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4-Component Spinors for SL(4,C) and Four Types of Transformations

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Abstract: We define a spinor-Minkowski metric for SL(4,C). It is not a trivial generalization of the SL(2,C) metric and it involves the Minkowski metric. We define 4x4 version of the Pauli matrices and their 4-component generalized eigenvectors. The generalized eigenvectors can be regarded as 4-component spinors and they can be grouped into four categories. Each category transforms in its own way. The outer products of pairwise combinations of 4-component spinors can be associated with 4-vectors.

Keywords: Lie Algebra; Particle Physics; quantum mechanics

0.1 Introduction

Let L_A be an element of $SL(2,C)$. In an exponential form with parameters θ and η:

$$L_A = exp(-\frac{i}{2}(\vec{\theta}\cdot\vec{\sigma} + i\vec{\eta}\cdot\vec{\sigma})) \tag{1}$$

$\vec{\sigma}$ is the Pauli vector with $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, $\sigma_3 = \sigma_z$. The subscript $_A$ is introduced in order to distinguish the other forms of L that will be introduced subsequently.

We rewrite L_A and its complex conjugate in the following compact forms:

$$L_A = exp(-\frac{i}{2}\vec{\pi}_A\cdot\vec{\sigma}), \quad L_A^* = exp(\frac{i}{2}\vec{\pi}_A^*\cdot\vec{\sigma}^*) \tag{2}$$

$(\pi_A)_i = \theta_i + i\eta_i$ and * denotes complex conjugation. L_A corresponds to the Lorentz transformation with θ_i and η_i being the rotation and boost parameters, respectively.

It is well known that the complex version of the 4×4 Lorentz transformation matrix can be written as a matrix direct product of L_A and L_A^*:

$$\lambda = L_A \otimes L_A^* \tag{3}$$

In order to obtain the familiar real matrix form of the Lorentz transformation it is enough to change the basis:

$$\Lambda = A(L_A \otimes L_A^*)A^{-1} \tag{4}$$

where

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & i & -i & 0\\ 1 & 0 & 0 & -1\end{pmatrix}, \quad A^{-1} = A^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & -i & 0\\ 0 & 1 & i & 0\\ 1 & 0 & 0 & -1\end{pmatrix} \tag{5}$$

Now, it is straightforward to show that $SO(3,1)$ can be written as a commutative product of $SL(4,C)$ and $SL(4,C)^*$ by simply rewriting Eq.(4) in a factorized form:

$$\Lambda = [A(L_A \otimes I)A^{-1}][A(I \otimes L_A^*)A^{-1}] = Z_A Z_A^* = Z_A^* Z_A, \tag{6}$$

$$Z_A = A(L_A \otimes I)A^{-1}, \quad Z_A^* = A(I \otimes L_A^*)A^{-1}. \tag{7}$$

Z_A and Z_A^* are the 4×4 versions of L_A and L_A^* matrices. They can be expressed in terms of Σ_i matrices:

$$Z_A = exp(-\frac{i}{2}\vec{\pi}_A \cdot \vec{\Sigma}), \quad Z_A^* = exp(\frac{i}{2}\vec{\pi}_A^* \cdot \vec{\Sigma}^*). \tag{8}$$

$\vec{\Sigma} = (\Sigma_1, \Sigma_2, \Sigma_3)$ and Σ_i are 4×4 versions of Pauli matrices:

$$\Sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \Sigma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \Sigma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{9}$$

These are traceless Hermitian matrices and they satisfy the same commutation relations as σ_i matrices

$$\left[\frac{1}{2}\Sigma_i, \frac{1}{2}\Sigma_j\right] = \frac{i}{2}\epsilon^{ijk}\Sigma_k. \tag{10}$$

By definition, $\Sigma_\mu = A(\sigma_\mu \otimes I)A^{-1}$, $(\mu = 0,1,2,3)$, Σ_0 is the 4×4 identity. Σ_μ basis do not form a complete set for 4×4 matrices, but the set of $\Sigma_\mu \Sigma_\nu^*$ does.

From the Eq.(7), Z_A can be found in terms of the elements of L_A:

$$Z_A = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_0 & -i\alpha_3 & i\alpha_2 \\ \alpha_2 & i\alpha_3 & \alpha_0 & -i\alpha_1 \\ \alpha_3 & -i\alpha_2 & i\alpha_1 & \alpha_0 \end{pmatrix} \tag{11}$$

where $\alpha_0 = \frac{1}{2}(L_{11} + L_{22})$, $\alpha_1 = \frac{1}{2}(L_{12} + L_{21})$, $\alpha_2 = \frac{i}{2}(L_{12} - L_{21})$, and $\alpha_3 = \frac{1}{2}(L_{11} - L_{22})$. Hence, L_A can be written in terms of α_μ as

$$L_A = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \tag{12}$$

We can write L_A and Z_A in terms of σ_μ and Σ_μ matrices:

$$L_A = \alpha_0\sigma_0 + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 \tag{13}$$

$$Z_A = \alpha_0\Sigma_0 + \alpha_1\Sigma_1 + \alpha_2\Sigma_2 + \alpha_3\Sigma_3 \tag{14}$$

Or, simply

$$L_A = (++++)_\sigma. \tag{15}$$

$$Z_A = (++++)_\Sigma. \tag{16}$$

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We also define the spinor metric g for $SL(4,C)$ that corresponds to the spinor metric ϵ of $SL(2,C)$:

$$g = g^{\mu\nu} = i\eta\Sigma_2^* = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \\ -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad g^{-1} = g_{\mu\nu} = g^\dagger. \tag{17}$$

η is the mostly minus Minkowski metric[1].

Z_A preserves the Minkowski metric:

$$Z_A^T \eta Z_A = \eta \tag{18}$$

Since η is real, $Z_A^T\eta Z_A = \eta$ directly entails $\Lambda^T\eta\Lambda = \eta$. In an analogy with $\epsilon\sigma_i\epsilon^{-1} = -\sigma_i^*$, we have the following very useful relation:

$$g\Sigma_i g^{-1} = -\Sigma_i^* \tag{19}$$

In this note we will show that there are eight generalized eigenvectors of Σ_3^* matrix that can be interpreted as 4-component covariant spinors. The generalized eigenvectors can be pairwise grouped into four categories. The first pair transforms in the usual way, but the other three transform in different ways.

In the following we will study the first and the second pairs in detail, and we will introduce the remaining two in the subsequent sections.

0.2 The first and the second pairs and their transformation properties

Let $L_A = exp(-\frac{i}{2}\vec{\pi}_A \cdot \vec{\sigma})$ be the $(\frac{1}{2}, 0)$ representation of the Lorentz group that acts on the 2-component left-chiral spinor ξ_L:

$$\xi_L \to \xi_L' = L_A\xi_L. \tag{20}$$

where

$$L_A = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \tag{21}$$

In terms of the components u, v of ξ_L:

$$u \to u' = L_{11}u + L_{12}v, \quad v \to v' = L_{21}u + L_{22}v. \tag{22}$$

Let us call this transformation scheme T_A.

Let $\dot{L}_A = exp(-\frac{i}{2}\vec{\pi}_A^* \cdot \vec{\sigma})$ be the dotted version corresponding to the $(0, \frac{1}{2})$ representation of the Lorentz group. $\dot{L}_A = (L_A^{-1})^\dagger$. Let ξ_R be the 2-component right-chiral spinor. $\xi_R = \epsilon\xi_L^*$, where

$$\epsilon = \epsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{-1} = \epsilon_{ab} = \epsilon^\dagger. \tag{23}$$

[1]We can define the spinor metric for $SL(4,C)$ as $i\eta\Sigma_1^*$ or $i\eta\Sigma_3^*$ if we like. These metrics also have the same properties of g.

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ξ_R transforms as

$$\xi_R \to \xi'_R = \dot{L}_A \xi_R \tag{24}$$

In terms of the components u, v, Eq.(24) is equivalent to the scheme T_A given in Eq.(22).

What happens when L_A acts on $\epsilon\xi_L$? In this case, in terms of the components

$$u \to u' = L_{22}u - L_{21}v, \quad v \to v' = -L_{12}u + L_{11}v \tag{25}$$

Let us call this transformation scheme T_B. We can write T_B in a matrix form:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} L_{22} & -L_{21} \\ -L_{12} & L_{11} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{26}$$

Let us name this transformation matrix as L_B. Note that, $L_B = (\dot{L}_A)^*$, and Eq.(26) is nothing but the transformation of ξ_L under the action of L_B, which is a type T_B transformation.

Now, let $Z_A = exp(-\frac{i}{2}\vec{\pi}_A \cdot \vec{\Sigma})$ be the $(\frac{1}{2}, 0)$ representation of $SL(4, C)$ that acts on the first pair of the 4-component undotted covariant spinors:

$$\chi_{(1)} \to Z_A \chi_{(1)}, \quad \chi_{(2)} \to Z_A \chi_{(2)} \tag{27}$$

where $\chi_{(1)}$ and $\chi_{(2)}$ are the generalized eigenvectors of Σ_3^* [2]:

$$\chi_{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ v \\ -iv \\ u \end{pmatrix}, \chi_{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -v \\ -u \\ -iu \\ v \end{pmatrix} \tag{28}$$

Indices in the parentheses are simply labels for 4-component spinors.

Now consider the second pair of the generalized eigenvectors of Σ_3^*:

$$\chi_{(3)} = \frac{1}{\sqrt{2}} \begin{pmatrix} -v \\ u \\ -iu \\ -v \end{pmatrix}, \chi_{(4)} = \frac{1}{\sqrt{2}} \begin{pmatrix} u \\ -v \\ -iv \\ -u \end{pmatrix} \tag{29}$$

Transformation scheme of $\chi_{(3)}$ and $\chi_{(4)}$ is different from that of $\chi_{(1)}$ and $\chi_{(2)}$. Under the action of Z_A, $\chi_{(1)}$ and $\chi_{(2)}$ transform according to the scheme T_A, but $\chi_{(3)}$ and $\chi_{(4)}$ transform according to the scheme T_B. However, we may think in an alternative way: Suppose that $\chi_{(3)}$ and $\chi_{(3)}$ are different kind of objects with different transformation properties, such that another transformation matrix, Z_B, acts on them and under the action of Z_B they transform according to the scheme T_A:

$$\chi_{(3)} \to Z_B \chi_{(3)}, \quad \chi_{(4)} \to Z_B \chi_{(4)}, \tag{30}$$

[2]We may use the generalized eigenvectors of Σ_1 or Σ_2 matrices as well, but, in that case, we have to employ the other forms of the spinor metric.

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By definition $Z_B = A(L_B \otimes I)A^{-1}$:

$$Z_B = \begin{pmatrix} \alpha_0 & -\alpha_1 & \alpha_2 & -\alpha_3 \\ -\alpha_1 & \alpha_0 & i\alpha_3 & i\alpha_2 \\ \alpha_2 & -i\alpha_3 & \alpha_0 & i\alpha_1 \\ -\alpha_3 & -i\alpha_2 & -i\alpha_1 & \alpha_0 \end{pmatrix} = \alpha_0\Sigma_0 - \alpha_1\Sigma_1 + \alpha_2\Sigma_2 - \alpha_3\Sigma_3. \tag{31}$$

Or, simply

$$Z_B = (+-+-)_\Sigma \tag{32}$$

Now let $\dot{Z}_A = exp(-\frac{i}{2}\vec{\pi}^*_A \cdot \vec{\Sigma})$ be the $(0, \frac{1}{2})$ representation. $\dot{Z}_A = (Z_A^{-1})^\dagger$. We regard the generalized eigenvectors of Σ_3 as 4-component undotted contravariant spinors and we define the first pair as follows:

$$\chi^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ -u \\ -iu \\ v \end{pmatrix}, \; \chi^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} u \\ -v \\ iv \\ -u \end{pmatrix} \tag{33}$$

Under the action of $\dot{Z}_A$, dotted versions of $\chi^{^{(1)}}$ and $\chi^{^{(1)}}$ transform according to the scheme T_A.

$$\dot{\chi}^{^{(1)}} \to \dot{Z}_A \dot{\chi}^{^{(1)}}, \quad \dot{\chi}^{^{(2)}} \to \dot{Z}_A \dot{\chi}^{^{(2)}} \tag{34}$$

The second pair of the generalized eigenvectors of Σ_3 is defined as

$$\chi^{(3)} = \frac{1}{\sqrt{2}}\begin{pmatrix} u \\ v \\ iv \\ u \end{pmatrix}, \; \chi^{(4)} = \frac{1}{\sqrt{2}}\begin{pmatrix} v \\ u \\ -iu \\ -v \end{pmatrix} \tag{35}$$

Under the action of $\dot{Z}_A$, the dotted versions of $\chi^{^{(3)}}$ and $\chi^{^{(4)}}$ transform according to the scheme T_B. But, they transform according to the scheme T_A under the action of $\dot{Z}_B$:

$$\dot{\chi}^{^{(3)}} \to \dot{Z}_B \dot{\chi}^{^{(3)}}, \quad \dot{\chi}^{^{(4)}} \to \dot{Z}_B \dot{\chi}^{^{(4)}} \tag{36}$$

where $\dot{Z}_B = (Z_B^{-1})^\dagger$ by definition. $\chi^{^{(a)}}$ is related to $\chi_{_{(a)}}$ by the $SL(4, C)$ metric, $\chi^{(a)} = g\chi_{_{(a)}}$, and its dotted version is defined as [3].

$$\dot{\chi}^{(a)} = (g\chi_{_{(a)}})^*. \tag{37}$$

We write various forms of Z and L matrices in compact notation to manifest the parallelism between them:

$$L_A = (++++)_\sigma, \quad L_B = (+-+-)_\sigma, \quad \dot{L}_A = (+---)^*_\sigma, \quad \dot{L}_B = (++-+)^*_\sigma. \tag{38}$$

$$Z_A = (++++)_\Sigma, \quad Z_B = (+-+-)_\Sigma, \quad \dot{Z}_A = (+---)^*_\Sigma, \quad \dot{Z}_B = (++-+)^*_\Sigma. \tag{39}$$

[3] The upper dot on a spinorial object simply means complex conjugation: $\dot{\chi}^{^{(a)}} = (\chi^{^{(a)}})^*$. But, the upper dot on an element of $SL(2, C)$ or $SL(4, C)$ has a particular meaning. $\dot{L}_A = exp(-\frac{i}{2}\vec{\pi}^*_A \cdot \vec{\sigma}) \neq L^*_A$. Similarly, $\dot{Z}_A = exp(-\frac{i}{2}\vec{\pi}^*_A \cdot \vec{\Sigma}) \neq Z^*_A$.

0.3 Outer products of 4-component spinors and null 4-vectors

Let us define the outer product $W_{_L} = \xi_{_L}\xi_{_L}^\dagger$ which transforms as

$$W_{_L} \to W'_{_L} = (L_A\xi_{_L})(L_A\xi_{_L})^\dagger = L_A W_{_L} L_A^\dagger \tag{40}$$

This is a type T_A transformation. Determinant of $W_{_L}$ is zero, hence $W_{_L}$ can be associated with a null 4-vector through the substitutions, $t = \frac{1}{2}(u\dot{u} + v\dot{v})$, $x = \frac{1}{2}(u\dot{v} + v\dot{u})$, $y = \frac{i}{2}(u\dot{v} - v\dot{u})$, $z = \frac{1}{2}(u\dot{u} - v\dot{v})$:

$$W_{_L} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \tag{41}$$

We also define the outer product $W_{_R} = \xi_{_R}\xi_{_R}^\dagger$ which transforms as

$$W_{_R} \to W'_{_R} = (\dot{L}_A\xi_{_R})(\dot{L}_A\xi_{_R})^\dagger = \dot{L}_A W_{_R} \dot{L}_A^\dagger \tag{42}$$

This is also a type T_A transformation. Determinant of $W_{_R}$ is zero and $W_{_R}$ can be associated with a null 4-vector:

$$W_{_R} = \begin{pmatrix} t-z & -x+iy \\ -x-iy & t+z \end{pmatrix} \tag{43}$$

Note that $W_{_R}$ can be obtained from $W_{_L}$ by parity inversion.

There are outer product forms of 4-component spinors that can be associated with null 4-vectors. $\mathcal{W}_{_{(11)}} = \chi_{_{(1)}}\chi_{_{(1)}}^\dagger$ and $\mathcal{W}_{_{(22)}} = \chi_{_{(2)}}\chi_{_{(2)}}^\dagger$ transform in a similar way with $W_{_L}$:

$$\mathcal{W}_{_{(11)}} = \begin{pmatrix} u\dot{u} & u\dot{v} & iu\dot{v} & u\dot{u} \\ v\dot{u} & v\dot{v} & iv\dot{v} & v\dot{u} \\ -iv\dot{u} & -iv\dot{v} & v\dot{v} & -iv\dot{u} \\ u\dot{u} & u\dot{v} & iu\dot{v} & u\dot{u} \end{pmatrix}, \mathcal{W}_{_{(22)}} = \begin{pmatrix} v\dot{v} & v\dot{u} & -iv\dot{u} & -v\dot{v} \\ u\dot{v} & u\dot{u} & -iu\dot{u} & -u\dot{v} \\ iu\dot{v} & iu\dot{u} & u\dot{u} & -iu\dot{v} \\ -v\dot{v} & -v\dot{u} & iv\dot{u} & v\dot{v} \end{pmatrix} \tag{44}$$

For $a = 1$ and $a = 2$, $\mathcal{W}_{_{(aa)}}$ transform according to the scheme T_A as

$$\mathcal{W}_{_{(aa)}} \to \mathcal{W}'_{_{(aa)}} = (Z_A\chi_{_{(a)}})(Z_A\chi_{_{(a)}})^\dagger = Z_A\mathcal{W}_{_{(aa)}}Z_A^\dagger. \tag{45}$$

This equation is equivalent to the Eq.(40), and it is the main motivation behind the interpretation of $\chi_{_{(a)}}$ as 4-component spinors for $SL(4, C)$.

$\dot{\mathcal{W}}^{^{(11)}} = \dot{\chi}^{^{(1)}}\dot{\chi}^{^{(1)\dagger}}$ and $\dot{\mathcal{W}}^{^{(22)}} = \dot{\chi}^{^{(2)}}\dot{\chi}^{^{(2)\dagger}}$ transform in a similar way with $W_{_R}$:

$$\dot{\mathcal{W}}^{^{(11)}} = \begin{pmatrix} v\dot{v} & -u\dot{v} & -iu\dot{v} & v\dot{v} \\ -v\dot{u} & u\dot{u} & iu\dot{u} & -v\dot{u} \\ iv\dot{u} & -iu\dot{u} & u\dot{u} & iv\dot{u} \\ v\dot{v} & -u\dot{v} & -iu\dot{v} & v\dot{v} \end{pmatrix}, \dot{\mathcal{W}}^{^{(22)}} = \begin{pmatrix} u\dot{u} & -v\dot{u} & iv\dot{u} & -u\dot{u} \\ -u\dot{v} & v\dot{v} & -iv\dot{v} & u\dot{v} \\ -iu\dot{v} & iv\dot{v} & v\dot{v} & iu\dot{v} \\ -u\dot{u} & v\dot{u} & -iv\dot{u} & u\dot{u} \end{pmatrix} \tag{46}$$

For $a = 1$ and $a = 2$, $\dot{\mathcal{W}}^{^{(aa)}}$ transform according to the scheme T_A as

$$\dot{\mathcal{W}}^{^{(aa)}} \to \dot{Z}_A\dot{\mathcal{W}}^{^{(aa)}}\dot{Z}_A^\dagger \tag{47}$$

$\mathcal{W}_{_{(aa)}}$ and $\dot{\mathcal{W}}^{^{(aa)}}$ are Hermitian and zero determinant matrices, hence they correspond to null 4-vectors.

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We also have outer products of 4-component spinors of the other kind. For $a = 3$ and $a = 4$, $\mathcal{W}_{(aa)}$ and $\dot{\mathcal{W}}^{(aa)}$ transform according to the scheme T_A under the action of Z_B and $\dot{Z}_B$:

$$\mathcal{W}_{(aa)} \to Z_B \mathcal{W}_{(aa)} Z_B^\dagger, \quad \dot{\mathcal{W}}^{(aa)} \to \dot{Z}_B \dot{\mathcal{W}}^{(aa)} \dot{Z}_B^\dagger \tag{48}$$

These are also Hermitian and zero determinant matrices and they correspond to null 4-vectors.

0.4 Quaternion forms and 4-vectors

In general, we can treat t, x, y and z as variables that do not depend on u and v. Then, we can associate the following matrices X_L and X_R with 4-vectors, which are not necessarily null:

$$W_L \to X_L = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3 = (++++)_\sigma \tag{49}$$

$$W_R \to X_R = \begin{pmatrix} t-z & -x+iy \\ -x-iy & t+z \end{pmatrix} = t\sigma_0 - x\sigma_1 - y\sigma_2 - z\sigma_3 = (+---)_\sigma \tag{50}$$

$\det X_L = \det X_R = t^2 - x^2 - y^2 - z^2$ and in general not zero. X_L and X_R transform as

$$X_L \to X'_L = L_A X_L L_A^\dagger, \quad X_R \to X'_R = \dot{L}_A X_R \dot{L}_A^\dagger \tag{51}$$

These are matrix representations of quaternions, because $-i\sigma_1, -i\sigma_2, -i\sigma_3$ matrices have the same properties as the Hamilton's quaternion basis, $\mathbf{i}, \mathbf{j}, \mathbf{k}$:

$$X_L = t\sigma_0 + ix(-i\sigma_1) + iy(-i\sigma_2) + iz(-i\sigma_3) = t\mathbb{1} + ix\mathbf{i} + iy\mathbf{j} + iz\mathbf{k}. \tag{52}$$

Similarly,

$$X_R = t\mathbb{1} - ix\mathbf{i} - iy\mathbf{j} - iz\mathbf{k}. \tag{53}$$

In order to make the analogy with $SL(4, C)$ we consider the following two column objects that are pairwise combinations of 4-component spinors:

$$\chi_A = \frac{1}{\sqrt{2}} \begin{pmatrix} u & -v \\ v & -u \\ -iv & -iu \\ u & v \end{pmatrix}, \chi_B = \frac{1}{\sqrt{2}} \begin{pmatrix} -v & u \\ u & -v \\ -iu & -iv \\ -v & -u \end{pmatrix}, \chi^A = \frac{1}{\sqrt{2}} \begin{pmatrix} v & u \\ -u & -v \\ -iu & iv \\ v & -u \end{pmatrix}, \chi^B = \frac{1}{\sqrt{2}} \begin{pmatrix} u & v \\ v & u \\ iv & -iu \\ u & -v \end{pmatrix} \tag{54}$$

where $\chi_A = (\chi_{(1)}, \chi_{(2)})$, $\chi_B = (\chi_{(3)}, \chi_{(4)})$, $\chi^A = (\chi^{(1)}, \chi^{(2)})$, $\chi^B = (\chi^{(3)}, \chi^{(4)})$.

We define an outer product of 4-component spinor pair in the form $\mathcal{W}_A = \chi_A \chi_A^\dagger$, which is formally a quaternion:

$$\mathcal{W}_A = \frac{1}{2} \begin{pmatrix} u\dot{u} + v\dot{v} & u\dot{v} + v\dot{u} & iu\dot{v} - iv\dot{u} & u\dot{u} - v\dot{v} \\ v\dot{u} + u\dot{v} & v\dot{v} + u\dot{u} & iv\dot{v} - iu\dot{u} & v\dot{u} - u\dot{v} \\ -iv\dot{u} + iu\dot{v} & -iv\dot{v} + iu\dot{u} & v\dot{v} + u\dot{u} & -iv\dot{u} - iu\dot{v} \\ u\dot{u} - v\dot{v} & u\dot{v} - v\dot{u} & iu\dot{v} + iv\dot{u} & u\dot{u} + v\dot{v} \end{pmatrix} \tag{55}$$

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$\mathcal{W}_A$ can be written as a sum of two basic forms: $\mathcal{W}_A = \mathcal{W}_{(11)} + \mathcal{W}_{(22)}$. In its present form $\det\mathcal{W}_A = 0$ and $\mathcal{W}_A$ corresponds to a null 4-vector, but we can associate $\mathcal{W}_A$ with an arbitrary 4-vector in terms of the variables t, x, y and z:

$$\mathcal{W}_A \to \mathcal{Q}_A = \begin{pmatrix} t & x & y & z \\ x & t & -iz & iy \\ y & iz & t & -ix \\ z & -iy & ix & t \end{pmatrix} = t\Sigma_0 + x\Sigma_1 + y\Sigma_2 + z\Sigma_3. \tag{56}$$

$\mathcal{Q}_A = (++++)_\Sigma$ and it is the 4×4 version of X_L:

$$\mathcal{Q}_A = A(X_L \otimes I)A^{-1} \tag{57}$$

Similarly, we define $\dot{\mathcal{W}}^A$:

$$\dot{\mathcal{W}}^A = \dot{\chi}^A \dot{\chi}^{A\dagger} = \dot{\mathcal{W}}^{(11)} + \dot{\mathcal{W}}^{(22)} = \frac{1}{2}\begin{pmatrix} \dot{v}v + \dot{u}u & -\dot{v}u - \dot{u}v & -i\dot{v}u + i\dot{u}v & \dot{v}v - \dot{u}u \\ -\dot{u}v - \dot{v}u & \dot{u}u + \dot{v}v & i\dot{u}u - i\dot{v}v & -\dot{u}v + \dot{v}u \\ i\dot{u}v - i\dot{v}u & -i\dot{u}u + i\dot{v}v & \dot{u}u + \dot{v}v & i\dot{u}v + i\dot{v}u \\ \dot{v}v - \dot{u}u & -\dot{v}u + \dot{u}v & -i\dot{v}u - i\dot{u}v & \dot{v}v + \dot{u}u \end{pmatrix} \tag{58}$$

In terms of the variables t, x, y and z:

$$\dot{\mathcal{W}}^A \to \dot{\mathcal{Q}}^A = \begin{pmatrix} t & -x & -y & -z \\ -x & t & iz & -iy \\ -y & -iz & t & ix \\ -z & iy & -ix & t \end{pmatrix} = t\Sigma_0 - x\Sigma_1 - y\Sigma_2 - z\Sigma_3. \tag{59}$$

$\dot{\mathcal{Q}}^A = (+---)_\Sigma$ and it is the 4×4 version of X_R:

$$\dot{\mathcal{Q}}^A = A(X_R \otimes I)A^{-1}. \tag{60}$$

$\dot{\mathcal{Q}}^A$ can be obtained from $\mathcal{Q}_A$ by parity inversion and they transform as

$$\mathcal{Q}_A \to Z_A \mathcal{Q}_A Z_A^\dagger, \quad \dot{\mathcal{Q}}^A \to \dot{Z}_A \dot{\mathcal{Q}}^A \dot{Z}_A^\dagger \tag{61}$$

These are type T_A transformations, hence these forms correspond to 4-vectors.

The outer product $\mathcal{W}_B = \chi_B \chi_B^\dagger$ is also a quaternion:

$$\mathcal{W}_B = \frac{1}{2}\begin{pmatrix} v\dot{v} + u\dot{u} & -v\dot{u} - u\dot{v} & -iv\dot{u} + iu\dot{v} & v\dot{v} - u\dot{u} \\ -u\dot{v} - v\dot{u} & u\dot{u} + v\dot{v} & iu\dot{u} - iv\dot{v} & -u\dot{v} + v\dot{u} \\ iu\dot{v} - iv\dot{u} & -iu\dot{u} + iv\dot{v} & u\dot{u} + v\dot{v} & iu\dot{v} + iv\dot{u} \\ v\dot{v} - u\dot{u} & -v\dot{u} + u\dot{v} & -iv\dot{u} - iu\dot{v} & v\dot{v} + u\dot{u} \end{pmatrix} \tag{62}$$

In terms of variables t, x, y and z:

$$\mathcal{W}_B \to \mathcal{Q}_B = \begin{pmatrix} t & -x & y & -z \\ -x & t & iz & iy \\ y & -iz & t & ix \\ -z & -iy & -ix & t \end{pmatrix} = t\Sigma_0 - x\Sigma_1 + y\Sigma_2 - z\Sigma_3. \tag{63}$$

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$\mathcal{Q}_B = (+-+-)_\Sigma$.

We also write $\dot{\mathcal{W}}^B$:

$$\dot{\mathcal{W}}^B = \dot{\chi}^B \dot{\chi}^{B\dagger} = \frac{1}{2}\begin{pmatrix} \dot{u}u+\dot{v}v & \dot{u}v+\dot{v}u & i\dot{u}v-i\dot{v}u & \dot{u}u-\dot{v}v \\ \dot{v}u+\dot{u}v & \dot{v}v+\dot{u}u & i\dot{v}v-i\dot{u}u & \dot{v}u-\dot{u}v \\ -i\dot{v}u+i\dot{u}v & -i\dot{v}v+i\dot{u}u & \dot{v}v+\dot{u}u & -i\dot{v}u-i\dot{u}v \\ \dot{u}u-\dot{v}v & \dot{u}v-\dot{v}u & i\dot{u}v+i\dot{v}u & \dot{u}u+\dot{v}v \end{pmatrix} \tag{64}$$

$$\dot{\mathcal{W}}^B \to \dot{\mathcal{Q}}^B = \begin{pmatrix} t & x & -y & z \\ x & t & -iz & -iy \\ -y & iz & t & -ix \\ z & iy & ix & t \end{pmatrix} = t\Sigma_0 + x\Sigma_1 - y\Sigma_2 + z\Sigma_3. \tag{65}$$

$\dot{\mathcal{Q}}^B = (++-+)_\Sigma$ and it can be obtained from $\mathcal{Q}_B$ by parity inversion. $\mathcal{Q}_B$ and $\dot{\mathcal{Q}}^B$ transform with Z_B and $\dot{Z}_B$:

$$\mathcal{Q}_B \to Z_B \mathcal{Q}_B Z_B^\dagger, \quad \dot{\mathcal{Q}}^B \to \dot{Z}_B \dot{\mathcal{Q}}^B \dot{Z}_B^\dagger \tag{66}$$

These transformations obey the scheme T_A also, hence they correspond to 4-vectors.

With the compact notation we can show a very nice symmetry: The form of the transformation matrix matches the form of the transformed object. For example, $Z_A = (++++)_\Sigma$ acts on the form $\mathcal{Q}_A = (++++)_\Sigma$, $Z_B = (+-+-)_\Sigma$ acts on the form $\mathcal{Q}_B = (+-+-)_\Sigma$, $\dot{Z}_A = (+---)^*_\Sigma$ acts on the form $\dot{\mathcal{Q}}^A = (+---)_\Sigma$, and $\dot{Z}_B = (++-+)^*_\Sigma$ acts on the form $\dot{\mathcal{Q}}^B = (++-+)_\Sigma$.

0.5 Two more pairs of spinors

There are four eigenvectors of Σ_3^* that constitute a complete orthonormal set of basis:

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}. \tag{67}$$

e_1 and e_3 correspond to $+1$ eigenvalue and e_2 and e_4 correspond to -1 eigenvalue. We obtain eight generalized eigenvectors by combining the basis corresponding to the same eigenvalue. For example, we can obtain the four generalized eigenvectors that we have previously studied as follows:

$$\chi_{(1)} = ue_1 + ve_3, \quad \chi_{(2)} = -ve_2 - ue_4, \tag{68}$$

$$\chi_{(3)} = -ve_1 + ue_3, \quad \chi_{(4)} = ue_2 - ve_4, \tag{69}$$

We can obtain four more generalized eigenvectors of Σ_3^* by changing the sign or swapping u and v:

$$\chi_{(5)} = -ue_1 + ve_3, \quad \chi_{(6)} = ve_2 - ue_4, \tag{70}$$

$$\chi_{(7)} = ve_1 + ue_3, \quad \chi_{(8)} = ue_2 + ve_4, \tag{71}$$

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Totally we get eight undotted covariant spinors:

$$\chi_{(1)} = \begin{pmatrix} u \\ v \\ -iv \\ u \end{pmatrix}, \chi_{(2)} = \begin{pmatrix} -v \\ -u \\ -iu \\ v \end{pmatrix}, \chi_{(3)} = \begin{pmatrix} -v \\ u \\ -iu \\ -v \end{pmatrix}, \chi_{(4)} = \begin{pmatrix} u \\ -v \\ -iv \\ -u \end{pmatrix}, \tag{72}$$

$$\chi_{(5)} = \begin{pmatrix} -u \\ v \\ -iv \\ -u \end{pmatrix}, \chi_{(6)} = \begin{pmatrix} v \\ -u \\ -iu \\ -v \end{pmatrix}, \chi_{(7)} = \begin{pmatrix} v \\ u \\ -iu \\ v \end{pmatrix}, \chi_{(8)} = \begin{pmatrix} u \\ v \\ iv \\ -u \end{pmatrix}. \tag{73}$$

We can group $\chi_{(a)}$ $(a = 1, 2, \cdots 8)$ pairwise:

$$P_A = \{\chi_{(1)}, \chi_{(2)}\},\ P_B = \{\chi_{(3)}, \chi_{(4)}\},\ P_C = \{\chi_{(5)}, \chi_{(6)}\},\ P_D = \{\chi_{(7)}, \chi_{(8)}\}, \tag{74}$$

We already know that P_A transforms with Z_A and P_B transforms with Z_B. Following the same procedure that we have applied in the previous sections we can show that P_C and P_D transform with Z_C and Z_D respectively:

$$Z_C = A(L_C \otimes I)A^{-1}, \quad Z_D = A(L_D \otimes I)A^{-1}, \tag{75}$$

where

$$L_C = \begin{pmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{pmatrix} = \begin{pmatrix} \alpha_0 + \alpha_3 & -\alpha_1 + i\alpha_2 \\ -\alpha_1 - i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} = (+--+)_\sigma \tag{76}$$

$$L_D = \begin{pmatrix} L_{22} & L_{21} \\ L_{12} & L_{11} \end{pmatrix} = \begin{pmatrix} \alpha_0 - \alpha_3 & \alpha_1 + i\alpha_2 \\ \alpha_1 - i\alpha_2 & \alpha_0 + \alpha_3 \end{pmatrix} = (++--)_\sigma \tag{77}$$

$$\dot{L}_C = (+++-)^*_\sigma = L^*_D \tag{78}$$

$$\dot{L}_D = (+-++)^*_\sigma = L^*_C \tag{79}$$

$$Z_C = \begin{pmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & \alpha_3 \\ -\alpha_1 & \alpha_0 & -i\alpha_3 & -i\alpha_2 \\ -\alpha_2 & i\alpha_3 & \alpha_0 & i\alpha_1 \\ \alpha_3 & i\alpha_2 & -i\alpha_1 & \alpha_0 \end{pmatrix} = (+--+)_\Sigma \tag{80}$$

$$Z_D = \begin{pmatrix} \alpha_0 & \alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & \alpha_0 & i\alpha_3 & -i\alpha_2 \\ -\alpha_2 & -i\alpha_3 & \alpha_0 & -i\alpha_1 \\ -\alpha_3 & i\alpha_2 & i\alpha_1 & \alpha_0 \end{pmatrix} = (++--)_\Sigma \tag{81}$$

There are also the dotted versions:

$$\dot{Z}_C = \begin{pmatrix} \alpha^*_0 & \alpha^*_1 & \alpha^*_2 & -\alpha^*_3 \\ \alpha^*_1 & \alpha^*_0 & i\alpha^*_3 & i\alpha^*_2 \\ \alpha^*_2 & -i\alpha^*_3 & \alpha^*_0 & -i\alpha^*_1 \\ -\alpha^*_3 & -i\alpha^*_2 & i\alpha^*_1 & \alpha^*_0 \end{pmatrix} = (+++-)^*_\Sigma \tag{82}$$

$$\dot{Z}_D = \begin{pmatrix} \alpha_0^* & -\alpha_1^* & \alpha_2^* & \alpha_3^* \\ -\alpha_1^* & \alpha_0^* & -i\alpha_3^* & i\alpha_2^* \\ \alpha_2^* & i\alpha_3^* & \alpha_0^* & i\alpha_1^* \\ \alpha_3^* & -i\alpha_2^* & -i\alpha_1^* & \alpha_0^* \end{pmatrix} = (+-++)^*_\Sigma \tag{83}$$

We also define the contravariat spinors $\chi^{(a)} = g\chi_{(a)}$ $(a = 1, 2, \cdots 8)$ that correspond to the generalized eigenvectors of Σ_3:

$$\chi^{(1)} = \begin{pmatrix} v \\ -u \\ -iu \\ v \end{pmatrix}, \chi^{(2)} = \begin{pmatrix} u \\ -v \\ iv \\ -u \end{pmatrix}, \chi^{(3)} = \begin{pmatrix} u \\ v \\ iv \\ u \end{pmatrix}, \chi^{(4)} = \begin{pmatrix} v \\ u \\ -iu \\ -v \end{pmatrix}, \tag{84}$$

$$\chi^{(5)} = \begin{pmatrix} v \\ u \\ iu \\ v \end{pmatrix}, \chi^{(6)} = \begin{pmatrix} u \\ v \\ -iv \\ -u \end{pmatrix}, \chi^{(7)} = \begin{pmatrix} u \\ -v \\ -iv \\ u \end{pmatrix}, \chi^{(8)} = \begin{pmatrix} -v \\ u \\ -iu \\ v \end{pmatrix}. \tag{85}$$

We group them pairwise:

$$P^A = \{\chi^{(1)}, \chi^{(2)}\},\ P^B = \{\chi^{(3)}, \chi^{(4)}\},\ P^C = \{\chi^{(5)}, \chi^{(6)}\},\ P^D = \{\chi^{(7)}, \chi^{(8)}\} \tag{86}$$

Each pair of the dotted contravariant spinors transform with the associated dotted Z matrix.

We define four two-column covariant objects:

$$\chi_A = (\chi_{(1)}, \chi_{(2)}),\ \chi_B = (\chi_{(3)}, \chi_{(4)}),\ \chi_C = (\chi_{(5)}, \chi_{(6)}),\ \chi_D = (\chi_{(7)}, \chi_{(8)}) \tag{87}$$

And we define the corresponding two-column contravariant objects

$$\chi^A = (\chi^{(1)}, \chi^{(2)}),\ \chi^B = (\chi^{(3)}, \chi^{(4)}),\ \chi^C = (\chi^{(5)}, \chi^{(6)}),\ \chi^D = (\chi^{(7)}, \chi^{(8)}) \tag{88}$$

Finally, we construct eight outer products that lead to the following quaternions:

$$\chi_A\chi_A^\dagger \to \mathcal{Q}_A = (++++)_\Sigma, \quad \dot{\chi}^A\dot{\chi}^{A\dagger} \to \dot{\mathcal{Q}}^A = (+---)_\Sigma. \tag{89}$$

$$\chi_B\chi_B^\dagger \to \mathcal{Q}_B = (+-+-)_\Sigma, \quad \dot{\chi}^B\dot{\chi}^{B\dagger} \to \dot{\mathcal{Q}}^B = (++-+)_\Sigma. \tag{90}$$

$$\chi_C\chi_C^\dagger \to \mathcal{Q}_C = (+--+)_\Sigma, \quad \dot{\chi}^C\dot{\chi}^{C\dagger} \to \dot{\mathcal{Q}}^C = (+++-)_\Sigma. \tag{91}$$

$$\chi_D\chi_D^\dagger \to \mathcal{Q}_D = (++--)_\Sigma, \quad \dot{\chi}^D\dot{\chi}^{D\dagger} \to \dot{\mathcal{Q}}^D = (+-++)_\Sigma. \tag{92}$$

Each form transforms in its own way with the matching Z or $\dot{Z}$ matrix.

0.6 Complex conjugated forms

Let us write the complex conjugates of the quaternion forms:

$$\mathcal{Q}_A = (++++)_\Sigma \xrightarrow{\text{c.c.}} \dot{\mathcal{Q}}_A = (++++)_{\Sigma^*} \tag{93}$$

$$\mathcal{Q}_B = (+-+-)_\Sigma \xrightarrow{\text{c.c.}} \dot{\mathcal{Q}}_B = (+-+-)_{\Sigma^*} \tag{94}$$

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$$\mathcal{Q}_C = (+--+)_\Sigma \xrightarrow{\text{c.c.}} \dot{\mathcal{Q}}_C = (+--+)_{\Sigma^*} \tag{95}$$

$$\mathcal{Q}_D = (++--)_\Sigma \xrightarrow{\text{c.c.}} \dot{\mathcal{Q}}_D = (++--)_{\Sigma^*} \tag{96}$$

$$\dot{\mathcal{Q}}^A = (+---)_\Sigma \xrightarrow{\text{c.c.}} \mathcal{Q}^A = (+---)_{\Sigma^*} \tag{97}$$

$$\dot{\mathcal{Q}}^B = (++-+)_\Sigma \xrightarrow{\text{c.c.}} \mathcal{Q}^B = (++-+)_{\Sigma^*} \tag{98}$$

$$\dot{\mathcal{Q}}^C = (+++-)_\Sigma \xrightarrow{\text{c.c.}} \mathcal{Q}^C = (+++-)_{\Sigma^*} \tag{99}$$

$$\dot{\mathcal{Q}}^D = (+-++)_\Sigma \xrightarrow{\text{c.c.}} \mathcal{Q}^D = (+-++)_{\Sigma^*} \tag{100}$$

Conjugate forms reside in the dual space that spanned by Σ^*_μ. They don't have any counterpart in $SL(2, C)$. Dotted lower indexed and undotted upper indexed forms transform with $Z^*_{...}$ or with $(\dot{Z}_{...})^*$ matrices respectively and all transformations obey the scheme T_A.

4-vector scalar product can be defined by using the dual forms in two equivalent ways. Let Q and P be two 4-vectors and let $\mathcal{Q}_X$ and $\mathcal{P}_X$ be the corresponding quaternions $(X = A, B, C, D)$:

$$\mathsf{Q} \cdot \mathsf{P} = \frac{1}{4}\mathrm{Tr}(\mathcal{Q}_X^T \mathcal{P}^X) \tag{101}$$

Or, noting that $\mathcal{P}^X = g\mathcal{P}_X g^{-1}$, we can write also as:

$$\mathsf{Q} \cdot \mathsf{P} = \frac{1}{4}(\mathcal{Q}_{\mu\dot{\nu}} \mathcal{P}^{\mu\dot{\nu}}) \tag{102}$$

where now indices refer to the components and the summation convention is implied.

In general, in order to get something real we have to use both Σ and Σ^*. As an example, $\Lambda = ZZ^* = Z^*Z$, is the real Lorentz transformation matrix.

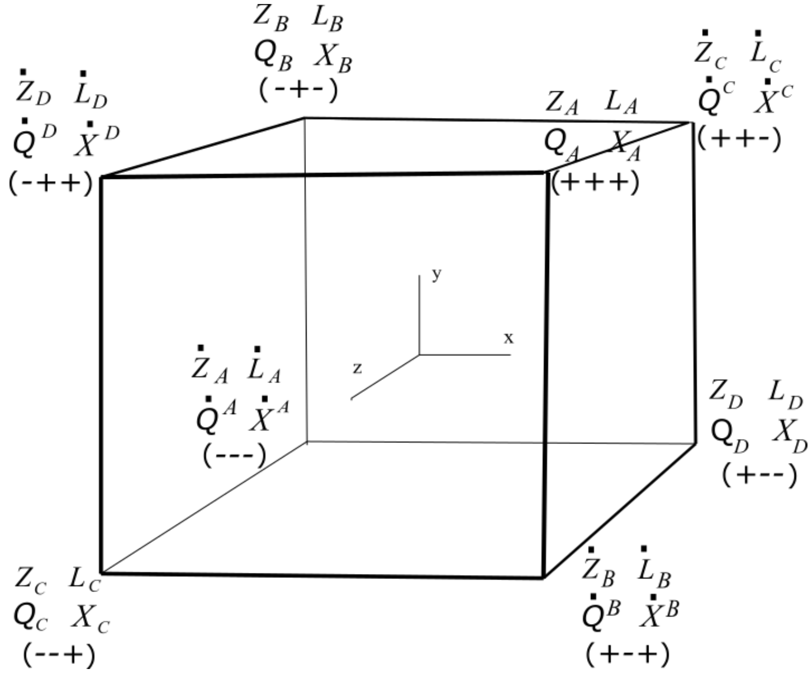

Figure 1: Reflections and inversions. Σ^* space is not shown.

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0.7 Four types of transformations for SL(2, C)

We can suggest a similar formalism for $SL(2,C)$. Let $\xi_{_A}, \xi_{_B}, \xi_{_C}$ and $\xi_{_D}$ be covariant spinors:

$$\xi_{_A} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \xi_{_B} = \begin{pmatrix} v \\ -u \end{pmatrix}, \quad \xi_{_C} = \begin{pmatrix} u \\ -v \end{pmatrix}, \quad \xi_{_D} = \begin{pmatrix} v \\ u \end{pmatrix} \tag{103}$$

and let $\xi^{^A}, \xi^{^B}, \xi^{^C}$ and $\xi^{^D}$ be contravariant spinors:

$$\xi^{^A} = \begin{pmatrix} v \\ -u \end{pmatrix}, \quad \xi^{^B} = \begin{pmatrix} -u \\ -v \end{pmatrix}, \quad \xi^{^C} = \begin{pmatrix} -v \\ -u \end{pmatrix}, \quad \xi_{_D} = \begin{pmatrix} u \\ -v \end{pmatrix} \tag{104}$$

where $\xi^{^A} = \xi_{_B}$, $\xi^{^B} = -\xi_{_A}$, $\xi^{^C} = -\xi_{_D}$ and $\xi^{^D} = \xi_{_C}$. This proliferation is necessary for the symmetry in Fig.1.

We have the following transformation properties:

$$\xi_{_A} \to L_A \xi_{_A}, \quad \xi_{_B} \to L_B \xi_{_B}, \quad \xi_{_C} \to L_C \xi_{_C}, \quad \xi_{_D} \to L_D \xi_{_D} \tag{105}$$

$$\dot{\xi}^{^A} \to \dot{L}_A \xi^{^A}, \quad \dot{\xi}^{^B} \to \dot{L}_B \xi^{^B}, \quad \dot{\xi}^{^C} \to \dot{L}_C \xi^{^C}, \quad \dot{\xi}^{^D} \to \dot{L}_D \xi^{^D}, \tag{106}$$

$$\dot{L}_A = L_B^*, \quad \dot{L}_B = L_A^*, \quad \dot{L}_C = L_D^*, \quad \dot{L}_D = L_C^* \tag{107}$$

All transformations obey the scheme T_A.

We have the following outer products:

$$\xi_{_A} \xi_{_A}^\dagger \to X_{_A} = (+ + + +)_\sigma, \quad \xi_{_B} \xi_{_B}^\dagger \to X_{_B} = (+ - + -)_\sigma \tag{108}$$

$$\xi_{_C} \xi_{_C}^\dagger \to X_{_C} = (+ - - +)_\sigma, \quad \xi_{_D} \xi_{_D}^\dagger \to X_{_D} = (+ + - -)_\sigma \tag{109}$$

$$\dot{\xi}^{^A} \dot{\xi}^{^A\dagger} \to \dot{X}^{^A} = (+ - - -)_\sigma, \quad \dot{\xi}^{^B} \dot{\xi}^{^B\dagger} \to \dot{X}^{^B} = (+ + - +)_\sigma, \tag{110}$$

$$\dot{\xi}^{^C} \dot{\xi}^{^C\dagger} \to \dot{X}^{^C} = (+ + + -)_\sigma, \quad \dot{\xi}^{^D} \dot{\xi}^{^D\dagger} \to \dot{X}^{^D} = (+ - + +)_\sigma, \tag{111}$$

It is worth noting that complex conjugating these forms does not yield anything new.

1 Appendix

Various forms of Z and L matrices

Let us begin with the exponential form $Z_A = e^R$, where $R = -\frac{i}{2}\vec{\pi} \cdot \vec{\Sigma}$, $\pi_i = \theta_i + i\eta_i$. Let ϕ be the complex angle defined as $\phi = \frac{1}{2}\sqrt{\pi_1^2 + \pi_2^2 + \pi_3^2}$. Using the property $R^2 = -\phi^2 I$:

$$Z_A = \cos\phi I - \frac{i \sin\phi}{2\phi} \vec{\pi} \cdot \vec{\Sigma} = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_0 & -i\alpha_3 & i\alpha_2 \\ \alpha_2 & i\alpha_3 & \alpha_0 & -i\alpha_1 \\ \alpha_3 & -i\alpha_2 & i\alpha_1 & \alpha_0 \end{pmatrix} \tag{112}$$

where

$$\alpha_0 = \cos\phi, \quad \alpha_1 = -\frac{i \sin\phi}{2\phi}\pi_1, \quad \alpha_2 = -\frac{i \sin\phi}{2\phi}\pi_2, \quad \alpha_3 = -\frac{i \sin\phi}{2\phi}\pi_3. \tag{113}$$

Or in a compact form

$$Z_A = \alpha_0\Sigma_0 + \alpha_1\Sigma_1 + \alpha_2\Sigma_2 + \alpha_3\Sigma_3 = (++++)_\Sigma. \tag{114}$$

It is easy to show that

$$Z_A^{-1} = \alpha_0\Sigma_0 - \alpha_1\Sigma_1 - \alpha_2\Sigma_2 - \alpha_3\Sigma_3 = (+---)_\Sigma. \tag{115}$$

and

$$Z_A^\dagger = \alpha_0^*\Sigma_0 + \alpha_1^*\Sigma_1 + \alpha_2^*\Sigma_2 + \alpha_3^*\Sigma_3 = (++++)_\Sigma^* \tag{116}$$

where complex conjugation is applied only to α_μ.

The corresponding L_A is

$$L_A = \begin{pmatrix} \alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_0 - \alpha_3 \end{pmatrix} \tag{117}$$

In terms of the Pauli matrices:

$$L_A = \alpha_0\sigma_0 + \alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3 = (++++)_\sigma. \tag{118}$$

In order to write Z_B we first find $L_B = (\dot{L}_A)^*$, where

$$\dot{L}_A = (L_A^{-1})^\dagger = \begin{pmatrix} \alpha_0^* - \alpha_3^* & -\alpha_1^* + i\alpha_2^* \\ -\alpha_1^* - i\alpha_2^* & \alpha_0^* + \alpha_3^* \end{pmatrix} = (+---)_\sigma^*. \tag{119}$$

From the definition $Z_B = A(L_B \otimes I)A^{-1}$:

$$Z_B = \begin{pmatrix} \alpha_0 & -\alpha_1 & \alpha_2 & -\alpha_3 \\ -\alpha_1 & \alpha_0 & i\alpha_3 & i\alpha_2 \\ \alpha_2 & -i\alpha_3 & \alpha_0 & i\alpha_1 \\ -\alpha_3 & -i\alpha_2 & -i\alpha_1 & \alpha_0 \end{pmatrix} = \alpha_0\Sigma_0 - \alpha_1\Sigma_1 + \alpha_2\Sigma_2 - \alpha_3\Sigma_3. \tag{120}$$

Or, simply

$$Z_B = (+-+-)_\Sigma \tag{121}$$

We write various forms of Z and L matrices in compact forms:

$$L_A = (++++)_\sigma, \quad L_B = (+-+-)_\sigma, \quad \dot{L}_A = (+---)_\sigma^*, \quad \dot{L}_B = (++-+)_\sigma^*. \tag{122}$$

$$Z_A = (++++)_\Sigma, \quad Z_B = (+-+-)_\Sigma, \quad \dot{Z}_A = (+---)_\Sigma^*, \quad \dot{Z}_B = (++-+)_\Sigma^*. \tag{123}$$

Although, all types of Z and L matrices are in the same form, there is a very important difference between them. Because of the particular property of the Pauli matrices, $\sigma_1^* = \sigma_1$, $\sigma_3^* = \sigma_3$, but $\sigma_2^* = -\sigma_2$, we have the following relations:

$$L_B = \dot{L}_A^*, \quad \dot{L}_B = L_A^*. \tag{124}$$

For example,

$$\dot{L}_A = (+---)_\sigma^* \rightarrow \dot{L}_A^* = (+---)_{\sigma^*} = (+-+-)_\sigma = L_B. \tag{125}$$

On the other hand we do not have a similar property with Σ matrices, hence

$$Z_B \neq \dot{Z}_A^*, \quad \dot{Z}_B \neq Z_A^*. \tag{126}$$

.

The structural difference between $SL(2,C)$ and $SL(4,C)$ becomes more apparent when we write the matrices in exponential forms. In order to do this we have to define two types of $\vec{\pi}$: $\vec{\pi}_A = (\pi_1, \pi_2, \pi_3)$ and $\vec{\pi}_B = (-\pi_1, \pi_2, -\pi_3)$.

$$L_A = exp(-\frac{i}{2}\vec{\pi}_A \cdot \vec{\sigma}), \quad L_B = exp(-\frac{i}{2}\vec{\pi}_B \cdot \vec{\sigma}), \quad \dot{L}_A = exp(-\frac{i}{2}\vec{\pi}_A^* \cdot \vec{\sigma}), \quad \dot{L}_B = exp(-\frac{i}{2}\vec{\pi}_B^* \cdot \vec{\sigma}). \tag{127}$$

$$Z_A = exp(-\frac{i}{2}\vec{\pi}_A \cdot \vec{\Sigma}), \quad Z_B = exp(-\frac{i}{2}\vec{\pi}_B \cdot \vec{\Sigma}), \quad \dot{Z}_A = exp(-\frac{i}{2}\vec{\pi}_A^* \cdot \vec{\Sigma}), \quad \dot{Z}_B = exp(-\frac{i}{2}\vec{\pi}_B^* \cdot \vec{\Sigma}). \tag{128}$$

Due to the properties, $-\vec{\pi}_B \cdot \vec{\sigma} = \vec{\pi}_A \cdot \vec{\sigma}^*$ and $-\vec{\pi}_B \cdot \vec{\sigma}^* = \vec{\pi}_A \cdot \vec{\sigma}$, the relations in Eq.(124) hold. But we do not have similar relations with the Σ matrices.

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