Lorentz Boosts and Rapidity for Inertial Observers in Kruskal-Szekeres Coordinates

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The Lorentz boost and rapidity for a radially falling frame are derived for the Schwarzschild metric in Kruskal-Szekeres coordinates. It is found through this analysis that the spacetime is radially length contracted toward the metric source in the falling frame as a result of the Lorentz boosting relative to the rest frame (which is shown to be consistent with the rest frame). When the falling frame reaches the event horizon, the Lorentz boost goes to infinity, causing the falling frame to become light-like, but trapped at the horizon. It is shown that in the falling frame at the horizon, the density of the source goes to infinity as a result of the length contraction.

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I. INTRODUCTION

The event horizon of the Schwarzschild metric is described as a coordinate singularity in Schwarzschild coordinates. This singularity is overcome using the Kruskal-Szekeres coordinates for the spacetime since the metric is regular at the horizon. However, the meaning of the the Kruskal-Szekeres coordinates in terms of spacelike and timelike basis vectors is not clear. For much of the spacetime, the Kruskal-Szekeres coordinates are mixtures of space and time such that the meaning of the slope of a worldline in these coordinates is not always clear.

In this paper, the Kruskal-Szekeres coordinates and the derivatives of worldlines in these coordinates is clarified. We find that we can define a Lorentz boost and rapidity in Kruskal-Szekeres coordinates for falling frames, and with those, it is shown that the worldlines of all frames approaching the event horizon become light-like there, even if the slope of the worldline in Kruskal-Szekeres coordinates is not ± 1 there. This is possible because the derivative of the worldline in Kruskal-Szekeres coordinates at the horizon is not a true velocity due to the fact that the coordinates are mixtures of space and time. But we can mathematically extract a derivative for the worldline in Kruskal-Szekeres coordinates which is a true velocity relative to the rest frame for any point on the worldline by removing the part of the derivative related to the rest frame at the same point, and we find that this velocity goes to 1 at the horizon for all worldlines.

II. THE FALLING FRAME IN KRUSKAL-SZEKERES COORDINATES

The Schwarzschild metric is the simplest non-trivial solution to Einstein's field equations. It is the metric that describes every spherically symmetric vacuum spacetime. The the external form of the metric can be expressed as:

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right) dt^2 - \frac{1}{1 - \frac{r_s}{r}} dr^2 - r^2 d\Omega^2 \qquad (1)$$

Equation 1 is the external metric with t being the timelike coordinate and r being the spacelike coordinate. The Schwarzschild radius of the metric is given by $r_s = 2GM$ in units with $c = 1$. The external metric is the metric for an eternally spherically-symmetric vacuum centered in space.

We can see in equation 1 that as $r \to r_s$, the basis vector ∂_t goes to zero while ∂_r goes to infinity. This location is called the 'Event Horizon' of the metric. The behaviour of the basis vectors at this location seem to imply that there is a coordinate singularity at this location since the spacetime curvature is not infinite there.

In order to overcome the behaviour of the basis vectors at this location, different coordinate systems have been developed which do not have degenerate behaviour at the Event Horizon. The most important of these coordinate systems are the Kruskal-Szekeres coordinates, which are the maximally extended coordinates for the Schwarzschild metric. The coordinate definitions and metric in Kruskal-Szekeres coordinates are given below (derivation of the coordinate definitions and metric can be found in reference [1] where $v = T$ and $u = X$).

$$\begin{aligned} T &= \sqrt{\left(\frac{r}{r_s} - 1\right) e^{\frac{r}{r_s}}} \sinh\left(\frac{t}{2r_s}\right) \\ X &= \sqrt{\left(\frac{r}{r_s} - 1\right) e^{\frac{r}{r_s}}} \cosh\left(\frac{t}{2r_s}\right) \end{aligned} \qquad (2)$$

With the full metric in Kruskal-Szekeres coordinates given by:

$$d\tau^2 = \frac{4r_s^3}{r} e^{-\frac{r}{r_s}} \left(dT^2 - dX^2\right) - r^2 d\Omega^2 \qquad (3)$$

Finally, we plot the metric on the Kruskal-Szekeres coordinate chart [2] in Figure 1:

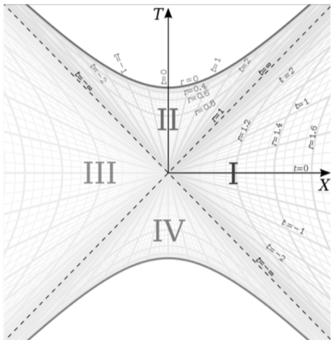

FIG. 1. Kruskal-Szekeres Coordinate Chart

In this paper, we will be focusing on region I in this chart, which is the spherically symmetric spacetime around a spherically symmetric source in space.

We can see in Figure 1 that for a rest frame ($r = const$), that $\frac{dX}{dT}$ depends on the value of t we evaluate the derivative at since the derivative for the rest frame is the tangent to the hyperbola at t. Since the metric is time-symmetric, we now that the actual physics does not depend on the value of t and therefore, we need a deeper understanding of the meaning of the changing $\frac{dX}{dT}$ in the rest frame and how it relates to the falling frame at the same point.

Let us first take the differentials of T and X in equations 2:

$$\begin{aligned} dX &= \frac{\partial X}{\partial r} dr + \frac{\partial X}{\partial t} dt \\ dT &= \frac{\partial T}{\partial r} dr + \frac{\partial T}{\partial t} dt \end{aligned} \tag{4}$$

Calculating the partial derivatives, rearranging and defining $R \equiv \frac{r e^{\frac{r}{r_s}}}{2r_s \sqrt{(\frac{r}{r_s}-1)e^{\frac{r}{r_s}}}}$:

$$\begin{aligned} \frac{dX}{dt} &= R\left[\frac{dr}{dt}\cosh\left(\frac{t}{2r_s}\right) + \left(1-\frac{r_s}{r}\right)\sinh\left(\frac{t}{2r_s}\right)\right] \\ \frac{dT}{dt} &= R\left[\frac{dr}{dt}\sinh\left(\frac{t}{2r_s}\right) + \left(1-\frac{r_s}{r}\right)\cosh\left(\frac{t}{2r_s}\right)\right] \end{aligned} \tag{5}$$

Next, we need to calculate $\frac{dX}{dT}$ from equations 5 by factoring out $\left(1-\frac{r_s}{r}\right)\cosh\left(\frac{t}{2r_s}\right)$ from each equation and dividing:

$$\frac{dX}{dT} = \frac{dX}{dt}\frac{dt}{dT} = \frac{\frac{dr}{dt}\left(1-\frac{r_s}{r}\right)^{-1} + \tanh\left(\frac{t}{2r_s}\right)}{\frac{dr}{dt}\left(1-\frac{r_s}{r}\right)^{-1}\tanh\left(\frac{t}{2r_s}\right) + 1} \tag{6}$$

Next, we make the following definitions:

$$\left(\frac{dX}{dT}\right)_0 \equiv \tanh\left(\frac{t}{2r_s}\right) \tag{7}$$

This is the derivative of the rest frame at t since plugging $\frac{dr}{dt} = 0$ into equation 6, we get $\frac{dX}{dT} = \tanh\left(\frac{t}{2r_s}\right)$. And we define the relative velocity of the frame in motion relative to the rest frame as:

$$\left(\frac{dX}{dT}\right)_{rel} \equiv \frac{dr}{dt}\left(1-\frac{r_s}{r}\right)^{-1} \tag{8}$$

This is the relative velocity in Kruskal-Szekeres coordinates between the frame in motion and the rest frame at r. This derivative is 0 for the rest frame since $\frac{dr}{dt} = 0$ in that frame.

Plugging these definitions into equation 6, we get:

$$\frac{dX}{dT} = \frac{\left(\frac{dX}{dT}\right)_{rel} + \left(\frac{dX}{dT}\right)_0}{\left(\frac{dX}{dT}\right)_{rel}\left(\frac{dX}{dT}\right)_0 + 1} \tag{9}$$

We recognize that equation 9 is the relativistic velocity addition formula giving us the total velocity as the relativistic sum of the rest frame velocity and the relative velocity between the moving frame and the rest frame. We can solve for $\left(\frac{dX}{dT}\right)_{rel}$ to get an expression for the relative velocity between a frame in motion and the rest frame in Kruskal-Szekeres coordinates:

$$\left(\frac{dX}{dT}\right)_{rel} = -\frac{\left(\frac{dX}{dT}\right)_0 - \frac{dX}{dT}}{1 - \left(\frac{dX}{dT}\right)_0 \frac{dX}{dT}} \tag{10}$$

Assuming $\frac{dX}{dT} \neq 1$, we see that the relative velocity approaches -1 for all $\frac{dX}{dT}$ as the horizon is approached since the horizon is at $t = \infty$, such that $\left(\frac{dX}{dT}\right)_0 = 1$ there. Equation 10 is also constant along a given hyperbola, meaning it is independent of t. In fact, it is the $\frac{dX}{dT}$ we get when setting $t = 0$ in equation 6 which makes sense because at $t = 0$, the Schwarzschild basis vectors and Kruskal-Szekeres basis vectors are aligned there, meaning that the X and T in $\left(\frac{dX}{dT}\right)_{rel}$ are pure spacelike and timelike basis vectors and thus the derivative gives a true velocity. This is in contrast to the general $\frac{dX}{dT}$ which is a derivative without a clear spacetime meaning since the X and T coordinates are mixtures of space and time everywhere else.

We will now get an expression for the Lorentz boosts of frames in motion relative to the rest frame. We can solve for $\frac{dt}{d\tau}$ in equation 1 as follows:

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{\left(1-\frac{r_s}{r}\right)\left(1-\left(\frac{dr}{dt}\right)^2\left(1-\frac{r_s}{r}\right)^{-2}\right)}} \tag{11}$$

We get the time dilation for the rest frame by setting $\frac{dr}{dt} = 0$. We can define the Lorentz boost of the moving frame by dividing $\frac{dt}{d\tau}$ in equation 11 by $\left(\frac{dt}{d\tau}\right)_0 = \frac{1}{\sqrt{\left(1-\frac{r_s}{r}\right)}}$, which is the time dilation in the rest frame at r:

$$\begin{aligned} \gamma = \frac{\frac{dt}{d\tau}}{\left(\frac{dt}{d\tau}\right)_0} &= \frac{1}{\sqrt{1-\left(\frac{dr}{dt}\right)^2\left(1-\frac{r_s}{r}\right)^{-2}}} \\ &= \frac{1}{\sqrt{1-\left(\frac{dX}{dT}\right)^2_{rel}}} \end{aligned} \tag{12}$$

The rapidity ω relative to the local rest observer in Kruskal-Szekeres coordinates is given by:

$$\omega = \tanh^{-1}\left(\left(\frac{dX}{dT}\right)_{rel}\right) \tag{13}$$

Finally, we can combine these to get the Lorentz factor as a function of rapidity:

$$\gamma = \frac{1}{\sqrt{1-\tanh^2(\omega)}} \tag{14}$$

We can begin to see that the Kruskal-Szekeres coordinates are similar to Minkowski coordinates for an inertial frame. For an inertial observer in freefall, the local speed of light is always 1, and in Kruskal-Szekeres coordinates, the speed of light is 1 everywhere. The difference here is, of course, that we have a curved spacetime such that the speed of light observed by rest frames will vary depending on the location relative to the black hole.

The Kruskal-Szekeres coordinate axes in Figure 1 represent the spacetime in the frame of the source of the metric. In this frame, rest frames accelerate away from the source, such that those lines are hyperbolas in the source frame. So $\left(\frac{dX}{dT}\right)_{rel}$ is the velocity of the falling frame as seen from the rest frame. But a falling frame will have an increasing velocity relative to the rest observers, and therefore, their reference frames will be Lorentz boosted relative to the rest frame per equation 12. Figure 2 shows three snapshots of boosted falling frames that started falling from infinity with the frame's r coordinate (the r the frame had fallen to when the boost is being depicted) labelled below each snapshot.

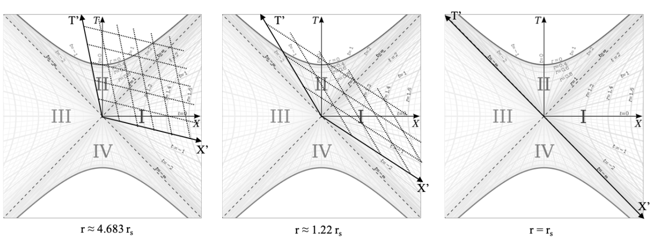

FIG. 2. Lorentz Boosted Frames During Freefall

In this figure, we can see that the basis vectors of the boosted frames are rotated relative to the rest frame (the X' and T' axes represent the boosted frames).

From this, we can also get a sense of length contraction in the falling frame. Firstly, we must note that the dX term of the metric really should be dR since it represents a change in the radial position of a particle. This means the Lorentz boost the falling frame experiences when it falls is radial in all directions, with the origin of the frame at the horizon (it is the radial $\partial_X = \partial_R$ basis vector that rotates in Figure 2). The planar surfaces perpendicular to a Cartesian coordinate axis in the Minkowski metric become spherical surfaces centered on the metric source in the Schwarzschild metric. So when we talk about length contraction in a radially falling frame, we are saying that circumferences around the source of the metric are length contracted in that frame and remain circular. This radial contraction is necessary for all inertial observers to see the spacetime as spherically symmetric. If the length contraction was dependant on the direction of the radial fall, then inertial observers would disagree on the spherical symmetry of the metric.

We can see the length contraction effect in Kruskal-Szekeres coordinates by looking at the X distance from the horizon to some radius r along a line of constant T for two different cases where the observers started falling from a different radius in each case such that when when we look at the frame of each observer at some time after they started falling, the Lorentz factor and rapidity have different values for each case.

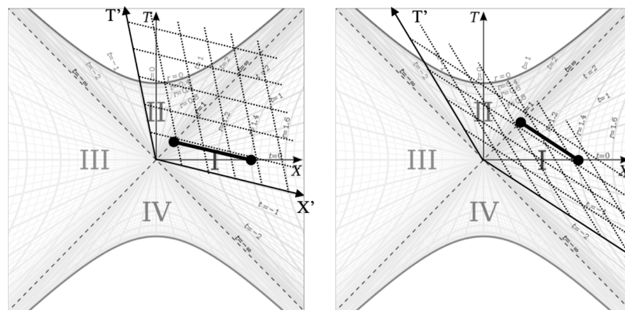

FIG. 3. Length Contraction in the Falling Frame

In the above figure, the left frame has a lower radial velocity and therefore a lower rapidity ($\omega = -0.25$ in this case) relative to the right frame, which has a rapidity of $\omega = -0.75$. We chose the same point in both frames for comparison to look at its distance from the event horizon in both frames. By comparing the dark lines in each picture, which represent the distance from the horizon to some r (the same r in both cases) measured in X' at constant T' in each frame, we clearly see that this distance is shorter in the frame on the right compared to the frame on the left.

We can calculate the actual ratio of the lengths in the two frames given the length contraction equation $X = \frac{X_0}{\gamma}$. The ratio of the lengths of the lines on the left (X_L) and right (X_R) sides of Figure 3 will be a ratio of Lorentz factors. We can see from the figure, that $\omega_L = -0.5$ and $\omega_R = -1.5$. Using equation 14, we can solve for the ratio of the lengths in the falling frames as:

$$\frac{X_L}{X_R} = \frac{\gamma_R}{\gamma_L} = \sqrt{\frac{1-\tanh^2(-0.25)}{1-\tanh^2(-0.75)}} \approx 1.26 \tag{15}$$

Note that we would get the same result regardless of which point on the r hyperbola we draw the line to, as long as we use the same point in all frames being compared and draw the lines to the horizon along a line of constant T in each frame.

As the rapidity increases further, we can see that this line will tend toward a null geodesic as the horizon is approached indicating that the spacetime contracts fully to the horizon in the falling frame, which is consistent with the argument that the falling frame becomes lightlike at the horizon.

The contraction of the r coordinate in these frames relative to the rest frame can be calculated by plugging the contracted X length into equation 2 and solving for r. By calculating the contracted r coordinate, we can calculate the contracted circumferences around the metric source in the falling frame and we can see from the above analysis that these circumferences will contract to zero when the frame reaches the horizon.

Given the length contraction observed, the observer should also see the volume of the source contract in the falling frame. Applying the length contraction equation to the volume of the source:

$$V' = \frac{V}{\gamma^3} = \frac{4\pi r_s^3}{3\gamma^3} \tag{16}$$

Where r_s is the Schwarzschild radius. The mass of the source is related to its Schwarzschild radius by $M = \frac{r_s c^2}{2G}$ If we assume the mass of the source is within the Schwarzschild radius, the density of the source in the falling frame is given by

$$\begin{aligned} \rho &= \frac{M}{V'} \\ &= \frac{3c^2}{8\pi G r_s^2}\gamma^3 \end{aligned} \tag{17}$$

So in the falling frame, γ goes to infinity as the horizon is approached such that the density there also goes to infinity. This shows us that due to the length contraction, there is no space beyond the event horizon in the falling frame, indicating that the event horizon is the end point of gravitational collapse. At the horizon, the falling frame will see the entire Universe radially length contracted to the horizon as well due to length contraction. This is reflected in the rest frame by the fact that all hyperbolas intersect with the horizon when $t = T = X = \infty$, which is the horizon itself. We see therefore that the Kruskal-Szekeres coordinates represent the spacetime in the frame of the source which is an infinitely dense point at $r = 1$. This point looks like a surface to a distant observer, but is length contracted to a point when the horizon is approached. In a followup work, it will be demonstrated that reaching the horizon in is not possible in the actual Universe.

If the horizon is the endpoint of gravitational collapse, then the question then remains, what is region II of Figure 1 describing. We know that the Schwarzschild solution describes all spherically symmetric vacua. In region I, the t coordinate is spacelike while the r coordinate is timelike. We see that at the the horizon, the spatial t coordinate density is infinite and the coordinate lines separate as one moves from the horizon to $r = 0$. It was also demonstrated that in the case of the external solution, the source of the metric is at $T = X = 0$. It stands to reason then that the source of the internal solution is also at that point. So the internal solution describes a spherically symmetric vacuum surrounded by a horizon which, from the perspective of an observer at some r between the horizon and $r = 0$, surrounds the vacuum infinitely far away in space and at some finite time in the past. And from the perspective of that observer, this horizon, which looks like a surrounding sphere, is a time where space is infinitely dense. A spacetime fitting this description would be any empty space in the Universe whose surrounding mass is spherically symmetric. Voids in the cosmic web would be an example of such a spacetime, and the horizon of the metric in this case would be the Big Bang, which is an event at some finite time in the past that surrounds all points in the Universe which has an infinite density. And an observer in the present Universe can never reach the Big Bang, no matter how far they travel through space, which is in alignment with the fact that the surface, from the perspective of a present observer, is infinitely far away from them in space. So we might think of the expanding Universe as baking bread where the air pockets that expand as the bread bakes give the bread a web-like structure over time, where the bread itself would be analogous to the cosmic filaments of matter in the Universe. A full cosmological analysis of the internal solution will be completed in a followup work which will also discuss all four regions of the Kruskal-Szekeres coordinate chart as well as the curvature singularity at $r = 0$.

III. CONCLUSIONS

An analysis of the Schwarzschild metric described with Kruskal-Szekeres coordinates has revealed the following:

- The Lorentz boost and rapidity for inertial, radially falling frames were derived and show that the falling frame will see the entire spacetime radially length contracted about the metric source as the frame approaches the event horizon.
- In the falling frame, the horizon becomes length contracted the closer the observer gets to the horizon, until it is infinitely contracted the horizon is approached such that the horizon must be the end point of gravitational collapse. The source of the metric is therefore at the horizon, not at $r = 0$.
- The consequences of these findings for the internal solution will be completed in a followup work.

[1] S. M. Carroll, Lecture notes on general relativity (1997), arXiv:9712019v1 [gr-qc].

[2] Figures 1, 2, and 3 are modifications of: 'kruskal diagram of schwarzschild chart' by dr greg. licensed under cc by-sa 3.0 via wikimedia commons, http://commons.wikimedia.org/wiki/File:Kruskal_diagram_of_Schwarzschild_chart.svg#/media/File:Kruskal_diagram_of_Schwarzschild_chart.svg (Accessed in 2017).