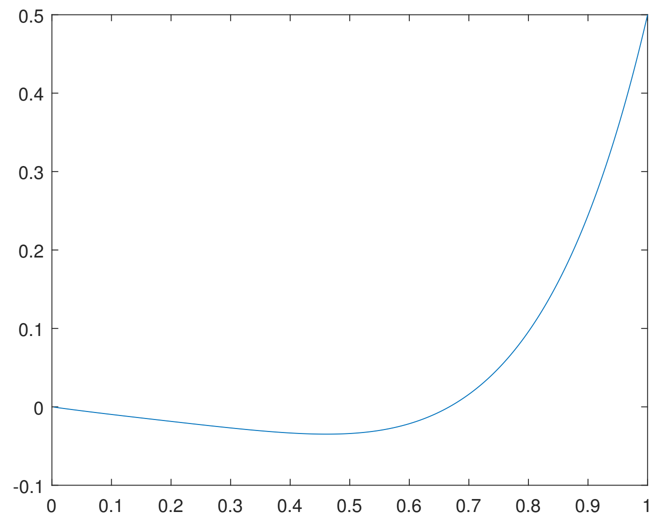

Figure 1. solution $u_0(x)$ for the case $f(x) = 0$.

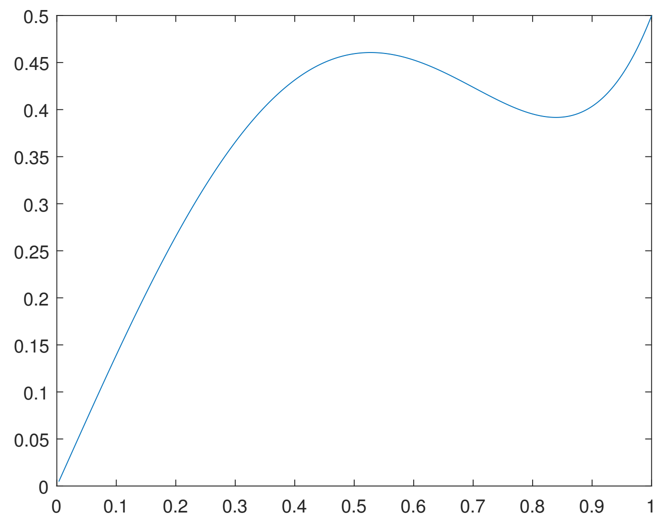

Figure 2. solution $u_0(x)$ for the case $f(x) = \sin(\pi x)/2$.

Remark 3.1. *Observe that the solutions obtained are approximate critical points. They are not, in a classical sense, the global solutions for the related optimization problems. Indeed, such solutions reflect the average behavior of weak cluster points for concerning minimizing sequences.*

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4. A convex dual variational formulation for a third similar model

In this section we present another duality principle for a third related model in phase transition. Let $\Omega = [0, 1] \subset \mathbb{R}$ and consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = \frac{1}{2}\int_\Omega \min\{(u'-1)^2, (u'+1)^2\}\,dx + \frac{1}{2}\int_\Omega u^2\,dx - \langle u, f\rangle_{L^2},$$

and where

$$V = \{u \in W^{1,2}(\Omega)\ :\ u(0) = 0 \text{ and } u(1) = 1/2\}$$

and $f \in L^2(\Omega)$.

A global optimum point is not attained for J so that the problem of finding a global minimum for J has no solution.

Anyway, one question remains, how the minimizing sequences behave close to the infimum of J.

We intend to use the duality theory to solve such a global optimization problem in an appropriate sense to be specified.

At this point we define, $F : V \to \mathbb{R}$ and $G : V \to \mathbb{R}$ by

$$\begin{aligned} F(u) &= \frac{1}{2}\int_\Omega \min\{(u'-1)^2, (u'+1)^2\}\,dx \\ &= \frac{1}{2}\int_\Omega (u')^2\,dx - \int_\Omega |u'|\,dx + 1/2 \\ &\equiv F_1(u'), \end{aligned} \tag{11}$$

and

$$G(u) = \frac{1}{2}\int_\Omega u^2\,dx - \langle u, f\rangle_{L^2}.$$

Denoting $Y = Y^* = L^2(\Omega)$ we also define the polar functional $F_1^* : Y^* \to \mathbb{R}$ and $G^* : Y^* \to \mathbb{R}$ by

$$\begin{aligned} F_1^*(v^*) &= \sup_{v\in Y}\{\langle v, v^*\rangle_{L^2} - F_1(v)\} \\ &= \frac{1}{2}\int_\Omega (v^*)^2\,dx + \int_\Omega |v^*|\,dx, \end{aligned} \tag{12}$$

and

$$\begin{aligned} G^*((v^*)') &= \sup_{u\in V}\{-\langle u', v^*\rangle_{L^2} - G(u)\} \\ &= \frac{1}{2}\int_\Omega ((v^*)' + f)^2\,dx - \frac{1}{2}v^*(1). \end{aligned} \tag{13}$$

Observe this is the scalar case of the calculus of variations, so that from the standard results on convex analysis, we have

$$\inf_{u\in V} J(u) = \max_{v^*\in Y^*}\{-F_1^*(v^*) - G^*(-(v^*)')\}.$$

Indeed, from the direct method of the calculus of variations, the maximum for the dual formulation is attained at some $\hat{v}^* \in Y^*$.

Moreover, the corresponding solution $u_0 \in V$ is obtained from the equation

$$u_0 = \frac{\partial G((\hat{v}^*)')}{\partial (v^*)'} = (\hat{v}^*)' + f.$$

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Finally, the Euler-Lagrange equations for the dual problem stands for

$$\begin{cases} (v^*)'' + f' - v^* - \text{ sign}(v^*) = 0, & \text{in } \Omega, \\ (v^*)'(0) + f(0) = 0, \; (v^*)'(1) + f(1) = 1/2, \end{cases} \tag{14}$$

where $\text{sign}(v^*(x)) = 1$ if $v^*(x) > 0$, $\text{sign}(v^*(x)) = -1$, if $v^*(x) < 0$ and

$$-1 \leq \text{ sign}(v^*(x)) \leq 1,$$

if $v^*(x) = 0$.

We have computed the solutions v^* and corresponding solutions $u_0 \in V$ for the cases in which $f(x) = 0$ and $f(x) = \sin(\pi x)/2$.

For the solution $u_0(x)$ for the case in which $f(x) = 0$, please see Figure 3.

For the solution $u_0(x)$ for the case in which $f(x) = \sin(\pi x)/2$, please see Figure 4.

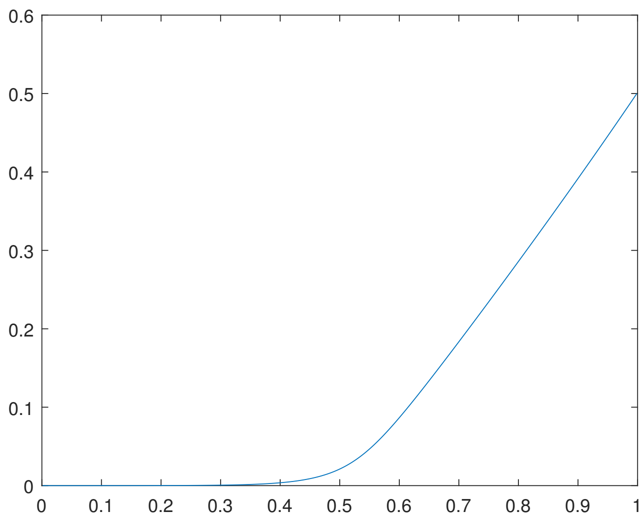

Figure 3. solution $u_0(x)$ for the case $f(x) = 0$.

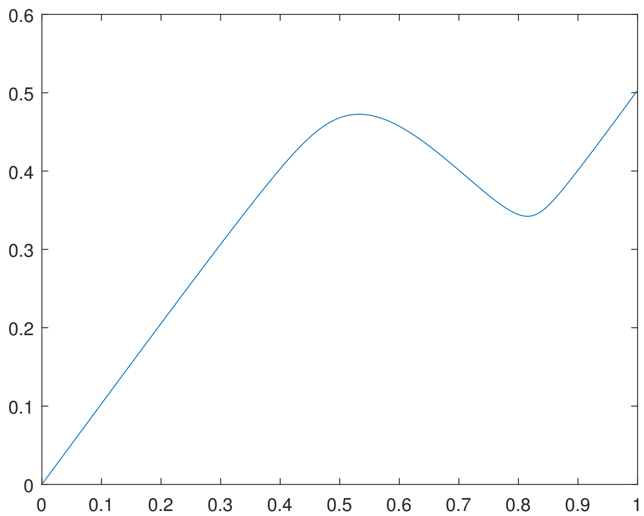

Figure 4. solution $u_0(x)$ for the case $f(x) = \sin(\pi x)/2$.

Remark 4.1. *Observe that such solutions u_0 obtained are not the global solutions for the related primal optimization problems. Indeed, such solutions reflect the average behavior of weak cluster points for concerning minimizing sequences.*

4.1. The algorithm through which we have obtained the numerical results

In this subsection we present the software in MATLAB through which we have obtained the last numerical results.

This algorithm is for solving the concerning Euler-Lagrange equations for the dual problem, that is, for solving the equation

$$\begin{cases} (v^*)'' + f' - v^* - \text{ sign}(v^*) = 0, & \text{in } \Omega, \\ (v^*)'(0) = 0, \ (v^*)'(1) = 1/2. \end{cases} \tag{15}$$

Here the concerning software in MATLAB. We emphasize to have used the smooth approximation

$$|v^*| \approx \sqrt{(v^*)^2 + e_1},$$

where a small value for e_1 is specified in the next lines.
****

1. clear all
2. $m_8 = 800$; (number of nodes)
3. $d = 1/m_8;$
4. $e_1 = 0.00001;$
5. $for\ i = 1 : m_8$

   $yo(i,1) = 0.01;$

   $y_1(i,1) = \sin(\pi * i/m_8)/2;$

   $end;$
6. $for\ i = 1 : m_8 - 1$

   $dy_1(i,1) = (y_1(i+1,1) - y_1(i,1))/d;$

   $end;$
7. $for\ k = 1 : 3000$ (we have fixed the number of iterations)

   $i = 1;$

   $h_3 = 1/\sqrt{vo(i,1)^2 + e_1};$

   $m_{12} = 1 + d^2 * h_3 + d^2;$

   $m_{50}(i) = 1/m_{12};$

   $z(i) = m_{50}(i) * (dy_1(i,1) * d^2);$
8. $for\ i = 2 : m_8 - 1$

   $h_3 = 1/\sqrt{vo(i,1)^2 + e_1};$

   $m_{12} = 2 + h_3 * d^2 + d^2 - m50(i-1);$

   $m50(i) = 1/m_{12};$

   $z(i) = m_{50}(i) * (z(i-1) + dy_1(i,1) * d^2);$

   $end;$
9. $v(m_8, 1) = (d/2 + z(m_8 - 1))/(1 - m_{50}(m_8 - 1));$
10. $for\ i = 1 : m_8 - 1$

    $v(m_8 - i, 1) = m_{50}(m_8 - i) * v(m_8 - i + 1) + z(m_8 - i);$

    $end;$

11. $v(m_8/2, 1)$
12. $vo = v;$

    $end;$
13. $for\ i = 1 : m_8 - 1$

    $u(i,1) = (v(i+1,1) - v(i,1))/d + y_1(i,1);$

    $end;$
14. $for\ i = 1 : m_8 - 1$

    $x(i) = i * d;$

    $end;$

    $plot(x, u(:,1))$

****

5. An improvement of the convexity conditions for a non-convex related model through an approximate primal formulation

In this section we develop an approximate primal dual formulation suitable for a large class of variational models.

Here, the applications are for the Kirchhoff-Love plate model, which may be found in Ciarlet, [11].

At this point we start to describe the primal variational formulation.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set which represents the middle surface of a plate of thickness h. The boundary of Ω, which is assumed to be regular (Lipschitzian), is denoted by $\partial\Omega$. The vectorial basis related to the cartesian system $\{x_1, x_2, x_3\}$ is denoted by $(\mathbf{a}_\alpha, \mathbf{a}_3)$, where $\alpha = 1, 2$ (in general Greek indices stand for 1 or 2), and where $\mathbf{a}_3$ is the vector normal to Ω, whereas $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal vectors parallel to Ω. Also, $\mathbf{n}$ is the outward normal to the plate surface.

The displacements will be denoted by

$$\hat{\mathbf{u}} = \{\hat{u}_\alpha, \hat{u}_3\} = \hat{u}_\alpha \mathbf{a}_\alpha + \hat{u}_3 \mathbf{a}_3.$$

The Kirchhoff-Love relations are

$$\begin{aligned} &\hat{u}_\alpha(x_1, x_2, x_3) = u_\alpha(x_1, x_2) - x_3 w(x_1, x_2)_{,\alpha} \\ &\text{and } \hat{u}_3(x_1, x_2, x_3) = w(x_1, x_2). \end{aligned} \tag{16}$$

Here $-h/2 \leq x_3 \leq h/2$ so that we have $u = (u_\alpha, w) \in U$ where

$$\begin{aligned} U &= \Big\{ u = (u_\alpha, w) \in W^{1,2}(\Omega; \mathbb{R}^2) \times W^{2,2}(\Omega), \\ &\quad u_\alpha = w = \frac{\partial w}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \} \\ &= W_0^{1,2}(\Omega; \mathbb{R}^2) \times W_0^{2,2}(\Omega). \end{aligned}$$

It is worth emphasizing that the boundary conditions here specified refer to a clamped plate.

We also define the operator $\Lambda : U \to Y \times Y$, where $Y = Y^* = L^2(\Omega; \mathbb{R}^{2\times 2})$, by

$$\Lambda(u) = \{\gamma(u), \kappa(u)\},$$

$$\gamma_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{w_{,\alpha} w_{,\beta}}{2},$$

$$\kappa_{\alpha\beta}(u) = -w_{,\alpha\beta}.$$

The constitutive relations are given by

$$N_{\alpha\beta}(u) = H_{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(u), \tag{17}$$

$$M_{\alpha\beta}(u) = h_{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}(u), \tag{18}$$

where: $\{H_{\alpha\beta\lambda\mu}\}$ and $\left\{h_{\alpha\beta\lambda\mu} = \frac{h^2}{12}H_{\alpha\beta\lambda\mu}\right\}$, are symmetric positive definite fourth order tensors. From now on, we denote $\{\overline{H}_{\alpha\beta\lambda\mu}\} = \{H_{\alpha\beta\lambda\mu}\}^{-1}$ and $\{\overline{h}_{\alpha\beta\lambda\mu}\} = \{h_{\alpha\beta\lambda\mu}\}^{-1}$.

Furthermore $\{N_{\alpha\beta}\}$ denote the membrane force tensor and $\{M_{\alpha\beta}\}$ the moment one. The plate stored energy, represented by $(G \circ \Lambda) : U \to \mathbb{R}$ is expressed by

$$(G \circ \Lambda)(u) = \frac{1}{2}\int_\Omega N_{\alpha\beta}(u)\gamma_{\alpha\beta}(u)\,dx + \frac{1}{2}\int_\Omega M_{\alpha\beta}(u)\kappa_{\alpha\beta}(u)\,dx \tag{19}$$

and the external work, represented by $F : U \to \mathbb{R}$, is given by

$$F(u) = \langle w, P\rangle_{L^2} + \langle u_\alpha, P_\alpha\rangle_{L^2}, \tag{20}$$

where $P, P_1, P_2 \in L^2(\Omega)$ are external loads in the directions $\mathbf{a}_3$, $\mathbf{a}_1$ and $\mathbf{a}_2$ respectively. The potential energy, denoted by $J : U \to \mathbb{R}$ is expressed by:

$$J(u) = (G \circ \Lambda)(u) - F(u)$$

Define now $J_3 : \tilde{U} \to \mathbb{R}$ by

$$J_3(u) = J(u) + J_5(w).$$

where

$$J_5(w) = 10\int_\Omega \frac{a^{K\,b\,w}}{\ln(a)\,K^{3/2}}\,dx + 10\int_\Omega \frac{a^{-K(b\,w-1/100)}}{\ln(a)\,K^{3/2}}\,dx.$$

In such a case for $a = 2.71$, $K = 185$, $b = P/|P|$ in Ω and

$$\tilde{U} = \{u \in U \;:\; \|w\|_\infty \leq 0.01 \text{ and } P\,w \geq 0 \text{ a.e. in } \Omega\},$$

we get

$$\begin{aligned}\frac{\partial J_3(u)}{\partial w} &= \frac{\partial J(u)}{\partial w} + \frac{\partial J_5(u)}{\partial w} \\ &\approx \frac{\partial J(u)}{\partial w} + \mathcal{O}(\pm 3.0),\end{aligned} \tag{21}$$

and

$$\begin{aligned}\frac{\partial^2 J_3(u)}{\partial w^2} &= \frac{\partial^2 J(u)}{\partial w^2} + \frac{\partial^2 J_5(u)}{\partial w^2} \\ &\approx \frac{\partial^2 J(u)}{\partial w^2} + \mathcal{O}(850).\end{aligned} \tag{22}$$

This new functional J_3 has a relevant improvement in the convexity conditions concerning the previous functional J.

Indeed, we have obtained a gain in positiveness for the second variation $\frac{\partial^2 J(u)}{\partial w^2}$, which has increased of order $\mathcal{O}(700 - 1000)$.

Moreover the difference between the approximate and exact equation

$$\frac{\partial J(u)}{\partial w} = \mathbf{0}$$

is of order $\mathcal{O}(\pm 3.0)$ which corresponds to a small perturbation in the original equation for a load of $P = 1500\ N/m^2$, for example. Summarizing, the exact equation may be approximately solved in an appropriate sense.

6. An exact convex dual variational formulation for a non-convex primal one

In this section we develop a convex dual variational formulation suitable to compute a critical point for the corresponding primal one.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = F(u_x, u_y) - \langle u, f \rangle_{L^2},$$

$V = W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.

Here we denote $Y = Y^* = L^2(\Omega)$ and $Y_1 = Y_1^* = L^2(\Omega) \times L^2(\Omega)$.

Defining

$$V_1 = \{u \in V \ : \ \|u\|_{1,\infty} \leq K_1\}$$

for some appropriate $K_1 > 0$, suppose also F is twice Fréchet differentiable and

$$\det\left\{\frac{\partial^2 F(u_x, u_y)}{\partial v_1 \partial v_2}\right\} \neq 0,$$

$\forall u \in V_1$.

Define now $F_1 : V \to \mathbb{R}$ and $F_2 : V \to \mathbb{R}$ by

$$F_1(u_x, u_y) = F(u_x, u_y) + \frac{\varepsilon}{2}\int_\Omega u_x^2\, dx + \frac{\varepsilon}{2}\int_\Omega u_y^2\, dx,$$

and

$$F_2(u_x, u_y) = \frac{\varepsilon}{2}\int_\Omega u_x^2\, dx + \frac{\varepsilon}{2}\int_\Omega u_y^2\, dx,$$

where here we denote $dx = dx_1 dx_2$.

Moreover, we define the respective Legendre transform functionals F_1^* and F_2^* as

$$F_1^*(v^*) = \langle v_1, v_1^* \rangle_{L^2} + \langle v_2, v_2^* \rangle_{L^2} - F_1(v_1, v_2),$$

where $v_1, v_2 \in Y$ are such that

$$v_1^* = \frac{\partial F_1(v_1, v_2)}{\partial v_1},$$

$$v_2^* = \frac{\partial F_1(v_1, v_2)}{\partial v_2},$$

and

$$F_2^*(v^*) = \langle v_1, v_1^* + f_1 \rangle_{L^2} + \langle v_2, v_2^* \rangle_{L^2} - F_2(v_1, v_2),$$

where $v_1, v_2 \in Y$ are such that

$$v_1^* + f_1 = \frac{\partial F_2(v_1, v_2)}{\partial v_1},$$

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$$v_2^* = \frac{\partial F_2(v_1, v_2)}{\partial v_2}.$$

Here f_1 is any function such that

$$(f_1)_x = f, \text{ in } \Omega.$$

Furthermore, we define

$$\begin{aligned} J^*(v^*) &= -F_1^*(v^*) + F_2^*(v^*) \\ &= -F_1^*(v^*) + \frac{1}{2\varepsilon}\int_\Omega (v_1^* + f_1)^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (v_2^*)^2\,dx. \end{aligned} \quad (23)$$

Observe that through the target conditions

$$v_1^* + f_1 = \varepsilon u_x,$$

$$v_2^* = \varepsilon u_y,$$

we may obtain the compatibility condition

$$(v_1^* + f_1)_y - (v_2^*)_x = 0.$$

Define now

$$A^* = \{v^* = (v_1^*, v_2^*) \in B_r(0,0) \subset Y_1^* \;:\; (v_1^* + f_1)_y - (v_2^*)_x = 0, \text{ in } \Omega\},$$

for some appropriate $r > 0$ such that J^* is convex in $B_r(0,0)$.

Consider the problem of minimizing J^* subject to $v^* \in A^*$.

Assuming $r > 0$ is large enough so that the restriction in r is not active, at this point we define the associated Lagrangian

$$J_1^*(v^*, \varphi) = J^*(v^*) + \langle \varphi, (v_1^* + f)_y - (v_2^*)_x \rangle_{L^2},$$

where φ is an appropriate Lagrange multiplier.

Therefore

$$\begin{aligned} J_1^*(v^*) &= -F_1^*(v^*) + \frac{1}{2\varepsilon}\int_\Omega (v_1^* + f_1)^2\,dx + \frac{1}{2\varepsilon}\int_\Omega (v_2^*)^2\,dx \\ &\quad + \langle \varphi, (v_1^* + f)_y - (v_2^*)_x \rangle_{L^2}. \end{aligned} \quad (24)$$

The optimal point in question will be a solution of the corresponding Euler-Lagrange equations for J_1^*.

From the variation of J_1^* in v_1^* we obtain

$$-\frac{\partial F_1^*(v^*)}{\partial v_1^*} + \frac{v_1^* + f}{\varepsilon} - \frac{\partial \varphi}{\partial y} = 0. \quad (25)$$

From the variation of J_1^* in v_2^* we obtain

$$-\frac{\partial F_1^*(v^*)}{\partial v_2^*} + \frac{v_2^*}{\varepsilon} + \frac{\partial \varphi}{\partial x} = 0. \quad (26)$$

From the variation of J_1^* in φ we have

$$(v_1^* + f)_y - (v_2^*)_x = 0.$$

From this last equation, we may obtain $u \in V$ such that

$$v_1^* + f = \varepsilon u_x,$$

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and
$$v_2^* = \varepsilon u_y.$$

From this and the previous extremal equations indicated we have
$$-\frac{\partial F_1^*(v^*)}{\partial v_1^*} + u_x - \frac{\partial \varphi}{\partial y} = 0,$$
and
$$-\frac{\partial F_1^*(v^*)}{\partial v_2^*} + u_y + \frac{\partial \varphi}{\partial x} = 0.$$
so that
$$v_1^* + f = \frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_1},$$
and
$$v_2^* = \frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_2}.$$

From this and equation (25) and (26) we have
$$\begin{aligned} & -\varepsilon\left(\frac{\partial F_1^*(v^*)}{\partial v_1^*}\right)_x - \varepsilon\left(\frac{\partial F_1^*(v^*)}{\partial v_2^*}\right)_y \\ & +(v_1^* + f_1)_x + (v_2^*)_y \\ = & -\varepsilon u_{xx} - \varepsilon u_{yy} + (v_1^*)_x + (v_2^*)_y + f = 0. \end{aligned} \tag{27}$$

Replacing the expressions of v_1^* and v_2^* into this last equation, we have
$$-\varepsilon u_{xx} - \varepsilon u_{yy} + \left(\frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_1}\right)_x + \left(\frac{\partial F_1(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_2}\right)_y + f = 0,$$
so that
$$\left(\frac{\partial F(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_1}\right)_x + \left(\frac{\partial F(u_x - \varphi_y, u_y + \varphi_x)}{\partial v_2}\right)_y + f = 0, \text{ in } \Omega. \tag{28}$$

Observe that if
$$\nabla^2 \varphi = 0$$
then there exists $\hat{u}$ such that u and φ are also such that
$$u_x - \varphi_y = \hat{u}_x$$
and
$$u_y + \varphi_x = \hat{u}_y.$$

The boundary conditions for φ must be such that $\hat{u} \in W_0^{1,2}$.

From this and equation (28) we obtain
$$\delta J(\hat{u}) = \mathbf{0}.$$

Summarizing, we may obtain a solution $\hat{u} \in W_0^{1,2}$ of equation $\delta J(\hat{u}) = \mathbf{0}$ by minimizing J^* on A^*.

Finally, observe that clearly J^* is convex in an appropriate large ball $B_r(0,0)$ for some appropriate $r > 0$

7. Another primal dual formulation for a related model

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular boundary denoted by $\partial\Omega$.

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Consider the functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u\, dx + \frac{\alpha}{2}\int_\Omega (u^2-\beta)^2\, dx \\ &\quad -\langle u, f\rangle_{L^2}, \end{aligned} \tag{29}$$

$\alpha > 0,\ \beta > 0,\ \gamma > 0$, $V = W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.

Denoting $Y = Y^* = L^2(\Omega)$, define now $J_1^* : V \times Y^* \to \mathbb{R}$ by

$$\begin{aligned} J_1^*(u, v_0^*) &= -\frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u\, dx - \langle u^2, v_0^*\rangle_{L^2} \\ &\quad +\frac{K_1}{2}\int_\Omega (-\gamma\nabla^2 u + 2v_0^* u - f)^2\, dx + \langle u, f\rangle_{L^2} \\ &\quad +\frac{1}{2\alpha}\int_\Omega (v_0^*)^2\, dx + \beta\int_\Omega v_0^*\, dx, \end{aligned} \tag{30}$$

Define also

$$A^+ = \{u \in V \ :\ u\, f \geq 0, \text{ a.e. in } \Omega\},$$

$$V_2 = \{u \in V \ :\ \|u\|_\infty \leq K_3\},$$

and

$$V_1 = V_2 \cap A^+$$

for some appropriate $K_3 > 0$ to be specified.

Moreover define

$$B^* = \{v_0^* \in Y^* \ :\ \|v_0^*\|_\infty \leq K\}$$

for some appropriate $K > 0$ to be specified.

Observe that, denoting

$$\varphi = -\gamma\nabla^2 u + 2v_0^* u - f$$

we have

$$\frac{\partial^2 J_1^*(u, v_0^*)}{\partial (v_0^*)^2} = \frac{1}{\alpha} + 4K_1 u^2$$

$$\frac{\partial^2 J_1^*(u, v_0^*)}{\partial u^2} = \gamma\nabla^2 - 2v_0^* + K_1(-\gamma\nabla^2 + 2v_0^*)^2$$

and

$$\frac{\partial^2 J_1^*(u, v_0^*)}{\partial u \partial v_0^*} = K_1(2\varphi + 2(-\gamma\nabla^2 u + 2v_0^* u)) - 2u$$

so that

$$\begin{aligned} &\det\{\delta^2 J_1^*(u, v_0^*)\} \\ &= \frac{\partial^2 J_1^*(u, v_0^*)}{\partial (v_0^*)^2}\frac{\partial^2 J_1^*(u, v_0^*)}{\partial u^2} - \left(\frac{\partial^2 J_1^*(u, v_0^*)}{\partial u \partial v_0^*}\right)^2 \\ &= \frac{K_1(-\gamma\nabla^2 + 2v_0^*)^2}{\alpha} - \frac{\gamma\nabla^2 + 2v_0^* + 4\alpha u^2}{\alpha} \\ &\quad -4K_1^2\varphi^2 - 8K_1\varphi(-\gamma\nabla^2 + 2v_0^*)u + 8K_1\varphi u \\ &\quad +4K_1(-\gamma\nabla^2 u + 2v_0^* u)u. \end{aligned} \tag{31}$$

Observe now that a critical point $\varphi = 0$ and $(-\gamma\nabla^2 u + 2v_0^* u)u = fu \geq 0$ in Ω.

Therefore, for an appropriate large $K_1 > 0$, also at a critical point, we have

$$\begin{aligned} &\det\{\delta^2 J_1^*(u, v_0^*)\} \\ = \; & 4K_1 f u - \frac{\delta^2 J(u)}{\alpha} + K_1 \frac{(-\gamma\nabla^2 + 2v_0^*)^2}{\alpha} > \mathbf{0}. \end{aligned} \tag{32}$$

Remark 7.1. *From this last equation we may observe that J_1^* has a large region of convexity about any critical point $(u_0, \hat{v}_0^*)$, that is, there exists a large $r > 0$ such that J_1^* is convex on $B_r(u_0, \hat{v}_0^*)$.*

With such results in mind, we may easily prove the following theorem.

Theorem 7.1. *Assume $K_1 \gg \max\{1, K, K_3\}$ and suppose $(u_0, \hat{v}_0^*) \in V_1 \times B^*$ is such that*

$$\delta J_1^*(u_0, \hat{v}_0^*) = \mathbf{0}.$$

Under such hypotheses, there exists $r > 0$ such that J_1^* is convex in $E^* = B_r(u_0, \hat{v}_0^*) \cap (V_1 \times B^*)$,

$$\delta J(u_0) = \mathbf{0},$$

and

$$-J(u_0) = J_1(u_0, \hat{v}_0^*) = \inf_{(u, v_0^*) \in E^*} J_1^*(u, v_0^*).$$

8. A third primal dual formulation for a related model

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular boundary denoted by $\partial\Omega$. Consider the functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) = \; & \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{2}\int_\Omega (u^2 - \beta)^2 \, dx \\ & -\langle u, f \rangle_{L^2}, \end{aligned} \tag{33}$$

$\alpha > 0$, $\beta > 0$, $\gamma > 0$, $V = W_0^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.

Denoting $Y = Y^* = L^2(\Omega)$, define now $J_1^* : V \times Y^* \times Y^* \to \mathbb{R}$ by

$$\begin{aligned} J_1^*(u, v_0^*, v_1^*) = \; & \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{1}{2}\int_\Omega K \, u^2 \, dx \\ & -\langle u, v_1^* \rangle_{L^2} + \frac{1}{2}\int_\Omega \frac{(v_1^*)^2}{(-2v_0^* + K)} \, dx \\ & + \frac{1}{2(\alpha + \varepsilon)}\int_\Omega (v_0^* - \alpha(u^2 - \beta))^2 \, dx + \langle u, f \rangle_{L^2} \\ & - \frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx - \beta\int_\Omega v_0^* \, dx, \end{aligned} \tag{34}$$

where $\varepsilon > 0$ is a small real constant.

Define also

$$A^+ = \{u \in V \; : \; u f \geq 0, \text{ a.e. in } \Omega\},$$

$$V_2 = \{u \in V \; : \; \|u\|_\infty \leq K_3\},$$

and

$$V_1 = V_2 \cap A^+$$

for some appropriate $K_3 > 0$ to be specified.

Moreover define

$$B^* = \{v_0^* \in Y^* \; : \; \|v_0^*\|_\infty \leq K_4\}$$

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and

$$D^* = \{v_1^* \in Y^* \; : \; \|v_1^*\| \leq K_5\},$$

for some appropriate real constants $K_4, K_5 > 0$ to be specified.

Remark 8.1. *Define now*

$$H_1(u, v_0^*) = -\gamma\nabla^2 + 2v_0^* + 4\alpha u^2$$

For an appropriate function (or, in a more general fashion, an appropriate bounded operator) M_1 define

$$E^* = \{(u, v_0^*) \in V_1 \times B^* \; : \; \sqrt{4\alpha}|u| \geq \sqrt{|M_1 + \gamma\nabla^2|} \text{ and } 2v_0^* + M_1 \geq \varepsilon_1\}.$$

Since for $(u, v_0^*) \in V_1 \times B^*$ we have $u\, f \geq 0$, in Ω, so that for $u_1, u_2 \in V_1$ we have

$$sign\ (u_1) = sign\ (u_2) \text{ in } \Omega,$$

we may infer that E^* is a convex set.

Moreover if $(u, v_0^*) \in E^*$, then

$$\sqrt{4\alpha}|u| \geq \sqrt{|M_1 + \gamma\nabla^2|}$$

so that

$$4\alpha u^2 \geq M_1 + \gamma\nabla^2$$

and

$$2v_0^* + M_1 \geq \varepsilon_1$$

so that

$$H_1(u, v_0^*) = -\gamma\nabla^2 + 2v_0^* + 4\alpha u^2 \geq \varepsilon_1.$$

Such a result we will be used many times in the next sections.

Observe that, defining

$$\varphi = v_0^* - \alpha(u^2 - \beta)$$

we may obtain

$$\frac{\partial^2 J_1^*(u, v_0^*, v_1^*)}{\partial u^2} = -\gamma\nabla^2 + K + \frac{\alpha}{\alpha + \varepsilon}4u^2 - 2\varphi\frac{\alpha}{\alpha + \varepsilon}$$

$$\frac{\partial^2 J_1^*(u, v_0^*, v_1^*)}{\partial (v_1^*)^2} = \frac{1}{-2v_0^* + K}$$

and

$$\frac{\partial^2 J_1^*(u, v_0^*, v_1^*)}{\partial u \partial v_1^*} = -1$$

so that

$$\begin{aligned}
&\det\left\{\frac{\partial^2 J_1^*(u, v_0^*, v_1^*)}{\partial u \partial v_1^*}\right\} \\
=\; & \frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial (v_1^*)^2}\frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial u^2} - \left(\frac{\partial^2 J_1^*(u, v_1^*, v_0^*)}{\partial u \partial v_1^*}\right)^2 \\
=\; & \frac{-\gamma\nabla^2 + 2v_0^* + 4\frac{\alpha^2}{\alpha+\varepsilon}u^2 - 2\frac{\alpha}{\alpha+\varepsilon}\varphi}{-2v_0^* + K} \\
\equiv\; & H(u, v_0^*).
\end{aligned} \tag{35}$$

However, at a critical point, we have $\varphi = \mathbf{0}$ so that, we define the non-active but convex restriction

$$C^* = \{(u, v_0^*) \in V_1 \times B \ : \ (\varphi)^2 \leq \varepsilon\},$$

for a small parameter $\varepsilon > 0$.

From such results, assuming $K \gg \max\{K_3, K_4, K_5\}$, and $0 < \varepsilon \ll \varepsilon_1 \ll 1$, we have that

$$H(u, v_0^*) > \mathbf{0}, \text{ in } C^* \cap E^*.$$

With such results in mind, we may easily prove the following theorem.

Theorem 8.1. *Suppose* $(u_0, \hat{v}_0^*, \hat{v}_1^*) \in E_1^* = (E^* \cap C^*) \times D^* \times B^*$ *is such that*

$$\delta J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) = \mathbf{0}.$$

Under such hypotheses, defining $C_{\hat{v}_0^*} = \{u \in V_1 \ : \ (u, \hat{v}_0^*) \in E_1^*\}$, *we have that*

$$\delta J(u_0) = \mathbf{0}$$

and

$$\begin{aligned} J(u_0) &= \inf_{u \in V_1 \cap C_{\hat{v}_0^*}} J(u) \\ &= J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) \\ &= \inf_{(u, v_1^*) \in (V_1 \cap C_{\hat{v}_0^*}) \times D^*} \left\{ \sup_{v_0^* \in B^*} J_1^*(u, v_1^*, v_0^*) \right\} \\ &= \sup_{v_0^* \in B^*} \left\{ \inf_{(u, v_1^*) \in (V_1 \cap C_{\hat{v}_0^*}) \times D^*} J_1^*(u, v_1^*, v_0^*) \right\}. \end{aligned} \tag{36}$$

Proof. The proof that

$$\delta J(u_0) = \mathbf{0}$$

and

$$J(u_0) = J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*)$$

may be easily made similarly as in the previous sections.

Moreover, from the hypotheses, we have

$$J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) = \inf_{(u, v_1^*) \in (V_1 \cap C_{\hat{v}_0^*}) \times D^*} J_1^*(u, v_1^*, \hat{v}_0^*)$$

and

$$J_1^*(u_0, \hat{v}_1^*, \hat{v}_0^*) = \sup_{v_0^* \in B^*} J_1^*(u_0, \hat{v}_1^*, v_0^*).$$

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From this, from a standard saddle point theorem and the remaining hypotheses, we may infer that

$$\begin{aligned} J(u_0) &= J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*) \\ &= \inf_{(u,v_1^*)\in(V_1\cap C_{\hat{v}_0^*})\times D^*}\left\{\sup_{v_0^*\in B^*} J_1^*(u,v_1^*,v_0^*)\right\} \\ &= \sup_{v_0^*\in B^*}\left\{\inf_{(u,v_1^*)\in(V_1\cap C_{\hat{v}_0^*})\times D^*} J_1^*(u,v_1^*,v_0^*)\right\}. \end{aligned} \tag{37}$$

Moreover, observe that

$$\begin{aligned} J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*) &= \inf_{(u,v_1^*)\in(V_1\cap C_{\hat{v}_0^*})\times D^*} J_1^*(u,v_1^*,\hat{v}_0^*) \\ &\leq \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\frac{K}{2}\int_\Omega u^2\,dx \\ &+\langle u^2,\hat{v}_0^*\rangle_{L^2}-\frac{K}{2}\int_\Omega u^2\,dx \\ &-\frac{1}{2\alpha}\int_\Omega(\hat{v}_0^*)^2\,dx-\beta\int_\Omega \hat{v}_0^*\,dx \\ &+\frac{1}{2(\alpha+\varepsilon)}\int_\Omega(\hat{v}_0^*-\alpha(u^2-\beta))^2\,dx-\langle u,f\rangle_{L^2} \\ &\leq \sup_{v_0^*\in Y^*}\left\{\frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\langle u^2,v_0^*\rangle\right. \\ &-\frac{1}{2\alpha}\int_\Omega(v_0^*)^2\,dx-\beta\int_\Omega v_0^*\,dx \\ &\left.+\frac{1}{2(\alpha+\varepsilon)}\int_\Omega(v_0^*-\alpha(u^2-\beta))^2\,dx-\langle u,f\rangle_{L^2}\right\} \\ &= \frac{\gamma}{2}\int_\Omega \nabla u\cdot\nabla u\,dx+\frac{\alpha}{2}\int_\Omega(u^2-\beta)^2\,dx \\ &-\langle u,f\rangle_{L^2},\ \forall u\in V_1\cap C_{\hat{v}_0^*}. \end{aligned} \tag{38}$$

Summarizing, we have got

$$J(u_0)=J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*)\leq \inf_{u\in V_1\cap C_{\hat{v}_0^*}} J(u).$$

From such results, we may infer that

$$\begin{aligned} J(u_0) &= \inf_{u\in V_1\cap C_{\hat{v}_0^*}} J(u) \\ &= J_1^*(u_0,\hat{v}_1^*,\hat{v}_0^*) \\ &= \inf_{(u,v_1^*)\in(V_1\cap C_{\hat{v}_0^*})\times D^*}\left\{\sup_{v_0^*\in B^*} J_1^*(u,v_1^*,v_0^*)\right\} \\ &= \sup_{v_0^*\in B^*}\left\{\inf_{(u,v_1^*)\in(V_1\cap C_{\hat{v}_0^*})\times D^*} J_1^*(u,v_1^*,v_0^*)\right\}. \end{aligned} \tag{39}$$

The proof is complete. □

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9. An algorithm for a related model in shape optimization

The next two subsections have been previously published by Fabio Silva Botelho and Alexandre Molter in [5], Chapter 21.

9.1. Introduction

Consider an elastic solid which the volume corresponds to an open, bounded, connected set, denoted by $\Omega \subset \mathbb{R}^3$ with a regular (Lipschitzian) boundary denoted by $\partial\Omega = \Gamma_0 \cup \Gamma_t$ where $\Gamma_0 \cap \Gamma_t = \emptyset$. Consider also the problem of minimizing the functional $\hat{J} : U \times B \to \mathbb{R}$ where

$$\hat{J}(u,t) = \frac{1}{2}\langle u_i, f_i\rangle_{L^2(\Omega)} + \frac{1}{2}\langle u_i, \hat{f}_i\rangle_{L^2(\Gamma_t)},$$

subject to

$$\begin{cases} (H_{ijkl}(t)e_{kl}(u))_{,j} + f_i = 0 \text{ in } \Omega, \\ \\ H_{ijkl}(t)e_{kl}(u)n_j - \hat{f}_i = 0, \text{ on } \Gamma_t, \ \forall i \in \{1,2,3\}. \end{cases} \tag{40}$$

Here $\mathbf{n} = (n_1, n_2, n_3)$ denotes the outward normal to $\partial\Omega$ and

$$U = \{u = (u_1, u_2, u_3) \in W^{1,2}(\Omega; \mathbb{R}^3) \ : \ u = (0,0,0) = \mathbf{0} \text{ on } \Gamma_0\},$$

$$B = \left\{ t : \Omega \to [0,1] \text{ measurable } \ : \ \int_\Omega t(x)\,dx = t_1|\Omega| \right\},$$

where

$$0 < t_1 < 1$$

and $|\Omega|$ denotes the Lebesgue measure of Ω.

Moreover $u = (u_1, u_2, u_3) \in W^{1,2}(\Omega; \mathbb{R}^3)$ is the field of displacements relating the cartesian system $(0, x_1, x_2, x_3)$, resulting from the action of the external loads $f \in L^2(\Omega; \mathbb{R}^3)$ and $\hat{f} \in L^2(\Gamma_t; \mathbb{R}^3)$.

We also define the stress tensor $\{\sigma_{ij}\} \in Y^* = Y = L^2(\Omega; \mathbb{R}^{3\times3})$, by

$$\sigma_{ij}(u) = H_{ijkl}(t)e_{kl}(u),$$

and the strain tensor $e : U \to L^2(\Omega; \mathbb{R}^{3\times3})$ by

$$e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \ \forall i,j \in \{1,2,3\}.$$

Finally,

$$\{H_{ijkl}(t)\} = \{tH^0_{ijkl} + (1-t)H^1_{ijkl}\},$$

where H^0 corresponds to a strong material and H^1 to a very soft material, intending to simulate voids along the solid structure.

The variable t is the design one, which the optimal distribution values along the structure are intended to minimize its inner work with a volume restriction indicated through the set B.

The duality principle obtained is developed inspired by the works in [2,3]. Similar theoretical results have been developed in [10], however we believe the proof here presented, which is based on the min-max theorem is easier to follow (indeed we thank an anonymous referee for his suggestion about applying the min-max theorem to complete the proof). We highlight throughout this text we have used the standard Einstein sum convention of repeated indices.

Moreover, details on the Sobolev spaces addressed may be found in [1]. In addition, the primal variational development of the topology optimization problem has been described in [10].

The main contributions of this work are to present the detailed development, through duality theory, for such a kind of optimization problems. We emphasize that to avoid the check-board standard and obtain appropriate robust optimized structures without the use of filters, it is necessary to discretize more in the load direction, in which the displacements are much larger.

9.2. Mathematical formulation of the topology optimization problem

Our mathematical topology optimization problem is summarized by the following theorem.

Theorem 9.1. *Consider the statements and assumptions indicated in the last section, in particular those refereing to Ω and the functional $\hat{J} : U \times B \to \mathbb{R}$.*

Define $J_1 : U \times B \to \mathbb{R}$ by

$$J_1(u,t) = -G(e(u),t) + \langle u_i, f_i \rangle_{L^2(\Omega)} + \langle u_i, \hat{f}_i \rangle_{L^2(\Gamma_t)},$$

where

$$G(e(u),t) = \frac{1}{2} \int_\Omega H_{ijkl}(t) e_{ij}(u) e_{kl}(u)\, dx,$$

and where

$$dx = dx_1 dx_2 dx_3.$$

Define also $J^* : U \to \mathbb{R}$ by

$$\begin{aligned} J^*(u) &= \inf_{t \in B} \{J_1(u,t)\} \\ &= \inf_{t \in B} \{-G(e(u),t) + \langle u_i, f_i \rangle_{L^2(\Omega)} + \langle u_i, \hat{f}_i \rangle_{L^2(\Gamma_t)}\}. \end{aligned} \tag{41}$$

Assume there exists $c_0, c_1 > 0$ such that

$$H^0_{ijkl} z_{ij} z_{kl} > c_0 z_{ij} z_{ij}$$

and

$$H^1_{ijkl} z_{ij} z_{kl} > c_1 z_{ij} z_{ij}, \ \forall z = \{z_{ij}\} \in \mathbb{R}^{3\times 3}, \ \text{ such that } z \neq \mathbf{0}.$$

Finally, define $J : U \times B \to \mathbb{R} \cup \{+\infty\}$ by

$$J(u,t) = \hat{J}(u,t) + Ind(u,t),$$

where

$$Ind(u,t) = \begin{cases} 0, & \text{if } (u,t) \in A^*, \\ +\infty, & \text{otherwise}, \end{cases} \tag{42}$$

where $A^* = A_1 \cap A_2$,

$$A_1 = \{(u,t) \in U \times B \ : \ (\sigma_{ij}(u))_{,j} + f_i = 0, \text{ in } \Omega, \ \forall i \in \{1,2,3\}\}$$

and

$$A_2 = \{(u,t) \in U \times B \ : \ \sigma_{ij}(u) n_j - \hat{f}_i = 0, \text{ on } \Gamma_t, \ \forall i \in \{1,2,3\}\}.$$

Under such hypotheses, there exists $(u_0, t_0) \in U \times B$ such that

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$$
\begin{aligned}
J(u_0,t_0) &= \inf_{(u,t)\in U\times B} J(u,t) \\
&= \sup_{\hat{u}\in U} J^*(\hat{u}) \\
&= J^*(u_0) \\
&= \hat{J}(u_0,t_0) \\
&= \inf_{(t,\sigma)\in B\times C^*} G^*(\sigma,t) \\
&= G^*(\sigma(u_0),t_0), \qquad (43)
\end{aligned}
$$

where

$$
\begin{aligned}
G^*(\sigma,t) &= \sup_{v\in Y}\{\langle v_{ij},\sigma_{ij}\rangle_{L^2(\Omega)} - G(v,t)\} \\
&= \frac{1}{2}\int_\Omega \overline{H}_{ijkl}(t)\sigma_{ij}\sigma_{kl}\,dx, \qquad (44)
\end{aligned}
$$

$$\{\overline{H}_{ijkl}(t)\} = \{H_{ijkl}(t)\}^{-1}$$

and $C^* = C_1 \cap C_2$, *where*

$$C_1 = \{\sigma \in Y^* \ :\ \sigma_{ij,j} + f_i = 0, \text{ in } \Omega, \ \forall i \in \{1,2,3\}\}$$

and

$$C_2 = \{\sigma \in Y^* \ :\ \sigma_{ij}n_j - \hat{f}_i = 0, \text{ on } \Gamma_t, \ \forall i \in \{1,2,3\}\}.$$

Proof. Observe that

$$
\begin{aligned}
\inf_{(u,t)\in U\times B} J(u,t) &= \inf_{t\in B}\left\{\inf_{u\in U} J(u,t)\right\} \\
&= \inf_{t\in B}\left\{\sup_{\hat{u}\in U}\left\{\inf_{u\in U}\left\{\frac{1}{2}\int_\Omega H_{ijkl}(t)e_{ij}(u)e_{kl}(u)\,dx\right.\right.\right. \\
&\quad +\langle \hat{u}_i,(H_{ijkl}(t)e_{kl}(u))_{,j}+f_i\rangle_{L^2(\Omega)} \\
&\quad \left.\left.\left. -\langle \hat{u}_i, H_{ijkl}(t)e_{kl}(u)n_j - \hat{f}_i\rangle_{L^2(\Gamma_t)}\right\}\right\}\right\} \\
&= \inf_{t\in B}\left\{\sup_{\hat{u}\in U}\left\{\inf_{u\in U}\left\{\frac{1}{2}\int_\Omega H_{ijkl}(t)e_{ij}(u)e_{kl}(u)\,dx\right.\right.\right. \\
&\quad -\int_\Omega H_{ijkl}(t)e_{ij}(\hat{u})e_{kl}(u)\,dx \\
&\quad \left.\left.\left. +\langle \hat{u}_i,f_i\rangle_{L^2(\Omega)} + \langle \hat{u}_i,\hat{f}_i\rangle_{L^2(\Gamma_t)}\right\}\right\}\right\} \\
&= \inf_{t\in B}\left\{\sup_{\hat{u}\in U}\left\{-\int_\Omega H_{ijkl}(t)e_{ij}(\hat{u})e_{kl}(\hat{u})\,dx\right.\right. \\
&\quad \left.\left. \langle \hat{u}_i,f_i\rangle_{L^2(\Omega)} + \langle \hat{u}_i,\hat{f}_i\rangle_{L^2(\Gamma_t)}\right\}\right\} \\
&= \inf_{t\in B}\left\{\inf_{\sigma\in C^*} G^*(\sigma,t)\right\}. \qquad (45)
\end{aligned}
$$

Also, from this and the min-max theorem, there exist $(u_0, t_0) \in U \times B$ such that

$$\begin{aligned}
\inf_{(u,t)\in U\times B} J(u,t) &= \inf_{t\in B}\left\{\sup_{\hat{u}\in U} J_1(u,t)\right\} \\
&= \sup_{u\in U}\left\{\inf_{t\in B} J_1(u,t)\right\} \\
&= J_1(u_0,t_0) \\
&= \inf_{t\in B} J_1(u_0,t) \\
&= J^*(u_0).
\end{aligned} \tag{46}$$

Finally, from the extremal necessary condition

$$\frac{\partial J_1(u_0,t_0)}{\partial u} = \mathbf{0}$$

we obtain

$$(H_{ijkl}(t_0)e_{kl}(u_0))_{,j} + f_i = 0 \text{ in } \Omega,$$

and

$$H_{ijkl}(t_0)e_{kl}(u_0)n_j - \hat{f}_i = 0 \text{ on } \Gamma_t,\ \forall i \in \{1,2,3\},$$

so that

$$G(e(u_0)) = \frac{1}{2}\langle (u_0)_i, f_i\rangle_{L^2(\Omega)} + \frac{1}{2}\langle (u_0)_i, \hat{f}_i\rangle_{L^2(\Gamma_t)}.$$

Hence $(u_0, t_0) \in A^*$ so that $Ind(u_0, t_0) = 0$ and $\sigma(u_0) \in C^*$.
Moreover

$$\begin{aligned}
J^*(u_0) &= -G(e(u_0)) + \langle (u_0)_i, f_i\rangle_{L^2(\Omega)} + \langle (u_0)_i, \hat{f}_i\rangle_{L^2(\Gamma_t)} \\
&= G(e(u_0)) \\
&= G(e(u_0)) + Ind(u_0,t_0) \\
&= J(u_0,t_0) \\
&= G^*(\sigma(u_0),t_0).
\end{aligned} \tag{47}$$

This completes the proof. □

9.3. *About a concerning algorithm and related numerical method*

For numerically solve this optimization problem in question, we present the following algorithm

1. Set $t_1 = 0.5$ in Ω and $n = 1$.
2. Calculate $u_n \in U$ such that
$$J_1(u_n, t_n) = \sup_{u\in U} J_1(u, t_n).$$
3. Calculate $t_{n+1} \in B$ such that
$$J_1(u_n, t_{n+1}) = \inf_{t\in B} J_1(u_n, t).$$
4. If $\|t_{n+1} - t_n\|_\infty < 10^{-4}$ or $n > 100$ then stop, else set $n := n+1$ and go to item 2.

We have developed a software in finite differences for solving such a problem.
Here the software.
****

1. clear all

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global P m8 d w u v Ea Eb Lo d1 z1 m9 du1 du2 dv1 dv2 c3

m8=27;

m9=24;

c3=0.95;

d=1.0/m8;

d1=0.5/m9;

Ea=210 $* 10^5$; (stronger material)

Eb=1000; (softer material simulating voids)

w=0.30;

P=-42000000;

z1=(m8-1)*(m9-1);

A3=zeros(z1,z1);

for i=1:z1

A3(1,i)=1.0;

end;

b=zeros(z1,1);

uo=0.000001*ones(z1,1);

u1=ones(z1,1);

b(1,1)=c3*z1;

for i=1:m9-1

for j=1:m8-1

Lo(i,j)=c3;

end; end;

for i=1:z1

x1(i)=c3*z1;

end;

for i=1:2*m8*m9

xo(i)=0.000;

end;

xw=xo;

xv=Lo;

for k2=1:24

c3=0.98*c3;

b(1,1)=c3*z1;

k2

b14=1.0;

k3=0;

while $(b14 > 10^{-3.5})$ and $(k3 < 5)$

k3=k3+1;

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b12=1.0;

k=0;

while $(b12 > 10^{-4.0})$ and $(k < 120)$

k=k+1;

k2

k3

k

X=fminunc('funbeam',xo);

xo=X;

b12=max(abs(xw-xo));

xw=X;

end;

for i=1:m9-1

for j=1:m8-1

$E1 = Lo(i,j)^2 * (Ea - Eb);$

ex=du1(i,j);

ey=dv2(i,j);

exy=1/2*(dv1(i,j)+du2(i,j));

$Sx = E1 * (ex + w * ey)/(1 - w^2);$

$Sy = E1 * (w * ex + ey)/(1 - w^2);$

Sxy=E1/(2*(1+w))*exy;

dc3(i,j)=-(Sx*ex+Sy*ey+2*Sxy*exy);

end;

end;

for i=1:m9-1

for j=1:m8-1

f(j+(i-1)*(m8-1))=dc3(i,j);

end;

end;

for k1=1:1

k1

X1=linprog(f,[ ],[ ],A3,b,uo,u1,x1);

x1=X1;

end;

for i=1:m9-1

for j=1:m8-1

Lo(i,j)=X1(j+(m8-1)*(i-1));

end;

end;

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```
b14=max(max(abs(Lo-xv)))
xv=Lo;
colormap(gray); imagesc(-Lo); axis equal; axis tight; axis off;pause(1e-6)
end;
end;
```

****

Here the auxiliary Function 'funbeam'

```
function S=funbeam(x)
global P m8 d w u v Ea Eb Lo d1 m9 du1 du2 dv1 dv2
for i=1:m9
for j=1:m8
u(i,j)=x(j+(m8)*(i-1));
v(i,j)=x(m8*m9+(i-1)*m8+j);
end;
end;
for i=1:m9
end;
u(m9-1,1)=0;
v(m9-1,1)=0;
u(m9-1,m8-1)=0;
v(m9-1,m8-1)=0;
for i=1:m9-1
for j=1:m8-1
du1(i,j)=(u(i,j+1)-u(i,j))/d;
du2(i,j)=(u(i+1,j)-u(i,j))/d1;
dv1(i,j)=(v(i,j+1)-v(i,j))/d;
dv2(i,j)=(v(i+1,j)-v(i,j))/d1;
end;
end;
S=0;
for i=1:m9-1
for j=1:m8-1
E1 = Lo(i,j)^3 * Ea + (1 - Lo(i,j)^3) * Eb;
ex=du1(i,j);
ey=dv2(i,j);
exy=1/2*(dv1(i,j)+du2(i,j));
Sx = E1 * (ex + w * ey)/(1 - w^2);
Sy = E1 * (w * ex + ey)/(1 - w^2);
Sxy=E1/(2*(1+w))*exy;
S=S+1/2*(Sx*ex+Sy*ey+2*Sxy*exy);
end;
end;
S=S*d*d1-P*v(2,(m8)/3)*d*d1;
```

****

For a two dimensional beam of dimensions $1m \times 0.5m$ and $t_1 = 0.63$ we have obtained the following results:

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1. Case A: For the optimal shape for a clamped beam at left (cantilever) and load $P = -4\ 10^6 N\mathbf{j}$ at $(x, y) = (1, 0.25)$, please Figure 5.
2. Case B :For the optimal shape for a simply supported beam at $(0, 0)$ and $(1, 0)$ and load $P = -4\ 10^6 N\mathbf{j}$ at $(x, y) = (1/3, 0.5)$, please Figure 6.

In the first case the mesh was 28×24. In the second one the mesh was 27×24

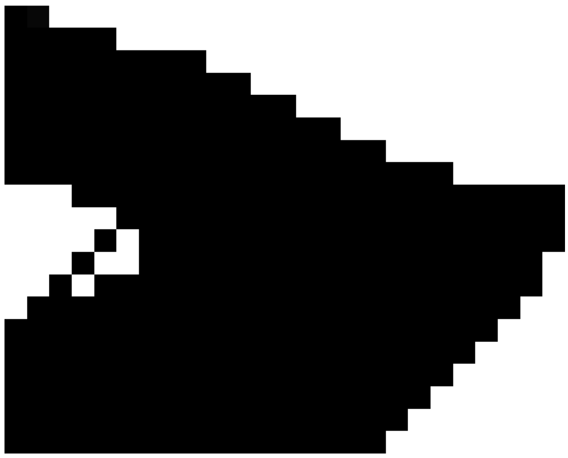

Figure 5. Density $t(x, y)$ for the Case A.

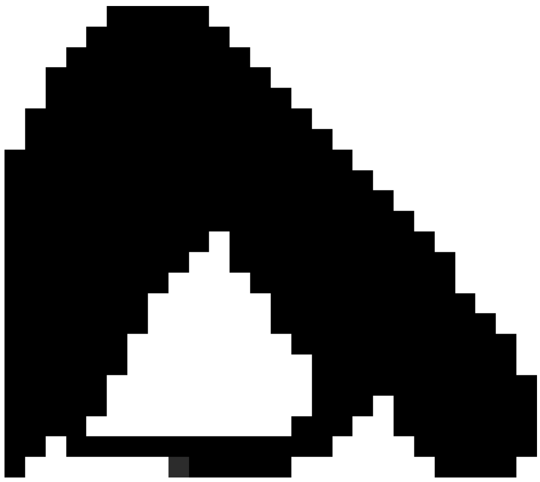

Figure 6. Density $t(x, y)$ for the Case B.

10. A duality principle for a general vectorial case in the calculus of variations

In this section we develop a duality principle for a general vectorial case in variational optimization.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$. Let $J : V \to \mathbb{R}$ be a functional where

$$J(u) = G(\nabla u_1, \cdots, \nabla u_N) - \langle u, f \rangle_{L^2},$$

where
$$V = W_0^{1,2}(\Omega; \mathbb{R}^N)$$
and
$$f = (f_1, \cdots, f_N) \in L^2(\Omega; \mathbb{R}^N).$$

Here we have denoted $u = (u_1, \cdots, u_N) \in V$ and
$$\langle u, f \rangle_{L^2} = \langle u_i, f_i \rangle_{L^2},$$
so that we may also denote
$$J(u) = G(\nabla u) - \langle u, f \rangle_{L^2}.$$

Assume
$$G(\nabla u) = \int_\Omega g(\nabla u)\, dx$$
where $g : \mathbb{R}^{3N} \to \mathbb{R}$ is a differentiable function such that
$$g(y) \to +\infty$$
as $|y| \to \infty$. Moreover, suppose there exists $\alpha \in \mathbb{R}$ such that
$$\alpha = \inf_{u \in V} J(u).$$

It is well known that
$$\begin{aligned} \alpha &= \inf_{u \in V} J(u) \\ &= \inf_{u \in V} J^{**}(u) \\ &= \inf_{u \in V} \{(G \circ \nabla)^{**}(u) - \langle u, f \rangle_{L^2}\}. \end{aligned} \tag{48}$$

Under some mild hypotheses, from convexity, we have that
$$\begin{aligned} &\inf_{u \in V} \{(G \circ \nabla)^{**}(u) - \langle u, f \rangle_{L^2}\} \\ = &\sup_{v^* \in A^*} \{-(G \circ \nabla)^*(-div\, v^*)\} = -(G \circ \nabla)^*(f), \end{aligned} \tag{49}$$
where
$$A^* = \{v^* \in Y = Y^* = L^2(\Omega; \mathbb{R}^{3N}) \;:\; div\, v^* + f = 0\}.$$

Now observe that the restriction $v = \nabla u$ for some $u \in V$ is equivalent to the restriction
$$\text{curl } v_i = \mathbf{0}, \text{ in } \Omega$$

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where $v = \{v_i\} = \{v_{ij}\}_{j=1}^3, \forall i \in \{1, \cdots, N\}$, with appropriate boundary conditions, so that with an appropriate Lagrange multiplier $\phi = \{\phi_i\}$, we obtain

$$\begin{aligned}
(G \circ \nabla)^*(-div\ v^*) &= \sup_{u \in V} \{\langle u, -div\ v^* \rangle_{L^2} - G(\nabla u)\} \\
&= \sup_{u \in V} \{\langle \nabla u, v^* \rangle_{L^2} - G(\nabla u)\} \\
&\leq \inf_{\phi \in Y^*} \left\{ \sup_{v \in Y} \{\langle v, v^* \rangle_{L^2} - G(v) + \langle \phi, \text{curl } v \rangle_{L^2} \right\} \\
&= \inf_{\phi \in Y^*} G^*(v^* + \text{curl } \phi). \qquad (50)
\end{aligned}$$

where we have denoted

$$\text{curl } v = \{\text{curl } v_i\}$$

and

$$\text{curl } \phi = \{\text{curl } \phi_i\}.$$

Joining the pieces, we have got

$$\begin{aligned}
\inf_{u \in V} J(u) &= \inf_{u \in V} \{G(\nabla u) - \langle u, f \rangle_{L^2}\} \\
&\geq \sup_{(v^*, \phi) \in A^* \times Y^*} \{-G^*(v^* + \text{curl } \phi)\}, \qquad (51)
\end{aligned}$$

where we recall that $Y = Y^* = L^2(\Omega; \mathbb{R}^{3N})$.

We emphasize such a dual formulation in (v^*, ϕ) is convex (in fact concave).

11. A note on the Galerkin Functional

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider the functional $J : V \to \mathbb{R}$ where

$$\begin{aligned}
J(u) &= \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u\, dx + \frac{\alpha}{4} \int_\Omega u^4\, dx \\
&\quad -\frac{\beta}{2} \int_\Omega u^2\, dx - \langle u, f \rangle_{L^2} \qquad (52)
\end{aligned}$$

Here $V = W_0^{1,2}(\Omega), \gamma > 0,\ \alpha > 0,\ \beta > 0$.

We denote also

$$Y = Y^* = L^2(\Omega).$$

At this point we define

$$A^+ = \{u \in V\ :\ u f \geq 0, \text{ in } \Omega\},$$

$$V_2 = \{u \in V\ :\ \|u\|_\infty \leq K_3\},$$

for some appropriate real constant $K_3 > 0$ and

$$V_1 = A^+ \cap V_2.$$

Observe that

$$J'(u) = -\gamma \nabla^2 u + \alpha u^3 - \beta - f,$$

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so that we define the Galerkin functional $J_1 : V \to \mathbb{R}$ by

$$J_1(u) = \frac{1}{2}\|J'(u)\|_2^2 = \frac{1}{2}\int_\Omega (-\gamma\nabla^2 u + \alpha u^3 - \beta u - f)^2\,dx.$$

From this, we get

$$\begin{aligned}\frac{\partial^2 J_1(u)}{\partial u^2} &= (-\gamma\nabla u + \alpha u^3 - \beta u - f)6\alpha u \\ &+(-\gamma\nabla^2 + 3\alpha u^2 - \beta)^2.\end{aligned} \tag{53}$$

Define now

$$\varphi_2 = (-\gamma\nabla^2 u + \alpha u^3 - \beta u - f)^2.$$

At this point, for an appropriate small real constant $\varepsilon_1 > 0$ and bounded constant operator $M_1 > \varepsilon_1$, we set the intended non-active restriction

$$\sqrt{3\alpha}|u| \geq \sqrt{|M_1 + \gamma\nabla^2 + \beta|},$$

and define

$$B_1 = \{u \in V_1 \ : \ \sqrt{3\alpha}|u| \geq \sqrt{|M_1 + \gamma\nabla^2 + \beta|}\}.$$

Observe that since for $u \in V_1$ we have $u\, f \geq 0$ in Ω so that if $u_1, u_2 \in V_1$ then

$$\text{sign}(u_1) = \text{sign}(u_2), \text{in } \Omega,$$

we may infer that B_1 is a convex set.

Furthermore, if $u \in B_1$, then

$$\sqrt{3\alpha}|u| \geq \sqrt{|M_1 + \gamma\nabla^2 + \beta|},$$

so that

$$3\alpha u^2 \geq M_1 + \gamma\nabla^2 + \beta,$$

and hence

$$\delta^2 J(u) = -\gamma\nabla^2 + 3\alpha u^2 - \beta \geq M_1 > \varepsilon_1 > \mathbf{0}.$$

For a small parameter $\varepsilon > 0$ we define the intended non-active restriction

$$\varphi_2 \leq \varepsilon, \text{ in } \Omega,$$

and define

$$B_2 = \{u \in V_1 \ : \ \varphi_2 \leq \varepsilon, \text{ in } \Omega\}.$$

Observe that for $\alpha > 0$ and $\beta > 0$ sufficiently large φ_2 is convex in V_1 (positive definite Hessian) so that B_2 is a convex set. Assuming $0 < \varepsilon \ll \varepsilon_1 \ll 1$, define $B_3 = B_1 \cap B_2$, which is a convex set.

Summarizing, if $u \in B_3$, then

$$\delta^2 J_1(u) \geq \mathbf{0}.$$

With such results in mind, we define the following convex optimization problem for finding a critical point of J.

Minimize

$$J_1(u) = \frac{1}{2}\|J'(u)\|_2^2 = \frac{1}{2}\int_\Omega (-\gamma\nabla^2 u + \alpha u^3 - \beta u - f)^2\,dx,$$

subject to

$$u \in B_3.$$

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Observe that a critical point $u_0 \in B_3$ of J_1, from such a concerning convexity of J_1 on the convex set B_1, is also such that

$$J(u_0) = \min_{u \in B_3} J_1(u).$$

Finally, we may also define the convex optimization problem of minimizing

$$\begin{aligned} J_3(u) &= K_1 J_1(u) + J(u) \\ &= \frac{K_1}{2} \int_\Omega (-\gamma \nabla^2 u + \alpha u^3 - \beta u - f)^2 \, dx \\ &\quad + \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{4} \int_\Omega u^4 \, dx \\ &\quad - \frac{\beta}{2} \int_\Omega u^2 \, dx - \langle u, f \rangle_{L^2}, \end{aligned} \tag{54}$$

subject to

$$u \in B_3.$$

Here $K_1 > 0$ is a large real constant.

Such a functional J_3 is also convex on B_3 so that a critical point $u_0 \in B_3$ of J is also a critical point of J_3, and thus

$$J_3(u_0) = \min_{u \in B_3} J_3(u).$$

12. A note on the Legendre-Galerkin functional

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider the functional $J : V \to \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx + \frac{\alpha}{4} \int_\Omega u^4 \, dx \\ &\quad - \frac{\beta}{2} \int_\Omega u^2 \, dx - \langle u, f \rangle_{L^2} \end{aligned} \tag{55}$$

Here $V = W_0^{1,2}(\Omega)$, $\gamma > 0$, $\alpha > 0$, $\beta > 0$.

We denote also

$$Y = Y^* = L^2(\Omega)$$

and $F_1 : V \to \mathbb{R}$, $F_2 : V \to \mathbb{R}$ and $F_3 : V \to \mathbb{R}$ by

$$F_1(u) = \frac{\gamma}{2} \int_\Omega \nabla u \cdot \nabla u \, dx,$$

$$F_2(u) = \frac{\alpha}{4} \int_\Omega u^4 \, dx,$$

$$F_3(u) = \frac{\beta}{2} \int_\Omega u^2 \, dx.$$

Moreover, we define $F_1^*, F_2^*, F_3^* : Y^* \to \mathbb{R}$ by

$$\begin{aligned} F_1^*(v_1^*) &= \sup_{u \in V} \{ \langle u, v_1^* \rangle_{L^2} - F_1(u) \} \\ &= \frac{1}{2} \int_\Omega \frac{(v_1^*)^2}{-\gamma \nabla^2} \, dx, \end{aligned} \tag{56}$$

$$\begin{aligned} F_2^*(v_2^*) &= \sup_{u\in V}\{\langle u, v_2^*\rangle_{L^2} - F_2(u)\} \\ &= \frac{3}{4}\int_\Omega \frac{(v_2^*)^{4/3}}{\alpha^{1/3}}\,dx, \end{aligned} \quad (57)$$

$$\begin{aligned} F_3^*(v_3^*) &= \sup_{u\in V}\{\langle u, v_3^*\rangle_{L^2} - F_3(u)\} \\ &= \frac{1}{2\beta}\int_\Omega (v_3^*)^2\,dx. \end{aligned} \quad (58)$$

Observe now that these three last suprema are attained through the equations,

$$v_1^* = \frac{\partial F_1(u)}{\partial u} = -\gamma\nabla^2 u,$$

$$v_2^* = \frac{\partial F_2(u)}{\partial u} = \alpha u^3$$

$$v_3^* = \frac{\partial F_3(u)}{\partial u} = \beta u.$$

From such results, at a critical point, we obtain the following compatibility conditions

$$u = \frac{v_1^*}{-\gamma\nabla^2} = \left(\frac{v_2^*}{\beta}\right)^{1/3} = \frac{v_3^*}{\beta}.$$

From such relations we have

$$\frac{v_1^*}{-\gamma\nabla^2} = \frac{v_3^*}{\beta},$$

and

$$v_2^* = \alpha\left(\frac{v_3^*}{\beta}\right)^3,$$

so that

$$v_1^* = -\gamma\nabla^2\left(\frac{v_3^*}{\beta}\right),$$

and

$$v_2^* = \alpha\left(\frac{v_3^*}{\beta}\right)^3.$$

Moreover, we define the functional $F_4^* : Y^* \to \mathbb{R}$, by

$$F_4^*(v^*) = \sup_{u\in V}\{\langle u, v_1^* + v_2^* - v_3^*\rangle_{L^2} - \langle u, f\rangle_{L^2}\}.$$

Therefore

$$F_4^*(v^*) = \begin{cases} 0, & \text{if } v_1^* + v_2^* - v_3^* - f = 0, \text{ in } \Omega, \\ +\infty, & \text{otherwise.} \end{cases} \quad (59)$$

Hence, a critical point of J corresponds to the solution of the following system of equations

$$v_1^* = -\gamma\nabla^2\left(\frac{v_3^*}{\beta}\right),$$

$$v_2^* = \alpha\left(\frac{v_3^*}{\beta}\right)^3,$$

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and

$$v_1^* + v_2^* - v_3^* - f = 0, \text{ in } \Omega.$$

From this last equation we may obtain

$$v_1^* = -v_2^* + v_3^* + f,$$

so that the final equations to be solved are

$$-v_2^* + v_3^* + f + \gamma\nabla^2\left(\frac{v_3^*}{\beta}\right) = 0$$

and

$$v_2^* - \alpha\left(\frac{v_3^*}{\beta}\right)^3 = 0, \text{ in } \Omega,$$

with the boundary conditions

$$u = \frac{v_3^*}{\beta} = 0, \text{ on } \partial\Omega.$$

With such results in mind, we define the Legendre-Galerkin functional $J^* : [Y^*]^2 \to \mathbb{R}$, where

$$\begin{aligned} J^*(v^*) &= \frac{1}{2}\int_\Omega \left(-v_2^* + v_3^* + f + \frac{\gamma\nabla^2 v_3^*}{\beta}\right)^2 dx \\ &+ \frac{1}{2}\int_\Omega \left(v_2^* - \alpha\left(\frac{v_3^*}{\beta}\right)^3\right)^2 dx. \end{aligned} \tag{60}$$

At this point, defining

$$\varphi = v_2^* - \alpha\left(\frac{v_3^*}{\beta}\right)^3,$$

we obtain

$$\frac{\partial^2 J^*(v^*)}{\partial (v_2^*)^2} = 2;$$

$$\frac{\partial^2 J^*(v^*)}{\partial (v_3^*)^2} = \left(-1 - \frac{\gamma\nabla^2}{\beta}\right)^2 + \frac{9\alpha^2 (v_3^*)^4}{\beta^6} + \mathcal{O}(\varphi),$$

$$\frac{\partial^2 J^*(v^*)}{\partial v_2^* \partial v_3^*} = \frac{-3\alpha(v_3^*)^2}{\beta^3} + \left(-1 - \frac{\gamma\nabla^2}{\beta}\right).$$

From such results we may infer that

$$\begin{aligned} \det\left(\frac{\partial^2 J^*(v^*)}{\partial v_2^* \partial v_3^*}\right) &= \frac{\partial^2 J^*(v^*)}{\partial (v_2^*)^2}\frac{\partial^2 J^*(v^*)}{\partial (v_3^*)^2} - \left(\frac{\partial^2 J^*(v^*)}{\partial v_2^* \partial v_3^*}\right)^2 \\ &= \left(-1 - \frac{\gamma\nabla^2}{\beta} + 3\alpha\frac{(v_3^*)^2}{\beta^3}\right)^2 + \mathcal{O}(\varphi) \end{aligned} \tag{61}$$

Observe that a critical point $\varphi = 0$ so that $\delta^2 J^*(v^*) > \mathbf{0}$ at a neighborhood of any critical point. At this point we define

$$A^+ = \left\{v^* = (v_2^*, v_3^*) \in [Y^*]^2 \ : \ \frac{v_3^*}{\beta} f \geq 0, \text{ in } \Omega\right\},$$

$$D^* = \{v^* = (v_2^*, v_3^*) \in [Y^*]^2 \ : \ \|v^*\|_\infty \leq K\},$$

for an appropriate real constant $K > 0$.

Define now $E^* = A^+ \cap D^*$,

$$C_1^* = \{v^* = (v_2^*, v_3^*) \in E^* \ : \ \varphi^2 \leq \varepsilon, \text{ in } \Omega\},$$

for a small real constant $\varepsilon > 0$,

$$C_2^* = \left\{ v^* = (v_2^*, v_3^*) \in E^* \ : \ \left(-1 - \frac{\gamma \nabla^2}{\beta} + 3\alpha \frac{(v_3^*)^2}{\beta^3} \right) \geq \varepsilon_1 \right\},$$

and

$$C^* = C_1^* \cap C_2^*.$$

Similarly as done in the previous section, we may prove that C^* is a convex set.

Furthermore, for $0 < \varepsilon \ll \varepsilon_1 \ll 1$, we have that J^* is convex on C^*.

Summarizing, we may define the following convex optimization problem to obtain a critical point of the primal functional J,

$$\text{Minimize } J^*(v_2^*, v_3^*) \text{ subject to } v^* = (v_2^*, v_3^*) \in C^*.$$

We call J^* the Legendre-Galerkin functional associated to J.

12.1. Numerical examples

We have obtained numerical solutions for two one-dimensional examples.

1. For $\gamma = 1.0$, $\alpha = 3.0$, $\beta = 30.0$, $f \equiv 10$, in $\Omega = [0, 1]$.

   For the respective solution please see Figure 7.
2. For $\gamma = 0.01$, $\alpha = 3.0$, $\beta = 30.0$, $f \equiv 10$, in $\Omega = [0, 1]$.

   For the respective solution please see Figure 8.

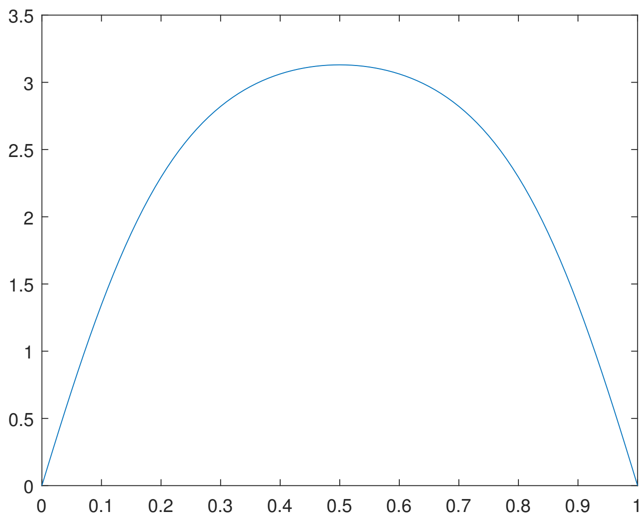

Figure 7. Solution $u(x) = v_3^*(x)/\beta$ for the example 1.

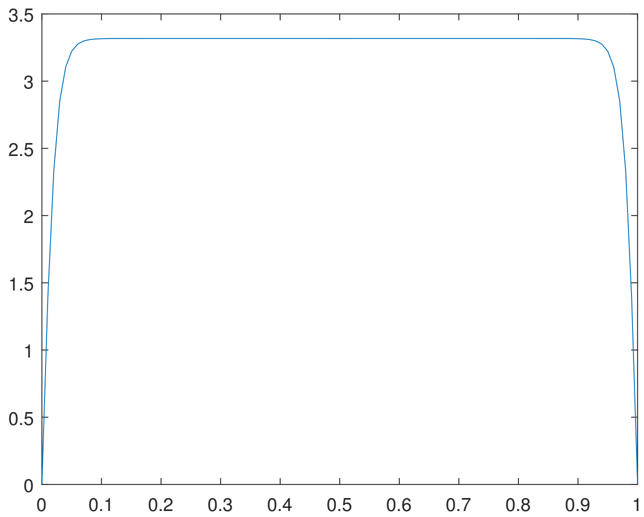

Figure 8. Solution $u(x) = v_3^*(x)/\beta$ for the example 2.

13. A general concave dual variational formulation for global optimization

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \to \mathbb{R}$ where

$$J(u) = G(u) - \langle u, f \rangle_{L^2},\ \forall u \in V.$$

Here $V = W_0^{1,2}(\Omega)$, $f \in L^2(\Omega)$ and we also denote $Y = Y^* = L^2(\Omega)$.

Assume there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = \inf_{u \in V} J(u).$$

Furthermore, suppose G is three times Fréchet differentiable and there exists $K > 0$ such that

$$\frac{\partial^2 G(u)}{\partial u^2} + K > \mathbf{0}, \forall u \in V.$$

Define now $J_1 : V \times Y \to \mathbb{R}$ where,

$$J_1(u, v) = G_1(u, v) + F(u),$$

where

$$G_1(u, v) = G(v) - \frac{\varepsilon}{2}\int_\Omega v^2\,dx + \frac{K}{2}\int_\Omega (v-u)^2\,dx,$$

and

$$F(u) = \frac{\varepsilon}{2}\int_\Omega u^2\,dx - \langle u, f \rangle_{L^2}.$$

Moreover, we define the polar functionals $G_1^* : Y^* \times V \to \mathbb{R}$ and $F^* : Y^* \to \mathbb{R}$, where

$$\begin{aligned} G_1^*(v^*, u) &= \sup_{v \in Y}\{\langle v, v^* \rangle_{L^2} - G_1(u, v)\} \\ &= -G_{K_\varepsilon}^*(v^* + Ku) + \frac{K}{2}\int_\Omega u^2\,dx, \end{aligned} \tag{62}$$

$$G^*_{K_\varepsilon}(v^*+Ku)=\sup_{v\in Y}\left\{\langle v,v^*\rangle_{L^2}-G(v)-\frac{K}{2}\int_\Omega v^2\,dx+\frac{\varepsilon}{2}\int_\Omega v^2\,dx\right\},$$

and

$$\begin{aligned}F^*(-v^*) &= \sup_{u\in V}\{-\langle u,v^*\rangle_{L^2}-F(u)\}\\ &= \frac{1}{2\varepsilon}\int_\Omega (v^*-f)^2\,dx. \end{aligned} \quad (63)$$

At this point we define the functional $J_2^*: Y^*\times V\to\mathbb{R}$ by

$$J_2^*(v^*,u)=-G^*_{K_\varepsilon}(v^*+Ku)+\frac{K}{2}\int_\Omega u^2\,dx-F^*(-v^*).$$

With such results in mind we define

$$V_1=\{u\in V\ :\ \|u\|_\infty\le K_3\},$$

and

$$D^*=\{v^*\in Y^*\ :\ \|v^*\|_\infty\le K_4\},$$

for appropriated real constants $K_3>0$ and $K_4>0$.

Moreover, we define also the penalized functional $J_3^*: Y^*\times V\to\mathbb{R}$ where

$$J_3^*(v^*,u)=J_2^*(v^*,u)-\frac{K_1}{2}\int_\Omega\left(v^*-\frac{\partial G(u)}{\partial u}+\varepsilon u\right)^2\,dx.$$

Finally, we remark that for $\varepsilon>0$ sufficiently small and $K_1>0$ sufficiently large, J_3^* is concave in $D^*\times V_1$ around a concerning critical point. We recall that a critical point

$$v^*-\frac{\partial G(u)}{\partial u}+\varepsilon u=0,\ \text{in }\Omega.$$

14. A related restricted problem in phase transition

In this section we develop a convex (in fact concave) dual variational for a model similar to those found in phase transition problems.

Let $\Omega=[0,1]\subset\mathbb{R}$. Consider the functional $J: V\to\mathbb{R}$ where

$$\begin{aligned}J(u) &= \frac{1}{2}\int_\Omega \min\{(u'+1)^2,(u'-1)^2\}\,dx\\ &\quad+\frac{1}{2}\int_\Omega u^2\,dx-\langle u,f\rangle_{L^2}\\ &= \frac{1}{2}\int_\Omega (u')^2\,dx-\int_\Omega |u'|\,dx+1/2\\ &\quad+\frac{1}{2}\int_\Omega u^2\,dx-\langle u,f\rangle_{L^2}. \end{aligned} \quad (64)$$

Here

$$V=\{u\in W^{1,2}(\Omega)\ :\ u(0)=0\text{ and }u(1)=1/2\}.$$

We also denote $V_1=W_0^{1,2}(\Omega)$, and $Y=Y^*=L^2(\Omega)$.

Furthermore, we define the relaxed functionals G and $F: V\times V_1\to\mathbb{R}$ by

$$G(u',v')=\frac{1}{2}\int_\Omega (u'+v')^2\,dx-\int_\Omega |u'+v'|\,dx+1/2,$$

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and

$$F(u,v) = \frac{1}{2}\int_\Omega u^2\,dx - \langle u, f\rangle_{L^2}.$$

Moreover we define $J_1 : V \times V_1 \to \mathbb{R}$ by

$$J_1(u,v) = G(u',v') + F(u,v),$$

and consider the problem of minimizing J_1 on the set

$$B = \{(u,v) \in V \times V_1 \;:\; (u')^2 \leq K_1 \text{ and } (v')^2 \leq K_2, \text{ in } \Omega\}.$$

Already including the Lagrange multipliers (ϕ_1, ϕ_2) concerning such restrictions, we define

$$J_2(u,v) = J_1(u,v) + \frac{1}{2}\langle \phi_1^2, (u')^2 - K_1\rangle_{L^2} + \frac{1}{2}\langle \phi_2^2, (v')^2 - K_2\rangle_{L^2}.$$

Observe now that

$$\begin{aligned}
J_2(u,v) &= J_1(u,v) + \frac{1}{2}\langle \phi_1^2, (u')^2 - K_1\rangle_{L^2} \\
&\quad + \frac{1}{2}\langle \phi_2^2, (v')^2 - K_2\rangle_{L^2} \\
&= G(u',v') + \frac{1}{2}\langle \phi_1^2, (u')^2 - K_1\rangle_{L^2} + \frac{1}{2}\langle \phi_2^2, (v')^2 - K_2\rangle_{L^2} \\
&\quad + F(u,v) \\
&= -\langle u', v_1^*\rangle_{L^2} - \langle v', v_2^*\rangle_{L^2} + G(u',v') \\
&\quad + \frac{1}{2}\langle \phi_1^2, (u')^2 - K_1\rangle_{L^2} + \frac{1}{2}\langle \phi_2^2, (v')^2 - K_2\rangle_{L^2} \\
&\quad \langle u', v_1^*\rangle_{L^2} + \langle v', v_2^*\rangle_{L^2} + F(u,v) \\
&\geq \inf_{(v_1,v_2)\in Y\times Y} \{-\langle v_1, v_1^*\rangle_{L^2} - \langle v_2, v_2^*\rangle_{L^2} + G_1(v_1,v_2,\phi) \\
&\quad + \frac{1}{2}\langle \phi_1^2, (v_1)^2 - K_1\rangle_{L^2} + \frac{1}{2}\langle \phi_2^2, (v_2)^2 - K_2\rangle_{L^2}\Big\} \\
&\quad + \inf_{(u,v)\in V\times V_1} \{\langle u', v_1^*\rangle_{L^2} + \langle v', v_2^*\rangle_{L^2} + F(u,v)\} \\
&= -G_1^*(v_1^*, v_2^*, \phi) - \tilde{F}^*(v_1^*, v_2^*),\; \forall (u,v) \in V \times V_1, (v_1^*, v_2^*, \phi) \in [Y^*]^4,
\end{aligned} \tag{65}$$

where

$$G_1(u',v',\phi) = G(u',v') + \frac{1}{2}\langle \phi_1^2, (u')^2 - K_1\rangle_{L^2} + \frac{1}{2}\langle \phi_2^2, (v')^2 - K_2\rangle_{L^2}.$$

Also,

$$\begin{aligned}
G_1^*(v_1^*, v_2^*, \phi) &= \sup_{(v_1,v_2)\in Y\times Y} \{\langle v_1, v_1^*\rangle_{L^2} + \langle v_1, v_1^*\rangle_{L^2} - G_1(v_1,v_2,\phi)\} \\
&= \frac{1}{2}\int_\Omega \frac{-\phi_1^2\,\phi_2^2 + (1+\phi_2^2)(v_1^*)^2 - 2v_1^*v_2^* + (1+\phi_1^2)(v_2^*)^2}{H}\,dx \\
&\quad + \frac{1}{2}\int_\Omega \frac{2|\phi_2^2 v_1^* + \phi_1^2 v_2^*|}{H}\,dx \\
&\quad + \int_\Omega \phi_1^2\,dx\,K_1/2 + \int_\Omega \phi_2^2\,dx\,K_2/2,
\end{aligned} \tag{66}$$

where

$$H = \phi_1^2 + \phi_2^2 + \phi_1^2\phi_2^2,$$

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and

$$\tilde{F}^*(v^*) = \begin{cases} \frac{1}{2}\int_\Omega ((v_1^*)' + f)^2\,dx - v_1^*(1)u(1), & \text{if } (v_2^*)' = 0, \text{ in } \Omega, \\ +\infty, & \text{otherwise.} \end{cases} \tag{67}$$

From this we may infer that $v_2^* = c$, in Ω, for some $c \in \mathbb{R}$.
Summarizing, denoting $v^* = (v_1^*, v_2^*)$, $\phi = (\phi_1, \phi_2)$ and

$$J^*(v^*, \phi) = -G_1^*(v^*, \phi) - \tilde{F}^*(v^*)$$

we have got

$$\inf_{(u,v)\in V\times V_1} J_1(u,v) \geq \sup_{(v^*,\phi)\in[Y^*]^4} J^*(v^*\phi).$$

We have developed numerical results by maximizing the dual functional J^* for two examples, namely.

1. Example A: In this case, we consider $f(x) = \cos(\pi x)/2$, $K_1 = 1$ and $K_2 = 1$.

   For the optimal
   $$u_0 = (v_1^*)' + f,$$
   please see Figure 9.
2. Example B: In this case, we consider $f(x) = \cos(\pi x)/2$, $K_1 = 50$ and $K_2 = 50$.

   For the optimal
   $$u_0 = (v_1^*)' + f,$$
   please see Figure 10.

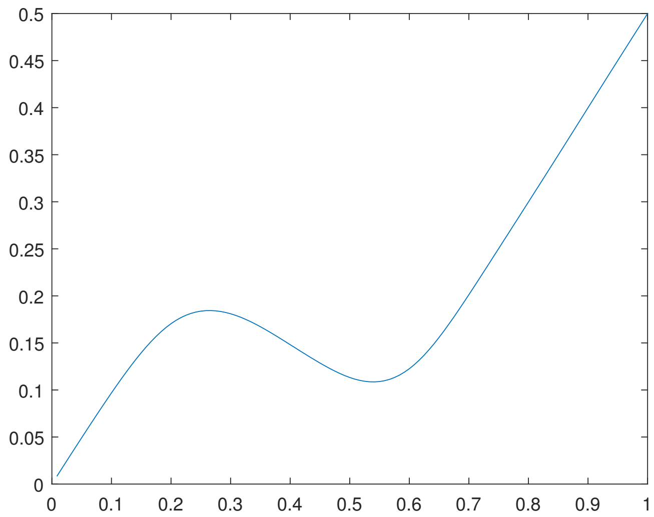

Figure 9. Solution $u_0(x)$ for the example A.

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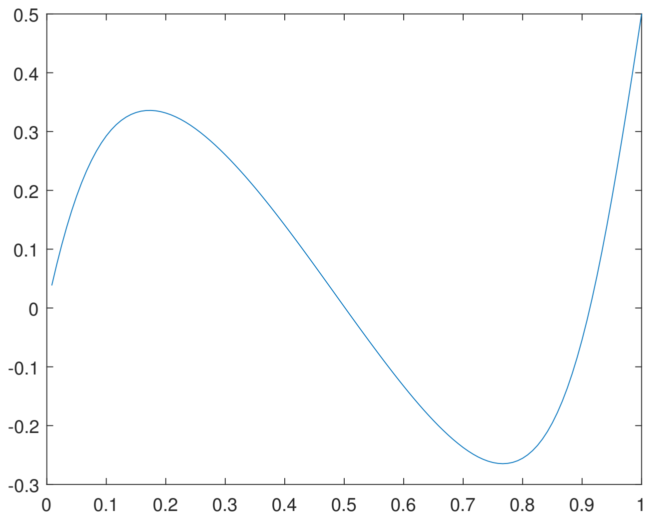

Figure 10. Solution $u_0(x)$ for the example B.

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