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Light-like Geodesics at the Event Horizon

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Abstract: This paper examines radially infalling worldlines in both Schwarzschild and Kruskal-Szekeres coordinates. A new Schwarzschild coordinate chart is presented which emphasizes the curvature of both the time and space coordinates and makes clear not only the mechanism by which the inertial observer is accelerated relative to the rest observers, but also clearly shows the light-like nature of the inertial worldline at the event horizon. The light-like nature of the falling frame is then supported by examining the falling worldlines in Kruskal-Szekeres coordinates. The effect of length contraction in the falling frame is also analyzed in both Schwarzschild and Kruskal-Szekeres coordinates.

Keywords: general relativity; Schwarzschild; Black Holes

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1. Introduction

The Schwarzschild metric in Schwarzschild coordinates has a well-known coordinate singularity at the event horizon. This singularity is removed from the metric using coordinate transformations, with the most popular transformation being the Kruskal-Szekeres coordinates, which are regular at the horizon. However, the meaning of the the Kruskal-Szekeres coordinates in terms of spacelike and timelike basis vectors is not clear. For much of the spacetime, the Kruskal-Szekeres coordinates are mixtures of space and time such that the meaning of the slope of a worldline in these coordinates is not always clear. Nonetheless, these coordinates have been used to argue that the event horizon can be crossed by falling observers since there is no coordinate singularity in the Kruskal-Szekeres system. This has led to the acceptance of the existence of Black Holes into which objects can fall and can never escape back out to the external Universe.

In the reference text 'Gravitation' [1], the authors depict a falling worldline crossing the event horizon in Kruskal-Szekeres coordinates in Figure 32.1(b) of the text, though there is no accompanying calculation of the worldline in Kruskal-Szekeres coordinates. The only such calculation found in the literature is in [2], where the author claims to demonstrate that all worldlines are light-like at the horizon in Kruskal-Szekeres coordinates regardless of their state of motion. However, as the author notes in that work, the proof actually demonstrates that the Kruskal-Szekeres derivative of the falling worldline is undefined at $T = X > 0$ in Kruskal-Szekeres coordinates. The author nonetheless concludes that in order for the extended spacetime to be consistent, the geodesic must in fact be light-like at the horizon.

The existence of Black Holes that can subsequently evaporate create an information paradox where information falling into a Black Hole is lost once the Black Hole evaporates due to Hawking radiation. Therefore, the possibility of objects falling into a Black Hole creates a physical paradox which has yet to be conclusively resolved. Therefore, this paper provides proof of the following hypothesis:

- *The worldline of radially falling frames become light-like at the event horizon and the horizon itself becomes length contracted to a point in the falling frame as the frame approaches the horizon.*

Therefore, in this work, the analysis done in [2] is extended, providing an unambiguous proof that the falling worldline is light-like at the event horizon without having an undefined Kruskal-Szekeres derivative at the horizon.

2. Modification of the Schwarzschild Coordinate Chart

The Schwarzschild metric is the simplest non-trivial solution to Einstein's field equations. It is the metric that describes every spherically symmetric vacuum spacetime. The the external form of the metric can be expressed as:

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right)dt^2 - \frac{1}{1 - \frac{r_s}{r}}dr^2 - r^2 d\Omega^2 \tag{1}$$

Equation 1 is the external metric (where $r > r_s$) with t being the timelike coordinate and r being the spacelike coordinate. The Schwarzschild radius of the metric is given by $r_s = 2GM$ in units with $c = 1$ and is commonly known as the Event Horizon. The external metric is the metric for an eternally spherically-symmetric vacuum centered in space.

The metric has a well known coordinate singularity at $r = r_s$ where the g_{rr} component goes to infinity and the g_{tt} component goes to zero when $r = r_s$. We can remove the problem with the g_{rr} component by making the following trivial coordinate transformation:

$$ds = \sqrt{\frac{1}{1 - \frac{r_s}{r}}}dr \tag{2}$$

This puts the metric in the following form (ignoring the angular term):

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right)dt^2 - ds^2 \tag{3}$$

Typically, the Schwarzschild coordinate chart is displayed with the r and t coordinates lines laid out as a flat grid. However, let us try to visualize the geometry from the point of view of observers at rest and observers beginning to fall from rest at $t = 0$. Figure 1 shows a modified coordinate chart that better captures the nature of the Schwarzschild geometry.
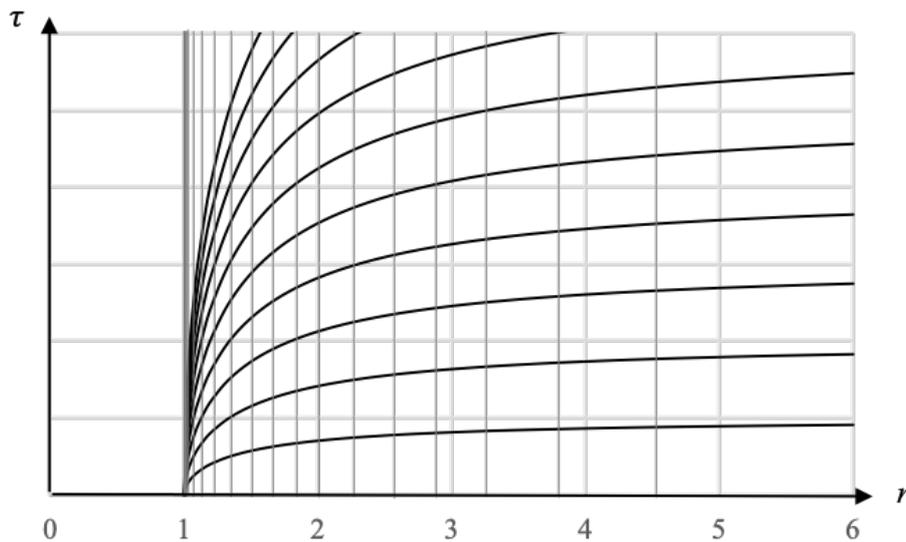
Figure 1. Modified Schwarzschild Coordinate Chart

On this chart, the t coordinate lines are the curved lines and come from solving the metric for rest observers ($dr = 0$) and integrating to get the following equation:

$$\tau = t\sqrt{1 - \frac{r_s}{r}} \tag{4}$$

Where each line corresponds to a fixed value of t, with $t = 0$ being the flat line on the r axis. Vertical worldlines on this coordinate chart are the worldlines of observers at rest and their height is the proper time elapsed. We can see that for a given Δt, less proper time passes for rest observers the closer they are to the horizon. Also plotted on this chart are the s coordinate lines from equation 2. We see that the density of these coordinate lines goes to infinity as the horizon is approached, reflecting the fact that $\frac{dr}{ds} = 0$ at the event horizon.

Now let us examine the inertial worldlines of falling observers falling from rest at $t = 0$ at different radii. The worldlines of observers falling from two different radii are shown in Figure 2.
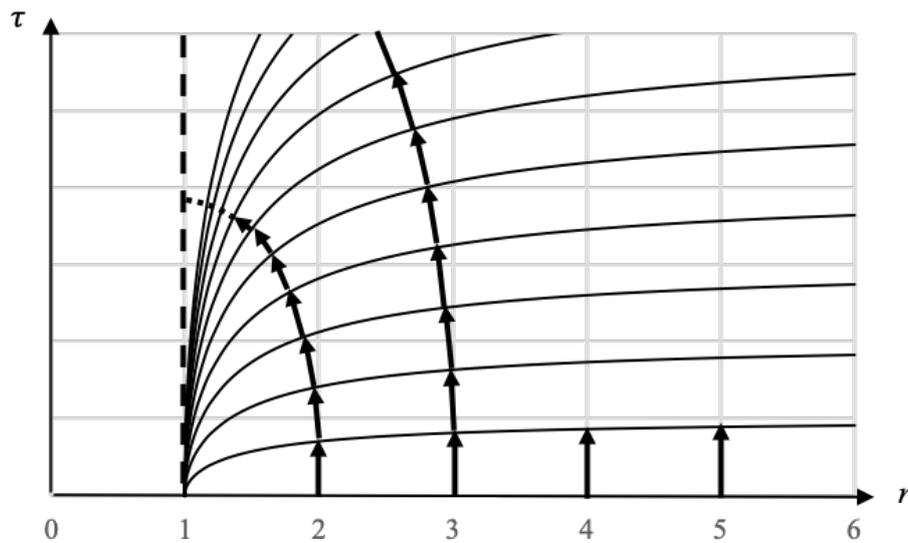

Figure 2. Falling Worldlines on the Modified Schwarzschild Coordinate Chart

We can think of the t coordinate lines as analogous to level curves on a contour chart where $t = 0$ is the highest level and $t = \infty$ is the lowest level. The falling observer takes the shortest path from the highest level to the lowest level and so their worldline is perpendicular to the t coordinate line at every point along the worldline. Thus, the worldlines of all falling observers start vertical at $t = 0$ and curve to remain perpendicular to the t coordinate lines at each point until they end horizontal at $t = \infty, r = r_s$.

We can see from Figure 2 the first evidence that the falling worldlines are light-like at the horizon since $\frac{d\tau}{dt}$ for the worldlines are zero there. But we can make the claims of Figure 2 more concrete by re-expressing the metric in equation 3 as follows:

$$\left(\frac{d\tau}{dt}\right)^2 + \left(\frac{ds}{dt}\right)^2 = c^2 \tag{5}$$

Where $c = \sqrt{1 - \frac{r_s}{r}}$. We can understand this expression geometrically with Figure 3 below:
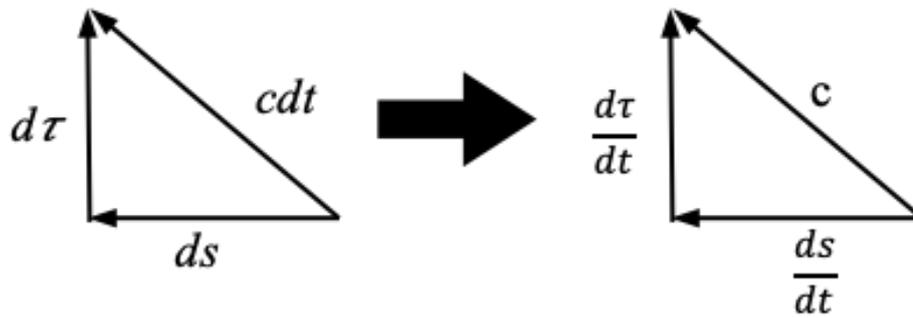
Figure 3. Components of ∂_t Basis Vector During a Fall

The hypotenuse of the triangle is the tangent of the falling worldline which is oriented in the ∂_t direction at every point along the worldline. At $t = 0$, the falling observer at r_0 is at rest, so $\frac{ds}{dt} = 0$ and therefore $\frac{d\tau}{dt}$ is equal to the speed of light at r_0. Due to the curvature of the t coordinate lines, the cdt vector rotates as the fall proceeds, giving it a growing $\frac{ds}{dt}$ component and reducing the magnitude of the $\frac{d\tau}{dt}$ component. Thus Figures 2 and 3 clearly illustrate the mechanism by which the falling observer is made to accelerate relative to the rest observers. The fall is caused by the orientation of the ∂_t basis vector changing relative to the ∂_r basis vector at different locations, resulting in the inertial acceleration of the falling observer.

Reference [3] tells us that $\frac{dr}{dt}$ for an observer falling from rest at $r = r_0$ is given by:

$$\frac{dr}{dt} = -\left(1 - \frac{r_s}{r}\right)\sqrt{\frac{\frac{r_s}{r} - \frac{r_s}{r_0}}{1 - \frac{r_s}{r_0}}} \tag{6}$$

Combining equations 6 and 2, we get the needed derivative:

$$\frac{ds}{dt} = \frac{dr}{dt}\frac{ds}{dr} = -\sqrt{1 - \frac{r_s}{r}}\sqrt{\frac{\frac{r_s}{r} - \frac{r_s}{r_0}}{1 - \frac{r_s}{r_0}}} \tag{7}$$

We can see that in equation 7, the expression $\sqrt{\frac{\frac{r_s}{r} - \frac{r_s}{r_0}}{1 - \frac{r_s}{r_0}}} = \frac{v}{c}$ since $\frac{ds}{dt} = v$ and $\sqrt{1 - \frac{r_s}{r}} = c$. We also see that $\frac{v}{c} = 1$ at the horizon, indicating that the worldline is light-like at the horizon. This is evident in Figure 2, where at the horizon, the worldline is horizontal meaning that the particle moves through spacetime without any proper time elapsing at that point.

Equation 7 is 0 at r_0, its magnitude reaches a maximum at some $r_s < r < r_0$, and then goes to 0 again at r_s. We can understand this with Figures 2 and 3. We note that the hypotenuse in Figure 3 both rotates and shrinks as the falling observer moves toward the horizon.

The rotation of the vector corresponds to the $\sqrt{\frac{\frac{r_s}{r} - \frac{r_s}{r_0}}{1 - \frac{r_s}{r_0}}}$ portion of equation 7. When the vector is vertical, the observer is at rest and $\frac{v}{c} = 0$. At the horizon, where the vector is horizontal the observer becomes light-like because $\frac{v}{c} = 1$ there.

The length of the vector corresponds to $\sqrt{1 - \frac{r_s}{r}}$, which is the speed of light c at r. So we see that this length is a maximum at the start of the fall and it goes to zero at the horizon. This can be seen in Figure 2 where the vectors between subsequent lines of t and s along the worldline become smaller and smaller as the horizon is approached as a result of the compression of the coordinates there. Thus the magnitude of $\frac{ds}{dt}$ increases up to a point because the contribution from the rotation of the hypotenuse is greater than the contribution from the shrinking length of the hypotenuse until both

effects balance, where the magnitude of $\frac{ds}{dt}$ is maximum. After that point, the effect of the shrinking length of the hypotenuse dominates and $\frac{ds}{dt}$ then decreases to zero at the horizon.

For completeness, we can also show the Schwarzschild coordinate chart in yet another form by choosing a falling worldline from Figure 2 and forcing it to be a straight vertical line on the coordinate chart. This is illustrated in Figure 4.
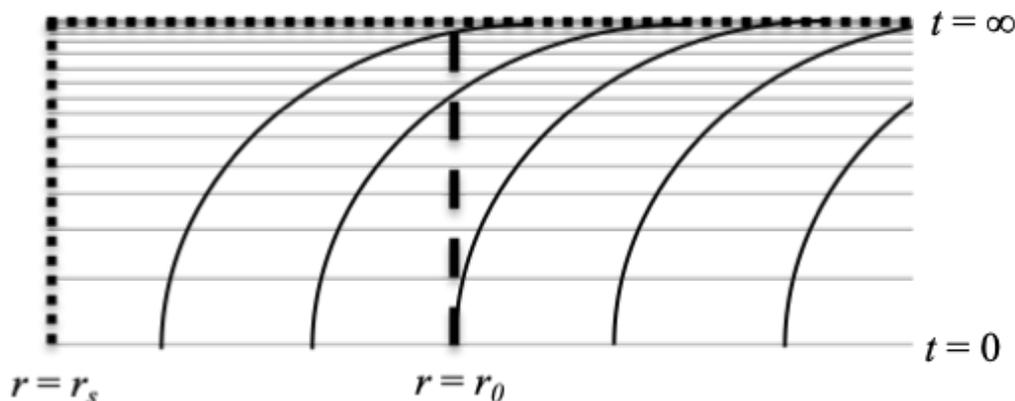

Figure 4. Schwarzschild Coordinates in Inertial Frame

Figure 4 shows the Schwarzschild geometry in the frame of a freefalling observer that starts falling from rest at $r = r_0$. The horizontal lines are the t coordinates and we see their spacing decrease as t goes to infinity. The curved lines are the r coordinate lines. So in the frame of the falling observer, it is the spatial coordinates that move relative to the observer, which is why they curve in this chart. The magnitude of the slope of the r coordinate lines at t in Figure 4 is equal to the magnitude of the slope of the worldline we are straightening from Figure 2 at the same t. The event horizon on this chart is represented by the dotted lines on the left side and top of the chart.

Note that falling worldlines that did not start falling at $r = r_0$ are not straight on this diagram. All frefalling worldlines will be vertical at $t = 0$ (because they start falling from rest there) and at $t = \infty$ (because all worldlines, including light-like worldlines on this chart will be vertical at the event horizon). But for $0 < t < \infty$, they will be curved because if we look at Figure 2, the slopes of the 2 displayed worldlines at a given value of t are different and so their slopes on this chart will also differ.

If we looked at the chart in Figure 4 for an observer falling from rest at some $r > r_0$, the curvature of the r coordinate lines would be less (they would still curve to be horizontal at the horizon, but they would be more vertical near $t = 0$) as shown in Figure 5.
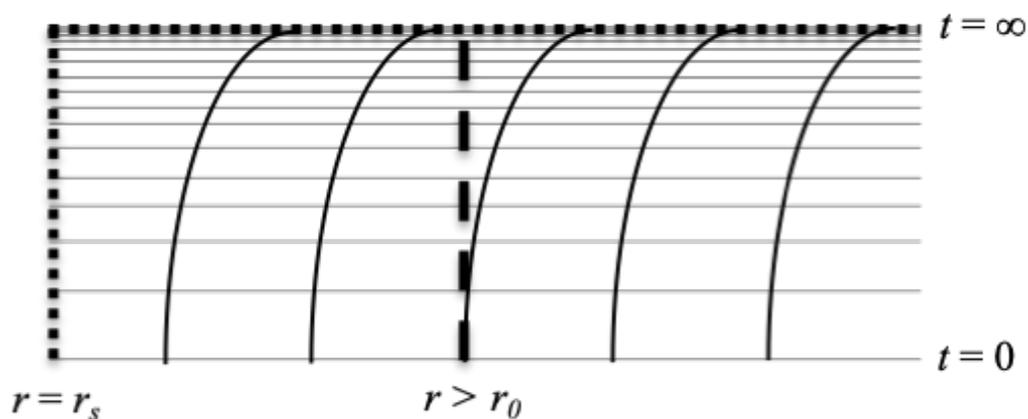

Figure 5. Schwarzschild Coordinates in Inertial Frame Part 2

As the radius of the start of the fall increases, the r coordinate lines get straighter until at $r = \infty$, the lines become vertical.

The frefalling observer should also see a length contraction effect since they have a velocity in Schwarzschild coordinates. In the rest frame, the Schwarzschild radius is r_s. In the falling frame, this radius will be length contracted as:

$$
\begin{aligned}
r'_s &= r_s \sqrt{1 - \left(\frac{1}{c} \frac{ds}{dt} \right)^2} \\
&= r_s \sqrt{\frac{1 - \frac{r_s}{r}}{1 - \frac{r_s}{r_0}}}
\end{aligned}
\tag{8}
$$

Which goes to zero at the event horizon. This implies that the event horizon itself becomes length contracted to 0 in the falling frame as the horizon is approached. The planar surfaces perpendicular to a Cartesian coordinate axis in the Minkowski metric become spherical surfaces centered on the metric source in the Schwarzschild metric. So when we talk about length contraction in a radially falling frame, we are saying that circumferences around the source of the metric are length contracted in that frame and remain circular. This radial contraction is necessary for all inertial observers to see the spacetime as spherically symmetric. If the length contraction was dependant on the direction of the radial fall, then inertial observers would disagree on the spherical symmetry of the metric.

So it has been demonstrated using the Schwarzschild coordinates that freefalling worldlines become light-like at the horizon. However, the speed of light in these coordinates is 0 at the horizon which is not ideal. We need to investigate the falling worldline in coordinates where the speed of light is non-zero at the horizon in order to make the claim perfectly clear. Looking at Figures 4 and 5, we see that the horizon is ill-defined as it is both the vertical and horizontal dotted lines. We can correct this by keeping the t coordinate lines straight while making $t = 0$ and $t = \infty$ coincident at $r = r_s$. By doing this, we create the top half of region I of the Kruskal-Szekeres coordinate chart, which is the chart we will analyze next.

## 3. The Falling Frame of the External Metric in Kruskal-Szekeres Coordinates

The Kruskal-Szekeres coordinates are the maximally extended coordinates for the Schwarzschild metric. The coordinate definitions and metric in Kruskal-Szekeres coordinates are given below (derivation of the coordinate definitions and metric can be found in reference [4] where $v = T$ and $u = X$).

$$
\begin{aligned}
T &= \sqrt{\left(\frac{r}{r_s} - 1 \right) e^{\frac{r}{r_s}}} \sinh \left(\frac{t}{2r_s} \right) \\
X &= \sqrt{\left(\frac{r}{r_s} - 1 \right) e^{\frac{r}{r_s}}} \cosh \left(\frac{t}{2r_s} \right)
\end{aligned}
\tag{9}
$$

With the full metric in Kruskal-Szekeres coordinates given by:

$$
d\tau^2 = \frac{4r_s^3}{r} e^{-\frac{r}{r_s}} \left(dT^2 - dX^2 \right) - r^2 d\Omega^2
\tag{10}
$$

Finally, we plot the metric on the Kruskal-Szekeres coordinate chart [5] in Figure 6:
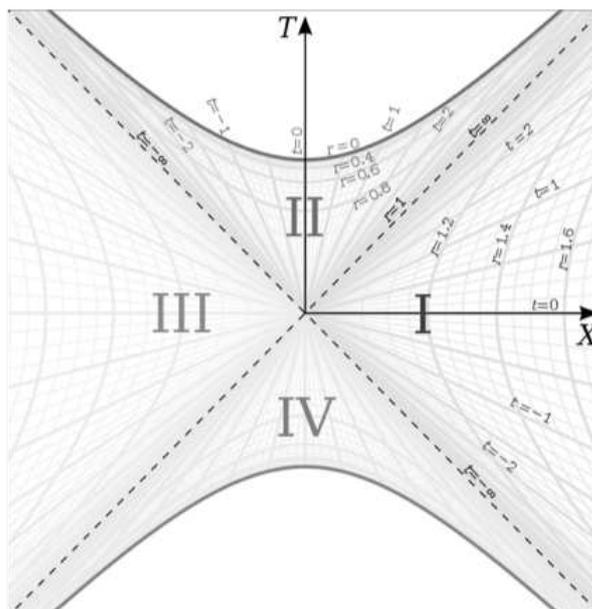
Figure 6. Kruskal-Szekeres Coordinate Chart

In this paper, we will be focusing on region I in this chart, which is the spherically symmetric spacetime around a spherically symmetric source in space.

Light-like geodesics are 45 degree lines on this diagram. So if we compare the top half of region I of this diagram to Figures 4 and 5, we can see that the falling worldlines should intersect the event horizon at the $T = X$ line at a 45 degree angle, meaning that it is indeed light-like there.

We can derive this mathematically as follows. Let us first take the differentials of T and X in equations 9:

$$dX = \frac{\partial X}{\partial r}dr + \frac{\partial X}{\partial t}dt$$
$$dT = \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial t}dt \tag{11}$$

Calculating the partial derivatives, rearranging and defining $R \equiv \frac{re^{\frac{r}{r_s}}}{2r_s^2\sqrt{(\frac{r}{r_s}-1)e^{\frac{r}{r_s}}}}$ we get:

$$\frac{dX}{dt} = R\left[\frac{dr}{dt}\cosh\left(\frac{t}{2r_s}\right) + \left(1 - \frac{r_s}{r}\right)\sinh\left(\frac{t}{2r_s}\right)\right]$$
$$\frac{dT}{dt} = R\left[\frac{dr}{dt}\sinh\left(\frac{t}{2r_s}\right) + \left(1 - \frac{r_s}{r}\right)\cosh\left(\frac{t}{2r_s}\right)\right] \tag{12}$$

Next, we need to calculate $\frac{dX}{dT}$ from equations 12 by factoring out $\left(1 - \frac{r_s}{r}\right)\cosh\left(\frac{t}{2r_s}\right)$ from each equation and dividing:

$$\frac{dX}{dT} = \frac{dX}{dt}\frac{dt}{dT} = \frac{\frac{dr}{dt}\left(1 - \frac{r_s}{r}\right)^{-1} + \tanh\left(\frac{t}{2r_s}\right)}{\frac{dr}{dt}\left(1 - \frac{r_s}{r}\right)^{-1}\tanh\left(\frac{t}{2r_s}\right) + 1} \tag{13}$$

This equation is the same equation derived in [2] but put in a slightly different form. Next, we make the following definitions:

$$\left(\frac{dX}{dT}\right)_0 \equiv \tanh\left(\frac{t}{2r_s}\right) \tag{14}$$

This is the derivative of the rest frame at t since plugging $\frac{dr}{dt} = 0$ into equation 13, we get $\frac{dX}{dT} = \tanh\left(\frac{t}{2r_s}\right)$. Since we know the Schwarzschild metric is independent of t, this derivative must be a

non-physical artifact of the Kruskal-Szekeres coordinates at fixed r and is not related to any actual change in motion through space and time.

And we define the relative velocity of the frame in motion relative to the rest frame as:

$$\left(\frac{dX}{dT}\right)_{rel} \equiv \frac{dr}{dt}\left(1 - \frac{r_s}{r}\right)^{-1} = \frac{v}{c} \tag{15}$$

This is the relative velocity in Kruskal-Szekeres coordinates between the frame in motion and the rest frame at r. This derivative is 0 for the rest frame since $\frac{dr}{dt} = 0$ in that frame. If we combine equations 15 and 6 we get:

$$\left(\frac{dX}{dT}\right)_{rel} = \frac{v}{c} = -\sqrt{\frac{\frac{r_s}{r} - \frac{r_s}{r_0}}{1 - \frac{r_s}{r_0}}} \tag{16}$$

Which is well behaved and equal to -1 when $r = r_s$. Plugging these definitions into equation 13, we get:

$$\frac{dX}{dT} = \frac{\left(\frac{dX}{dT}\right)_{rel} + \left(\frac{dX}{dT}\right)_0}{\left(\frac{dX}{dT}\right)_{rel}\left(\frac{dX}{dT}\right)_0 + 1} \tag{17}$$

We recognize that equation 17 is the relativistic velocity addition formula giving us the total velocity as the relativistic sum of the rest frame velocity and the relative velocity between the moving frame and the rest frame. We can solve for $\left(\frac{dX}{dT}\right)_{rel}$ to get an expression for the relative velocity between a frame in motion and the rest frame in Kruskal-Szekeres coordinates:

$$\left(\frac{dX}{dT}\right)_{rel} = -\frac{\left(\frac{dX}{dT}\right)_0 - \frac{dX}{dT}}{1 - \left(\frac{dX}{dT}\right)_0 \frac{dX}{dT}} \tag{18}$$

Assuming that $\left(\frac{dX}{dT}\right)_0$ ranges from -1 to 1 and $1 > \frac{dX}{dT} > -1$, we see that the relative velocity approaches 1 or -1 for all $\frac{dX}{dT}$ as the horizon is approached since the horizon is at $t = \pm\infty$, such that $\left(\frac{dX}{dT}\right)_0 = 1$ there. Equation 18 is also constant along a given hyperbola (i.e it is independent of t) since it represents the relative velocity between the moving and rest frames.

It is notable that equation 17 is undefined when $t = \infty$ because since $\left(\frac{dX}{dT}\right)_{rel} = -1$ there and $\left(\frac{dX}{dT}\right)_0 = 1$ there, we get:

$$\left(\frac{dX}{dT}\right)_{t=\infty} = \frac{-1 + 1}{-1 + 1} = \frac{0}{0} \tag{19}$$

This is also what was demonstrated in [2]. Therefore, under these conditions, $\frac{d\tau}{dT}$ and $\frac{d\tau}{dX}$ are undefined at the horizon when T and X are greater than 0. However, it is notable that if $t = 0$ and $r = r_s$, then $\frac{dX}{dT} = \left(\frac{dX}{dT}\right)_{rel} = -1$, indicating that under that condition, the worldline is not undefined, but clearly light-like at the horizon.

To understand the undefined nature of the worldline at the horizon in these coordinates, let us examine the rest observer. In these coordinates, rest observers accelerate over time as evidenced by the fact that their worldlines are hyperbolas in these coordinates. If we consider the X position of a rest observer at some $t > 0$, we see that its X-coordinate is given by $X = X_0 \cosh\left(\frac{t}{2r_s}\right)$ where X_0 is the X coordinate of the rest observer at r when $t = 0$ $\left(X_0 = \sqrt{\left(\frac{r}{r_s} - 1\right)e^{\frac{r}{r_s}}}\right)$. At $t > 0$, the rest observer is moving with some velocity relative to itself at $t = 0$ on this coordinate chart. Therefore, we should expect that the X coordinate will be length contracted as t increases in the rest frame in

Kruskal-Szekeres coordinates. The length contracted value of X in the rest frame at r and $t > 0$ is given by:

$$
\begin{aligned}
X' &= X\sqrt{1 - \left(\frac{dX}{dT}\right)_0^2} \\
&= X_0 \cosh\left(\frac{t}{2r_s}\right) \sqrt{1 - \tanh^2\left(\frac{t}{2r_s}\right)} \\
&= X_0 \cosh\left(\frac{t}{2r_s}\right) \frac{1}{\cosh\left(\frac{t}{2r_s}\right)} \\
&= X_0
\end{aligned}
\tag{20}
$$

We use $\left(\frac{dX}{dT}\right)_0$ in the equation because we are looking at rest observers, so according to equation 17, $\frac{dX}{dT} = \left(\frac{dX}{dT}\right)_0$ in this case.

So even though the X coordinate grows for an observer at rest in the Kruskal-Szekeres coordinate chart, when we shift to the frame of the rest observer by taking into account the length contraction of the Kruskal-Szekeres coordinates in that frame, we see that we end up back at $t = 0$. This is a very important finding. It means that region II in the coordinate chart, which only exists for $T > 0$ cannot actually be reached as a result of the length contraction.

If the rest frame has the length contraction, then the falling frame, which is moving relative to the rest frame will have an even greater contraction. So the rest frame at r sees their distance from $X = 0$ in their frame as X_0. This distance in the falling frame will become:

$$
\begin{aligned}
X &= X_0 \sqrt{1 - \left(\frac{dX}{dT}\right)_{rel}^2} \\
&= X_0 \sqrt{\frac{1 - \frac{r_s}{r}}{1 - \frac{r_s}{r_0}}}
\end{aligned}
\tag{21}
$$

We can use these facts to change how we visualize the worldline of an observer falling toward the horizon. Rather than drawing the line from $t = 0$ at some r to $t = \infty$ at the horizon, we can continuously hyperbolically rotate the space as the observer falls such that the 'present' state of the falling frame is always at $t = 0$. An example of this is given in Figure 7.
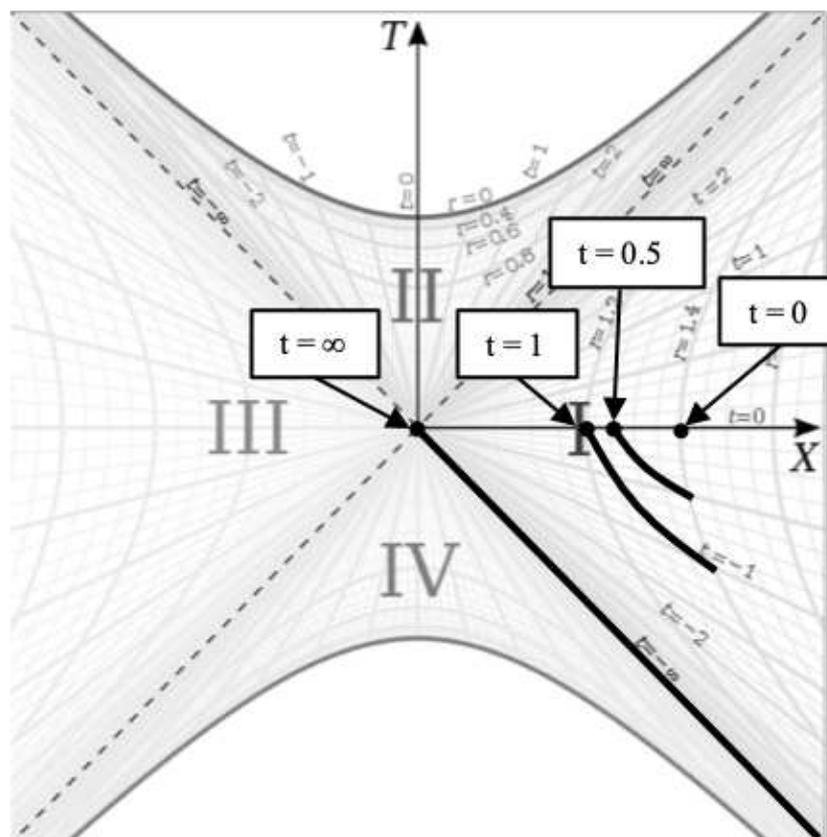
Figure 7. Worldline of Falling Frame on the Kruskal-Szekeres Coordinate Chart with Continuous
Hyperbolic Rotation

In Figure 7, the observer begins falling at $r = 1.4r_s$. This is represented by the rightmost point on
the X axis in the diagram. After $\Delta t = 0.5$, the observer has fallen to a lower radius represented by the
point to the left of the rightmost point. So rather than having the worldline grow up from $t = 0$ as time
passes, we hyperbolically rotated the worldline points down as time passes to keep the present point
of the worldline on the X axis (accounting for the length contraction of the rest frame). This is a valid
way of visualizing the worldline as a result of the time symmetry of the metric (the geometry has a
Killing vector in the ∂_t direction, meaning that we can hyperbolically rotate the space as much as we
like without changing any of the physics).

When using this construction, we see that the falling worldline reaches the $T = X = 0$ point
on the diagram. And if the worldline does indeed become light-like at the horizon, this means the
worldline remains on the $T = -X$ line since that is the light-like geodesic representing the horizon.

Now let's calculate the situation described in Figure 7 where we calculate the worldline falling
along the X axis as the past worldline is hyperbolically rotated down. For this construction, we set
$t = 0$ in the equations since the derivative is always taken on the X axis. For this calculation, we put
the metric in the following form (we will be examining radial infall so $d\Omega = 0$):

$$d\tau^2 = \frac{4r_s^3}{re^{\frac{r}{r_s}}} \left(\left(\frac{dT}{dX} \right)^2 - 1 \right) dX^2 \tag{22}$$

Since we are keeping $t = 0$, we can use the inverse of equation 16 for $\frac{dT}{dX}$ in the equation. We can solve
for r in terms of X by setting $t = 0$ for the X equation in equation 9 and solving for r:

$$r = r_s \left(W \left(\frac{X^2}{e} \right) + 1 \right) \tag{23}$$

Where W is the Product Log function. Substituting equations 23 and 16 into equation 22 gives us an equation for the falling worldline along the X axis:

$$d\tau^2 = \frac{4r_s^2 e^{-\left(W\left(\frac{X^2}{e}\right)+1\right)}}{W\left(\frac{X^2}{e}\right)+1} \left(\frac{1-\frac{r_s}{r_0}}{\frac{1}{W\left(\frac{X^2}{e}\right)+1}-\frac{r_s}{r_0}}-1\right)dX^2 \tag{24}$$

Which goes to 0 as X goes to 0, meaning, once again, the worldline becomes light-like at the horizon. But we can now show that this is exactly equivalent to falling in Schwarzschild coordinates by first using equation 12 to solve for dX when $t = 0$:

$$dX = \frac{re^{\frac{r}{r_s}}}{2r_s^2\sqrt{(\frac{r}{r_s}-1)e^{\frac{r}{r_s}}}}dr$$

$$= \sqrt{\frac{re^{\frac{r}{r_s}}}{4r_s^3}}\frac{1}{\sqrt{1-\frac{r_s}{r}}}dr \tag{25}$$

Substituting equations 25 and the inverse of 15 into equation 22 gives:

$$d\tau^2 = \left(\left(\left(1-\frac{r_s}{r}\right)\frac{dt}{dr}\right)^2-1\right)\frac{1}{1-\frac{r_s}{r}}dr^2$$

$$= \left(1-\frac{r_s}{r}\right)dt^2 - \frac{1}{1-\frac{r_s}{r}}dr^2 \tag{26}$$

And we can see that Equation 26 is in fact the Schwarzschild metric in Schwarzschild coordinates.

Therefore it has been proven that the worldline construction shown in Figure 7 is equivalent to falling in Schwarzschild coordinates and it has been demonstrated that the worldline in that construction is light-like at the horizon. When we couple this finding with the fact that the Kruskal-Szekeres derivative is undefined at the horizon for any construction in which the worldline reaches the horizon at $T = X > 0$, we can conclude that the event horizon is length contracted to a point in a falling frame approaching the horizon as a result of the fact that the worldline becomes light-like there.

Furthermore, we see from equation 25 that $\frac{dr}{dX}$ is zero at $r = r_s$. Therefore, the falling frame remains at the horizon along the $T = -X$ line in the Kruskal-Szekeres coordinate chart when it reaches the horizon.

4. Conclusion

Using a modified version of the Schwarzschild coordinate chart which displays the Schwarzschild geometry in a more accurate way than the typical rectangular coordinate layout, it has been demonstrated that infalling geodesics become light-like at the event horizon. The same fact was demonstrated once more by analyzing the infalling worldline in Kruskal-Szekeres coordinates. Coordinate transformations cannot turn a light-like worldline into a timelike or spacelike worldline, and therefore it must be true that the infalling worldline must be light-like in all coordinate systems.

Furthermore, it has been demonstrated in both Schwarzschild and Kruskal-Szekeres coordinates that the infalling observer will see the event horizon length contracted as they approach the event horizon using elementary relativity theory. This contraction suggests that the event horizon may be the endpoint of gravitational collapse, though that has not been proven in the current work.

Detailed analysis of the internal metric and the three other regions of the Kruskal-Szekeres coordinate chart will be the topic of a followup work. It will be shown that the internal metric describes a spatially homogeneous spherically symmetric space that expands at an initially decelerated and then

accelerated rate over time. From the perspective of an observer in this space, the source of the metric is an infinite mass located infinitely far away in space at a finite time in the past.

Data Availability Statement: All data generated or analysed during this study are included in this published article [and its supplementary information files.

Conflicts of Interest: There are no competing interests.

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5. Figures 4, 5, 6, and 7 are modifications of: 'Kruskal diagram of Schwarzschild chart' by Dr Greg. Licensed under CC BY-SA 3.0 via Wikimedia Commons. http://commons.wikimedia.org/wiki/File:Kruskal_diagram_of_Schwarzschild_chart.svg#/media/File:Kruskal_diagram_of_Schwarzschild_chart.svg, Accessed in 2017.