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Article

Towards Finding Equalities Involving Mixed Products of the Moore–Penrose and Group Inverses of a Matrix

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Abstract: Given a square matrix A, we are able to construct numerous equalities that involve reasonable mixed operations of A and its conjugate transpose A^*, Moore–Penrose inverse $A^\dagger$, and group inverse $A^\#$. Such kind of equalities can be generally represented in the equation form $f(A, A^*, A^\dagger, A^\#) = 0$. In this article, the author constructs a series of simple or complicated matrix equalities, as well as matrix rank equalities involving the mixed operations of the four matrices. As applications, we give a sequence of necessary and sufficient conditions for a square matrix to be range-Hermitian.

Keywords: block matrix; group inverse; Moore–Penrose inverse; range; rank; reverse-order law

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1. Introduction

Throughout this article, let $\mathbb{C}^{m\times n}$ stand for the set of all $m \times n$ matrices over the field of complex numbers; A^* denote the conjugate transpose of $A \in \mathbb{C}^{m\times n}$; $r(A)$ stand for the rank of $A \in \mathbb{C}^{m\times n}$; I_m stand for the identity matrix of order m, $[\,A,\, B]$ stand for a columnwise partitioned matrix consisting of two submatrices A and B. We introduce the concepts of generalized inverses of a matrix. For an $A \in \mathbb{C}^{m\times n}$, the Moore–Penrose generalized inverse of A is defined to be the unique matrix $X \in \mathbb{C}^{n\times m}$ that satisfies the four Penrose equations

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA. \tag{1.1}$$

A matrix $X \in \mathbb{C}^{n\times m}$ is called a $\{i,\ldots,j\}$-generalized inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^{(i,\ldots,j)}$, if it satisfies the ith,$\ldots$, jth matrix equations in (1.1). Notice that solutions of each of the four equations in (1.1) are not necessarily unique. Hence, the set of all $\{i,\ldots,j\}$-generalized inverses of the matrix A is usually denoted by the symbol $\{A^{(i,\ldots,j)}\}$. There are altogether fifteen sorts of $\{i,\ldots,j\}$-generalized inverses of the matrix A by the above definition. In addition, a square matrix $A \in \mathbb{C}^{m\times m}$ is said to be group invertible if and only if there exists a matrix $X \in \mathbb{C}^{m\times m}$ that satisfies the following three matrix equations

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (5)\ AX = XA. \tag{1.2}$$

In such a case, the matrix X, called the group inverse of A, is unique and is denoted by $X = A^\#$. The Moore–Penrose inverse and the group inverse of a matrix were defined and approached in 1950s, which now are well known in matrix theory and its applications due to their unreplaceable role in dealing with singular matrices. For more basic results and facts concerning generalized inverses of matrices and their properties, we refer the reader to the three references [2–4].

As we know in matrix algebra that constructions, classifications, and characterizations of various algebraic matrix equalities have been attractive issues because their unreplaceable roles in in matrix analysis and its applications. Here, we mention an illustrated example. For a given a square matrix A, we are able to construct any possible matrix equalities involving reasonable operations of A, the conjugate transpose A^*, the Moore–Penrose inverse $A^\dagger$, and the group inverse $A^\#$. As usual in algebra theory, we can represent them in an general equation form

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$$f(A, A^*, A^\dagger, A^\#) = 0. \tag{1.3}$$

Clearly, matrix equalities as such provide a gold mine of fascinating research problems in the theory of generalized inverses, which was realized and studied since the establishments of the theory of generalized inverses. For the sake of exposition, let us mention several simple and nontrivial matrix equalities that are constructed from the ordinary algebraic operations of a matrix and its generalized inverses:

$$(A^2)^\dagger = (A^\dagger)^2,\ (A^2)^\dagger = A^*(A^*A^2A^*)^\dagger A^*,\ (A^3)^\dagger = (A^2)^\dagger A(A^2)^\dagger,\ AA^\dagger = A^\dagger A,$$

$$A^\dagger = A^\#,\ A^\dagger A^\# = A^\# A^\dagger,\ (A^2)^\dagger = A^*(A^*A^2A^*)^\# A^*,\ (AA^\#)^* = AA^\#,$$

$$(A^\# A^\dagger)^\dagger = A(A^\#)^\dagger,\ (A^\# A^\dagger A^\#)^\dagger = (A^\#)^\dagger A(A^\#)^\dagger,\ (A^\dagger A^\# A^\dagger)^\dagger = A(A^\#)^\dagger A$$

as reasonable representatives in the context of (1.3). As illustrated, algebraists can construct a tremendous number of simple or complicated matrix equalities with the form in (1.3) from theoretical points of view. Once a matrix equality as such is formulated, a subsequent work is to determine whether it holds. However, it can be figured out from the facts that the matrix algebras are noncommutative and a given matrix may be singular. Thus, characterizations of matrix equalities as such involve lots of difficult and divisive issues related to simplifications of various algebraic operations of matrices and their generalized inverses, and many preparations with matrix analysis tools are needed to finish the task. It should be pointed out that there are numerous results and facts that can be found in the literature concerning matrix equalities for generalized inverses and related issues, see e.g., [6–10]. The the present author realized the importance of this fundamental work in 1980s and has been paying attention to the constructions, classifications, and characterizations of many kinds of algebraic matrix equalities that involve generalized inverses, see e.g., [5,11–14]). Especially in a recent article [15], the present author revisited the characterization problem of the group invertibility of a matrix and presented several hundreds of formulas, results, and facts regarding mixed operations of a matrix and its Moore–Penrose inverses and group inverses.

The objective of this paper is to construct and classify a diversity of reasonable matrix equalities composed of A, A^*, $A^\dagger$, $A^\#$ and their algebraic operations. The rest of this article is organized as follows. In Section 2, we introduce some preliminary results and facts on ranks, ranges, and generalized inverses of matrices. In Section 3, we show how to construct a diverse of reasonable matrix equalities involving mixed products of A, A^*, $A^\dagger$, and $A^\#$ by definitions of generalized inverses and matrix rank equalities. As applications, we give a sequence of necessary and sufficient conditions for a square matrix to be range-Hermitian. Some conclusions and remarks are given Section 4.

2. Some preliminary results

In this section, we introduce a selection of existing formulas and facts related to ranks, ranges, and generalized inverses of matrices (cf. [2–4,15]) and shall use them in establishments and characterizations of matrix equalities as described in Section 1.

Lemma 2.1. *Let $A \in \mathbb{C}^{m\times n}$. Then,*

$$(A^\dagger)^* = (A^*)^\dagger, \quad (A^\dagger)^\dagger = A, \tag{2.1}$$

$$A^\dagger = A^*(AA^*)^\dagger = (A^*A)^\dagger A^* = A^*(A^*AA^*)^\dagger A^*, \tag{2.2}$$

$$(A^*)^\dagger A^* = (AA^\dagger)^* = AA^\dagger,\ A^*(A^*)^\dagger = (A^\dagger A)^* = A^\dagger A, \tag{2.3}$$

$$(AA^*)^\dagger = (A^\dagger)^* A^\dagger,\ (A^*A)^\dagger = A^\dagger (A^\dagger)^*,\ (AA^*A)^\dagger = A^\dagger (A^\dagger)^* A^\dagger. \tag{2.4}$$

Lemma 2.2. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$, $P \in \mathbb{C}^{p\times m}$, and $Q \in \mathbb{C}^{q\times n}$. Then,*

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$$
\begin{aligned}
& r(AA^*A) = r(AA^*) = r(A^*A) = r(A^\dagger) = r(A), && (2.5)\\
& \mathscr{R}(AA^*A) = \mathscr{R}(AA^*) = \mathscr{R}(AA^\dagger) = \mathscr{R}((A^\dagger)^*) = \mathscr{R}(A), && (2.6)\\
& \mathscr{R}(A^*AA^*) = \mathscr{R}(A^*A) = \mathscr{R}(A^\dagger A) = \mathscr{R}(A^\dagger) = \mathscr{R}(A^*), && (2.7)\\
& \mathscr{R}(AQ^\dagger Q) = \mathscr{R}(AQ^\dagger) = \mathscr{R}(AQ^*Q) = \mathscr{R}(AQ^*), && (2.8)\\
& \mathscr{R}(A) \subseteq \mathscr{R}(B) \ \text{and} \ r(A) = r(B) \Rightarrow \mathscr{R}(A) = \mathscr{R}(B), && (2.9)\\
& \mathscr{R}(A) \subseteq \mathscr{R}(B) \Rightarrow \mathscr{R}(PA) \subseteq \mathscr{R}(PB), && (2.10)\\
& \mathscr{R}(A) = \mathscr{R}(B) \Rightarrow \mathscr{R}(PA) = \mathscr{R}(PB), && (2.11)\\
& \mathscr{R}(A) \cap \mathscr{R}(I_m - AA^\dagger) = \{0\}, && (2.12)\\
& \mathscr{R}(A^\dagger) \cap \mathscr{R}(I_n - A^\dagger A) = \{0\}. && (2.13)
\end{aligned}
$$

Lemma 2.3. *Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{m\times n}$. Then, the following matrix rank and range inequalities*

$$
\begin{aligned}
& r(A) \geq r(A^2) \geq \cdots \geq r(A^k), && (2.14)\\
& \mathscr{R}(A) \supseteq \mathscr{R}(A^2) \supseteq \cdots \supseteq \mathscr{R}(A^k) && (2.15)
\end{aligned}
$$

hold for any integer $k \geq 2$, and the following matrix rank equalities

$$r(A^2) = r((A^2)^\dagger) = r((A^\dagger)^2) \tag{2.16}$$

hold. In particular, the following facts

$$
\begin{aligned}
& r(A) = r(A^2) \Leftrightarrow r(A) = r(A^k) \Leftrightarrow r((A^\dagger)^2) = r(A) \Leftrightarrow r((A^\dagger)^k) = r(A), && (2.17)\\
& r(A) = r(A^2) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(A^2) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(A^k), && (2.18)\\
& r(A) = r(A^2) \Leftrightarrow \mathscr{R}(A^*) = \mathscr{R}((A^2)^*) \Leftrightarrow \mathscr{R}(A^*) = \mathscr{R}((A^k)^*) && (2.19)
\end{aligned}
$$

hold for any integer $k \geq 2$, and the following fact

$$r(A) = r(A^2) \Rightarrow r(AB) = r(A^kB) \tag{2.20}$$

holds for any integer $k \geq 2$.

Lemma 2.4. *Let $A \in \mathbb{C}^{m\times m}$. Then, A is group invertible if and only if $r(A^2) = r(A)$. In this case, the following equalities hold:*

$$
\begin{aligned}
& (A^\#)^* = (A^*)^\#, && (2.21)\\
& (A^\#)^\# = A, && (2.22)\\
& A^\# = AA^\dagger A^\# = A^\# A^\dagger A = AA^\dagger A^\# A^\dagger A, && (2.23)
\end{aligned}
$$

and

$$
\begin{aligned}
& A^\# = A(A^2)^\dagger A(A^2)^\dagger A, && (2.24)\\
& A^\# = A(A^3)^\dagger A, && (2.25)\\
& A^\# = (A^\dagger A^3 A^\dagger)^\dagger = AA^\dagger (A^\dagger A^3 A^\dagger)^\dagger A^\dagger A, && (2.26)\\
& A^\# = (A^2A^\dagger)^\dagger A(A^\dagger A^2)^\dagger = AA^\dagger (A^2A^\dagger)^\dagger A(A^\dagger A^2)^\dagger A^\dagger A, && (2.27)\\
& A^\# = (A^\dagger A^2 A^\dagger)^\dagger A^\dagger (A^\dagger A^2 A^\dagger)^\dagger = AA^\dagger (A^\dagger A^2 A^\dagger)^\dagger A^\dagger (A^\dagger A^2 A^\dagger)^\dagger A^\dagger A, && (2.28)\\
& A^\# = (A^3A^\dagger)^\dagger A^3 (A^\dagger A^3)^\dagger = AA^\dagger (A^3A^\dagger)^\dagger A^3 (A^\dagger A^3)^\dagger A^\dagger A, && (2.29)\\
& A^\# = A(A^\dagger A^3)^\dagger A(A^3A^\dagger)^\dagger A, && (2.30)\\
& AA^\# = A(A^2)^\dagger A; && (2.31)
\end{aligned}
$$

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the following rank and range equalities hold:

$$r(AA^{\#}) = r(A^{\#}A) = r(A^{\#}) = r(A), \tag{2.32}$$
$$r(A^{\dagger}A^{\#}) = r(A^{\#}A^{\dagger}) = r(A^{\dagger}AA^{\#}) = r(A^{\#}AA^{\dagger}) = r((A^{\dagger}AA^{\#})^2) = r((A^{\#}AA^{\dagger})^2) = r(A), \tag{2.33}$$
$$\mathscr{R}(AA^{\#}) = \mathscr{R}(A^{\#}A) = \mathscr{R}(A^{\#}) = \mathscr{R}(A), \tag{2.34}$$
$$\mathscr{R}((AA^{\#})^*) = \mathscr{R}((A^{\#}A)^*) = \mathscr{R}((A^{\#})^*) = \mathscr{R}((A^*)^{\#}) = \mathscr{R}(A^*), \tag{2.35}$$
$$\mathscr{R}(A^{\#}A^{\dagger}) = \mathscr{R}(A^{\#}AA^{\dagger}) = \mathscr{R}((A^{\#}AA^{\dagger})^2) = \mathscr{R}(A), \tag{2.36}$$
$$\mathscr{R}(A^{\dagger}A^{\#}) = \mathscr{R}(A^{\dagger}AA^{\#}) = \mathscr{R}((A^{\dagger}AA^{\#})^2) = \mathscr{R}(A^*), \tag{2.37}$$

and the following equalities hold:

$$\mathscr{R}(A) \cap \mathscr{R}(I_m - AA^{\#}) = \{0\}, \tag{2.38}$$
$$\mathscr{R}(A^*) \cap \mathscr{R}(I_m - (AA^{\#})^*) = \{0\}. \tag{2.39}$$

Lemma 2.5. *Let $A_1 \in \mathbb{C}^{m\times n_1}$, $A_2 \in \mathbb{C}^{m\times n_2}$, $B_1 \in \mathbb{C}^{m\times p_1}$, and $B_2 \in \mathbb{C}^{m\times p_2}$. If $\mathscr{R}(A_1) = \mathscr{R}(B_1)$ and $\mathscr{R}(A_2) = \mathscr{R}(B_2)$, then two equalities $\mathscr{R}[A_1,\ A_2] = \mathscr{R}[B_1,\ B_2]$ and $r[A_1,\ A_2] = r[B_1,\ B_2]$ hold.*

Recall a basic fact in linear algebra that a matrix is null if and only if its rank is zero. As a direct consequence of this assertion, we see that two matrices A and B of the same size coincide if and only if $r(A - B) = 0$, in other words, this equivalent statement connects the matrix equality $A = B$ alternatively to the matrix rank equality $r(A - B) = 0$. In view of this obvious fact, we may figure out that if certain nontrivial analytical formulas for calculating the rank of $A - B$ are obtained, they can reasonably be utilized to interpret essential links between the two matrices and to characterize the matrix quality $A = B$ in a convenient manner. The usefulness of this proposed methodology is based on the fact that we are able to precisely calculate the rank of matrix by elementary operations of matrices. Now matrix rank formulas have been highly recognized as skillful and reliable techniques to construct, characterize, and understand intrinsic structures and properties of various simple or complex algebraic equalities for matrices and their operations. According to the above assertions, we see that $f(A,\ A^*, A^{\dagger}, A^{\#}) = 0$ if and only if $r(f(A,\ A^*, A^{\dagger}, A^{\#})) = 0$. Thus, if certain analytical formulas are developed for calculating the rank of $f(A,\ A^*, A^{\dagger}, A^{\#})$, we can derive necessary and sufficient conditions for $f(A,\ A^*, A^{\dagger}, A^{\#}) = 0$ to hold from the rank formulas. This fact was proved to be a magnificent and effective mathematical analysis tool to construct and describe tremendous matrix equalities that involve generalized inverses of matrices. As a matter of fact, algebraists have developed since 1960s a large number of matrix rank equalities with positive perspective in the constructions, classifications, and characterizations of matrix equalities that involve generalized inverses of matrices (cf. [5,10,11,14–16]). Now matrix rank formulas have been already regarded as a useful and effective methodology to construct, understand, and reveal intrinsic structures and performances of algebraic expressions and equalities for matrices and their operations. As necessary preparations used in the sequel, we give below a selection of existing matrix rank formulas in the following two lemmas (cf. [10,11]).

Lemma 2.6. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, and $D \in \mathbb{C}^{l\times k}$. Then,*

$$r[A,\ B] = r(A) + r(B - AA^{\dagger}B) = r(B) + r(A - BB^{\dagger}A), \tag{2.40}$$
$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C - CA^{\dagger}A) = r(C) + r(A - AC^{\dagger}C), \tag{2.41}$$
$$r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} = r(A) + r(D - CA^{\dagger}B). \tag{2.42}$$

Therefore,

$$r\begin{bmatrix} A^*AA^* & A^*B \\ CA^* & D \end{bmatrix} = r(A) \Leftrightarrow CA^{\dagger}B = D.$$

In particular, if $\mathscr{R}(A) \supseteq \mathscr{R}(B)$ *and* $\mathscr{R}(A^*) \supseteq \mathscr{R}(C^*)$*, then*

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) + r(D - CA^{\dagger}B). \tag{2.43}$$

Therefore,

$$r\begin{bmatrix} A & B \\ C & D \end{bmatrix} = r(A) \Leftrightarrow CA^{\dagger}B = D.$$

Lemma 2.7. *Let* $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times k}$*, and* $C \in \mathbb{C}^{l\times m}$*, and assume that* A *is group invertible. Then,*

$$r[A,\, B] = r(A) + r(B - AA^{\#}B), \tag{2.44}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(C - CA^{\#}A). \tag{2.45}$$

Lemma 2.8 [10]**).** *Let* $A,\, B \in \mathbb{C}^{m\times n}$*. Then, the following matrix rank inequality*

$$r(A+B) \geq r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\, B] - r(A) - r(B) \tag{2.46}$$

holds. Especially,

$$r(A+B) = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) \cap \mathscr{R}(B) = \{0\} \text{ and } \mathscr{R}(A^*) \cap \mathscr{R}(B^*) = \{0\}. \tag{2.47}$$

Lemma 2.9 [13]**).** *Let* $A,\, B \in \mathbb{C}^{m\times n}$*, and assume that* $AXA = A$ *and* $BXB = B$ *hold for an* $X \in \mathbb{C}^{n\times m}$*, namely,* $A,\, B \in \{X^{(2)}\}$*. Then, the following matrix rank equality*

$$r(A-B) = r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\, B] - r(A) - r(B) \tag{2.48}$$

holds. Therefore,

$$A = B \Leftrightarrow r\begin{bmatrix} A \\ B \end{bmatrix} + r[A,\, B] = r(A) + r(B) \Leftrightarrow \mathscr{R}(A) = \mathscr{R}(B) \text{ and } \mathscr{R}(A^*) = \mathscr{R}(B^*). \tag{2.49}$$

Lemma 2.10 [11]**).** *Let* $A \in \mathbb{C}^{m\times n}$ *and* $B \in \mathbb{C}^{n\times p}$*. Then*

$$r(AB - ABB^{\dagger}A^{\dagger}AB) = r[A^*,\, B] - r(A) - r(B) + r(AB), \tag{2.50}$$

$$r((AB)^{\dagger} - B^{\dagger}A^{\dagger}) = r\left(\begin{bmatrix} ABB^* \\ A \end{bmatrix}[A^*AB,\, B]\right) - r(AB). \tag{2.51}$$

Hence,

$$ABB^{\dagger}A^{\dagger}AB = AB \Leftrightarrow r[A^*,\, B] = r(A) + r(B) - r(AB), \tag{2.52}$$

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow r\left(\begin{bmatrix} ABB^* \\ A \end{bmatrix}[A^*AB,\, B]\right) = r(AB). \tag{2.53}$$

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Corollary 2.11. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Then, the following two rank equalities*

$$r((A^*A)^{1/2}(BB^*)^{1/2} - (A^*A)^{1/2}(BB^*)^{1/2}((BB^*)^{1/2})^{\dagger}((A^*A)^{1/2})^{\dagger}(A^*A)^{1/2}(BB^*)^{1/2}) = r[A^*,\, B] - r(A) - r(B) + r(AB), \tag{2.54}$$

$$r(((A^*A)^{1/2}(BB^*)^{1/2})^{\dagger} - ((BB^*)^{1/2})^{\dagger}((A^*A)^{1/2})^{\dagger}) = r\left(\begin{bmatrix} ABB^* \\ A \end{bmatrix}[A^*AB,\, B]\right) - r(AB) \tag{2.55}$$

hold. Hence,

$$(A^*A)^{1/2}(BB^*)^{1/2}((BB^*)^{1/2})^{\dagger}((A^*A)^{1/2})^{\dagger}(A^*A)^{1/2}(BB^*)^{1/2} = (A^*A)^{1/2}(BB^*)^{1/2} \Leftrightarrow r[A^*,\, B] = r(A) + r(B) - r(AB), \tag{2.56}$$

$$((A^*A)^{1/2}(BB^*)^{1/2})^{\dagger} = ((BB^*)^{1/2})^{\dagger}((A^*A)^{1/2})^{\dagger} \Leftrightarrow r\left(\begin{bmatrix} ABB^* \\ A \end{bmatrix}[A^*AB,\, B]\right) = r(AB). \tag{2.57}$$

The following two rank equalities

$$r(ABB^*A^*AB - ABB^*A^*AB(A^*AB)^{\dagger}(ABB^*)^{\dagger}ABB^*A^*AB) = r((ABB^*A^*AB)^{\dagger} - (A^*AB)^{\dagger}(ABB^*)^{\dagger}) = r[A^*ABB^*, BB^*A^*A] - r(AB) \tag{2.58}$$

hold. Hence,

$$ABB^*A^*AB(A^*AB)^{\dagger}(ABB^*)^{\dagger}ABB^*A^*AB = ABB^*A^*AB \Leftrightarrow (ABB^*A^*AB)^{\dagger} = (A^*AB)^{\dagger}(ABB^*)^{\dagger} \Leftrightarrow \mathscr{R}(A^*ABB^*) = \mathscr{R}(BB^*A^*A). \tag{2.59}$$

The following two rank equalities

$$r((A^*ABB^*)^2 - (A^*ABB^*)^2((A^*ABB^*)^{\dagger})^2(A^*ABB^*)^2) = r(((A^*ABB^*)^2)^{\dagger} - ((A^*ABB^*)^{\dagger})^2) = r[A^*ABB^*, BB^*A^*A] - r(AB) \tag{2.60}$$

hold. Hence,

$$(A^*ABB^*)^2((A^*ABB^*)^{\dagger})^2(A^*ABB^*)^2 = (A^*ABB^*)^2 \Leftrightarrow ((A^*ABB^*)^2)^{\dagger} = ((A^*ABB^*)^{\dagger})^2 \Leftrightarrow \mathscr{R}(A^*ABB^*) = \mathscr{R}(BB^*A^*A). \tag{2.61}$$

Lemma 2.12 [2]). *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times p}$. Then*

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} \Leftrightarrow \mathscr{R}(A^*ABB^*) = \mathscr{R}(BB^*A^*A). \tag{2.62}$$

3. Constructions of matrix equalities for the Moore–Penrose inverse and the group inverse of a matrix

In this section, we first show a group of fundamental equalities for mixed products of the Moore–Penrose inverse and the group inverse of a matrix.

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Theorem 3.1. *Let $A \in \mathbb{C}^{m\times m}$ and assume that $r(A^2) = r(A)$. Then, the following equalities*

$$A^\dagger AA^\# \in \{A^{(1,2)}\}, \tag{3.1}$$
$$A^\# AA^\dagger \in \{A^{(1,2)}\}, \tag{3.2}$$
$$A^\dagger A^\# A^\dagger \in \{(A^3)^{(1,2)}\}, \tag{3.3}$$
$$A^\# A^\dagger A^\# \in \{(A^3)^{(1,2)}\}, \tag{3.4}$$
$$A^\dagger(A^\# AA^\dagger)^\dagger A^\# \in \{A^{(1,2)}\}, \tag{3.5}$$
$$A^\#(A^\dagger AA^\#)^\dagger A^\dagger \in \{A^{(1,2)}\}, \tag{3.6}$$
$$A^\dagger(A^\# AA^\dagger)^\# A^\# \in \{A^{(1,2)}\}, \tag{3.7}$$
$$A^\#(A^\dagger AA^\#)^\# A^\dagger \in \{A^{(1,2)}\}, \tag{3.8}$$
$$(A^\dagger AA^\#)^2 \in \{(A^2)^{(1,2)}\}, \tag{3.9}$$
$$(A^\# AA^\dagger)^2 \in \{(A^2)^{(1,2)}\}, \tag{3.10}$$
$$A^\#(A^2)^\dagger A^\# \in \{(A^4)^{(1,2)}\}, \tag{3.11}$$
$$A^\dagger(A^2)^\# A^\dagger \in \{(A^4)^{(1,2)}\}, \tag{3.12}$$
$$(A^2)^\# A^\dagger(A^2)^\# \in \{(A^5)^{(1,2)}\}, \tag{3.13}$$
$$(A^2)^\dagger A^\#(A^2)^\dagger \in \{(A^5)^{(1,2)}\}, \tag{3.14}$$
$$A^\#(A^3)^\dagger A^\# \in \{(A^5)^{(1,2)}\}, \tag{3.15}$$
$$A^\dagger(A^3)^\# A^\dagger \in \{(A^5)^{(1,2)}\}, \tag{3.16}$$
$$A^\#(A^2)^\dagger A^\#(A^2)^\dagger A^\# \in \{(A^7)^{(1,2)}\}, \tag{3.17}$$
$$A^\dagger(A^2)^\# A^\dagger(A^2)^\# A^\dagger \in \{(A^7)^{(1,2)}\}, \tag{3.18}$$
$$A^\#(A^3)^\dagger A^\#(A^3)^\dagger A^\# \in \{(A^9)^{(1,2)}\}, \tag{3.19}$$
$$A^\dagger(A^3)^\# A^\dagger(A^3)^\# A^\dagger \in \{(A^9)^{(1,2)}\}, \tag{3.20}$$

$$A^\dagger(A^\dagger A^2 A^\dagger)^\dagger A^\dagger \in \{(A^2)^{(1,2)}\}, \tag{3.21}$$
$$A^\dagger(A^\dagger A^2 A^\dagger)^\# A^\dagger \in \{(A^2)^{(1,2)}\}, \tag{3.22}$$
$$A^*(A^* A^2 A^*)^\dagger A^* \in \{(A^2)^{(1,2)}\}, \tag{3.23}$$
$$A^*(A^* A^2 A^*)^\# A^* \in \{(A^2)^{(1,2)}\} \tag{3.24}$$

hold, and the following equalities

$$(A^\dagger A^\#)^k \in \{(A^{2k})^{(1,2)}\}, \tag{3.25}$$
$$(A^\# A^\dagger)^k \in \{(A^{2k})^{(1,2)}\}, \tag{3.26}$$
$$(A^\dagger AA^\#)^k \in \{(A^k)^{(1,2)}\}, \tag{3.27}$$
$$(A^\# AA^\dagger)^k \in \{(A^k)^{(1,2)}\}, \tag{3.28}$$
$$(A^k)^\dagger(A^s)^\#(A^t)^\dagger \in \{(A^{k+s+t})^{(1,2)}\}, \tag{3.29}$$
$$(A^k)^\#(A^s)^\dagger(A^t)^\# \in \{(A^{k+s+t})^{(1,2)}\} \tag{3.30}$$

hold for any integers $k, s, t \geq 1$.

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Proof. It is easy to verify by the definitions of the Moore–Penrose inverse and group inverse, (2.25), and (2.23) that

$$
\begin{aligned}
&A(A^{\dagger}AA^{\#})A = AA^{\#}A = A,\\
&(A^{\dagger}AA^{\#})A(A^{\dagger}AA^{\#}) = A^{\dagger}AA^{\dagger}AA^{\#} = A^{\dagger}AA^{\#},\\
&A(A^{\#}AA^{\dagger})A = AA^{\#}A = A,\\
&(A^{\#}AA^{\dagger})A(A^{\#}AA^{\dagger}) = A^{\#}AA^{\#}AA^{\dagger} = A^{\#}AA^{\dagger},\\
&A^3A^{\dagger}A^{\#}A^{\dagger}A^3 = A^2A^{\#}A^2 = A^3,\\
&A^{\dagger}A^{\#}A^{\dagger}A^3A^{\dagger}A^{\#}A^{\dagger} = A^{\dagger}A^{\#}AA^{\#}A^{\dagger} = A^{\dagger}A^{\#}A^{\dagger},\\
&A^3A^{\#}A^{\dagger}A^{\#}A^3 = A^2A^{\dagger}A^2 = A^3,\\
&A^{\#}A^{\dagger}A^{\#}A^3A^{\#}A^{\dagger}A^{\#} = A^{\#}A^{\dagger}AA^{\dagger}A^{\#} = A^{\#}A^{\dagger}A^{\#},\\
&AA^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#}A = AA^{\#}AA^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#}AA^{\dagger}A = AA^{\#}AA^{\dagger}A = A,\\
&A^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#}AA^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#} = A^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#},\\
&AA^{\#}(A^{\dagger}AA^{\#})^{\dagger}A^{\dagger}A = AA^{\dagger}AA^{\#}(A^{\dagger}AA^{\#})^{\dagger}A^{\dagger}AA^{\#}A = AA^{\dagger}AA^{\#}A = A,\\
&A^{\#}(A^{\dagger}AA^{\#})^{\dagger}A^{\dagger}AA^{\#}(A^{\dagger}AA^{\#})^{\dagger}A^{\dagger} = A^{\#}(A^{\dagger}AA^{\#})^{\dagger}A^{\dagger},\\
&AA^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#}A = AA^{\#}AA^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#}AA^{\dagger}A = AA^{\#}AA^{\dagger}A = A,\\
&A^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#}AA^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#} = A^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#},\\
&AA^{\#}(A^{\dagger}AA^{\#})^{\#}A^{\dagger}A = AA^{\dagger}AA^{\#}(A^{\dagger}AA^{\#})^{\#}A^{\dagger}AA^{\#}A = AA^{\dagger}AA^{\#}A = A,\\
&A^{\#}(A^{\dagger}AA^{\#})^{\#}A^{\dagger}AA^{\#}(A^{\dagger}AA^{\#})^{\#}A^{\dagger} = A^{\#}(A^{\dagger}AA^{\#})^{\#}A^{\dagger},\\
&A^2(A^{\dagger}AA^{\#})^2A^2 = A^2A^{\#}A^{\dagger}A^2 = AA^{\dagger}A^2 = A^2,\\
&(A^{\dagger}AA^{\#})^2A^2(A^{\dagger}AA^{\#})^2 = A^{\dagger}AA^{\#}A^{\#}A^2A^{\#}A^{\#} = A^{\dagger}A^{\#} = (A^{\dagger}AA^{\#})^2,\\
&A^2(A^{\#}AA^{\dagger})^2A^2 = A^2A^{\dagger}A^{\#}A^2 = A^2A^{\dagger}A = A^2,\\
&(A^{\#}AA^{\dagger})^2A^2(A^{\#}AA^{\dagger})^2 = A^{\#}A^{\#}A^2A^{\#}AA^{\#}AA^{\dagger} = A^{\#}A^{\dagger} = (A^{\#}AA^{\dagger})^2,\\
&(A^{\dagger}A^{\#})^2A^4(A^{\dagger}A^{\#})^2 = A^{\dagger}(A^{\#})^3A^4A^{\dagger}(A^{\#})^3 = A^{\dagger}(A^{\#})^3 = (A^{\dagger}A^{\#})^2,\\
&(A^{\#}A^{\dagger})^2A^4(A^{\#}A^{\dagger})^2 = (A^{\#})^3A^{\dagger}A^4(A^{\#})^3A^{\dagger} = (A^{\#})^3A^{\dagger} = (A^{\#}A^{\dagger})^2,\\
&A^4A^{\#}(A^2)^{\dagger}A^{\#}A^4 = A^3(A^2)^{\dagger}A^3 = A^4,\\
&A^{\#}(A^2)^{\dagger}A^{\#}A^4A^{\#}(A^2)^{\dagger}A^{\#} = A^{\#}(A^2)^{\dagger}A^2(A^2)^{\dagger}A^{\#} = A^{\#}(A^2)^{\dagger}A^{\#},\\
&A^4A^{\dagger}(A^2)^{\#}A^{\dagger}A^4 = A^4A^{\dagger}(A^2)^{\#}A^{\dagger}A^4 = A^3(A^2)^{\#}A^3 = A^4,\\
&A^{\dagger}(A^2)^{\#}A^{\dagger}A^4A^{\dagger}(A^2)^{\#}A^{\dagger} = A^{\dagger}(A^2)^{\#}A^2(A^2)^{\#}A^{\dagger}A^{\dagger}(A^2)^{\#}A^{\dagger},\\
&A^5(A^2)^{\#}A^{\dagger}(A^2)^{\#}A^5 = A^3A^{\dagger}A^3 = A^5,\\
&(A^2)^{\#}A^{\dagger}(A^2)^{\#}A^5(A^2)^{\#}A^{\dagger}(A^2)^{\#} = (A^2)^{\#}A^{\dagger}AA^{\dagger}(A^2)^{\#} = (A^2)^{\#}A^{\dagger}(A^2)^{\#},\\
&A^5(A^2)^{\dagger}A^{\#}(A^2)^{\dagger}A^5 = A^4A^{\dagger}A^{\#}A^{\dagger}A^4 = A^3A^{\#}A^3 = A^5,\\
&(A^2)^{\dagger}A^{\#}(A^2)^{\dagger}A^5(A^2)^{\dagger}A^{\#}(A^2)^{\dagger} = (A^2)^{\dagger}A^{\#}A^{\dagger}A^3A^{\dagger}A^{\#}(A^2)^{\dagger} = (A^2)^{\dagger}A^{\#}AA^{\#}(A^2)^{\dagger} = (A^2)^{\dagger}A^{\#}(A^2)^{\dagger}.
\end{aligned}
$$

These matrix equalities imply (3.1)–(3.16) by the definition in (1.1). Eqs. (3.17)–(3.30) can be shown by the definitions in (1.1), (1.2), as well as (2.25) and (2.23). The details are omitted. □

In the following, we first give a list of matrix rank equalities for the differences of two matrix expressions involving algebraic operations of a matrix and the Moore–Penrose inverses induced from the matrix. As direct consequences of the matrix rank equalities, we also derive a group of known or novel identifying conditions for a complex square matrix to be range-Hermitian.

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Theorem 3.2. *Let $A \in \mathbb{C}^{m\times m}$. Then the following matrix rank equalities hold*:

$$r(A - A^2A^\dagger) = r(A - A^\dagger A^2) = r[A,\, A^*] - r(A), \tag{3.31}$$

$$r(AA^\dagger - A^\dagger A) = 2r[A,\, A^*] - 2r(A), \tag{3.32}$$

$$r((I_m - AA^\dagger) - (I_m - A^\dagger A)) = 2r[I_m - AA^\dagger,\, I_m - A^\dagger A] - 2m + 2r(A), \tag{3.33}$$

$$r[(I_m - AA^\dagger)A^\dagger A,\, (I_m - A^\dagger A)AA^\dagger] = 2r[A,\, A^*] - 2r(A), \tag{3.34}$$

$$r(A^2 - A^2(A^\dagger)^2A^2) = r[A,\, A^*] - 2r(A) + r(A^2), \tag{3.35}$$

$$r((A^\dagger)^2 - (A^\dagger)^2A^2(A^\dagger)^2) = r[A,\, A^*] - 2r(A) + r(A^2), \tag{3.36}$$

$$r(A^\dagger A^2A^\dagger - (A^\dagger A^2A^\dagger)^2) = r[A,\, A^*] - 2r(A) + r(A^2), \tag{3.37}$$

$$r(A(A^\dagger)^2A - (A(A^\dagger)^2A)^2) = r[A,\, A^*] - 2r(A) + r(A^2), \tag{3.38}$$

$$r(A^4 - A^4((A^2)^\dagger)^2A^4) = r[A^2,\, (A^2)^*] - 2r(A^2) + r(A^4), \tag{3.39}$$

$$r(A - (A(A^\dagger)^3A)^\dagger) = r\begin{bmatrix} A \\ ((A^\dagger)^*)^3A^* \end{bmatrix} + r[A,\, A^*((A^\dagger)^*)^3] - r(A) - r((A^\dagger)^3), \tag{3.40}$$

$$r(A^\dagger - (A^\dagger A^3A^\dagger)^\dagger) = r\begin{bmatrix} A \\ A^3A^\dagger \end{bmatrix} + r[A,\, A^\dagger A^3] - r(A) - r(A^3), \tag{3.41}$$

$$r((A^2)^\dagger - (A^\dagger A^4A^\dagger)^\dagger) = r\begin{bmatrix} A^2 \\ A^4A^\dagger \end{bmatrix} + r[A^2,\, A^\dagger A^4] - r(A^2) - r(A^4), \tag{3.42}$$

$$r(A^\dagger - A(A^3)^\dagger A) = r\begin{bmatrix} A \\ A^3A^* \end{bmatrix} + r[A,\, A^*A^3] - r(A) - r(A^3), \tag{3.43}$$

$$r((A^2)^\dagger - A(A^4)^\dagger A) = r\begin{bmatrix} A^2 \\ A^4A^* \end{bmatrix} + r[A^2,\, A^*A^4] - r(A^2) - r(A^4), \tag{3.44}$$

$$r((A^2)^\dagger - (A^\dagger)^2) = r\left(\begin{bmatrix} (A^2)^*A \\ A^* \end{bmatrix}[A(A^2)^*, A^*]\right) - r(A^2), \tag{3.45}$$

$$r(A(A^2)^\dagger A - A(A^\dagger)^2A) = r\left(\begin{bmatrix} (A^2)^*A \\ AA^\dagger \end{bmatrix}[A(A^2)^*, A^\dagger A]\right) - r(A^2), \tag{3.46}$$

$$r(A^*A(A^2)^\dagger AA^* - (A^2)^*) = r\left(\begin{bmatrix} (A^2)^*A \\ A^* \end{bmatrix}[A(A^2)^*, A^*]\right) - r(A^2), \tag{3.47}$$

$$r((A^3)^\dagger - A^\dagger(A^2)^\dagger) = r\left(\begin{bmatrix} A^3A^* \\ A^2 \end{bmatrix}[(A^2)^*A^3,\, A]\right) - r(A^3), \tag{3.48}$$

$$r((A^3)^\dagger - (A^2)^\dagger A^\dagger) = r\left(\begin{bmatrix} A^3(A^2)^* \\ A \end{bmatrix}[A^*A^3,\, A^2]\right) - r(A^3), \tag{3.49}$$

$$r((A^4)^\dagger - ((A^2)^\dagger)^2) = r\left(\begin{bmatrix} (A^4)^*A^2 \\ (A^2)^\dagger \end{bmatrix}[A^2(A^4)^*, (A^2)^\dagger]\right) - r(A^4), \tag{3.50}$$

$$r(A(A^4)^\dagger A - A((A^2)^\dagger)^2A) = r\left(\begin{bmatrix} (A^4)^*A^2 \\ A(A^2)^\dagger \end{bmatrix}[A^2(A^4)^*, (A^2)^\dagger A]\right) - r(A^4), \tag{3.51}$$

$$r(A^2(A^4)^\dagger A^2 - A^2((A^2)^\dagger)^2A^2) = r\left(\begin{bmatrix} (A^4)^*A^2 \\ A^2(A^2)^\dagger \end{bmatrix}[A^2(A^4)^*, (A^2)^\dagger A^2]\right) - r(A^4), \tag{3.52}$$

$$r(A^3(A^4)^\dagger A^3 - A^3((A^2)^\dagger)^2A^3) = r\left(\begin{bmatrix} (A^4)^*A^2 \\ A^3(A^2)^\dagger \end{bmatrix}[A^2(A^4)^*, (A^2)^\dagger A^3]\right) - r(A^4), \tag{3.53}$$

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$$r(A^*A^2(A^4)^\dagger A^2A^* - A^*A^2((A^2)^\dagger)^2A^2A^*) = r\left(\begin{bmatrix}(A^4)^*A^2\\ A^*A^2(A^2)^\dagger\end{bmatrix}[A^2(A^4)^*, (A^2)^\dagger A^2A^*]\right) - r(A^4), \quad (3.54)$$

$$r(A^*A^2(A^4)^\dagger A^2A^* - (A^2)^*) = r\left(\begin{bmatrix}(A^4)^*A^2\\ A^*\end{bmatrix}[A^2(A^4)^*, A^*]\right) - r(A^4), \quad (3.55)$$

and

$$r(A^2(A^2)^*A^2 - A^2(A^2)^*A^2(A^2(A^2)^*A^2)^\dagger A^2(A^2)^*A^2) = r[A^*A^2A^*,\ A(A^*)^2A] - r(A^2), \quad (3.56)$$
$$r((A^2(A^2)^*A^2)^\dagger - (A^*A^2)^\dagger(A^2A^*)^\dagger) = r[A^*A^2A^*,\ A(A^*)^2A] - r(A^2), \quad (3.57)$$
$$r((A^*A^2A^*)^2 - (A^*A^2A^*)^2((A^*A^2A^*)^\dagger)^2(A^*A^2A^*)^2) = r[A^*A^2A^*,\ A(A^*)^2A] - r(A^2), \quad (3.58)$$
$$r(((A^*A^2A^*)^2)^\dagger - ((A^*A^2A^*)^\dagger)^2) = r[A^*A^2A^*,\ A(A^*)^2A] - r(A^2) \quad (3.59)$$

$$r((A^*A)^{1/2}(AA^*)^{1/2} - (A^*A)^{1/2}(AA^*)^{1/2}((AA^*)^{1/2})^\dagger((A^*A)^{1/2})^\dagger(A^*A)^{1/2}(AA^*)^{1/2})$$
$$= r[A,\ A^*] - 2r(A) + r(A^2), \quad (3.60)$$
$$r(((A^*A)^{1/2}(AA^*)^{1/2})^\dagger - ((AA^*)^{1/2})^\dagger((A^*A)^{1/2})^\dagger) = r\left(\begin{bmatrix}(A^2)^*A\\ A^*\end{bmatrix}[A(A^2)^*, A^*]\right) - r(A^2), \quad (3.61)$$

hold. Consequently, the following 32 conditions are equivalent:

⟨1⟩ $A^2A^\dagger = A$.

⟨2⟩ $A^\dagger A^2 = A$.

⟨3⟩ $AA^\dagger = A^\dagger A$.

⟨4⟩ $A^2(A^\dagger)^2A^2 = A^2$ *and* $r(A^2) = r(A)$.

⟨5⟩ $(A^\dagger)^2A^2(A^\dagger)^2 = (A^\dagger)^2$ *and* $r(A^2) = r(A)$.

⟨6⟩ $(A^\dagger A^2A^\dagger)^2 = A^\dagger A^2A^\dagger$ *and* $r(A^2) = r(A)$.

⟨7⟩ $(A(A^\dagger)^2A)^2 = A(A^\dagger)^2A$ *and* $r(A^2) = r(A)$.

⟨8⟩ $A^\dagger = A(A^3)^\dagger A$.

⟨9⟩ $(A^2)^\dagger = A(A^4)^\dagger A$ *and* $r(A^2) = r(A)$.

⟨10⟩ $(A^2)^\dagger = (A^\dagger)^2$ *and* $r(A^2) = r(A)$.

⟨11⟩ $A(A^2)^\dagger A = A(A^\dagger)^2A$ *and* $r(A^2) = r(A)$.

⟨12⟩ $A^*A(A^2)^\dagger AA^* = (A^2)^*$ *and* $r(A^2) = r(A)$.

⟨13⟩ $(A^3)^\dagger = A^\dagger(A^2)^\dagger$ *and* $r(A^2) = r(A)$.

⟨14⟩ $(A^3)^\dagger = (A^2)^\dagger A^\dagger$ *and* $r(A^2) = r(A)$.

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⟨15⟩ $(A^4)^\dagger = ((A^2)^\dagger)^2$ *and* $r(A^2) = r(A)$.

⟨16⟩ $A(A^4)^\dagger A = A((A^2)^\dagger)^2 A$ *and* $r(A^2) = r(A)$.

⟨17⟩ $A^2(A^4)^\dagger A^2 = A^2((A^2)^\dagger)^2 A^2$ *and* $r(A^2) = r(A)$.

⟨18⟩ $A^3(A^4)^\dagger A^3 = A^3((A^2)^\dagger)^2 A^3$ *and* $r(A^2) = r(A)$.

⟨19⟩ $A^*A^2(A^4)^\dagger A^2 A^* = (A^2)^*$ *and* $r(A^2) = r(A)$.

⟨20⟩ $A^\dagger A^3 A^\dagger = A$.

⟨21⟩ $A^\dagger A^4 A^\dagger = A^2$ *and* $r(A^2) = r(A)$.

⟨22⟩ $A^2(A^2)^*A^2(A^2(A^2)^*A^2)^\dagger A^2(A^2)^*A^2 = A^2(A^2)^*A^2$ *and* $r(A^2) = r(A)$.

⟨23⟩ $(A^2(A^2)^*A^2)^\dagger = (A^*A^2)^\dagger(A^2A^*)^\dagger$ *and* $r(A^2) = r(A)$.

⟨24⟩ $(A^*A^2A^*)^2((A^*A^2A^*)^\dagger)^2(A^*A^2A^*)^2 = (A^*A^2A^*)^2$ *and* $r(A^2) = r(A)$.

⟨25⟩ $((A^*A^2A^*)^2)^\dagger = ((A^*A^2A^*)^\dagger)^2$ *and* $r(A^2) = r(A)$.

⟨26⟩ $(A^*A)^{1/2}(AA^*)^{1/2}((AA^*)^{1/2})^\dagger((A^*A)^{1/2})^\dagger(A^*A)^{1/2}(AA^*)^{1/2} = (A^*A)^{1/2}(AA^*)^{1/2}$ *and* $r(A^2) = r(A)$.

⟨27⟩ $((A^*A)^{1/2}(AA^*)^{1/2})^\dagger = ((AA^*)^{1/2})^\dagger((A^*A)^{1/2})^\dagger$ *and* $r(A^2) = r(A)$.

⟨28⟩ $(I_m - AA^\dagger)A^\dagger A = (I_m - A^\dagger A)AA^\dagger = 0$.

⟨29⟩ $r[A,\, A^*] = r(A)$.

⟨30⟩ $r[I_m - AA^\dagger,\, I_m - A^\dagger A] = m - r(A)$.

⟨31⟩ $\mathscr{R}(A) = \mathscr{R}(A^*)$.

⟨32⟩ $\mathscr{R}(I_m - AA^\dagger) = \mathscr{R}(I_m - A^\dagger A)$.

Proof. The three rank equalities in (3.31) and (3.32) were shown in [12]. Notice by the definition in (1.1) that that $AA^\dagger$, $A^\dagger A$, $I_m - AA^\dagger$, and $I_m - A^\dagger A$ are idempotent matrices. In this case, we obtain from Lemma 2.9 and the corresponding matrix operations that

$$\begin{aligned} & r((I_m - AA^\dagger) - (I_m - A^\dagger A)) \\ & = r\begin{bmatrix} I_m - AA^\dagger \\ I_m - A^\dagger A \end{bmatrix} + r[I_m - AA^\dagger, I_m - A^\dagger A] - r(I_m - AA^\dagger) - r(I_m - A^\dagger A) \\ & = 2r[I_m - AA^\dagger,\, I_m - A^\dagger A] - 2m + 2r(A), \end{aligned}$$

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as required for (3.33). Since $AA^{\dagger} - A^{\dagger}A$ is Hermitian, it follows that $\mathscr{R}((AA^{\dagger} - A^{\dagger}A)^2) = \mathscr{R}(AA^{\dagger} - A^{\dagger}A)$. In this case, we obtain from Lemma 2.5 and (3.32) that

$$\begin{aligned}
& r[(I_m - AA^{\dagger})A^{\dagger}A,\ (I_m - A^{\dagger}A)AA^{\dagger}] \\
&= r[(I_m - AA^{\dagger})A^{\dagger}A + (I_m - A^{\dagger}A)AA^{\dagger},\ (I_m - A^{\dagger}A)AA^{\dagger}] \\
&= r[(AA^{\dagger} - A^{\dagger}A)^2,\ AA^{\dagger} - A^{\dagger}A^2A^{\dagger}] \\
&= r[AA^{\dagger} - A^{\dagger}A,\ (AA^{\dagger} - A^{\dagger}A)AA^{\dagger}] \\
&= r[AA^{\dagger} - A^{\dagger}A,\ 0] = 2r[A,\ A^*] - 2r(A),
\end{aligned}$$

as required for (3.34). Replacing B with A in (2.50) leads to (3.35); replacing A with $A^{\dagger}$ in (2.50) and simplifying by (2.1) and (2.16) lead to (3.36); replacing A with A^2 in (2.50) leads to (3.36).

Pre- and post-multiplying $A^2 - A^2(A^{\dagger})^2A^2$ with $A^{\dagger}$ leads to the rank equality $r(A^2 - A^2(A^{\dagger})^2A^2) = r(A^{\dagger}A^2A^{\dagger} - (A^{\dagger}A^2A^{\dagger})^2)$, thus establishing (3.37) from (3.35). Pre- and post-multiplying $(A^{\dagger})^2 - (A^{\dagger})^2A^2(A^{\dagger})^2$ with A leads to the rank equality $r((A^{\dagger})^2 - (A^{\dagger})^2A^2(A^{\dagger})^2) = r(A(A^{\dagger})^2A - (A(A^{\dagger})^2A)^2)$, thus establishing (3.38) from (3.36).

It is easy to verify by the definition in (1.1) that $A,\ (A(A^{\dagger})^3A)^{\dagger} \in \{(A^{\dagger})^{(2)}\}$. Then, we obtain from Lemma 2.9 and the corresponding matrix operations that

$$\begin{aligned}
r(A - (A(A^{\dagger})^3A)^{\dagger}) &= r\begin{bmatrix} A \\ (A(A^{\dagger})^3A)^{\dagger} \end{bmatrix} + r[A,\ (A(A^{\dagger})^3A)^{\dagger}] - r(A) - r((A(A^{\dagger})^3A)^{\dagger}) \\
&= r\begin{bmatrix} A \\ (A(A^{\dagger})^3A)^{*} \end{bmatrix} + r[A,\ (A(A^{\dagger})^3A)^{*}] - r(A) - r(A(A^{\dagger})^3A) \\
&= r\begin{bmatrix} A \\ ((A^{\dagger})^{*})^3A^{*} \end{bmatrix} + r[A,\ A^{*}((A^{\dagger})^{*})^3] - r(A) - r((A^{\dagger})^3),
\end{aligned}$$

as required for (3.40). Replacing A with $A^{\dagger}$ in (3.40) and simplifying lead to (3.41).

It is easy to verify by the definition in (1.1) that $(A^2)^{\dagger},\ (A(A^{\dagger})^3A)^{\dagger} \in \{(A^2)^{(2)}\}$. Then, we obtain from Lemma 2.9 and the corresponding matrix operations that

$$\begin{aligned}
r((A^2)^{\dagger} - (A^{\dagger}A^4A^{\dagger})^{\dagger}) &= r\begin{bmatrix} (A^2)^{\dagger} \\ (A^{\dagger}A^4A^{\dagger})^{\dagger}A^4A^{\dagger} \end{bmatrix} + r[(A^2)^{\dagger},\ (A^{\dagger}A^4A^{\dagger})^{\dagger}] - r((A^2)^{\dagger}) - r((A^{\dagger}A^4A^{\dagger})^{\dagger}) \\
&= r\begin{bmatrix} (A^2)^{*} \\ (A^{\dagger}A^4A^{\dagger})^{*}A^4A^{*} \end{bmatrix} + r[(A^2)^{*},\ (A^{\dagger}A^4A^{\dagger})^{*}] - r(A^2) - r(A^4) \\
&= r\begin{bmatrix} A^2 \\ A^4A^{\dagger} \end{bmatrix} + r[A^2,\ A^{\dagger}A^4] - r(A^2) - r(A^4),
\end{aligned}$$

as required for (3.42). It is easy to verify by the definition in (1.1) that $A^{\dagger},\ A(A^3)^{\dagger}A \in \{A^{(2)}\}$, and $(A^2)^{\dagger},\ A(A^4)^{\dagger}A \in \{(A^2)^{(2)}\}$. Then, we obtain from Lemma 2.9 and the corresponding matrix operations that

$$\begin{aligned}
r(A^{\dagger} - A(A^3)^{\dagger}A) &= r\begin{bmatrix} A^{\dagger} \\ A(A^3)^{\dagger}A \end{bmatrix} + r[A^{\dagger}, A(A^3)^{\dagger}A] - r(A^{\dagger}) - r(A(A^3)^{\dagger}A) \\
&= r\begin{bmatrix} A^{*} \\ (A^3)^{*}A \end{bmatrix} + r[A^{*}, A(A^3)^{*}] - r(A) - r((A^3)^{\dagger}) \\
&= r\begin{bmatrix} A \\ A^3A^{*} \end{bmatrix} + r[A,\ A^{*}A^3] - r(A) - r(A^3),
\end{aligned}$$

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and

$$\begin{aligned} r((A^2)^{\dagger} - A(A^4)^{\dagger}A) &= r\begin{bmatrix} (A^2)^{\dagger} \\ A(A^4)^{\dagger}A \end{bmatrix} + r[(A^2)^{\dagger},\ A(A^4)^{\dagger}A] - r((A^2)^{\dagger}) - r(A(A^4)^{\dagger}A) \\ &= r\begin{bmatrix} (A^2)^{*} \\ (A^4)^{*}A \end{bmatrix} + r[(A^2)^{*}, A(A^4)^{*}] - r(A^2) - r((A^4)^{\dagger}) \\ &= r\begin{bmatrix} A^2 \\ A^4A^{*} \end{bmatrix} + r[A^2,\ A^{*}A^4] - r(A^2) - r(A^4), \end{aligned}$$

as required for (3.43) and (3.44). Eqs. (3.45)–(3.61) follow directly from (2.42), as well as Lemma 2.10 and Corollary 2.11. Setting all sides of (3.35)–(3.61) equal to zero leads to the equivalences of $\langle 1\rangle$–$\langle 32\rangle$. □

Consequently, we present a series of existing and new matrix rank equalities for the differences of two matrix expressions involving the Moore–Penrose inverse and the group inverse of a matrix, and then develop a sequence of identifying conditions for a complex square matrix to be range-Hermitian.

Theorem 3.3. *Let $A \in \mathbb{C}^{m\times m}$ and assume that $r(A^2) = r(A)$. Then, the following rank equalities hold*:

$$r(A^{\dagger} - A^{\#}) = 2r[A,\ A^{*}] - 2r(A), \tag{3.62}$$
$$r(A - (A^{\dagger})^{\#}) = 2r[A,\ A^{*}] - 2r(A), \tag{3.63}$$
$$r(A - (A^{\#})^{\dagger}) = 2r[A,\ A^{*}] - 2r(A), \tag{3.64}$$
$$r(A^{\dagger} - A^{\dagger}AA^{\#}) = r[A,\ A^{*}] - r(A), \tag{3.65}$$
$$r(A^{\dagger} - A^{\#}AA^{\dagger}) = r[A,\ A^{*}] - r(A), \tag{3.66}$$
$$r(A^{\#} - A^{\dagger}AA^{\#}) = r[A,\ A^{*}] - r(A), \tag{3.67}$$
$$r(A^{\#} - A^{\#}AA^{\dagger}) = r[A,\ A^{*}] - r(A), \tag{3.68}$$
$$r(A^{*} - AA^{\#}A^{*}) = r[A,\ A^{*}] - r(A), \tag{3.69}$$
$$r(A^{*} - A^{*}A^{\#}A) = r[A,\ A^{*}] - r(A), \tag{3.70}$$
$$r(AA^{\dagger} - AA^{\#}) = r(A^{\dagger}A - A^{\#}A) = r[A,\ A^{*}] - r(A), \tag{3.71}$$
$$r(AA^{\dagger} - (AA^{\#})^{\dagger}) = r(A^{\dagger}A - (A^{\#}A)^{\dagger}) = r[A,\ A^{*}] - r(A), \tag{3.72}$$
$$r((AA^{\#})^{*} - AA^{\#}) = 2r[A,\ A^{*}] - 2r(A), \tag{3.73}$$
$$r(A^{\dagger}A^{\#} - A^{\#}A^{\dagger}) = 2r[A,\ A^{*}] - 2r(A), \tag{3.74}$$
$$r(A(A^{\dagger})^{\#} - (A^{\dagger})^{\#}A) = 2r[A,\ A^{*}] - 2r(A), \tag{3.75}$$
$$r(A(A^{\#})^{\dagger} - (A^{\#})^{\dagger}A) = 2r[A,\ A^{*}] - 2r(A), \tag{3.76}$$
$$r((A^2)^{\dagger} - A^{\dagger}A^{\#}) = r((A^2)^{\dagger} - A^{\#}A^{\dagger}) = r[A,\ A^{*}] - r(A), \tag{3.77}$$
$$r((A^2)^{\#} - A^{\dagger}A^{\#}) = r((A^2)^{\#} - A^{\#}A^{\dagger}) = r[A,\ A^{*}] - r(A), \tag{3.78}$$
$$r((AA^{\dagger} - AA^{\#})(A^{\dagger}A - A^{\#}A)) = r(A(A^{\dagger})^2A - AA^{\#}) = r[A,\ A^{*}] - r(A), \tag{3.79}$$
$$r(AA^{\#}(AA^{\#})^{*}AA^{\#} - AA^{\#}) = r(A^{*}AA^{\#}A^{*} - (A^2)^{*}) = r[A,\ A^{*}] - r(A), \tag{3.80}$$
$$r(A^{\dagger}AA^{\#} - A^{\#}AA^{\dagger}) = 2r[A,\ A^{*}] - 2r(A), \tag{3.81}$$
$$r(AA^{\dagger}(A^{\dagger})^{\#} - (A^{\dagger})^{\#}A^{\dagger}A) = 2r[A,\ A^{*}] - 2r(A), \tag{3.82}$$
$$r(AA^{\#}(A^{\#})^{\dagger} - (A^{\#})^{\dagger}A^{\#}A) = 2r[A,\ A^{*}] - 2r(A), \tag{3.83}$$
$$r(A^{\dagger}A^{\#}A^{\dagger} - A^{\#}A^{\dagger}A^{\#}) = 2r[A,\ A^{*}] - 2r(A), \tag{3.84}$$
$$r((A^{\dagger}A^{\#})^2 - (A^{\#}A^{\dagger})^2) = 2r[A,\ A^{*}] - 2r(A), \tag{3.85}$$
$$r((A^3)^{\dagger} - A^{\#}A^{\dagger}A^{\#}) = 2r[A,\ A^{*}] - 2r(A), \tag{3.86}$$

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$$r((A^3)^{\#} - A^{\dagger}A^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.87}$$
$$r((A^4)^{\dagger} - A^{\#}(A^2)^{\dagger}A^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.88}$$
$$r((A^4)^{\#} - A^{\dagger}(A^2)^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.89}$$
$$r(A^{\dagger}(A^2)^{\#}A^{\dagger} - A^{\#}(A^2)^{\dagger}A^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.90}$$
$$r((A^5)^{\dagger} - (A^2)^{\#}A^{\dagger}(A^2)^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.91}$$
$$r((A^5)^{\#} - (A^2)^{\dagger}A^{\#}(A^2)^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.92}$$
$$r((A^2)^{\dagger}A^{\#}(A^2)^{\dagger} - (A^2)^{\#}A^{\dagger}(A^2)^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.93}$$
$$r((A^5)^{\dagger} - A^{\#}(A^3)^{\dagger}A^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.94}$$
$$r((A^5)^{\#} - A^{\dagger}(A^3)^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.95}$$
$$r(A^{\dagger}(A^3)^{\#}A^{\dagger} - A^{\#}(A^3)^{\dagger}A^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.96}$$
$$r((A^7)^{\dagger} - A^{\#}(A^2)^{\dagger}A^{\#}(A^2)^{\dagger}A^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.97}$$
$$r((A^7)^{\#} - A^{\dagger}(A^2)^{\#}A^{\dagger}(A^2)^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.98}$$
$$r(A^{\#}(A^2)^{\dagger}A^{\#}(A^2)^{\dagger}A^{\#} - A^{\dagger}(A^2)^{\#}A^{\dagger}(A^2)^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.99}$$
$$r((A^9)^{\dagger} - A^{\#}(A^3)^{\dagger}A^{\#}(A^3)^{\dagger}A^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.100}$$
$$r((A^9)^{\#} - A^{\dagger}(A^3)^{\#}A^{\dagger}(A^3)^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.101}$$
$$r(A^{\#}(A^3)^{\dagger}A^{\#}(A^3)^{\dagger}A^{\#} - A^{\dagger}(A^3)^{\#}A^{\dagger}(A^3)^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.102}$$
$$r(A^{\dagger} - A^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#}) = r[A,\ A^*] - r(A), \tag{3.103}$$
$$r(A^{\dagger} - A^{\#}(A^{\dagger}AA^{\#})^{\dagger}A^{\dagger}) = r[A,\ A^*] - r(A), \tag{3.104}$$
$$r(A^{\dagger} - A^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#}) = r[A,\ A^*] - r(A), \tag{3.105}$$
$$r(A^{\dagger} - A^{\#}(A^{\dagger}AA^{\#})^{\#}A^{\dagger}) = r[A,\ A^*] - r(A), \tag{3.106}$$
$$r(A^{\#} - A^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#}) = r[A,\ A^*] - r(A), \tag{3.107}$$
$$r(A^{\#} - A^{\#}(A^{\dagger}AA^{\#})^{\dagger}A^{\dagger}) = r[A,\ A^*] - r(A), \tag{3.108}$$
$$r(A^{\#} - A^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#}) = r[A,\ A^*] - r(A), \tag{3.109}$$
$$r(A^{\#} - A^{\#}(A^{\dagger}AA^{\#})^{\#}A^{\dagger}) = r[A,\ A^*] - r(A), \tag{3.110}$$
$$r(A - A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.111}$$
$$r(A - A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.112}$$
$$r((A^{\dagger})^{\#} - A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.113}$$
$$r((A^{\dagger})^{\#} - A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.114}$$
$$r(A^{\dagger} - A^{\#}A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\dagger}A^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.115}$$
$$r(A^{\dagger} - A^{\#}A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\dagger}A^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.116}$$
$$r(A^{\#} - A^{\dagger}A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.117}$$
$$r(A^{\#} - A^{\dagger}A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.118}$$
$$r((A^2)^{\#} - A^{\dagger}(A^{\dagger}A^2A^{\dagger})^{\dagger}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.119}$$
$$r((A^2)^{\#} - A^{\dagger}(A^{\dagger}A^2A^{\dagger})^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.120}$$
$$r((A^{\#}AA^{\dagger})^{\dagger} - (A^{\dagger}AA^{\#})^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.121}$$
$$r(A^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#} - A^{\#}(A^{\dagger}AA^{\#})^{\dagger}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.122}$$
$$r(A^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#} - A^{\#}(A^{\dagger}AA^{\#})^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A); \tag{3.123}$$

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the following rank equalities hold:

$$r((A^k)^{\dagger} - (A^k)^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.124}$$
$$r((A^{\dagger}A^{\#})^k - (A^{\#}A^{\dagger})^k) = 2r[A,\ A^*] - 2r(A), \tag{3.125}$$
$$r((A^{\dagger}AA^{\#})^k - (A^{\#}AA^{\dagger})^k) = 2r[A,\ A^*] - 2r(A), \tag{3.126}$$
$$r((A^{k+s})^{\dagger} - (A^k)^{\#}(A^s)^{\dagger}) = r[A,\ A^*] - r(A), \tag{3.127}$$
$$r((A^{k+s})^{\dagger} - (A^k)^{\dagger}(A^s)^{\#}) = r[A,\ A^*] - r(A), \tag{3.128}$$
$$r((A^{k+s})^{\#} - (A^k)^{\#}(A^s)^{\dagger}) = r[A,\ A^*] - r(A), \tag{3.129}$$
$$r((A^{k+s})^{\#} - (A^k)^{\dagger}(A^s)^{\#}) = r[A,\ A^*] - r(A), \tag{3.130}$$
$$r((A^{k+s+t})^{\#} - (A^k)^{\dagger}(A^s)^{\#}(A^t)^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.131}$$
$$r((A^{k+s+t})^{\dagger} - (A^k)^{\#}(A^s)^{\dagger}(A^t)^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.132}$$
$$r((A^k)^{\dagger}(A^s)^{\#} - (A^s)^{\#}(A^k)^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.133}$$
$$r((A^k)^{\dagger}(A^s)^{\#}(A^t)^{\dagger} - (A^k)^{\#}(A^s)^{\dagger}(A^t)^{\#}) = 2r[A,\ A^*] - 2r(A) \tag{3.134}$$

for any integers $k, s, t \geq 2$; *the following rank equalities hold*:

$$r(2AA^{\#} - AA^{\dagger} - A^{\dagger}A) = 2r[A,\ A^*] - 2r(A), \tag{3.135}$$
$$r(2AA^{\dagger} - AA^{\#} - (AA^{\#})^*) = 2r[A,\ A^*] - 2r(A), \tag{3.136}$$
$$r(2A^{\dagger}A - AA^{\#} - (AA^{\#})^*) = 2r[A,\ A^*] - 2r(A), \tag{3.137}$$
$$r(2AA^{\dagger} - A^{\dagger}A - A^{\#}A) = 2r[A,\ A^*] - 2r(A), \tag{3.138}$$
$$r(2A^{\dagger}A - AA^{\dagger} - AA^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.139}$$
$$r((2AA^{\#})^* - AA^{\dagger} - AA^{\#}) = 2r[A,\ A^*] - 2r(A), \tag{3.140}$$
$$r((2A^{\#}A)^* - A^{\dagger}A - A^{\#}A) = 2r[A,\ A^*] - 2r(A), \tag{3.141}$$
$$r(2A^{\dagger} - A^{\dagger}AA^{\#} - A^{\#}AA^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.142}$$
$$r(2A^{\#} - A^{\dagger}AA^{\#} - A^{\#}AA^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.143}$$
$$r(2(A^2)^{\dagger} - A^{\dagger}A^{\#} - A^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.144}$$
$$r(2(A^2)^{\#} - A^{\dagger}A^{\#} - A^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.145}$$
$$r(2(A^3)^{\dagger} - A^{\dagger}(A^2)^{\#} - (A^2)^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A), \tag{3.146}$$
$$r(2(A^3)^{\#} - A^{\dagger}(A^2)^{\#} - (A^2)^{\#}A^{\dagger}) = 2r[A,\ A^*] - 2r(A); \tag{3.147}$$

and the following rank equalities hold:

$$r[I_m - AA^{\dagger},\ I_m - AA^{\#}] = m - 2r(A) + r[A,\ A^*], \tag{3.148}$$
$$r[I_m - A^{\dagger}A,\ I_m - (AA^{\#})^*] = m - 2r(A) + r[A,\ A^*], \tag{3.149}$$
$$r[I_m - AA^{\#},\ I_m - (AA^{\#})^*] = m - 2r(A) + r[A,\ A^*], \tag{3.150}$$
$$r[I_m - A^{\dagger}A,\ I_m - AA^{\#}] = m - r(A), \tag{3.151}$$
$$r[I_m - AA^{\dagger},\ I_m - (AA^{\#})^*] = m - r(A). \tag{3.152}$$

Consequently, the following 351 conditions are equivalent:

$\langle 1 \rangle$ $A^{\dagger} = A^{\#}$.
$\langle 2 \rangle$ $(A^{\dagger})^{\#} = A$.
$\langle 3 \rangle$ $(A^{\#})^{\dagger} = A$.
$\langle 4 \rangle$ $AA^{\#}A^{\dagger} = A^{\dagger}$.
$\langle 5 \rangle$ $A^{\dagger}A^{\#}A = A^{\dagger}$.
$\langle 6 \rangle$ $A^{\dagger}A^{\#}A = A^{\#}$.

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⟨7⟩ $AA^{\#}A^{\dagger} = A^{\#}$.
⟨8⟩ $(A^{\dagger})^{\#}AA^{\#} = (A^{\dagger})^{\#}$.
⟨9⟩ $(A^{\dagger})^{\#}AA^{\#} = (A^{\#})^{\dagger}$.
⟨10⟩ $AA^{\#}(A^{\dagger})^{\#} = (A^{\dagger})^{\#}$.
⟨11⟩ $AA^{\#}(A^{\dagger})^{\#} = (A^{\#})^{\dagger}$.
⟨12⟩ $(A^{\dagger})^{\#}AA^{\#} = A$.
⟨13⟩ $AA^{\#}(A^{\dagger})^{\#} = A$.
⟨14⟩ $(A^{\#})^{\dagger}A^{\#}A = A$.
⟨15⟩ $AA^{\#}(A^{\#})^{\dagger} = A$.
⟨16⟩ $AA^{\dagger}A^{\#}A^{\dagger}A = A^{\#}$.
⟨17⟩ $AA^{\#}(A^{\dagger})^{\#}A^{\#}A = (A^{\dagger})^{\#}$.
⟨18⟩ $AA^{\#}(A^{\dagger})^{\#}A^{\#}A = (A^{\#})^{\dagger}$.
⟨19⟩ $AA^{\#}(A^{\dagger})^{\#}A^{\#}A = (A^{\dagger})^{\#}$.
⟨20⟩ $AA^{\#}(A^{\dagger})^{\#}A^{\#}A = (A^{\#})^{\dagger}$.
⟨21⟩ $(AA^{\#})^{\dagger}A(A^{\#}A)^{\dagger} = A$.
⟨22⟩ $(AA^{\#})^{\dagger}A^{\#}(A^{\#}A)^{\dagger} = A^{\#}$.
⟨23⟩ $A^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#} = A^{\dagger}$.
⟨24⟩ $A^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#} = A^{\#}$.
⟨25⟩ $A^{\#}(A^{\#}AA^{\dagger})^{\dagger}A^{\dagger} = A^{\dagger}$.
⟨26⟩ $A^{\#}(A^{\#}AA^{\dagger})^{\dagger}A^{\dagger} = A^{\#}$.
⟨27⟩ $A^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#} = A^{\dagger}$.
⟨28⟩ $A^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#} = A^{\#}$.
⟨29⟩ $A^{\#}(A^{\#}AA^{\dagger})^{\#}A^{\dagger} = A^{\dagger}$.
⟨30⟩ $A^{\#}(A^{\#}AA^{\dagger})^{\#}A^{\dagger} = A^{\#}$.
⟨31⟩ $A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\dagger} = A$.
⟨32⟩ $A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\dagger} = A$.
⟨33⟩ $A^{\dagger}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\dagger} = A$.
⟨34⟩ $A^{\dagger}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\dagger} = A$.
⟨35⟩ $A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\#} = A$.
⟨36⟩ $A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\#} = A$.
⟨37⟩ $A^{\dagger}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\#} = A$.
⟨38⟩ $A^{\dagger}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\#} = A$.
⟨39⟩ $A^{\#}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\dagger} = A$.
⟨40⟩ $A^{\#}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\dagger} = A$.
⟨41⟩ $A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\dagger} = A$.
⟨42⟩ $A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\dagger} = A$.
⟨43⟩ $A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\#} = (A^{\dagger})^{\#}$.
⟨44⟩ $A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\#} = (A^{\#})^{\dagger}$.
⟨45⟩ $A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\#} = (A^{\dagger})^{\#}$.
⟨46⟩ $A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\#} = (A^{\#})^{\dagger}$.
⟨47⟩ $A^{\#}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\#} = (A^{\dagger})^{\#}$.
⟨48⟩ $A^{\#}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\#} = (A^{\#})^{\dagger}$.
⟨49⟩ $A^{\#}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\#} = (A^{\dagger})^{\#}$.
⟨50⟩ $A^{\#}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\#} = (A^{\#})^{\dagger}$.
⟨51⟩ $A^{\#}A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\dagger}A^{\#} = A^{\dagger}$.
⟨52⟩ $A^{\#}A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\dagger}A^{\#} = A^{\dagger}$.
⟨53⟩ $A^{\#}A^{\dagger}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\dagger}A^{\#} = A^{\dagger}$.
⟨54⟩ $A^{\#}A^{\dagger}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\dagger}A^{\#} = A^{\dagger}$.
⟨55⟩ $A^{\dagger}A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\#}A^{\dagger} = A^{\#}$.
⟨56⟩ $A^{\dagger}A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\#}A^{\dagger} = A^{\#}$.
⟨57⟩ $A^{\dagger}A^{\#}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\#}A^{\dagger} = A^{\#}$.
⟨58⟩ $A^{\dagger}A^{\#}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\#}A^{\dagger} = A^{\#}$.
⟨59⟩ $AA^{\#}A^{*} = A^{*}$.
⟨60⟩ $A^{*}A^{\#}A = A^{*}$.
⟨61⟩ $AA^{\#}A^{*}A^{\#}A = A^{*}$.

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⟨62⟩ $AA^{\#}(A^{\#})^{*} = (A^{\#})^{*}$.
⟨63⟩ $(A^{\#})^{*}A^{\#}A = (A^{\#})^{*}$.
⟨64⟩ $AA^{\#}(A^{\#})^{*}A^{\#}A = (A^{\#})^{*}$.
⟨65⟩ $AA^{\#}(AA^{\#})^{*}A = A$.
⟨66⟩ $A(A^{\#}A)^{*}A^{\#}A = A$.
⟨67⟩ $AA^{\#}(AA^{\#})^{*}A^{\#} = A^{\#}$.
⟨68⟩ $A^{\#}(A^{\#}A)^{*}A^{\#}A = A^{\#}$.
⟨69⟩ $AA^{\#}(AA^{\#})^{*}A^{\dagger} = A^{\dagger}$.
⟨70⟩ $A^{\dagger}(A^{\#}A)^{*}A^{\#} = A^{\dagger}$.
⟨71⟩ $AA^{\#}(AA^{\#})^{*}(A^{\dagger})^{\#} = (A^{\dagger})^{\#}$.
⟨72⟩ $(A^{\dagger})^{\#}(A^{\#}A)^{*}A^{\#}A = (A^{\dagger})^{\#}$.
⟨73⟩ $AA^{\#}(AA^{\#})^{*}(A^{\#})^{\dagger} = (A^{\#})^{\dagger}$.
⟨74⟩ $(A^{\#})^{\dagger}(A^{\#}A)^{*}A^{\#}A = (A^{\#})^{\dagger}$.
⟨75⟩ $AA^{\#}(AA^{\#})^{*}A(A^{\#}A)^{*}A^{\#}A = A$.
⟨76⟩ $AA^{\#}(AA^{\#})^{*}A^{\#}(A^{\#}A)^{*}A^{\#}A = A^{\#}$.
⟨77⟩ $AA^{\#}(AA^{\#})^{*}A^{\dagger}(AA^{\#})^{*}AA^{\#} = A^{\dagger}$.
⟨78⟩ $AA^{\#}(AA^{\#})^{*}(A^{\dagger})^{\#}(AA^{\#})^{*}AA^{\#} = (A^{\dagger})^{\#}$.
⟨79⟩ $AA^{\#}(AA^{\#})^{*}(A^{\dagger})^{\#}(AA^{\#})^{*}AA^{\#} = (A^{\#})^{\dagger}$.
⟨80⟩ $AA^{\#}(AA^{\#})^{*}(A^{\#})^{\dagger}(AA^{\#})^{*}AA^{\#} = (A^{\dagger})^{\#}$.
⟨81⟩ $AA^{\#}(AA^{\#})^{*}(A^{\#})^{\dagger}(AA^{\#})^{*}AA^{\#} = (A^{\#})^{\dagger}$.
⟨82⟩ $(AA^{\#})^{*} = AA^{\#}$.
⟨83⟩ $(AA^{\#})^{*} = (AA^{\#})^{\dagger}$.
⟨84⟩ $AA^{\#}(AA^{\#})^{*}AA^{\#} = AA^{\#}$.
⟨85⟩ $AA^{\#}(AA^{\#})^{*}AA^{\#} = (AA^{\#})^{*}$.
⟨86⟩ $AA^{\#}(AA^{\#})^{*} = (AA^{\#})^{*}AA^{\#}$.
⟨87⟩ $AA^{\dagger} = AA^{\#}$.
⟨88⟩ $A^{\dagger}A = A^{\#}A$.
⟨89⟩ $A^{\dagger}A^{\#} = A^{\#}A^{\dagger}$.
⟨90⟩ $A(A^{\dagger})^{\#} = (A^{\dagger})^{\#}A$.
⟨91⟩ $A(A^{\#})^{\dagger} = (A^{\#})^{\dagger}A$.
⟨92⟩ $A(A^{\dagger})^{\#} = (A^{\#})^{\dagger}A$.
⟨93⟩ $A(A^{\#})^{\dagger} = (A^{\dagger})^{\#}A$.
⟨94⟩ $(AA^{\#})^{\dagger} = AA^{\#}$.
⟨95⟩ $(AA^{\#})^{\dagger} = (A^{\#})^{\dagger}A^{\dagger}$.
⟨96⟩ $(A^{\#}A)^{\dagger} = A^{\dagger}(A^{\#})^{\dagger}$.
⟨97⟩ $(A^{\dagger}A^{\#})^{\dagger} = (A^{\#}A^{\dagger})^{\#}$.
⟨98⟩ $(A^{\dagger}A^{\#})^{\#} = (A^{\#}A^{\dagger})^{\dagger}$.
⟨99⟩ $((A^{\dagger}A^{\#})^{\dagger})^{\#} = ((A^{\#}A^{\dagger})^{\#})^{\dagger}$.
⟨100⟩ $((A^{\dagger}A^{\#})^{\#})^{\dagger} = ((A^{\#}A^{\dagger})^{\dagger})^{\#}$.
⟨101⟩ $(A(A^{\dagger})^{\#})^{\dagger} = ((A^{\dagger})^{\#}A)^{\#}$.
⟨102⟩ $(A(A^{\dagger})^{\#})^{\#} = ((A^{\dagger})^{\#}A)^{\dagger}$.
⟨103⟩ $(A(A^{\#})^{\dagger})^{\dagger} = ((A^{\#})^{\dagger}A)^{\#}$.
⟨104⟩ $(A(A^{\#})^{\dagger})^{\#} = ((A^{\#})^{\dagger}A)^{\dagger}$.
⟨105⟩ $A^{\#}(A^{\dagger})^{\#} = (A^{\dagger})^{\#}A^{\#}$.
⟨106⟩ $((A^{\dagger})^{\#}(A^{\#})^{\dagger})^{\dagger} = ((A^{\#})^{\dagger}(A^{\dagger})^{\#})^{\#}$.
⟨107⟩ $((A^{\#})^{\dagger}(A^{\dagger})^{\#})^{\dagger} = ((A^{\dagger})^{\#}(A^{\#})^{\dagger})^{\#}$.
⟨108⟩ $A^{\dagger}A^{*}A^{\dagger} = A^{\#}A^{*}A^{\#}$.
⟨109⟩ $A^{\dagger}A^{*}A^{\#} = A^{\#}A^{*}A^{\dagger}$.
⟨110⟩ $AA^{\#}A^{*} = A^{*}A^{\#}A$.
⟨111⟩ $AA^{\#}A^{\dagger} = A^{\dagger}A^{\#}A$.
⟨112⟩ $AA^{\dagger}(A^{\dagger})^{\#} = (A^{\dagger})^{\#}A^{\dagger}A$.
⟨113⟩ $AA^{\dagger}(A^{\#})^{\dagger} = (A^{\#})^{\dagger}A^{\dagger}A$.
⟨114⟩ $AA^{\#}(A^{\dagger})^{\#} = (A^{\dagger})^{\#}A^{\#}A$.
⟨115⟩ $(A^{\dagger}AA^{\#})^{\dagger} = (A^{\#}AA^{\dagger})^{\#}$.
⟨116⟩ $(A^{\dagger}AA^{\#})^{\#} = (A^{\#}AA^{\dagger})^{\dagger}$.

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⟨117⟩ $(AA^{\dagger}(A^{\dagger})^{\#})^{\dagger} = ((A^{\dagger})^{\#}A^{\dagger}A)^{\#}$.
⟨118⟩ $(AA^{\dagger}(A^{\dagger})^{\#})^{\#} = ((A^{\dagger})^{\#}A^{\dagger}A)^{\dagger}$.
⟨119⟩ $(AA^{\#}(A^{\#})^{\dagger})^{\dagger} = ((A^{\#})^{\dagger}A^{\#}A)^{\#}$.
⟨120⟩ $(AA^{\#}(A^{\#})^{\dagger})^{\#} = ((A^{\#})^{\dagger}A^{\#}A)^{\dagger}$.
⟨121⟩ $A^{\dagger}A^{\#}A^{\dagger} = A^{\#}A^{\dagger}A^{\#}$.
⟨122⟩ $A(A^{\dagger})^{\#}A = (A^{\dagger})^{\#}A(A^{\dagger})^{\#}$.
⟨123⟩ $(A^{\#})^{\dagger}A(A^{\#})^{\dagger} = A(A^{\#})^{\dagger}A$.
⟨124⟩ $A^{\dagger}(A^{\#})^{\dagger}A^{\dagger} = A^{\#}(A^{\dagger})^{\#}A^{\#}$.
⟨125⟩ $AA^{\#}(AA^{\#})^{*} = (A^{\#}A)^{*}A^{\#}A$.
⟨126⟩ $(AA^{*})(AA^{\#}) = (AA^{\#})(AA^{*})$.
⟨127⟩ $(A^{*}A)(A^{\#}A) = (A^{\#}A)(A^{*}A)$.
⟨128⟩ $AA^{\dagger}(A^{\dagger})^{\#}(A^{\#})^{\dagger} = (A^{\#})^{\dagger}(A^{\dagger})^{\#}A^{\dagger}A$.
⟨129⟩ $AA^{\dagger}(A^{\#})^{\dagger}(A^{\dagger})^{\#} = (A^{\dagger})^{\#}(A^{\#})^{\dagger}A^{\dagger}A$.
⟨130⟩ $AA^{\#}(A^{\dagger})^{\#}(A^{\#})^{\dagger} = (A^{\#})^{\dagger}(A^{\dagger})^{\#}A^{\#}A$.
⟨131⟩ $A^{\dagger}A^{*}AA^{*}A^{\dagger} = A^{\#}A^{*}AA^{*}A^{\#}$.
⟨132⟩ $A^{\dagger}A^{*}AA^{*}A^{\#} = A^{\#}A^{*}AA^{*}A^{\dagger}$.
⟨133⟩ $AA^{\#}A^{*}AA^{*} = A^{*}AA^{*}A^{\#}A$.
⟨134⟩ $AA^{\#}A^{*}AA^{*}A^{\#}A = A^{*}AA^{*}$.
⟨135⟩ $(AA^{\#})^{\dagger}AA^{*}A(A^{\#}A)^{\dagger} = AA^{*}A$.
⟨136⟩ $A^{\dagger}(A^{\#}AA^{\dagger})^{\dagger}A^{\#} = A^{\#}(A^{\dagger}AA^{\#})^{\dagger}A^{\dagger}$.
⟨137⟩ $A^{\dagger}(A^{\#}AA^{\dagger})^{\#}A^{\#} = A^{\#}(A^{\dagger}AA^{\#})^{\#}A^{\dagger}$.
⟨138⟩ $A^{\dagger}(A^{\#}A^{*}A^{\dagger})^{\dagger}A^{\#} = A^{\#}(A^{\dagger}A^{*}A^{\#})^{\dagger}A^{\dagger}$.
⟨139⟩ $A^{\dagger}(A^{\#}A^{*}A^{\dagger})^{\#}A^{\#} = A^{\#}(A^{\dagger}A^{*}A^{\#})^{\#}A^{\dagger}$.
⟨140⟩ $A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\dagger} = A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\#}$.
⟨141⟩ $A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\dagger}A^{\dagger} = A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\#}$.
⟨142⟩ $A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\dagger} = A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\dagger}A^{\#}$.
⟨143⟩ $A^{\dagger}(A^{\dagger}A^{\#}A^{\dagger})^{\#}A^{\dagger} = A^{\#}(A^{\#}A^{\dagger}A^{\#})^{\#}A^{\#}$.
⟨144⟩ $A^{2} = ((A^{\dagger})^{\#})^{2}$.
⟨145⟩ $A^{2} = ((A^{\#})^{\dagger})^{2}$.
⟨146⟩ $A^{2} = (A^{\dagger})^{\#}(A^{\#})^{\dagger}$.
⟨147⟩ $A^{2} = (A^{\#})^{\dagger}(A^{\dagger})^{\#}$.
⟨148⟩ $(A^{2})^{\dagger} = A^{\dagger}A^{\#}$.
⟨149⟩ $(A^{2})^{\dagger} = A^{\#}A^{\dagger}$.
⟨150⟩ $(A^{\dagger})^{2} = A^{\dagger}A^{\#}$.
⟨151⟩ $(A^{\dagger})^{2} = A^{\#}A^{\dagger}$.
⟨152⟩ $(A^{2})^{\#} = A^{\dagger}A^{\#}$.
⟨153⟩ $(A^{2})^{\#} = A^{\#}A^{\dagger}$.
⟨154⟩ $A(A^{2})^{\dagger} = A^{\#}$.
⟨155⟩ $(A^{2})^{\dagger}A = A^{\#}$.
⟨156⟩ $A(A^{\dagger})^{2} = A^{\#}$.
⟨157⟩ $(A^{\dagger})^{2}A = A^{\#}$.
⟨158⟩ $A(A^{\dagger})^{2}A = AA^{\#}$.
⟨159⟩ $A^{\dagger}A^{2}A^{\dagger} = AA^{\#}$.
⟨160⟩ $(A^{2})^{*} = A^{*}AA^{\#}A^{*}$.
⟨161⟩ $(A^{\dagger})^{2} = A^{\dagger}AA^{\#}A^{\dagger}$.
⟨162⟩ $(A^{2})^{\dagger} = A^{\#}(A^{\dagger}A^{2}A^{\dagger})^{\dagger}A^{\#}$.
⟨163⟩ $(A^{2})^{\dagger} = A^{\#}(A^{\dagger}A^{2}A^{\dagger})^{\#}A^{\#}$.
⟨164⟩ $(A^{\dagger})^{2} = A^{\#}(A^{\dagger}A^{2}A^{\dagger})^{\dagger}A^{\#}$.
⟨165⟩ $(A^{\dagger})^{2} = A^{\#}(A^{\dagger}A^{2}A^{\dagger})^{\#}A^{\#}$.
⟨166⟩ $(A^{2})^{\#} = A^{\dagger}(A^{\dagger}A^{2}A^{\dagger})^{\dagger}A^{\dagger}$.
⟨167⟩ $(A^{2})^{\#} = A^{\dagger}(A^{\dagger}A^{2}A^{\dagger})^{\#}A^{\dagger}$.
⟨168⟩ $(A^{2})^{\dagger} = A^{\#}(A(A^{2})^{\dagger}A)^{\dagger}A^{\#}$.
⟨169⟩ $(A^{2})^{\dagger} = A^{\#}(A(A^{2})^{\dagger}A)^{\#}A^{\#}$.
⟨170⟩ $(A^{\dagger})^{2} = A^{\#}(A(A^{\dagger})^{2}A)^{\dagger}A^{\#}$.
⟨171⟩ $(A^{\dagger})^{2} = A^{\#}(A(A^{\dagger})^{2}A)^{\#}A^{\#}$.

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⟨172⟩ $(A^2)^\# = A^\dagger(A(A^2)^\dagger A)^\dagger A^\dagger$.
⟨173⟩ $(A^2)^\# = A^\dagger(A(A^2)^\dagger A)^\# A^\dagger$.
⟨174⟩ $(A^2)^\# = A^\dagger(A(A^\dagger)^2 A)^\dagger A^\dagger$.
⟨175⟩ $(A^2)^\# = A^\dagger(A(A^\dagger)^2 A)^\# A^\dagger$.
⟨176⟩ $(A^2)^\# = A^*(A^* A^2 A^*)^\dagger A^*$.
⟨177⟩ $(A^2)^\# = A^*(A^* A^2 A^*)^\# A^*$.
⟨178⟩ $(A^\dagger A^\#)^2 = (A^\# A^\dagger)^2$.
⟨179⟩ $((A^\dagger A^\#)^2)^\dagger = ((A^\# A^\dagger)^2)^\#$.
⟨180⟩ $((A^\dagger A^\#)^2)^\# = ((A^\# A^\dagger)^2)^\dagger$.
⟨181⟩ $(((A^\dagger A^\#)^2)^\dagger)^\# = (((A^\# A^\dagger)^2)^\#)^\dagger$.
⟨182⟩ $(((A^\dagger A^\#)^2)^\#)^\dagger = (((A^\# A^\dagger)^2)^\dagger)^\#$.
⟨183⟩ $A^\dagger(A^2)^\# A^\dagger = A^\#(A^2)^\dagger A^\#$.
⟨184⟩ $A^\dagger(A^2)^\# A^\dagger = A^\#(A^\dagger)^2 A^\#$.
⟨185⟩ $A((A^\dagger)^\#)^2 A = (A^\dagger)^\# A^2 (A^\dagger)^\#$.
⟨186⟩ $A((A^\#)^\dagger)^2 A = (A^\#)^\dagger A^2 (A^\#)^\dagger$.
⟨187⟩ $A((A^2)^\dagger)^\# A = (A^\dagger)^\# A^2 (A^\dagger)^\#$.
⟨188⟩ $A((A^2)^\#)^\dagger A = (A^\#)^\dagger A^2 (A^\#)^\dagger$.
⟨189⟩ $(A^2)^\dagger A^\# (A^2)^\dagger = (A^2)^\# A^\dagger (A^2)^\#$.
⟨190⟩ $A^3 = ((A^\dagger)^\#)^3$.
⟨191⟩ $A^3 = ((A^\#)^\dagger)^3$.
⟨192⟩ $A^3 = (A^\dagger)^\# A (A^\dagger)^\#$.
⟨193⟩ $A^3 = (A^\#)^\dagger A (A^\#)^\dagger$.
⟨194⟩ $A^3 = (A^\dagger)^\# A (A^\#)^\dagger$.
⟨195⟩ $A^3 = (A^\#)^\dagger A (A^\dagger)^\#$.
⟨196⟩ $A^3 = (A^\dagger)^\# (A^\#)^\dagger (A^\dagger)^\#$.
⟨197⟩ $A^3 = (A^\#)^\dagger (A^\dagger)^\# (A^\dagger)^\#$.
⟨198⟩ $(A^3)^\dagger = A^\# A^\dagger A^\#$.
⟨199⟩ $(A^\dagger)^3 = A^\# A^\dagger A^\#$.
⟨200⟩ $(A^3)^\# = A^\dagger A^\# A^\dagger$.
⟨201⟩ $A^\dagger (A^3)^\# A^\dagger = A^\# (A^3)^\dagger A^\#$.
⟨202⟩ $A^4 = (A^\dagger)^\# A^2 (A^\dagger)^\#$.
⟨203⟩ $A^4 = (A^\#)^\dagger A^2 (A^\#)^\dagger$.
⟨204⟩ $A^4 = (A^\dagger)^\# A^2 (A^\#)^\dagger$.
⟨205⟩ $A^4 = (A^\#)^\dagger A^2 (A^\dagger)^\#$.
⟨206⟩ $A^4 = ((A^\dagger)^\#)^2 ((A^\#)^\dagger)^2$.
⟨207⟩ $A^4 = ((A^\#)^\dagger)^2 ((A^\dagger)^\#)^2$.
⟨208⟩ $(A^4)^\dagger = (A^\dagger A^\#)^2$.
⟨209⟩ $(A^4)^\dagger = (A^\# A^\dagger)^2$.
⟨210⟩ $(A^\dagger)^4 = (A^\dagger A^\#)^2$.
⟨211⟩ $(A^\dagger)^4 = (A^\# A^\dagger)^2$.
⟨212⟩ $(A^4)^\# = (A^\dagger A^\#)^2$.
⟨213⟩ $(A^4)^\# = (A^\# A^\dagger)^2$.
⟨214⟩ $(A^4)^\dagger = A^\# (A^2)^\dagger A^\#$.
⟨215⟩ $(A^\dagger)^4 = A^\# (A^\dagger)^2 A^\#$.
⟨216⟩ $(A^4)^\# = A^\dagger (A^2)^\# A^\dagger$.
⟨217⟩ $A^5 = ((A^\dagger)^\#)^2 A ((A^\#)^\dagger)^2$.
⟨218⟩ $A^5 = ((A^\#)^\dagger)^2 A ((A^\dagger)^\#)^2$.
⟨219⟩ $A^5 = (A^\dagger)^\# A^3 (A^\dagger)^\#$.
⟨220⟩ $A^5 = (A^\#)^\dagger A^3 (A^\#)^\dagger$.
⟨221⟩ $(A^5)^\dagger = (A^2)^\# A^\dagger (A^2)^\#$.
⟨222⟩ $(A^\dagger)^5 = (A^2)^\# A^\dagger (A^2)^\#$.
⟨223⟩ $(A^5)^\# = (A^2)^\dagger A^\# (A^2)^\dagger$.
⟨224⟩ $(A^5)^\# = (A^\dagger)^2 A^\# (A^\dagger)^2$.
⟨225⟩ $(A^5)^\dagger = A^\# (A^3)^\dagger A^\#$.
⟨226⟩ $(A^5)^\dagger = A^\# (A^\dagger)^3 A^\#$.

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⟨227⟩ $(A^{\dagger})^5 = A^{\#}(A^3)^{\dagger}A^{\#}$.
⟨228⟩ $(A^{\dagger})^5 = A^{\#}(A^{\dagger})^3A^{\#}$.
⟨229⟩ $(A^5)^{\#} = A^{\dagger}(A^3)^{\#}A^{\dagger}$.
⟨230⟩ $(A^7)^{\dagger} = A^{\#}(A^2)^{\dagger}A^{\#}(A^2)^{\dagger}A^{\#}$.
⟨231⟩ $(A^7)^{\dagger} = A^{\#}(A^{\dagger})^2A^{\#}(A^{\dagger})^2A^{\#}$.
⟨232⟩ $(A^{\dagger})^7 = A^{\#}(A^2)^{\dagger}A^{\#}(A^2)^{\dagger}A^{\#}$.
⟨233⟩ $(A^{\dagger})^7 = A^{\#}(A^{\dagger})^2A^{\#}(A^{\dagger})^2A^{\#}$.
⟨234⟩ $(A^{\#})^7 = A^{\dagger}(A^2)^{\#}A^{\dagger}(A^2)^{\#}A^{\dagger}$.
⟨235⟩ $(A^{\dagger})^9 = A^{\#}(A^3)^{\dagger}A^{\#}(A^3)^{\dagger}A^{\#}$.
⟨236⟩ $(A^{\dagger})^9 = A^{\#}(A^{\dagger})^3A^{\#}(A^{\dagger})^3A^{\#}$.
⟨237⟩ $(A^9)^{\dagger} = A^{\#}(A^3)^{\dagger}A^{\#}(A^3)^{\dagger}A^{\#}$.
⟨238⟩ $(A^9)^{\dagger} = A^{\#}(A^{\dagger})^3A^{\#}(A^{\dagger})^3A^{\#}$.
⟨239⟩ $(A^{\#})^9 = A^{\dagger}(A^3)^{\#}A^{\dagger}(A^3)^{\#}A^{\dagger}$.
⟨240⟩ $(AA^{\dagger} - AA^{\#})(A^{\dagger}A - A^{\#}A) = 0$.
⟨241⟩ $(A^k)^{\dagger} = (A^k)^{\#}$ *for some/any integers* $k \geq 2$.
⟨242⟩ $(A^{\dagger})^k = (A^{\#})^k$ *for some/any integers* $k \geq 2$.
⟨243⟩ $(A^{\dagger})^k(A^{\#})^s = (A^{\#})^s(A^{\dagger})^k$ *for some/any integers* $k, s \geq 2$.
⟨244⟩ $(A^k)^{\dagger}(A^s)^{\#} = (A^s)^{\#}(A^k)^{\dagger}$ *for some/any integers* $k, s \geq 2$.
⟨245⟩ $(A^k)^{\dagger}AA^{\#} = A^{\#}A(A^k)^{\dagger}$ *for some/any integers* $k \geq 2$.
⟨246⟩ $(A^{\dagger})^kAA^{\#} = A^{\#}A(A^{\dagger})^k$ *for some/any integers* $k \geq 2$.
⟨247⟩ $(A^{\dagger}A^{\#})^k = (A^{\#}A^{\dagger})^k$ *for some/any integers* $k \geq 3$.
⟨248⟩ $(A^{k+s})^{\dagger} = (A^k)^{\#}(A^s)^{\dagger}$ *for some/any integers* $k, s \geq 2$.
⟨249⟩ $(A^{k+s})^{\dagger} = (A^k)^{\dagger}(A^s)^{\#}$ *for some/any integers* $k, s \geq 2$.
⟨250⟩ $(A^{\dagger})^{k+s} = (A^k)^{\#}(A^s)^{\dagger}$ *for some/any integers* $k, s \geq 2$.
⟨251⟩ $(A^{\dagger})^{k+s} = (A^k)^{\dagger}(A^s)^{\#}$ *for some/any integers* $k, s \geq 2$.
⟨252⟩ $(A^{k+s})^{\#} = (A^k)^{\#}(A^s)^{\dagger}$ *for some/any integers* $k, s \geq 2$.
⟨253⟩ $(A^{k+s})^{\#} = (A^k)^{\dagger}(A^s)^{\#}$ *for some/any integers* $k, s \geq 2$.
⟨254⟩ $(A^{k+s+t})^{\#} = (A^k)^{\dagger}(A^s)^{\#}(A^t)^{\dagger}$ *for some/any integers* $k, s, t \geq 2$.
⟨255⟩ $(A^{k+s+t})^{\dagger} = (A^k)^{\#}(A^s)^{\dagger}(A^t)^{\#}$ *for some/any integers* $k, s \geq 2$.
⟨256⟩ $(A^{\dagger})^{k+s+t} = (A^k)^{\#}(A^s)^{\dagger}(A^t)^{\#}$ *for some/any integers* $k, s \geq 2$.
⟨257⟩ $(A^k)^{\dagger}(A^s)^{\#}(A^t)^{\dagger} = (A^k)^{\#}(A^s)^{\dagger}(A^t)^{\#}$ *for some/any integers* $k, s, t \geq 2$.
⟨258⟩ $(A^k)^{\dagger}(A^s)^{\#} = (A^s)^{\#}(A^k)^{\dagger}$ *for some/any integers* $k, s, t \geq 2$.
⟨259⟩ $(A^k)^{\dagger}(A^s)^{\#}(A^t)^{\dagger} = (A^k)^{\#}(A^s)^{\dagger}(A^t)^{\#}$ *for some/any integers* $k, s, t \geq 2$.
⟨260⟩ $(A^{2k+1})^{\dagger} = A^{\#}(A^{\dagger}A^{\#})^k$ *for some/any integers* $k \geq 2$.
⟨261⟩ $(A^{\dagger})^{2k+1} = A^{\#}(A^{\dagger}A^{\#})^k$ *for some/any integers* $k \geq 2$.
⟨262⟩ $(A^{2k+1})^{\#} = A^{\dagger}(A^{\#}A^{\dagger})^k$ *for some/any integers* $k \geq 2$.
⟨263⟩ $A^{\dagger}(A^{\#}A^{\dagger})^k = A^{\#}(A^{\dagger}A^{\#})^k$ *for some/any integers* $k \geq 2$.
⟨264⟩ $(A^k)^{\dagger} = A^{\#}(A^{\dagger}A^kA^{\dagger})^{\dagger}A^{\#}$ *for some/any integers* $k \geq 3$.
⟨265⟩ $(A^k)^{\dagger} = A^{\#}(A^{\dagger}A^kA^{\dagger})^{\#}A^{\#}$ *for some/any integers* $k \geq 3$.
⟨266⟩ $(A^{\dagger})^k = A^{\#}(A^{\dagger}A^kA^{\dagger})^{\dagger}A^{\#}$ *for some/any integers* $k \geq 3$.
⟨267⟩ $(A^{\dagger})^k = A^{\#}(A^{\dagger}A^kA^{\dagger})^{\#}A^{\#}$ *for some/any integers* $k \geq 3$.
⟨268⟩ $(A^k)^{\#} = A^{\dagger}(A^{\dagger}A^kA^{\dagger})^{\dagger}A^{\dagger}$ *for some/any integers* $k \geq 3$.
⟨269⟩ $(A^k)^{\#} = A^{\dagger}(A^{\dagger}A^kA^{\dagger})^{\#}A^{\dagger}$ *for some/any integers* $k \geq 3$.
⟨270⟩ $(A^k)^{\dagger} = A^{\#}(A(A^k)^{\dagger}A)^{\dagger}A^{\#}$ *for some/any integers* $k \geq 3$.
⟨271⟩ $(A^k)^{\dagger} = A^{\#}(A(A^k)^{\dagger}A)^{\#}A^{\#}$ *for some/any integers* $k \geq 3$.
⟨272⟩ $(A^{\dagger})^k = A^{\#}(A(A^{\dagger})^kA)^{\dagger}A^{\#}$ *for some/any integers* $k \geq 3$.
⟨273⟩ $(A^{\dagger})^k = A^{\#}(A(A^{\dagger})^kA)^{\#}A^{\#}$ *for some/any integers* $k \geq 3$.
⟨274⟩ $(A^k)^{\#} = A^{\dagger}(A(A^k)^{\dagger}A)^{\dagger}A^{\dagger}$ *for some/any integers* $k \geq 3$.
⟨275⟩ $(A^k)^{\#} = A^{\dagger}(A(A^k)^{\dagger}A)^{\#}A^{\dagger}$ *for some/any integers* $k \geq 3$.
⟨276⟩ $(A^k)^{\#} = A^{\dagger}(A(A^{\dagger})^kA)^{\dagger}A^{\dagger}$ *for some/any integers* $k \geq 3$.
⟨277⟩ $(A^k)^{\#} = A^{\dagger}(A(A^{\dagger})^kA)^{\#}A^{\dagger}$ *for some/any integers* $k \geq 3$.
⟨278⟩ $(A^k)^{\#} = A^*(A^*A^kA^*)^{\dagger}A^*$ *for some/any integers* $k \geq 3$.
⟨279⟩ $(A^k)^{\#} = A^*(A^*A^kA^*)^{\#}A^*$ *for some/any integers* $k \geq 3$.
⟨280⟩ $(A^{\dagger}AA^{\#})^k = (A^{\#}AA^{\dagger})^k$ *for some/any integers* $k \geq 2$.
⟨281⟩ $(A^{\dagger}A^{\#}A^{\dagger})^k = (A^{\#}A^{\dagger}A^{\#})^k$ *for some/any integers* $k \geq 2$.

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⟨282⟩ $(A(A^\dagger)^\# A)^k = ((A^\dagger)^\# A(A^\dagger)^\#)^k$ *for some/any integers* $k \geq 2$.
⟨283⟩ $((A^\#)^\dagger A(A^\#)^\dagger)^k = (A(A^\#)^\dagger A)^k$ *for some/any integers* $k \geq 2$.
⟨284⟩ $(A^\dagger (A^\#)^\dagger A^\dagger)^k = (A^\# (A^\dagger)^\# A^\#)^k$ *for some/any integers* $k \geq 2$.
⟨285⟩ $(A^\dagger A^* A^\dagger)^k = (A^\# A^* A^\#)^k$ *for some/any integers* $k \geq 2$.
⟨286⟩ $(A^\dagger A^* A^\#)^k = (A^\# A^* A^\dagger)^k$ *for some/any integers* $k \geq 2$.
⟨287⟩ $(AA^\# A^*)^k = (A^* A^\# A)^k$ *for some/any integers* $k \geq 2$.
⟨288⟩ $2AA^\# = AA^\dagger + A^\dagger A$.
⟨289⟩ $2AA^\dagger = AA^\# + (AA^\#)^*$.
⟨290⟩ $2A^\dagger A = AA^\# + (AA^\#)^*$.
⟨291⟩ $2AA^\dagger = A^\dagger A + A^\# A$.
⟨292⟩ $2A^\dagger A = AA^\dagger + AA^\#$.
⟨293⟩ $2(AA^\#)^* = AA^\dagger + AA^\#$.
⟨294⟩ $2(A^\# A)^* = A^\dagger A + A^\# A$.
⟨295⟩ $2A^\dagger = A^\dagger AA^\# + A^\# AA^\dagger$.
⟨296⟩ $2A^\# = A^\dagger AA^\# + A^\# AA^\dagger$.
⟨297⟩ $2A = AA^\dagger (A^\dagger)^\# + (A^\dagger)^\# A^\dagger A$.
⟨298⟩ $2A = (A^\#)^\dagger AA^\# + A^\# A(A^\#)^\dagger$.
⟨299⟩ $2A^\dagger A = AA^\# (AA^\#)^* + (AA^\#)^* AA^\#$.
⟨300⟩ $2AA^\dagger = AA^\# (AA^\#)^* + (AA^\#)^* AA^\#$.
⟨301⟩ $2(A^2)^\dagger = A^\dagger A^\# + A^\# A^\dagger$.
⟨302⟩ $2(A^\dagger)^2 = A^\dagger A^\# + A^\# A^\dagger$.
⟨303⟩ $2(A^2)^\# = A^\dagger A^\# + A^\# A^\dagger$.
⟨304⟩ $(A^2)^\dagger + (A^\dagger)^2 = A^\dagger A^\# + A^\# A^\dagger$.
⟨305⟩ $(A^2)^\dagger + A^\dagger A^\# = (A^\dagger)^2 + A^\# A^\dagger$.
⟨306⟩ $(A^2)^\dagger + A^\# A^\dagger = (A^\dagger)^2 + A^\dagger A^\#$.
⟨307⟩ $3(A^2)^\dagger = (A^\dagger)^2 + A^\# A^\dagger + A^\dagger A^\#$.
⟨308⟩ $3(A^\dagger)^2 = (A^2)^\dagger + A^\# A^\dagger + A^\dagger A^\#$.
⟨309⟩ $2(A^3)^\dagger = A^\dagger A^\# A^\dagger + A^\# A^\dagger A^\#$.
⟨310⟩ $2(A^\dagger)^3 = A^\dagger A^\# A^\dagger + A^\# A^\dagger A^\#$.
⟨311⟩ $2(A^\#)^3 = A^\dagger A^\# A^\dagger + A^\# A^\dagger A^\#$.
⟨312⟩ $2(A^3)^\dagger = (A^2)^\dagger A^\# + A^\# (A^2)^\dagger$.
⟨313⟩ $2(A^3)^\dagger = (A^\dagger)^2 A^\# + A^\# (A^\dagger)^2$.
⟨314⟩ $2(A^3)^\dagger = A^\dagger (A^2)^\# + (A^2)^\# A^\dagger$.
⟨315⟩ $2(A^\dagger)^3 = (A^2)^\dagger A^\# + A^\# (A^2)^\dagger$.
⟨316⟩ $2(A^\dagger)^3 = (A^\dagger)^2 A^\# + A^\# (A^\dagger)^2$.
⟨317⟩ $2(A^\dagger)^3 = A^\dagger (A^2)^\# + (A^2)^\# A^\dagger$.
⟨318⟩ $2(A^3)^\# = (A^2)^\dagger A^\# + A^\# (A^2)^\dagger$.
⟨319⟩ $2(A^3)^\# = (A^\dagger)^2 A^\# + A^\# (A^\dagger)^2$.
⟨320⟩ $2(A^3)^\# = A^\dagger (A^2)^\# + (A^2)^\# A^\dagger$.
⟨321⟩ $3(A^3)^\dagger = (A^2)^\dagger A^\# + A^\dagger A^\# A^\dagger + A^\# (A^2)^\dagger$.
⟨322⟩ $3(A^3)^\dagger = (A^\dagger)^2 A^\# + A^\dagger A^\# A^\dagger + A^\# (A^\dagger)^2$.
⟨323⟩ $3(A^3)^\dagger = A^\dagger (A^2)^\# + A^\dagger A^\# A^\dagger + (A^2)^\# A^\dagger$.
⟨324⟩ $3(A^\dagger)^3 = (A^2)^\dagger A^\# + A^\dagger A^\# A^\dagger + A^\# (A^2)^\dagger$.
⟨325⟩ $3(A^\dagger)^3 = (A^\dagger)^2 A^\# + A^\dagger A^\# A^\dagger + A^\# (A^\dagger)^2$.
⟨326⟩ $3(A^\dagger)^3 = A^\dagger (A^2)^\# + A^\dagger A^\# A^\dagger + (A^2)^\# A^\dagger$.
⟨327⟩ $3(A^3)^\# = (A^2)^\dagger A^\# + A^\# A^\dagger A^\# + A^\# (A^2)^\dagger$.
⟨328⟩ $3(A^3)^\# = (A^\dagger)^2 A^\# + A^\# A^\dagger A^\# + A^\# (A^\dagger)^2$.
⟨329⟩ $3(A^3)^\# = A^\dagger (A^2)^\# + A^\# A^\dagger A^\# + (A^2)^\# A^\dagger$.
⟨330⟩ $4(A^3)^\dagger = (A^\dagger)^3 + A^\dagger (A^2)^\# + (A^2)^\dagger A^\# + (A^3)^\#$.
⟨331⟩ $4(A^3)^\dagger = (A^\dagger)^3 + (A^2)^\# A^\dagger + A^\# (A^2)^\dagger + (A^3)^\#$.
⟨332⟩ $4(A^3)^\# = (A^\dagger)^3 + (A^\dagger)^2 A^\dagger + A^\dagger (A^\dagger)^2 + (A^\dagger)^3$.
⟨333⟩ $4(A^3)^\# = (A^\dagger)^3 + A^\dagger (A^2)^\# + (A^2)^\dagger A^\# + (A^3)^\dagger$.
⟨334⟩ $4(A^3)^\# = (A^\dagger)^3 + (A^2)^\# A^\dagger + A^\# (A^2)^\dagger + (A^3)^\dagger$.
⟨335⟩ $(A^3)^\dagger + (A^\dagger)^3 = A^\dagger A^\# A^\dagger + A^\# A^\dagger A^\#$.
⟨336⟩ $(A^3)^\dagger + (A^\dagger)^3 = (A^2)^\dagger A^\# + A^\# (A^2)^\dagger$.

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⟨337⟩ $(A^3)^\dagger + (A^\dagger)^3 = (A^\dagger)^2A^\# + A^\#(A^\dagger)^2$.
⟨338⟩ $(A^3)^\dagger + (A^\dagger)^3 = A^\dagger(A^2)^\# + (A^2)^\# A^\dagger$.
⟨337⟩ $(A^3)^\dagger + (A^\dagger)^3 = (A^\dagger)^2A^\# + A^\#(A^\dagger)^2$.
⟨339⟩ $I_m - AA^\# = (I_m - AA^\#)^\dagger$.
⟨340⟩ $(I_m - AA^\#)^* = (I_m - AA^\#)^\dagger$.
⟨341⟩ $(I_m - AA^\#)^\dagger = ((I_m - AA^\#)^\dagger)^2$.
⟨342⟩ $AA^\#(I_m - AA^\#)^\dagger = (I_m - AA^\#)^\dagger AA^\#$.
⟨343⟩ $(I_m - AA^\#)(I_m - AA^\#)^* = (I_m - AA^\#)^*(I_m - AA^\#)$.
⟨344⟩ $(I_m - AA^\#)(I_m - AA^\#)^\dagger = (I_m - AA^\#)^\dagger(I_m - AA^\#)$.
⟨345⟩ $AA^\dagger + AA^\# = A^\dagger A + (AA^\#)^*$.
⟨346⟩ $(AA^\dagger - AA^\#)(A^\dagger A - A^\# A) = 0$.
⟨347⟩ $r[A,\ A^*] = r(A)$.
⟨348⟩ $\mathscr{R}(I_m - AA^\dagger) = \mathscr{R}(I_m - AA^\#)$.
⟨349⟩ $\mathscr{R}(I_m - A^\dagger A) = \mathscr{R}(I_m - (AA^\#)^*)$.
⟨350⟩ $\mathscr{R}(I_m - AA^\#) = \mathscr{R}(I_m - (AA^\#)^*)$.
⟨351⟩ $\mathscr{R}(A) = \mathscr{R}(A^*)$, *namely, A is range-Hermitian.*

Proof. Note that $A^\dagger, A^\# \in \{A^{(2)}\}$ by the definitions in (1.1) and (1.2). In this case, applying (2.48) to the difference $A^\dagger - A^\#$ and simplifying by Lemma 2.5, (2.5), and (2.32), we obtain

$$r(A^\dagger - A^\#) = r\begin{bmatrix} A^\dagger \\ A^\# \end{bmatrix} + r[A^\dagger,\ A^\#] - r(A^\dagger) - r(A^\#)$$
$$= r\begin{bmatrix} A^* \\ A \end{bmatrix} + r[A^*,\ A] - 2r(A) = 2r[A,\ A^*] - 2r(A),$$

thus establishing (3.62).

Replacing A in (3.62) with $A^\dagger$ and $A^\#$ respectively and applying (2.1), (2.5)–(2.7), (2.22), and (2.32)–(2.35) leads to (3.63) and (3.64).

By (2.40), (2.41), (2.44), and (2.45),

$$r(A^\dagger - A^\dagger AA^\#) = r\begin{bmatrix} A^\dagger \\ A \end{bmatrix} - r(A) = r[A,\ A^*] - r(A),$$
$$r(A^\dagger - A^\# AA^\dagger) = r[A,\ A^\dagger] - r(A) = r[A,\ A^*] - r(A),$$
$$r(A^\# - A^\dagger AA^\#) = r[A^\dagger,\ A^\#] - r(A^\dagger) = r[A,\ A^*] - r(A),$$
$$r(A^\# - A^\# AA^\dagger) = r\begin{bmatrix} A^\dagger \\ A^\# \end{bmatrix} - r(A^\dagger) = r[A,\ A^*] - r(A),$$

thus establishing (3.64)–(3.68). Eqs. (3.69) and (3.70) follow directly from (2.44) and (2.45).

It is easy to verify $AA^\dagger$, $AA^\#$, and $A^\# A$ are three idempotent matrices by the definitions in (1.1) and (1.2). In this case, applying (2.48) to the differences $AA^\dagger - AA^\#$ and $A^\dagger A - A^\# A$, and then simplifying by Lemma 2.5, (2.5), and (2.32), we obtain

$$r(AA^\dagger - AA^\#) = r\begin{bmatrix} AA^\dagger \\ AA^\# \end{bmatrix} + r[AA^\dagger,\ AA^\#] - r(AA^\dagger) - r(AA^\#)$$
$$= r\begin{bmatrix} A^* \\ A \end{bmatrix} + r[A,\ A] - 2r(A) = r[A,\ A^*] - r(A),$$
$$r(A^\dagger A - A^\# A) = r\begin{bmatrix} A^\dagger A \\ A^\# A \end{bmatrix} + r[A^\dagger A,\ A^\# A] - r(A^\dagger A) - r(A^\# A)$$
$$= r\begin{bmatrix} A \\ A \end{bmatrix} + r[A^* A,\ A] - 2r(A) = r[A,\ A^*] - r(A),$$

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thus establishing the two rank equalities in (3.71).

It is easy to verify by the definitions in (1.1) and (1.2) that $AA^{\dagger}, (AA^{\#})^{\dagger} \in \{AA^{\#})^{(2)}\}$. In this case, applying (2.48) to the differences $AA^{\dagger} - (AA^{\#})^{\dagger}$ and $A^{\dagger}A - (A^{\#}A)^{\dagger}$, and then simplifying by Lemma 2.5, (2.5), and (2.32), we obtain

$$\begin{aligned} r(AA^{\dagger} - (AA^{\#})^{\dagger}) &= r\begin{bmatrix} AA^{\dagger} \\ (AA^{\#})^{\dagger} \end{bmatrix} + r[AA^{\dagger}, (AA^{\#})^{\dagger}] - r(AA^{\dagger}) - r((AA^{\#})^{\dagger}) \\ &= r\begin{bmatrix} A^* \\ A^* \end{bmatrix} + r[A,\, A^*] - 2r(A) = r[A,\, A^*] - r(A), \\ r(A^{\dagger}A - (AA^{\#})^{\dagger}) &= r\begin{bmatrix} A^{\dagger}A \\ (A^{\#}A)^{\dagger} \end{bmatrix} + r[A^{\dagger}A, (A^{\#}A)^{\dagger}] - r(AA^{\dagger}) - r((AA^{\#})^{\dagger}) \\ &= r\begin{bmatrix} A \\ A^* \end{bmatrix} + r[A^*,\, A^*] - 2r(A) = r[A,\, A^*] - r(A), \end{aligned}$$

thus establishing the two rank equalities in (3.72). It is easy to verify that $AA^{\#}$ and $(AA^{\#})^*$ are two idempotent matrices by the definition of the group inverse. In this case, applying (2.48) to the difference $(AA^{\#})^* - AA^{\#}$, and then simplifying by Lemma 2.5, (2.32)–(2.35), we obtain

$$\begin{aligned} r((AA^{\#})^* - AA^{\#}) &= r\begin{bmatrix} (AA^{\#})^* \\ AA^{\#} \end{bmatrix} + r[(AA^{\#})^*,\, AA^{\#}] - r((AA^{\#})^*) - r(AA^{\#}) \\ &= r\begin{bmatrix} A^* \\ A \end{bmatrix} + r[A^*,\, A] - 2r(A) = 2r[A,\, A^*] - 2r(A). \end{aligned}$$

thus establishing the rank equality in (3.73).

Note that $A^{\dagger}A^{\#}, A^{\#}A^{\dagger} \in \{(A^2)^{(2)}\}$ by (3.25) and (3.26) for $k = 1$. In this case, applying (2.48) to the difference $A^{\dagger}A^{\#} - A^{\#}A^{\dagger}$ and simplifying by Lemma 2.5, (2.17), and (2.33)–(2.37), we obtain

$$\begin{aligned} r(A^{\dagger}A^{\#} - A^{\#}A^{\dagger}) &= r\begin{bmatrix} A^{\dagger}A^{\#} \\ A^{\#}A^{\dagger} \end{bmatrix} + r[A^{\dagger}A^{\#}\, A^{\#}A^{\dagger}] - r(A^{\dagger}A^{\#}) - r(A^{\#}A^{\dagger}) \\ &= r\begin{bmatrix} A \\ A^* \end{bmatrix} + r[A^*,\, A] - 2r(A) = 2r[A,\, A^*] - 2r(A), \end{aligned}$$

thus establishing the rank equality in (3.74). Replacing A in (3.74) with $A^{\dagger}$ and $A^{\#}$ respectively and applying (2.1), (2.5)–(2.7), (2.22), and (2.32)–(2.35) leads to (3.75) and (3.76).

Note that $(A^2)^{\dagger}, (A^2)^{\#}, A^{\dagger}A^{\#}, A^{\#}A^{\dagger} \in \{A^{(2)}\}$ by the definitions in in (1.1) and (1.2), as well as (3.25) and (3.26). In this case, applying (2.48) to the difference $(A^2)^{\dagger} - A^{\dagger}A^{\#}$ and $(A^2)^{\dagger} - A^{\#}A^{\dagger}$ and simplifying by Lemma 2.5, (2.17), and (2.33)–(2.37), we obtain

$$\begin{aligned} r((A^2)^{\dagger} - A^{\dagger}A^{\#}) &= r\begin{bmatrix} (A^2)^{\dagger} \\ A^{\dagger}A^{\#} \end{bmatrix} + r[(A^2)^{\dagger},\, A^{\dagger}A^{\#}] - r((A^2)^{\dagger}) - r(A^{\dagger}A^{\#}) \\ &= r\begin{bmatrix} A^* \\ A \end{bmatrix} + r[A^*,\, A^*] - 2r(A) = r[A,\, A^*] - r(A), \\ r((A^2)^{\dagger} - A^{\#}A^{\dagger}) &= r\begin{bmatrix} (A^2)^{\dagger} \\ A^{\#}A^{\dagger} \end{bmatrix} + r[(A^2)^{\dagger},\, A^{\#}A^{\dagger}] - r((A^2)^{\dagger}) - r(A^{\#}A^{\dagger}) \\ &= r\begin{bmatrix} A^* \\ A^* \end{bmatrix} + r[A^*,\, A] - 2r(A) = r[A,\, A^*] - r(A), \end{aligned}$$

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and

$$r((A^2)^{\#} - A^{\dagger}A^{\#}) = r\begin{bmatrix}(A^2)^{\#}\\ A^{\dagger}A^{\#}\end{bmatrix} + r[(A^2)^{\#},\, A^{\dagger}A^{\#}] - r((A^2)^{\#}) - r(A^{\dagger}A^{\#})$$
$$= r\begin{bmatrix}A\\ A\end{bmatrix} + r[A\, A^*] - 2r(A) = r[A,\, A^*] - r(A),$$
$$r((A^2)^{\#} - A^{\#}A^{\dagger}) = r\begin{bmatrix}(A^2)^{\#}\\ A^{\#}A^{\dagger}\end{bmatrix} + r[(A^2)^{\#},\, A^{\#}A^{\dagger}] - r((A^2)^{\#}) - r(A^{\#}A^{\dagger})$$
$$= r\begin{bmatrix}A\\ A^*\end{bmatrix} + r[A\, A] - 2r(A) = r[A,\, A^*] - r(A),$$

thus establishing the rank equalities in (3.77) and (3.78).

By (2.31) and (3.46),

$$\begin{aligned}
r((AA^{\dagger} - AA^{\#})(A^{\dagger}A - A^{\#}A)) &= r(A(A^{\dagger})^2A - AA^{\#}) \\
&= r(A(A^{\dagger})^2A - A(A^2)^{\dagger}A) \\
&= r\left(\begin{bmatrix}(A^2)^*A\\ AA^{\dagger}\end{bmatrix}[A(A^2)^*, A^{\dagger}A]\right) - r(A^2) \\
&= r\left(\begin{bmatrix}A\\ A^*\end{bmatrix}[A,\, A^*]\right) - r(A) = r[A,\, A^*] - r(A),
\end{aligned}$$

thus establishing the rank equalities in (3.79).

By (2.31) and (3.47),

$$\begin{aligned}
r(AA^{\#} - AA^{\#}(AA^{\#})^*AA^{\#}) &= r(A^2 - A(AA^{\#})^*A) \\
&= r((A^2)^* - A^*AA^{\#}A^*) \\
&= r((A^2)^* - A^*A(A^2)^{\dagger}AA^*) \\
&= r\left(\begin{bmatrix}(A^2)^*A\\ A^*\end{bmatrix}[A(A^2)^*, A^*]\right) - r(A^2) \\
&= r\left(\begin{bmatrix}A\\ A^*\end{bmatrix}[A,\, A^*]\right) - r(A) = r[A,\, A^*] - r(A),
\end{aligned}$$

thus establishing the rank equalities in (3.80).

Based on the two facts in (3.1) and (3.2), we apply (2.48) to the difference $A^{\dagger}AA^{\#} - A^{\#}AA^{\dagger}$ and simplify by Lemma 2.5, (2.17), and (2.33)–(2.37) lead to

$$r(A^{\dagger}AA^{\#} - A^{\#}AA^{\dagger}) = r\begin{bmatrix}A^{\dagger}AA^{\#}\\ A^{\#}AA^{\dagger}\end{bmatrix} + r[A^{\dagger}AA^{\#},\, A^{\#}AA^{\dagger}] - r(A^{\dagger}AA^{\#}) - r(A^{\#}AA^{\dagger})$$
$$= r\begin{bmatrix}A\\ A^*\end{bmatrix} + r[A^*,\, A] - 2r(A) = 2r[A,\, A^*] - 2r(A),$$

thus establishing the rank equality in (3.81).

Replacing A in (3.81) with $A^{\dagger}$ and $A^{\#}$ respectively and applying (2.1), (2.5)–(2.7), (2.22), and (2.32)–(2.35) leads to (3.82) and (3.83).

Based on the facts in (3.3)–(3.30) and simplifying by Lemma 2.5, we are able to obtain (3.84)–(3.134), but the details are omitted.

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Rewrite $2AA^{\#} - AA^{\dagger} - A^{\dagger}A$ as

$$2AA^{\#} - AA^{\dagger} - A^{\dagger}A = AA^{\dagger}(I_m - A^{\#}A) + (I_m - AA^{\#})A^{\dagger}A,$$

where we see from (2.38) and (2.39) that

$$\mathscr{R}(AA^{\dagger}(I_m - A^{\#}A)) \cap \mathscr{R}((I_m - AA^{\#})A^{\dagger}A) = \{0\},$$

$$\mathscr{R}((AA^{\dagger}(I_m - A^{\#}A))^*) \cap \mathscr{R}(((I_m - AA^{\#})A^{\dagger}A)^*) = \{0\}$$

hold. Then, we obtain from (2.6), (2.7), (2.44), (2.45), and (2.47) that

$$\begin{aligned} r(2AA^{\#} - AA^{\dagger} - A^{\dagger}A) &= r(AA^{\dagger}(I_m - A^{\#}A)) + r((I_m - AA^{\#})A^{\dagger}A) \\ &= r\begin{bmatrix} AA^{\dagger} \\ A \end{bmatrix} + r[A,\ A^{\dagger}A] - 2r(A) \\ &= 2r[A,\ A^*] - 2r(A), \end{aligned}$$

thus establishing the rank equality in (3.135). Note that

$$\begin{aligned} 2AA^{\dagger} - AA^{\#} - (AA^{\#})^* &= -AA^{\#}(I_m - AA^{\dagger}) - (I_m - AA^{\dagger})(AA^{\#})^*, \\ 2A^{\dagger}A - AA^{\#} - (AA^{\#})^* &= -(AA^{\#})^*(I_m - A^{\dagger}A) - (I_m - A^{\dagger}A)AA^{\#}. \end{aligned}$$

Then, we obtain from (2.12), (2.13), (2.34), (2.35), (2.40), (2.41), and (2.47) that

$$\begin{aligned} r(2AA^{\dagger} - AA^{\#} - (AA^{\#})^*) &= r(AA^{\#}(I_m - AA^{\dagger})) + r((I_m - AA^{\dagger})(AA^{\#})^*) \\ &= r\begin{bmatrix} AA^{\#} \\ A^* \end{bmatrix} + r[A,\ (AA^{\#})^*] - 2r(A) \\ &= 2r[A,\ A^*] - 2r(A), \end{aligned}$$

$$\begin{aligned} r(2A^{\dagger}A - AA^{\#} - (AA^{\#})^*) &= r((AA^{\#})^*(I_m - A^{\dagger})A)) + r((I_m - A^{\dagger}A)AA^{\#}) \\ &= r\begin{bmatrix} (AA^{\#})^* \\ A \end{bmatrix} + r[A^*,\ AA^{\#}] - 2r(A) \\ &= 2r[A,\ A^*] - 2r(A), \end{aligned}$$

thus establishing the two rank equalities in (3.136) and (3.137). Observe that

$$\begin{aligned} AA^{\dagger} + AA^{\#} &= AA^{\dagger} + AA^{\dagger}AA^{\#} = AA^{\dagger}(I_m + AA^{\#}), \\ A^{\dagger}A + (A^{\#}A)^* &= A^{\dagger}A + A^{\dagger}A(A^{\#}A)^* = A^{\dagger}A(I_m + (A^{\#}A)^*), \end{aligned}$$

where $I_m + AA^{\#}$ and $I_m + (A^{\#}A)^*$ are two invertible matrices because $AA^{\#}$ and $(A^{\#}A)^*$ are idempotent. In this case, we have

$$r(AA^{\dagger} + AA^{\#}) = r(A), \quad r(A^{\dagger}A + (A^{\#}A)^*) = r(A), \tag{3.153}$$

$$\mathscr{R}(AA^{\dagger} + AA^{\#}) = \mathscr{R}(A), \ \mathscr{R}(A^{\dagger}A + (A^{\#}A)^*) = \mathscr{R}(A^*). \tag{3.154}$$

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Also it is easy to verify by the definitions in (1.1) and (1.2) that $(A^{\dagger}A + A^{\#}A)/2$ and $(AA^{\dagger} + AA^{\#})/2$ are two idempotent matrices. In this case, we obtain from (2.48), (3.153), and (3.154) that

$$\begin{aligned}
& r(2AA^{\dagger} - A^{\dagger}A - A^{\#}A) \\
&= r(AA^{\dagger} - (A^{\dagger}A + A^{\#}A)/2) \\
&= r\begin{bmatrix} AA^{\dagger} \\ (A^{\dagger}A + A^{\#}A)/2 \end{bmatrix} + r[AA^{\dagger},\ (A^{\dagger}A + A^{\#}A)/2] - r(AA^{\dagger}) - r((A^{\dagger}A + A^{\#}A)/2) \\
&= r\begin{bmatrix} AA^{\dagger} \\ A \end{bmatrix} + r[AA^{\dagger},\ A^{\dagger}A] - 2r(A) \\
&= 2r[A,\ A^{*}] - 2r(A),
\end{aligned}$$

$$\begin{aligned}
& r(2A^{\dagger}A - AA^{\dagger} - AA^{\#}) \\
&= r(A^{\dagger}A - (AA^{\dagger} + AA^{\#})/2) \\
&= r\begin{bmatrix} A^{\dagger}A \\ (AA^{\dagger} + AA^{\#})/2 \end{bmatrix} + r[A^{\dagger}A,\ (AA^{\dagger} + AA^{\#})/2] - r(A^{\dagger}A) - r((AA^{\dagger} + AA^{\#})/2) \\
&= r[AA^{\dagger},\ A^{\dagger}A] + r[A^{\dagger}A,\ A] - 2r(A) \\
&= 2r[A,\ A^{*}] - 2r(A),
\end{aligned}$$

$$\begin{aligned}
& r(2(AA^{\#})^{*} - AA^{\dagger} - AA^{\#}) \\
&= r((AA^{\#})^{*} - (AA^{\dagger} + AA^{\#})/2) \\
&= r\begin{bmatrix} (AA^{\#})^{*} \\ (AA^{\dagger} + AA^{\#})/2 \end{bmatrix} + r[(AA^{\#})^{*},\ (AA^{\dagger} + AA^{\#})/2] - r((AA^{\#})^{*}) - r((AA^{\dagger} + AA^{\#})/2) \\
&= r\begin{bmatrix} (AA^{\#})^{*} \\ AA^{\#} \end{bmatrix} + r[(AA^{\#})^{*},\ A] - 2r(A) \\
&= 2r[A,\ A^{*}] - 2r(A),
\end{aligned}$$

$$\begin{aligned}
& r(2(A^{\#}A)^{*} - A^{\dagger}A - A^{\#}A) \\
&= r((A^{\#}A)^{*} - (A^{\dagger}A + A^{\#}A)/2) \\
&= r\begin{bmatrix} (A^{\#}A)^{*} \\ (A^{\dagger}A + A^{\#}A)/2 \end{bmatrix} + r[(A^{\#}A)^{*},\ (A^{\dagger}A + A^{\#}A)/2] - r((A^{\#}A)^{*}) - r((A^{\dagger}A + A^{\#}A)/2) \\
&= r\begin{bmatrix} (AA^{\#})^{*} \\ A \end{bmatrix} + r[(AA^{\#})^{*},\ AA^{\#}] - 2r(A) \\
&= 2r[A,\ A^{*}] - 2r(A),
\end{aligned}$$

thus establishing (3.138)–(3.141). Note that

$$2A^{\dagger} - A^{\dagger}AA^{\#} - A^{\#}AA^{\dagger} = -A^{\dagger}AA^{\#}(I_m - AA^{\dagger}) - (I_m - A^{\dagger}A)A^{\#}AA^{\dagger},$$

where we obtain from (2.12), (2.13), (2.36), (2.37), (2.40), (2.41), and (2.47) that

$$\begin{aligned} r(2A^{\dagger} - A^{\dagger}AA^{\#} - A^{\#}AA^{\dagger}) &= r(A^{\dagger}AA^{\#}(I_m - AA^{\dagger})) + r((I_m - A^{\dagger}A)A^{\#}AA^{\dagger})) \\ &= r\begin{bmatrix} A^* \\ A^{\dagger}AA^{\#} \end{bmatrix} + r[A^*, A^{\#}AA^{\dagger}] - 2r(A) \\ &= 2r[A,\ A^*] - 2r(A), \end{aligned}$$

thus establishing (3.142). Note that

$$2A^{\#} - A^{\dagger}AA^{\#} - A^{\#}AA^{\dagger} = -A^{\#}AA^{\dagger}(I_m - AA^{\#}) - (I_m - A^{\#}A)A^{\dagger}AA^{\#},$$

where we obtain from (2.36), (2.37), (2.38), (2.39), (2.44), (2.45), and (2.47) that

$$\begin{aligned} r(2A^{\#} - A^{\dagger}AA^{\#} - A^{\#}AA^{\dagger}) &= r(A^{\#}AA^{\dagger}(I_m - AA^{\#})) + r((I_m - A^{\#}A)A^{\dagger}AA^{\#}) \\ &= r\begin{bmatrix} A \\ A^{\#}AA^{\dagger} \end{bmatrix} + r[A, A^{\dagger}AA^{\#}] - 2r(A) \\ &= 2r[A,\ A^*] - 2r(A), \end{aligned}$$

thus establishing (3.143). Replacing A with A^2 in (3.142) and (3.143) and simplifying with the condition $r(A^2) = r(A)$ lead to (3.144) and (3.145), respectively. Replacing A with A^3 in (3.142) and (3.143) and simplifying with the condition $r(A^2) = r(A)$ lead to (3.146) and (3.147), respectively.

By (2.40), (2.41), (2.44), and (2.45), we obtain

$$\begin{aligned} r[I_m - AA^{\dagger},\ I_m - AA^{\#}] &= r(I_m - AA^{\dagger}) + r(AA^{\dagger} - AA^{\dagger}A^{\#}A) \\ &= m - r(A) + r\begin{bmatrix} AA^{\dagger} \\ A \end{bmatrix} - r(A) \\ &= m - 2r(A) + r[A,\ A^*], \end{aligned}$$

$$\begin{aligned} r[I_m - A^{\dagger}A,\ I_m - (AA^{\#})^*] &= r(I_m - A^{\dagger}A) + r(A^{\dagger}A - A^{\dagger}A(AA^{\#})^*) \\ &= m - r(A) + r(A^{\dagger}A - AA^{\#}A^{\dagger}A) \\ &= m - r(A) + r[A,\ A^{\dagger}A] - r(A) \\ &= m - 2r(A) + r[A,\ A^*], \end{aligned}$$

$$\begin{aligned} r[I_m - AA^{\#},\ I_m - (AA^{\#})^*] &= r(I_m - AA^{\#}) + r(AA^{\#} - AA^{\#}(AA^{\#})^*) \\ &= m - r(A) + r((AA^{\#})^* - AA^{\#}(AA^{\#})^*) \\ &= m - r(A) + r((AA^{\#})^* - AA^{\#}(AA^{\#})^*) \\ &= m - r(A) + r[AA^{\#},\ (AA^{\#})^*] - r(A) \\ &= m - 2r(A) + r[A,\ A^*], \end{aligned}$$

$$\begin{aligned} r[I_m - A^{\dagger}A,\ I_m - AA^{\#}] &= r(I_m - A^{\dagger}A) + r(A^{\dagger}A - A^{\dagger}AAA^{\#}) \\ &= m - r(A), \end{aligned}$$

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$$
\begin{aligned}
r[I_m - AA^\dagger,\, I_m - (AA^\#)^*] &= r(I_m - AA^\dagger) + r(AA^\dagger - AA^\dagger(AA^\#)^*) \\
&= m - r(A) + r(AA^\dagger - AA^\dagger(AA^\#)^*) \\
&= m - r(A) + r(AA^\dagger - AA^\# AA^\dagger) \\
&= m - r(A),
\end{aligned}
$$

thus establishing (3.148)–(3.152).

The equivalences of Conditions ⟨1⟩–⟨351⟩ are direct consequences of the matrix rank equalities in (3.62)–(3.152), as well as various variations of the conditions, and therefore, the details are omitted due to lack of space. □

4. Characterizations of a reverer-order law for the Moore–Penrose inverse of a matrix product

As we know in matrix analysis that the range-Hermitian property of a matrix is a commonly-used concept that can be used to describe hidden features of matrix operations. Here, we mention the well-known reverse-order law $(AB)^\dagger = B^\dagger A^\dagger$ for the the Moore–Penrose inverse of matrices, where $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. This reverse-order law does not necessarily hold because of the possible singularity of the two given matrices, so that a problem encountered in dealing with the Moore–Penrose inverse of a matrix product is to establish identifying conditions under which the reverse-order law holds. Here, we mention a highly-recognized result:

$$
(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow A^*ABB^* \text{ is range-Hermitian,} \tag{4.1}
$$

namely, the reverse-order law in (4.1) can be characterized by the fact that the matrix product A^*ABB^* is a range-Hermitian matrix (see [2, p. 161]). This fact helps establish connections between the basic reverse-order law and many other matrix equalities that are composed of mixed products of A, B, and their conjugate transposes. As direct applications of Theorem 3.3 to the mixed product A^*ABB^*, we obtain the following conclusions.

Theorem 4.1. *Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$, and denote $M = A^*ABB^*$. Then, $r(M^2) = r(M)$ always holds, and the following 352 conditions are equivalent:*

⟨0⟩ $(AB)^\dagger = B^\dagger A^\dagger$.
⟨1⟩ $M^\dagger = M^\#$.
⟨2⟩ $(M^\dagger)^\# = M$.
⟨3⟩ $(M^\#)^\dagger = M$.
⟨4⟩ $MM^\#M^\dagger = M^\dagger$.
⟨5⟩ $M^\dagger M^\# M = M^\dagger$.
⟨6⟩ $M^\dagger M^\# M = M^\#$.
⟨7⟩ $MM^\#M^\dagger = M^\#$.
⟨8⟩ $(M^\dagger)^\# MM^\# = (M^\dagger)^\#$.
⟨9⟩ $(M^\dagger)^\# MM^\# = (M^\#)^\dagger$.
⟨10⟩ $MM^\#(M^\dagger)^\# = (M^\dagger)^\#$.
⟨11⟩ $MM^\#(M^\dagger)^\# = (M^\#)^\dagger$.
⟨12⟩ $(M^\dagger)^\# MM^\# = M$.
⟨13⟩ $MM^\#(M^\dagger)^\# = M$.
⟨14⟩ $(M^\#)^\dagger M^\# M = M$.
⟨15⟩ $MM^\#(M^\#)^\dagger = M$.
⟨16⟩ $MM^\dagger M^\# M^\dagger M = M^\#$.
⟨17⟩ $MM^\#(M^\dagger)^\# M^\# M = (M^\dagger)^\#$.
⟨18⟩ $MM^\#(M^\dagger)^\# M^\# M = (M^\#)^\dagger$.
⟨19⟩ $MM^\#(M^\dagger)^\# M^\# M = (M^\dagger)^\#$.
⟨20⟩ $MM^\#(M^\dagger)^\# M^\# M = (M^\#)^\dagger$.
⟨21⟩ $(MM^\#)^\dagger M(M^\# M)^\dagger = M$.

⟨22⟩ $(MM^{\#})^{\dagger}M^{\#}(M^{\#}M)^{\dagger} = M^{\#}$.
⟨23⟩ $M^{\dagger}(M^{\#}MM^{\dagger})^{\dagger}M^{\#} = M^{\dagger}$.
⟨24⟩ $M^{\dagger}(M^{\#}MM^{\dagger})^{\dagger}M^{\#} = M^{\#}$.
⟨25⟩ $M^{\#}(M^{\#}MM^{\dagger})^{\dagger}M^{\dagger} = M^{\dagger}$.
⟨26⟩ $M^{\#}(M^{\#}MM^{\dagger})^{\dagger}M^{\dagger} = M^{\#}$.
⟨27⟩ $M^{\dagger}(M^{\#}MM^{\dagger})^{\#}M^{\#} = M^{\dagger}$.
⟨28⟩ $M^{\dagger}(M^{\#}MM^{\dagger})^{\#}M^{\#} = M^{\#}$.
⟨29⟩ $M^{\#}(M^{\#}MM^{\dagger})^{\#}M^{\dagger} = M^{\dagger}$.
⟨30⟩ $M^{\#}(M^{\#}MM^{\dagger})^{\#}M^{\dagger} = M^{\#}$.
⟨31⟩ $M^{\dagger}(M^{\dagger}M^{\#}M^{\dagger})^{\dagger}M^{\dagger} = M$.
⟨32⟩ $M^{\dagger}(M^{\dagger}M^{\#}M^{\dagger})^{\#}M^{\dagger} = M$.
⟨33⟩ $M^{\dagger}(M^{\#}M^{\dagger}M^{\#})^{\dagger}M^{\dagger} = M$.
⟨34⟩ $M^{\dagger}(M^{\#}M^{\dagger}M^{\#})^{\#}M^{\dagger} = M$.
⟨35⟩ $M^{\dagger}(M^{\dagger}M^{\#}M^{\dagger})^{\dagger}M^{\#} = M$.
⟨36⟩ $M^{\dagger}(M^{\dagger}M^{\#}M^{\dagger})^{\#}M^{\#} = M$.
⟨37⟩ $M^{\dagger}(M^{\#}M^{\dagger}M^{\#})^{\dagger}M^{\#} = M$.
⟨38⟩ $M^{\dagger}(M^{\#}M^{\dagger}M^{\#})^{\#}M^{\#} = M$.
⟨39⟩ $M^{\#}(M^{\dagger}M^{\#}M^{\dagger})^{\dagger}M^{\dagger} = M$.
⟨40⟩ $M^{\#}(M^{\dagger}M^{\#}M^{\dagger})^{\#}M^{\dagger} = M$.
⟨41⟩ $M^{\#}(M^{\#}M^{\dagger}M^{\#})^{\dagger}M^{\dagger} = M$.
⟨42⟩ $M^{\#}(M^{\#}M^{\dagger}M^{\#})^{\#}M^{\dagger} = M$.
⟨43⟩ $M^{\#}(M^{\#}M^{\dagger}M^{\#})^{\dagger}M^{\#} = (M^{\dagger})^{\#}$.
⟨44⟩ $M^{\#}(M^{\#}M^{\dagger}M^{\#})^{\dagger}M^{\#} = (M^{\#})^{\dagger}$.
⟨45⟩ $M^{\#}(M^{\#}M^{\dagger}M^{\#})^{\#}M^{\#} = (M^{\dagger})^{\#}$.
⟨46⟩ $M^{\#}(M^{\#}M^{\dagger}M^{\#})^{\#}M^{\#} = (M^{\#})^{\dagger}$.
⟨47⟩ $M^{\#}(M^{\dagger}M^{\#}M^{\dagger})^{\dagger}M^{\#} = (M^{\dagger})^{\#}$.
⟨48⟩ $M^{\#}(M^{\dagger}M^{\#}M^{\dagger})^{\dagger}M^{\#} = (M^{\#})^{\dagger}$.
⟨49⟩ $M^{\#}(M^{\dagger}M^{\#}M^{\dagger})^{\#}M^{\#} = (M^{\dagger})^{\#}$.
⟨50⟩ $M^{\#}(M^{\dagger}M^{\#}M^{\dagger})^{\#}M^{\#} = (M^{\#})^{\dagger}$.
⟨51⟩ $M^{\#}M^{\dagger}(M^{\dagger}M^{\#}M^{\dagger})^{\dagger}M^{\dagger}M^{\#} = M^{\dagger}$.
⟨52⟩ $M^{\#}M^{\dagger}(M^{\dagger}M^{\#}M^{\dagger})^{\#}M^{\dagger}M^{\#} = M^{\dagger}$.
⟨53⟩ $M^{\#}M^{\dagger}(M^{\#}M^{\dagger}M^{\#})^{\dagger}M^{\dagger}M^{\#} = M^{\dagger}$.
⟨54⟩ $M^{\#}M^{\dagger}(M^{\#}M^{\dagger}M^{\#})^{\#}M^{\dagger}M^{\#} = M^{\dagger}$.
⟨55⟩ $M^{\dagger}M^{\#}(M^{\#}M^{\dagger}M^{\#})^{\dagger}M^{\#}M^{\dagger} = M^{\#}$.
⟨56⟩ $M^{\dagger}M^{\#}(M^{\#}M^{\dagger}M^{\#})^{\#}M^{\#}M^{\dagger} = M^{\#}$.
⟨57⟩ $M^{\dagger}M^{\#}(M^{\dagger}M^{\#}M^{\dagger})^{\dagger}M^{\#}M^{\dagger} = M^{\#}$.
⟨58⟩ $M^{\dagger}M^{\#}(M^{\dagger}M^{\#}M^{\dagger})^{\#}M^{\#}M^{\dagger} = M^{\#}$.
⟨59⟩ $MM^{\#}M^{*} = M^{*}$.
⟨60⟩ $M^{*}M^{\#}M = M^{*}$.
⟨61⟩ $MM^{\#}M^{*}M^{\#}M = M^{*}$.
⟨62⟩ $MM^{\#}(M^{\#})^{*} = (M^{\#})^{*}$.
⟨63⟩ $(M^{\#})^{*}M^{\#}M = (M^{\#})^{*}$.
⟨64⟩ $MM^{\#}(M^{\#})^{*}M^{\#}M = (M^{\#})^{*}$.
⟨65⟩ $MM^{\#}(MM^{\#})^{*}M = M$.
⟨66⟩ $M(M^{\#}M)^{*}M^{\#}M = M$.
⟨67⟩ $MM^{\#}(MM^{\#})^{*}M^{\#} = M^{\#}$.
⟨68⟩ $M^{\#}(M^{\#}M)^{*}M^{\#}M = M^{\#}$.
⟨69⟩ $MM^{\#}(MM^{\#})^{*}M^{\dagger} = M^{\dagger}$.
⟨70⟩ $M^{\dagger}(M^{\#}M)^{*}M^{\#} = M^{\dagger}$.
⟨71⟩ $MM^{\#}(MM^{\#})^{*}(M^{\dagger})^{\#} = (M^{\dagger})^{\#}$.
⟨72⟩ $(M^{\dagger})^{\#}(M^{\#}M)^{*}M^{\#}M = (M^{\dagger})^{\#}$.
⟨73⟩ $MM^{\#}(MM^{\#})^{*}(M^{\#})^{\dagger} = (M^{\#})^{\dagger}$.
⟨74⟩ $(M^{\#})^{\dagger}(M^{\#}M)^{*}M^{\#}M = (M^{\#})^{\dagger}$.
⟨75⟩ $MM^{\#}(MM^{\#})^{*}M(M^{\#}M)^{*}M^{\#}M = M$.
⟨76⟩ $MM^{\#}(MM^{\#})^{*}M^{\#}(M^{\#}M)^{*}M^{\#}M = M^{\#}$.

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⟨77⟩ $MM^{\#}(MM^{\#})^{*}M^{\dagger}(MM^{\#})^{*}MM^{\#} = M^{\dagger}$.
⟨78⟩ $MM^{\#}(MM^{\#})^{*}(M^{\dagger})^{\#}(MM^{\#})^{*}MM^{\#} = (M^{\dagger})^{\#}$.
⟨79⟩ $MM^{\#}(MM^{\#})^{*}(M^{\dagger})^{\#}(MM^{\#})^{*}MM^{\#} = (M^{\#})^{\dagger}$.
⟨80⟩ $MM^{\#}(MM^{\#})^{*}(M^{\#})^{\dagger}(MM^{\#})^{*}MM^{\#} = (M^{\dagger})^{\#}$.
⟨81⟩ $MM^{\#}(MM^{\#})^{*}(M^{\#})^{\dagger}(MM^{\#})^{*}MM^{\#} = (M^{\#})^{\dagger}$.
⟨82⟩ $(MM^{\#})^{*} = MM^{\#}$.
⟨83⟩ $(MM^{\#})^{*} = (MM^{\#})^{\dagger}$.
⟨84⟩ $MM^{\#}(MM^{\#})^{*}MM^{\#} = MM^{\#}$.
⟨85⟩ $MM^{\#}(MM^{\#})^{*}MM^{\#} = (MM^{\#})^{*}$.
⟨86⟩ $MM^{\#}(MM^{\#})^{*} = (MM^{\#})^{*}MM^{\#}$.
⟨87⟩ $MM^{\dagger} = MM^{\#}$.
⟨88⟩ $M^{\dagger}M = M^{\#}M$.
⟨89⟩ $M^{\dagger}M^{\#} = M^{\#}M^{\dagger}$.
⟨90⟩ $M(M^{\dagger})^{\#} = (M^{\dagger})^{\#}M$.
⟨91⟩ $M(M^{\#})^{\dagger} = (M^{\#})^{\dagger}M$.
⟨92⟩ $M(M^{\dagger})^{\#} = (M^{\#})^{\dagger}M$.
⟨93⟩ $M(M^{\#})^{\dagger} = (M^{\dagger})^{\#}M$.
⟨94⟩ $(MM^{\#})^{\dagger} = MM^{\#}$.
⟨95⟩ $(MM^{\#})^{\dagger} = (M^{\#})^{\dagger}M^{\dagger}$.
⟨96⟩ $(M^{\#}M)^{\dagger} = M^{\dagger}(M^{\#})^{\dagger}$.
⟨97⟩ $(M^{\dagger}M^{\#})^{\dagger} = (M^{\#}M^{\dagger})^{\#}$.
⟨98⟩ $(M^{\dagger}M^{\#})^{\#} = (M^{\#}M^{\dagger})^{\dagger}$.
⟨99⟩ $((M^{\dagger}M^{\#})^{\dagger})^{\#} = ((M^{\#}M^{\dagger})^{\#})^{\dagger}$.
⟨100⟩ $((M^{\dagger}M^{\#})^{\#})^{\dagger} = ((M^{\#}M^{\dagger})^{\dagger})^{\#}$.
⟨101⟩ $(M(M^{\dagger})^{\#})^{\dagger} = ((M^{\dagger})^{\#}M)^{\#}$.
⟨102⟩ $(M(M^{\dagger})^{\#})^{\#} = ((M^{\dagger})^{\#}M)^{\dagger}$.
⟨103⟩ $(M(M^{\#})^{\dagger})^{\dagger} = ((M^{\#})^{\dagger}M)^{\#}$.
⟨104⟩ $(M(M^{\#})^{\dagger})^{\#} = ((M^{\#})^{\dagger}M)^{\dagger}$.
⟨105⟩ $M^{\#}(M^{\dagger})^{\#} = (M^{\dagger})^{\#}M^{\#}$.
⟨106⟩ $((M^{\dagger})^{\#}(M^{\#})^{\dagger})^{\dagger} = ((M^{\#})^{\dagger}(M^{\dagger})^{\#})^{\#}$.
⟨107⟩ $((M^{\#})^{\dagger}(M^{\dagger})^{\#})^{\dagger} = ((M^{\dagger})^{\#}(M^{\#})^{\dagger})^{\#}$.
⟨108⟩ $M^{\dagger}M^{*}M^{\dagger} = M^{\#}M^{*}M^{\#}$.
⟨109⟩ $M^{\dagger}M^{*}M^{\#} = M^{\#}M^{*}M^{\dagger}$.
⟨110⟩ $MM^{\#}M^{*} = M^{*}M^{\#}M$.
⟨111⟩ $MM^{\#}M^{\dagger} = M^{\dagger}M^{\#}M$.
⟨112⟩ $MM^{\dagger}(M^{\dagger})^{\#} = (M^{\dagger})^{\#}M^{\dagger}M$.
⟨113⟩ $MM^{\dagger}(M^{\#})^{\dagger} = (M^{\#})^{\dagger}M^{\dagger}M$.
⟨114⟩ $MM^{\#}(M^{\dagger})^{\#} = (M^{\dagger})^{\#}M^{\#}M$.
⟨115⟩ $(M^{\dagger}MM^{\#})^{\dagger} = (M^{\#}MM^{\dagger})^{\#}$.
⟨116⟩ $(M^{\dagger}MM^{\#})^{\#} = (M^{\#}MM^{\dagger})^{\dagger}$.
⟨117⟩ $(MM^{\dagger}(M^{\dagger})^{\#})^{\dagger} = ((M^{\dagger})^{\#}M^{\dagger}M)^{\#}$.
⟨118⟩ $(MM^{\dagger}(M^{\dagger})^{\#})^{\#} = ((M^{\dagger})^{\#}M^{\dagger}M)^{\dagger}$.
⟨119⟩ $(MM^{\#}(M^{\#})^{\dagger})^{\dagger} = ((M^{\#})^{\dagger}M^{\#}M)^{\#}$.
⟨120⟩ $(MM^{\#}(M^{\#})^{\dagger})^{\#} = ((M^{\#})^{\dagger}M^{\#}M)^{\dagger}$.
⟨121⟩ $M^{\dagger}M^{\#}M^{\dagger} = M^{\#}M^{\dagger}M^{\#}$.
⟨122⟩ $M(M^{\dagger})^{\#}M = (M^{\dagger})^{\#}M(M^{\dagger})^{\#}$.
⟨123⟩ $(M^{\#})^{\dagger}M(M^{\#})^{\dagger} = M(M^{\#})^{\dagger}M$.
⟨124⟩ $M^{\dagger}(M^{\#})^{\dagger}M^{\dagger} = M^{\#}(M^{\dagger})^{\#}M^{\#}$.
⟨125⟩ $MM^{\#}(MM^{\#})^{*} = (M^{\#}M)^{*}M^{\#}M$.
⟨126⟩ $(MM^{*})(MM^{\#}) = (MM^{\#})(MM^{*})$.
⟨127⟩ $(M^{*}M)(M^{\#}M) = (M^{\#}M)(M^{*}M)$.
⟨128⟩ $MM^{\dagger}(M^{\dagger})^{\#}(M^{\#})^{\dagger} = (M^{\#})^{\dagger}(M^{\dagger})^{\#}M^{\dagger}M$.
⟨129⟩ $MM^{\dagger}(M^{\#})^{\dagger}(M^{\dagger})^{\#} = (M^{\dagger})^{\#}(M^{\#})^{\dagger}M^{\dagger}M$.
⟨130⟩ $MM^{\#}(M^{\dagger})^{\#}(M^{\#})^{\dagger} = (M^{\#})^{\dagger}(M^{\dagger})^{\#}M^{\#}M$.
⟨131⟩ $M^{\dagger}M^{*}MM^{*}M^{\dagger} = M^{\#}M^{*}MM^{*}M^{\#}$.

⟨132⟩ $M^\dagger M^* M M^* M^\# = M^\# M^* M M^* M^\dagger$.
⟨133⟩ $M M^\# M^* M M^* = M^* M M^* M^\# M$.
⟨134⟩ $M M^\# M^* M M^* M^\# M = M^* M M^*$.
⟨135⟩ $(M M^\#)^\dagger M M^* M (M^\# M)^\dagger = M M^* M$.
⟨136⟩ $M^\dagger (M^\# M M^\dagger)^\dagger M^\# = M^\# (M^\dagger M M^\#)^\dagger M^\dagger$.
⟨137⟩ $M^\dagger (M^\# M M^\dagger)^\# M^\# = M^\# (M^\dagger M M^\#)^\# M^\dagger$.
⟨138⟩ $M^\dagger (M^\# M^* M^\dagger)^\dagger M^\# = M^\# (M^\dagger M^* M^\#)^\dagger M^\dagger$.
⟨139⟩ $M^\dagger (M^\# M^* M^\dagger)^\# M^\# = M^\# (M^\dagger M^* M^\#)^\# M^\dagger$.
⟨140⟩ $M^\dagger (M^\dagger M^\# M^\dagger)^\dagger M^\dagger = M^\# (M^\# M^\dagger M^\#)^\dagger M^\#$.
⟨141⟩ $M^\dagger (M^\dagger M^\# M^\dagger)^\dagger M^\dagger = M^\# (M^\# M^\dagger M^\#)^\# M^\#$.
⟨142⟩ $M^\dagger (M^\dagger M^\# M^\dagger)^\# M^\dagger = M^\# (M^\# M^\dagger M^\#)^\dagger M^\#$.
⟨143⟩ $M^\dagger (M^\dagger M^\# M^\dagger)^\# M^\dagger = M^\# (M^\# M^\dagger M^\#)^\# M^\#$.
⟨144⟩ $M^2 = ((M^\dagger)^\#)^2$.
⟨145⟩ $M^2 = ((M^\#)^\dagger)^2$.
⟨146⟩ $M^2 = (M^\dagger)^\# (M^\#)^\dagger$.
⟨147⟩ $M^2 = (M^\#)^\dagger (M^\dagger)^\#$.
⟨148⟩ $(M^2)^\dagger = M^\dagger M^\#$.
⟨149⟩ $(M^2)^\dagger = M^\# M^\dagger$.
⟨150⟩ $(M^\dagger)^2 = M^\dagger M^\#$.
⟨151⟩ $(M^\dagger)^2 = M^\# M^\dagger$.
⟨152⟩ $(M^2)^\# = M^\dagger M^\#$.
⟨153⟩ $(M^2)^\# = M^\# M^\dagger$.
⟨154⟩ $M(M^2)^\dagger = M^\#$.
⟨155⟩ $(M^2)^\dagger M = M^\#$.
⟨156⟩ $M(M^\dagger)^2 = M^\#$.
⟨157⟩ $(M^\dagger)^2 M = M^\#$.
⟨158⟩ $M(M^\dagger)^2 M = M M^\#$.
⟨159⟩ $M^\dagger M^2 M^\dagger = M M^\#$.
⟨160⟩ $(M^2)^* = M^* M M^\# M^*$.
⟨161⟩ $(M^\dagger)^2 = M^\dagger M M^\# M^\dagger$.
⟨162⟩ $(M^2)^\dagger = M^\# (M^\dagger M^2 M^\dagger)^\dagger M^\#$.
⟨163⟩ $(M^2)^\dagger = M^\# (M^\dagger M^2 M^\dagger)^\# M^\#$.
⟨164⟩ $(M^\dagger)^2 = M^\# (M^\dagger M^2 M^\dagger)^\dagger M^\#$.
⟨165⟩ $(M^\dagger)^2 = M^\# (M^\dagger M^2 M^\dagger)^\# M^\#$.
⟨166⟩ $(M^2)^\# = M^\dagger (M^\dagger M^2 M^\dagger)^\dagger M^\dagger$.
⟨167⟩ $(M^2)^\# = M^\dagger (M^\dagger M^2 M^\dagger)^\# M^\dagger$.
⟨168⟩ $(M^2)^\dagger = M^\# (M(M^2)^\dagger M)^\dagger M^\#$.
⟨169⟩ $(M^2)^\dagger = M^\# (M(M^2)^\dagger M)^\# M^\#$.
⟨170⟩ $(M^\dagger)^2 = M^\# (M(M^\dagger)^2 M)^\dagger M^\#$.
⟨171⟩ $(M^\dagger)^2 = M^\# (M(M^\dagger)^2 M)^\# M^\#$.
⟨172⟩ $(M^2)^\# = M^\dagger (M(M^2)^\dagger M)^\dagger M^\dagger$.
⟨173⟩ $(M^2)^\# = M^\dagger (M(M^2)^\dagger M)^\# M^\dagger$.
⟨174⟩ $(M^2)^\# = M^\dagger (M(M^\dagger)^2 M)^\dagger M^\dagger$.
⟨175⟩ $(M^2)^\# = M^\dagger (M(M^\dagger)^2 M)^\# M^\dagger$.
⟨176⟩ $(M^2)^\# = M^* (M^* M^2 M^*)^\dagger M^*$.
⟨177⟩ $(M^2)^\# = M^* (M^* M^2 M^*)^\# M^*$.
⟨178⟩ $(M^\dagger M^\#)^2 = (M^\# M^\dagger)^2$.
⟨179⟩ $((M^\dagger M^\#)^2)^\dagger = ((M^\# M^\dagger)^2)^\#$.
⟨180⟩ $((M^\dagger M^\#)^2)^\# = ((M^\# M^\dagger)^2)^\dagger$.
⟨181⟩ $(((M^\dagger M^\#)^2)^\dagger)^\# = (((M^\# M^\dagger)^2)^\#)^\dagger$.
⟨182⟩ $(((M^\dagger M^\#)^2)^\#)^\dagger = (((M^\# M^\dagger)^2)^\dagger)^\#$.
⟨183⟩ $M^\dagger (M^2)^\# M^\dagger = M^\# (M^2)^\dagger M^\#$.
⟨184⟩ $M^\dagger (M^2)^\# M^\dagger = M^\# (M^\dagger)^2 M^\#$.
⟨185⟩ $M((M^\dagger)^\#)^2 M = (M^\dagger)^\# M^2 (M^\dagger)^\#$.
⟨186⟩ $M((M^\#)^\dagger)^2 M = (M^\#)^\dagger M^2 (M^\#)^\dagger$.

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⟨187⟩ $M((M^2)^\dagger)^\# M = (M^\dagger)^\# M^2 (M^\dagger)^\#$.
⟨188⟩ $M((M^2)^\#)^\dagger M = (M^\#)^\dagger M^2 (M^\#)^\dagger$.
⟨189⟩ $(M^2)^\dagger M^\# (M^2)^\dagger = (M^2)^\# M^\dagger (M^2)^\#$.
⟨190⟩ $M^3 = ((M^\dagger)^\#)^3$.
⟨191⟩ $M^3 = ((M^\#)^\dagger)^3$.
⟨192⟩ $M^3 = (M^\dagger)^\# M (M^\dagger)^\#$.
⟨193⟩ $M^3 = (M^\#)^\dagger M (M^\#)^\dagger$.
⟨194⟩ $M^3 = (M^\dagger)^\# M (M^\#)^\dagger$.
⟨195⟩ $M^3 = (M^\#)^\dagger M (M^\dagger)^\#$.
⟨196⟩ $M^3 = (M^\dagger)^\# (M^\#)^\dagger (M^\dagger)^\#$.
⟨197⟩ $M^3 = (M^\#)^\dagger (M^\dagger)^\# (M^\dagger)^\#$.
⟨198⟩ $(M^3)^\dagger = M^\# M^\dagger M^\#$.
⟨199⟩ $(M^\dagger)^3 = M^\# M^\dagger M^\#$.
⟨200⟩ $(M^3)^\# = M^\dagger M^\# M^\dagger$.
⟨201⟩ $M^\dagger (M^3)^\# M^\dagger = M^\# (M^3)^\dagger M^\#$.
⟨202⟩ $M^4 = (M^\dagger)^\# M^2 (M^\dagger)^\#$.
⟨203⟩ $M^4 = (M^\#)^\dagger M^2 (M^\#)^\dagger$.
⟨204⟩ $M^4 = (M^\dagger)^\# M^2 (M^\#)^\dagger$.
⟨205⟩ $M^4 = (M^\#)^\dagger M^2 (M^\dagger)^\#$.
⟨206⟩ $M^4 = ((M^\dagger)^\#)^2 ((M^\#)^\dagger)^2$.
⟨207⟩ $M^4 = ((M^\#)^\dagger)^2 ((M^\dagger)^\#)^2$.
⟨208⟩ $(M^4)^\dagger = (M^\dagger M^\#)^2$.
⟨209⟩ $(M^4)^\dagger = (M^\# M^\dagger)^2$.
⟨210⟩ $(M^\dagger)^4 = (M^\dagger M^\#)^2$.
⟨211⟩ $(M^\dagger)^4 = (M^\# M^\dagger)^2$.
⟨212⟩ $(M^4)^\# = (M^\dagger M^\#)^2$.
⟨213⟩ $(M^4)^\# = (M^\# M^\dagger)^2$.
⟨214⟩ $(M^4)^\dagger = M^\# (M^2)^\dagger M^\#$.
⟨215⟩ $(M^\dagger)^4 = M^\# (M^\dagger)^2 M^\#$.
⟨216⟩ $(M^4)^\# = M^\dagger (M^2)^\# M^\dagger$.
⟨217⟩ $M^5 = ((M^\dagger)^\#)^2 M ((M^\#)^\dagger)^2$.
⟨218⟩ $M^5 = ((M^\#)^\dagger)^2 M ((M^\dagger)^\#)^2$.
⟨219⟩ $M^5 = (M^\dagger)^\# M^3 (M^\dagger)^\#$.
⟨220⟩ $M^5 = (M^\#)^\dagger M^3 (M^\#)^\dagger$.
⟨221⟩ $(M^5)^\dagger = (M^2)^\# M^\dagger (M^2)^\#$.
⟨222⟩ $(M^\dagger)^5 = (M^2)^\# M^\dagger (M^2)^\#$.
⟨223⟩ $(M^5)^\# = (M^2)^\dagger M^\# (M^2)^\dagger$.
⟨224⟩ $(M^5)^\# = (M^\dagger)^2 M^\# (M^\dagger)^2$.
⟨225⟩ $(M^5)^\dagger = M^\# (M^3)^\dagger M^\#$.
⟨226⟩ $(M^5)^\dagger = M^\# (M^\dagger)^3 M^\#$.
⟨227⟩ $(M^\dagger)^5 = M^\# (M^3)^\dagger M^\#$.
⟨228⟩ $(M^\dagger)^5 = M^\# (M^\dagger)^3 M^\#$.
⟨229⟩ $(M^5)^\# = M^\dagger (M^3)^\# M^\dagger$.
⟨230⟩ $(M^7)^\dagger = M^\# (M^2)^\dagger M^\# (M^2)^\dagger M^\#$.
⟨231⟩ $(M^7)^\dagger = M^\# (M^\dagger)^2 M^\# (M^\dagger)^2 M^\#$.
⟨232⟩ $(M^\dagger)^7 = M^\# (M^2)^\dagger M^\# (M^2)^\dagger M^\#$.
⟨233⟩ $(M^\dagger)^7 = M^\# (M^\dagger)^2 M^\# (M^\dagger)^2 M^\#$.
⟨234⟩ $(M^\#)^7 = M^\dagger (M^2)^\# M^\dagger (M^2)^\# M^\dagger$.
⟨235⟩ $(M^\dagger)^9 = M^\# (M^3)^\dagger M^\# (M^3)^\dagger M^\#$.
⟨236⟩ $(M^\dagger)^9 = M^\# (M^\dagger)^3 M^\# (M^\dagger)^3 M^\#$.
⟨237⟩ $(M^9)^\dagger = M^\# (M^3)^\dagger M^\# (M^3)^\dagger M^\#$.
⟨238⟩ $(M^9)^\dagger = M^\# (M^\dagger)^3 M^\# (M^\dagger)^3 M^\#$.
⟨239⟩ $(M^\#)^9 = M^\dagger (M^3)^\# M^\dagger (M^3)^\# M^\dagger$.
⟨240⟩ $(MM^\dagger - MM^\#)(M^\dagger M - M^\# M) = 0$.
⟨241⟩ $(M^k)^\dagger = (M^k)^\#$ *for some/any integers* $k \geq 2$.

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⟨242⟩ $(M^\dagger)^k = (M^\#)^k$ *for some/any integers* $k \geq 2$.
⟨243⟩ $(M^\dagger)^k (M^\#)^s = (M^\#)^s (M^\dagger)^k$ *for some/any integers* $k, s \geq 2$.
⟨244⟩ $(M^k)^\dagger (M^s)^\# = (M^s)^\# (M^k)^\dagger$ *for some/any integers* $k, s \geq 2$.
⟨245⟩ $(M^k)^\dagger M M^\# = M^\# M (M^k)^\dagger$ *for some/any integers* $k \geq 2$.
⟨246⟩ $(M^\dagger)^k M M^\# = M^\# M (M^\dagger)^k$ *for some/any integers* $k \geq 2$.
⟨247⟩ $(M^\dagger M^\#)^k = (M^\# M^\dagger)^k$ *for some/any integers* $k \geq 3$.
⟨248⟩ $(M^{k+s})^\dagger = (M^k)^\# (M^s)^\dagger$ *for some/any integers* $k, s \geq 2$.
⟨249⟩ $(M^{k+s})^\dagger = (M^k)^\dagger (M^s)^\#$ *for some/any integers* $k, s \geq 2$.
⟨250⟩ $(M^\dagger)^{k+s} = (M^k)^\# (M^s)^\dagger$ *for some/any integers* $k, s \geq 2$.
⟨251⟩ $(M^\dagger)^{k+s} = (M^k)^\dagger (M^s)^\#$ *for some/any integers* $k, s \geq 2$.
⟨252⟩ $(M^{k+s})^\# = (M^k)^\# (M^s)^\dagger$ *for some/any integers* $k, s \geq 2$.
⟨253⟩ $(M^{k+s})^\# = (M^k)^\dagger (M^s)^\#$ *for some/any integers* $k, s \geq 2$.
⟨254⟩ $(M^{k+s+t})^\# = (M^k)^\dagger (M^s)^\# (M^t)^\dagger$ *for some/any integers* $k, s, t \geq 2$.
⟨255⟩ $(M^{k+s+t})^\dagger = (M^k)^\# (M^s)^\dagger (M^t)^\#$ *for some/any integers* $k, s \geq 2$.
⟨256⟩ $(M^\dagger)^{k+s+t} = (M^k)^\# (M^s)^\dagger (M^t)^\#$ *for some/any integers* $k, s \geq 2$.
⟨257⟩ $(M^k)^\dagger (M^s)^\# (M^t)^\dagger = (M^k)^\# (M^s)^\dagger (M^t)^\#$ *for some/any integers* $k, s, t \geq 2$.
⟨258⟩ $(M^k)^\dagger (M^s)^\# = (M^s)^\# (M^k)^\dagger$ *for some/any integers* $k, s, t \geq 2$.
⟨259⟩ $(M^k)^\dagger (M^s)^\# (M^t)^\dagger = (M^k)^\# (M^s)^\dagger (M^t)^\#$ *for some/any integers* $k, s, t \geq 2$.
⟨260⟩ $(M^{2k+1})^\dagger = M^\# (M^\dagger M^\#)^k$ *for some/any integers* $k \geq 2$.
⟨261⟩ $(M^\dagger)^{2k+1} = M^\# (M^\dagger M^\#)^k$ *for some/any integers* $k \geq 2$.
⟨262⟩ $(M^{2k+1})^\# = M^\dagger (M^\# M^\dagger)^k$ *for some/any integers* $k \geq 2$.
⟨263⟩ $M^\dagger (M^\# M^\dagger)^k = M^\# (M^\dagger M^\#)^k$ *for some/any integers* $k \geq 2$.
⟨264⟩ $(M^k)^\dagger = M^\# (M^\dagger M^k M^\dagger)^\dagger M^\#$ *for some/any integers* $k \geq 3$.
⟨265⟩ $(M^k)^\dagger = M^\# (M^\dagger M^k M^\dagger)^\# M^\#$ *for some/any integers* $k \geq 3$.
⟨266⟩ $(M^\dagger)^k = M^\# (M^\dagger M^k M^\dagger)^\dagger M^\#$ *for some/any integers* $k \geq 3$.
⟨267⟩ $(M^\dagger)^k = M^\# (M^\dagger M^k M^\dagger)^\# M^\#$ *for some/any integers* $k \geq 3$.
⟨268⟩ $(M^k)^\# = M^\dagger (M^\dagger M^k M^\dagger)^\dagger M^\dagger$ *for some/any integers* $k \geq 3$.
⟨269⟩ $(M^k)^\# = M^\dagger (M^\dagger M^k M^\dagger)^\# M^\dagger$ *for some/any integers* $k \geq 3$.
⟨270⟩ $(M^k)^\dagger = M^\# (M (M^k)^\dagger M)^\dagger M^\#$ *for some/any integers* $k \geq 3$.
⟨271⟩ $(M^k)^\dagger = M^\# (M (M^k)^\dagger M)^\# M^\#$ *for some/any integers* $k \geq 3$.
⟨272⟩ $(M^\dagger)^k = M^\# (M (M^\dagger)^k M)^\dagger M^\#$ *for some/any integers* $k \geq 3$.
⟨273⟩ $(M^\dagger)^k = M^\# (M (M^\dagger)^k M)^\# M^\#$ *for some/any integers* $k \geq 3$.
⟨274⟩ $(M^k)^\# = M^\dagger (M (M^k)^\dagger M)^\dagger M^\dagger$ *for some/any integers* $k \geq 3$.
⟨275⟩ $(M^k)^\# = M^\dagger (M (M^k)^\dagger M)^\# M^\dagger$ *for some/any integers* $k \geq 3$.
⟨276⟩ $(M^k)^\# = M^\dagger (M (M^\dagger)^k M)^\dagger M^\dagger$ *for some/any integers* $k \geq 3$.
⟨277⟩ $(M^k)^\# = M^\dagger (M (M^\dagger)^k M)^\# M^\dagger$ *for some/any integers* $k \geq 3$.
⟨278⟩ $(M^k)^\# = M^* (M^* M^k M^*)^\dagger M^*$ *for some/any integers* $k \geq 3$.
⟨279⟩ $(M^k)^\# = M^* (M^* M^k M^*)^\# M^*$ *for some/any integers* $k \geq 3$.
⟨280⟩ $(M^\dagger M M^\#)^k = (M^\# M M^\dagger)^k$ *for some/any integers* $k \geq 2$.
⟨281⟩ $(M^\dagger M^\# M^\dagger)^k = (M^\# M^\dagger M^\#)^k$ *for some/any integers* $k \geq 2$.
⟨282⟩ $(M (M^\dagger)^\# M)^k = ((M^\dagger)^\# M (M^\dagger)^\#)^k$ *for some/any integers* $k \geq 2$.
⟨283⟩ $((M^\#)^\dagger M (M^\#)^\dagger)^k = (M (M^\#)^\dagger M)^k$ *for some/any integers* $k \geq 2$.
⟨284⟩ $(M^\dagger (M^\#)^\dagger M^\dagger)^k = (M^\# (M^\dagger)^\# M^\#)^k$ *for some/any integers* $k \geq 2$.
⟨285⟩ $(M^\dagger M^* M^\dagger)^k = (M^\# M^* M^\#)^k$ *for some/any integers* $k \geq 2$.
⟨286⟩ $(M^\dagger M^* M^\#)^k = (M^\# M^* M^\dagger)^k$ *for some/any integers* $k \geq 2$.
⟨287⟩ $(M M^\# M^*)^k = (M^* M^\# M)^k$ *for some/any integers* $k \geq 2$.
⟨288⟩ $2MM^\# = MM^\dagger + M^\dagger M$.
⟨289⟩ $2MM^\dagger = MM^\# + (MM^\#)^*$.
⟨290⟩ $2M^\dagger M = MM^\# + (MM^\#)^*$.
⟨291⟩ $2MM^\dagger = M^\dagger M + M^\# M$.
⟨292⟩ $2M^\dagger M = MM^\dagger + MM^\#$.
⟨293⟩ $2(MM^\#)^* = MM^\dagger + MM^\#$.
⟨294⟩ $2(M^\# M)^* = M^\dagger M + M^\# M$.
⟨295⟩ $2M^\dagger = M^\dagger M M^\# + M^\# M M^\dagger$.
⟨296⟩ $2M^\# = M^\dagger M M^\# + M^\# M M^\dagger$.

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⟨297⟩ $2M = MM^{\dagger}(M^{\dagger})^{\#} + (M^{\dagger})^{\#}M^{\dagger}M$.
⟨298⟩ $2M = (M^{\#})^{\dagger}MM^{\#} + M^{\#}M(M^{\#})^{\dagger}$.
⟨299⟩ $2M^{\dagger}M = MM^{\#}(MM^{\#})^{*} + (MM^{\#})^{*}MM^{\#}$.
⟨300⟩ $2MM^{\dagger} = MM^{\#}(MM^{\#})^{*} + (MM^{\#})^{*}MM^{\#}$.
⟨301⟩ $2(M^2)^{\dagger} = M^{\dagger}M^{\#} + M^{\#}M^{\dagger}$.
⟨302⟩ $2(M^{\dagger})^2 = M^{\dagger}M^{\#} + M^{\#}M^{\dagger}$.
⟨303⟩ $2(M^2)^{\#} = M^{\dagger}M^{\#} + M^{\#}M^{\dagger}$.
⟨304⟩ $(M^2)^{\dagger} + (M^{\dagger})^2 = M^{\dagger}M^{\#} + M^{\#}M^{\dagger}$.
⟨305⟩ $(M^2)^{\dagger} + M^{\dagger}M^{\#} = (M^{\dagger})^2 + M^{\#}M^{\dagger}$.
⟨306⟩ $(M^2)^{\dagger} + M^{\#}M^{\dagger} = (M^{\dagger})^2 + M^{\dagger}M^{\#}$.
⟨307⟩ $3(M^2)^{\dagger} = (M^{\dagger})^2 + M^{\#}M^{\dagger} + M^{\dagger}M^{\#}$.
⟨308⟩ $3(M^{\dagger})^2 = (M^2)^{\dagger} + M^{\#}M^{\dagger} + M^{\dagger}M^{\#}$.
⟨309⟩ $2(M^3)^{\dagger} = M^{\dagger}M^{\#}M^{\dagger} + M^{\#}M^{\dagger}M^{\#}$.
⟨310⟩ $2(M^{\dagger})^3 = M^{\dagger}M^{\#}M^{\dagger} + M^{\#}M^{\dagger}M^{\#}$.
⟨311⟩ $2(M^{\#})^3 = M^{\dagger}M^{\#}M^{\dagger} + M^{\#}M^{\dagger}M^{\#}$.
⟨312⟩ $2(M^3)^{\dagger} = (M^2)^{\dagger}M^{\#} + M^{\#}(M^2)^{\dagger}$.
⟨313⟩ $2(M^3)^{\dagger} = (M^{\dagger})^2M^{\#} + M^{\#}(M^{\dagger})^2$.
⟨314⟩ $2(M^3)^{\dagger} = M^{\dagger}(M^2)^{\#} + (M^2)^{\#}M^{\dagger}$.
⟨315⟩ $2(M^{\dagger})^3 = (M^2)^{\dagger}M^{\#} + M^{\#}(M^2)^{\dagger}$.
⟨316⟩ $2(M^{\dagger})^3 = (M^{\dagger})^2M^{\#} + M^{\#}(M^{\dagger})^2$.
⟨317⟩ $2(M^{\dagger})^3 = M^{\dagger}(M^2)^{\#} + (M^2)^{\#}M^{\dagger}$.
⟨318⟩ $2(M^3)^{\#} = (M^2)^{\dagger}M^{\#} + M^{\#}(M^2)^{\dagger}$.
⟨319⟩ $2(M^3)^{\#} = (M^{\dagger})^2M^{\#} + M^{\#}(M^{\dagger})^2$.
⟨320⟩ $2(M^3)^{\#} = M^{\dagger}(M^2)^{\#} + (M^2)^{\#}M^{\dagger}$.
⟨321⟩ $3(M^3)^{\dagger} = (M^2)^{\dagger}M^{\#} + M^{\dagger}M^{\#}M^{\dagger} + M^{\#}(M^2)^{\dagger}$.
⟨322⟩ $3(M^3)^{\dagger} = (M^{\dagger})^2M^{\#} + M^{\dagger}M^{\#}M^{\dagger} + M^{\#}(M^{\dagger})^2$.
⟨323⟩ $3(M^3)^{\dagger} = M^{\dagger}(M^2)^{\#} + M^{\dagger}M^{\#}M^{\dagger} + (M^2)^{\#}M^{\dagger}$.
⟨324⟩ $3(M^{\dagger})^3 = (M^2)^{\dagger}M^{\#} + M^{\dagger}M^{\#}M^{\dagger} + M^{\#}(M^2)^{\dagger}$.
⟨325⟩ $3(M^{\dagger})^3 = (M^{\dagger})^2M^{\#} + M^{\dagger}M^{\#}M^{\dagger} + M^{\#}(M^{\dagger})^2$.
⟨326⟩ $3(M^{\dagger})^3 = M^{\dagger}(M^2)^{\#} + M^{\dagger}M^{\#}M^{\dagger} + (M^2)^{\#}M^{\dagger}$.
⟨327⟩ $3(M^3)^{\#} = (M^2)^{\dagger}M^{\#} + M^{\#}M^{\dagger}M^{\#} + M^{\#}(M^2)^{\dagger}$.
⟨328⟩ $3(M^3)^{\#} = (M^{\dagger})^2M^{\#} + M^{\#}M^{\dagger}M^{\#} + M^{\#}(M^{\dagger})^2$.
⟨329⟩ $3(M^3)^{\#} = M^{\dagger}(M^2)^{\#} + M^{\#}M^{\dagger}M^{\#} + (M^2)^{\#}M^{\dagger}$.
⟨330⟩ $4(M^3)^{\dagger} = (M^{\dagger})^3 + M^{\dagger}(M^2)^{\#} + (M^2)^{\dagger}M^{\#} + (M^3)^{\#}$.
⟨331⟩ $4(M^3)^{\dagger} = (M^{\dagger})^3 + (M^2)^{\#}M^{\dagger} + M^{\#}(M^2)^{\dagger} + (M^3)^{\#}$.
⟨332⟩ $4(M^3)^{\#} = (M^{\dagger})^3 + (M^{\dagger})^2M^{\dagger} + M^{\dagger}(M^{\dagger})^2 + (M^{\dagger})^3$.
⟨333⟩ $4(M^3)^{\#} = (M^{\dagger})^3 + M^{\dagger}(M^2)^{\#} + (M^2)^{\dagger}M^{\#} + (M^3)^{\#}$.
⟨334⟩ $4(M^3)^{\#} = (M^{\dagger})^3 + (M^2)^{\#}M^{\dagger} + M^{\#}(M^2)^{\dagger} + (M^3)^{\#}$.
⟨335⟩ $(M^3)^{\dagger} + (M^{\dagger})^3 = M^{\dagger}M^{\#}M^{\dagger} + M^{\#}M^{\dagger}M^{\#}$.
⟨336⟩ $(M^3)^{\dagger} + (M^{\dagger})^3 = (M^2)^{\dagger}M^{\#} + M^{\#}(M^2)^{\dagger}$.
⟨337⟩ $(M^3)^{\dagger} + (M^{\dagger})^3 = (M^{\dagger})^2M^{\#} + M^{\#}(M^{\dagger})^2$.
⟨338⟩ $(M^3)^{\dagger} + (M^{\dagger})^3 = M^{\dagger}(M^2)^{\#} + (M^2)^{\#}M^{\dagger}$.
⟨339⟩ $I_m - MM^{\#} = (I_m - MM^{\#})^{\dagger}$.
⟨340⟩ $(I_m - MM^{\#})^{*} = (I_m - MM^{\#})^{\dagger}$.
⟨341⟩ $(I_m - MM^{\#})^{\dagger} = ((I_m - MM^{\#})^{\dagger})^2$.
⟨342⟩ $MM^{\#}(I_m - MM^{\#})^{\dagger} = (I_m - MM^{\#})^{\dagger}MM^{\#}$.
⟨343⟩ $(I_m - MM^{\#})(I_m - MM^{\#})^{*} = (I_m - MM^{\#})^{*}(I_m - MM^{\#})$.
⟨344⟩ $(I_m - MM^{\#})(I_m - MM^{\#})^{\dagger} = (I_m - MM^{\#})^{\dagger}(I_m - MM^{\#})$.
⟨345⟩ $MM^{\dagger} + MM^{\#} = M^{\dagger}M + (MM^{\#})^{*}$.
⟨346⟩ $(MM^{\dagger} - MM^{\#})(M^{\dagger}M - M^{\#}M) = 0$.
⟨347⟩ $r[M, M^{*}] = r(M)$.
⟨348⟩ $\mathscr{R}(I_m - MM^{\dagger}) = \mathscr{R}(I_m - MM^{\#})$.
⟨349⟩ $\mathscr{R}(I_m - M^{\dagger}M) = \mathscr{R}(I_m - (MM^{\#})^{*})$.
⟨350⟩ $\mathscr{R}(I_m - MM^{\#}) = \mathscr{R}(I_m - (MM^{\#})^{*})$.
⟨351⟩ $\mathscr{R}(M) = \mathscr{R}(M^{*})$, *namely, M is range-Hermitian.*

5. Conclusions

We constructed a compilation of matrix equalities involving mixed products of the Moore–Penrose inverse and the group inverse of a matrix, and obtained many basic and novel formulas, results, and facts related to matrix expressions and matrix equalities. This work reveals a variety of intrinsic properties and links between different matrix expressions and matrix equalities. Since the whole study is explicitly conducted by means of ordinary algebraic operations of matrices and their generalized inverses, including the active and cogent use of the block matrix representation method and the matrix rank method, they are easy to understand within the ordinary discipline of matrix theory and generalized inverses. It can be figured out from a large number of findings in the preceding and current papers by the present author that we are able to construct many new and valuable matrix equalities that are composed of mixed operations of the four matrices A, A^*, $A^\dagger$ and $A^\#$ from algebraic and applied points of view, and can find feasible ways to characterize the matrix equalities in most cases using the powerful block matrix representation method and the matrix rank method.

Furthermore, we remark that algebraists love to establish, classify, and characterize various equalities or equations in a given algebraic framework. If they manage, by means of certain new tricks and methods, to prove some formulas, results, and facts, they always endeavour to get more mileage out of the ideas and findings by searching for other situations in which the work can be used or extended. As one of such reasonable instances, it is not difficult to symbolically extend the main results and facts in this paper to some general algebraic settings, such as, rings and operator algebras, in which generalized inverses of elements, as well as reverse-order laws for generalized inverses can be defined accordingly.

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