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Article

On the Question of Fermat's Last Theorem

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Abstract: This paper presents a new approach to a different proof of "Fermat's Last Theorem." For the proof of the theorem, it is proposed to use a more straightforward geometrical approach. A special family of curves, the Elba curves, is introduced to facilitate the proof. This approach makes the proof easier to comprehend by mathematicians and those interested in the subject.

Keywords: Fermat's last theorem; triangle inequalities; Elba curves; ellipse; geometric proof; circle

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1. Introduction

The article shows the application of some simple geometric methods to Fermat's last theorem. In particular, the application of the method of n-lines to the solution of this problem is shown [1,2,5], that is, the division of the sides of a triangle is proportional to the n-th powers of the adjacent sides. As is known before Andrew Wiles [6], many people tried to prove this theorem, as a result of which many interesting results were obtained [7], which contributed to the development of mathematics in general. Here, as in many authors, an attempt is made to prove by contradiction. That is, the attempt begins with the assumption that there are positive integers a, b, c satisfying the equality $a^n + b^n = c^n$, where n (n > 2) is an integer, in order to show that this is impossible and leads to a contradiction.

2. A Brief Literature Review

It is stated that the proof of the last Fermat's theorem, which has not been solved for nearly three centuries, is self-evident only when the power is 4 (Edwards, 1980). Some also note that although Fermat did not write the proof of the theorem, he knew the proof himself (Poulkas, 2020). About a century later, Euler was able to prove the theorem for the case where the power was 3, and two centuries later, Dirichlet and Legendre proved the theorem for the case when the power was 5. Lame demonstrated the theorem for the case where the power was 7. Wels announced that he had proved the theorem in 1993, but after the presentation, it was revealed that there was a flaw in the proof. Wiles presented the final proof of the theorem two years later, in 1995, together with his former student Taylor (Taylor, Wiles, 1995). Wiles and Taylor's proof is very long, some 130 pages. Considering this, later, many mathematicians worked on the existence of more straightforward alternative ways of proof. As an example, we will mention several approaches that have been proposed in recent times. Agofontsev (2012) and Gevorkyan (2020a, 2020b) investigated the theorem for certain cases and obtained some general conclusions. Poulkas (2020) states his results regarding the proof of both LFT and his proposed Generalized Fermat's Theorem (the case when there are n terms in the sum) using number theory.

3. Methodology and Used Mathematical Tools

Although certain tools of mathematical analysis and number theory are used to prove the theorem in this article, the geometrical approach is the main focus. A special family of curves called Elba curves is introduced to facilitate the proof.

In this article, many interesting results have been obtained that have not been previously published.

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4. Determining the length of n - lines in a triangle and their properties

Fermat's Last Theorem states that for any natural number n>2 the equation $a^n + b^n = c^n$ has no solutions in non-zero integers a, b, c.

In the article [3] "Towards the proof of Fermat's theorem", we showed that the solution of this problem is adequate to finding ΔABC with integer sides a, b, c, for which

$a^n + b^n = c^n$, where natural n more than two (n > 2).

Consider the problem of dividing the sides of a triangle proportionally n-th powers of adjacent sides [1,2,5].

Let ΔABC be given with corresponding sides a, b, c. Let's draw a straight-line CD from vertex C to side AB.

If:

1) CD-median, then $\frac{AD}{DB} = \frac{b^0}{a^0} = 1$ (1)

2) CD-bisector, then $\frac{AD}{DB} = \frac{b^1}{a^1}$ (2)

3) CD-symmedian, then $\frac{AD}{DB} = \frac{b^2}{a^2}$ (3)

CD is called and direct if $\frac{AD}{DB} = \frac{b^n}{a^n}$ (4)

All three n-lines intersect at one point, which is easily proved by the Ceva's theorem.

Consider the following problem. Let the lengths of the sides of the triangle be integers. Let's determine for what values of n the points of intersection of the lines of n - straight lines lie on the midline.

As is known, the point "k" belongs to the n-line CD if and only if the distances from "k" to sides *a* and *b* are proportional to the *(n-1)-th* powers of these sides (see Figure 1), i.e.

$$\frac{KL}{KM} = \frac{b^{n-1}}{a^{n-1}} \quad (5)$$

For clarity, we present the proof of this statement.

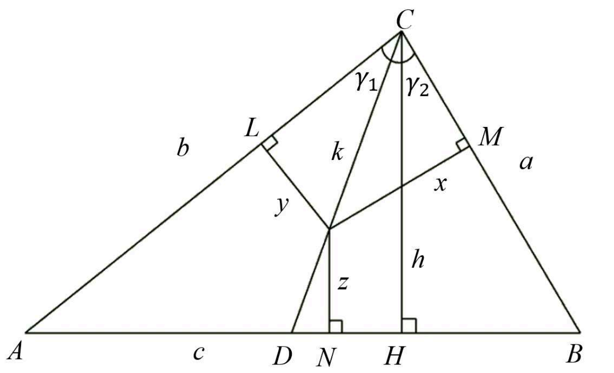

Figure 1

Let the angles at the vertex C be γ_1 and γ_2

Then

$$\frac{KL}{KM} = \frac{\frac{KL}{KC}}{\frac{KM}{KC}} = \frac{\sin\gamma_1}{\sin\gamma_2} \quad (6)$$

Let's carry out the following transformations:

Then

$$\frac{KL}{KM} = \frac{\frac{1}{2}CD\sin\gamma_1}{\frac{1}{2}CD\sin\gamma_2} = \frac{a}{b}\frac{\frac{1}{2}CD \cdot b \cdot \sin\gamma_1}{\frac{1}{2}CD \cdot a \cdot \sin\gamma_2} = \frac{a}{b} \cdot \frac{S_{\Delta ACD}}{S_{\Delta BCD}}$$

On the other hand, $S_{\Delta ACD}$=1/2·h·AD, $S_{\Delta BCD}$=1/2·h·BD, where *h* is the height drawn from vertex C to side AB. Then, using the relations $\frac{AD}{DB} = \frac{b^n}{a^n}$, we get

$$\frac{KL}{KM} = \frac{a}{b} \cdot \frac{\frac{1}{2} \cdot h \cdot AD}{\frac{1}{2} \cdot h \cdot BD} = \frac{a}{b}\frac{AD}{BD} = \frac{a}{b} \cdot \frac{b^n}{a^n} = \frac{b^{n-1}}{a^{n-1}}$$

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And now let the point K be the point of intersection of n-lines drawn from different vertices, and the points M, L, N based on the perpendiculars drawn to the corresponding sides a, b, c. Using the previous formula, we can get the following formula

$$\frac{KM}{a^{n-1}} = \frac{KL}{b^{n-1}} = \frac{KN}{c^{n-1}} \quad \text{or} \quad \frac{x}{a^{n-1}} = \frac{y}{b^{n-1}} = \frac{z}{c^{n-1}} \tag{7}$$

After doing some transformations, we get:

$$\frac{aKM}{a^n} = \frac{bKL}{b^n} + \frac{cKN}{c^n} = \frac{aKM+bKL+cKN}{a^n+b^n+c^n} = \frac{2S_{\Delta BKC}+2S_{\Delta AKC}+2S_{\Delta AKB}}{a^n+b^n+c^n} = \frac{2S_{\Delta ABC}}{a^n+b^n+c^n} \tag{8}$$

Let this point lie on the midline parallel to side c. Then

$$KN = \frac{1}{2}h \text{ и } \frac{cKN}{c^n} = \frac{2\cdot\frac{1}{2}ch}{a^n+b^n+c^n} + \frac{c\cdot\frac{1}{2}\cdot h}{c^n} \Rightarrow a^n + b^n + c^n = 2c^n \Rightarrow a^n + b^n = c^n \tag{9}$$

What does the last formula mean? For n=0 it has 1+1≠1, i.e. the point of intersection of the medians cannot lie on the midline. For n=1 we have *a+b=c*, which contradicts the triangle inequality. That is, the point of intersection of the bisectors also cannot lie on the midline.

For *n*=2 we get a right triangle because

$$a^2 + b^2 = c^2 \tag{10}$$

As is known, Only the Simedians of the rectangular triangle intersect in the middle of the height and $\frac{b^2}{a^2} = \frac{AD}{DB}$, where $CD = h = \frac{ab}{c}$

If we assume that the point K is in the middle of the CD-line of n-lines, then we can write the following equality: (see Figure 2).

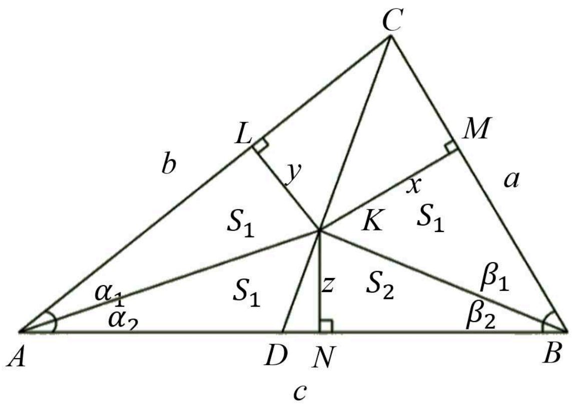

Figure 2

$$S_{\Delta AKB} = S_{\Delta AKC} + S_{\Delta BKC} \tag{11}$$

Where

$$\frac{1}{2}cz = \frac{1}{2}ax + \frac{1}{2}by \tag{12}$$

$$\text{or} \quad ax + by = cz \Rightarrow a\frac{x}{z} + b\frac{y}{z} = c \tag{13}$$

On the other hand, by the property of the n-straight lines

$$\frac{x}{a^{n-1}} = \frac{y}{b^{n-1}} = \frac{z}{c^{n-1}} \Rightarrow \frac{x}{z} = \frac{a^{n-1}}{c^{n-1}}, \frac{y}{z} = \frac{b^{n-1}}{c^{n-1}}, \frac{x}{y} = \frac{a^{n-1}}{b^{n-1}} \tag{14}$$

Given the formulas (14) in (13), we get: $a\cdot\frac{a^{n-1}}{c^{n-1}} + b\cdot\frac{b^{n-1}}{c^{n-1}} = c$ or $a^n + b^n = c^n$

It should be noted that the same result can be obtained by comparing the formulas:

$$a\cos\beta + b\cos\alpha = c \tag{15}$$

and $a^n + b^n = c^n$.

As you can see with

$$cos\beta = \frac{a^{n-1}}{c^{n-1}}, cos\alpha = \frac{b^{n-1}}{c^{n-1}} \tag{16}$$

we get: $a^n + b^n = c^n$

And for all values n formulas (16) are true? With n = 2, as we have already noted, a rectangular triangle is obtained.

And formulas $cos\alpha = \frac{b^{2-1}}{c^{2-1}} = \frac{b}{c}, cos\beta = cos\alpha = \frac{a^{2-1}}{c^{2-1}} = \frac{a}{c}$ are true.

To get a unambiguous answer to this question with $n > 2$, we first get formulas for N-straight lengths. To do this, we use the Stuart formula.

Let $AD = m_1 = kb^n, DB = m_1 ka^n$

Then according to the Stuart formula (see Figure 3)

$$L_c^2 = \frac{m_1 a^2 + n_1 b^2}{m_1 + n_1} - m_1 n_1 \tag{17}$$

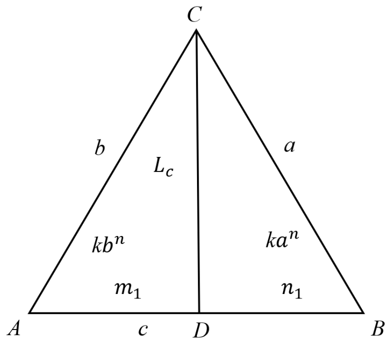

Figure 3

On the other hand, after the substitution of values for AD and DB we have:

$$L_c^2 = \frac{k(b^n a^2 + a^n b^2)}{k(b^n + a^n)} - k^2 a^n b^n = \frac{b^n a^2 + a^n b^2}{b^n + a^n} - k^2 a^n b^n \tag{18}$$

$$\text{Here } k(a^n + b^n) = c \Rightarrow k = \frac{c}{a^n + b^n}, AD = \frac{cb^n}{a^n + b^n}, DB = \frac{ca^n}{a^n + b^n} \tag{19}$$

By substituting (19) in (18) and conducting some transformations, we have:

$$L_c^2 = \frac{a^2 b^n + a^n b^2}{a^n + b^n} - \frac{c^2 \cdot a^n b^n}{(a^n + b^n)^2} = \frac{(a^2 b^n + a^n b^2)(a^n + b^n) - c^2 a^n b^n}{(a^n + b^n)^2}$$

$$\text{Or } L_c = \frac{ab}{a^n + b^n}\sqrt{(a^{n-2} + b^{n-2})(a^n + b^n) - a^{n-2} b^{n-2} c^2} \tag{20}$$

Similar formulas can be obtained for the other two Simedians:

$$L_a = \frac{bc}{b^n + c^n}\sqrt{(b^{n-2} + c^{n-2})(b^n + c^n) - b^{n-2} c^{n-2} a^2} \tag{21}$$

$$L_b = \frac{ac}{a^n + c^n}\sqrt{(a^{n-2} + c^{n-2})(a^n + c^n) - a^{n-2} c^{n-2} b^2} \tag{22}$$

Let us show that these formulas can be used to obtain formulas for the median, bisector and symmedian of the triangle ΔABC

For n=0, from formula (20) we obtain the median formula:

$$L_a = \frac{bc}{2}\sqrt{2\left(\frac{1}{b^2} + \frac{1}{c^2}\right) - \frac{a^2}{b^2 c^2}} = \frac{1}{2} bc\sqrt{\frac{2(b^2 + c^2) - a^2}{b^2 c^2}} = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2} = m_a. \tag{23}$$

For n=1, we get the bisector formula:

$$L_a = \frac{bc}{b+c}\sqrt{\left(\frac{1}{b} + \frac{1}{c}\right)(b + c) - \frac{a^2}{bc}} = \frac{bc}{b+c}\sqrt{\frac{(b+c)^2 - a^2}{bc}} = \frac{bc}{b+c}\sqrt{\frac{(b+c+a)(b+c-a)}{bc}} =$$

$$\frac{1}{b+c}\sqrt{\frac{b^2 c^2 (2p - 2a) 2p}{bc}} = \frac{2}{b+c}\sqrt{pbc(p - a)} \tag{24}$$

For n=2, we get the formula for the symmedian:

$$L_a = \frac{bc}{b^2+c^2}\sqrt{(a^0+b^0)(a^2+b^2)-a^2b^0c^0} = \frac{bc}{b^2+c^2} \quad (25)$$

The formula for the length of n-lines in a triangle in which $a^n+b^n=c^n$ can be written as follows:

$$L_c = \frac{ab}{c^n}\sqrt{(a^{n-2}+b^{n-2})c^n - a^{n-2}b^{n-2}c^2} = \frac{abc}{c^n}\sqrt{(ac)^{n-2}+(bc)^{n-2}-(ab)^{n-2}} \quad (26)$$

$$L_a = \frac{abc}{a^n}\sqrt{(ab)^{n-2}+(ac)^{n-2}-(bc)^{n-2}} \quad (27)$$

$$L_b = \frac{abc}{b^n}\sqrt{(ab)^{n-2}+(bc)^{n-2}-(ac)^{n-2}} \quad (28)$$

And now we will show that in a triangle in which the n-line CD=L_c is drawn, where the equality $a^n+b^n=c^n$ holds, formulas (16) are true only for n=2.

Indeed, this can be obtained if we consider ΔACD and ΔBCD

First consider the triangle ΔACD

For the triangle ΔACD, we write the cosine theorem and instead of cosα, we substitute the value $cos\alpha = \frac{b^{n-1}}{c^{n-1}}$ from formula (19) substituting $a^n+b^n=c^n$ we get $AD = \frac{cb^n}{c^n} = \frac{b^n}{c^{n-1}}$, $DB = \frac{ca^n}{c^n} = \frac{a^n}{c^{n-1}}$

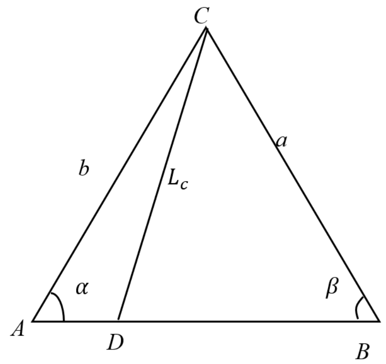

Figure 4

The cosine theorem has the following form:

$$L_c^2 = b^2 + AD^2 - 2b\cdot AD\cdot cos\alpha$$

Substituting the values of AD and cosα, we obtain:

$$L_c^2 = b^2 + \left(\frac{b^n}{c^{n-1}}\right)^2 - 2b\cdot\frac{b^n}{c^{n-1}}\cdot\frac{b^{n-1}}{c^{n-1}} = b^2 + \frac{b^{2n}}{c^{2n-2}} - \frac{2b^{2n}}{c^{2n-2}} = b^2 - \frac{b^{2n}}{c^{2n-2}} \quad (29)$$

Similarly, from the triangle ΔACB we have:

$$L_c^2 = a^2 + \frac{a^{2n}}{c^{2n-2}} - 2a\cdot\frac{a^n}{c^{n-1}}\cdot\frac{a^{n-1}}{c^{n-1}} = a^2 + \frac{a^{2n}}{c^{2n-2}} - \frac{2a^{2n}}{c^{2n-2}} = a^2 - \frac{a^{2n}}{c^{2n-2}} \quad (30)$$

Equating equations (29) and (30), we obtain:

$$a^2 - \frac{a^{2n}}{c^{2n-2}} = b^2 - \frac{b^{2n}}{c^{2n-2}} \Rightarrow a^2 - b^2 = \frac{a^{2n}-b^{2n}}{c^{2n-2}} \Rightarrow a^2 - b^2 = \frac{(a^n+b^n)(a^n-b^n)}{c^{2n-2}}$$

$$\Rightarrow (a^2-b^2) = \frac{c^n(a^n-b^n)}{c^{2n-2}} \Rightarrow a^2-b^2 = \frac{a^n-b^n}{c^{n-2}} \Rightarrow c^{n-2}(a^2-b^2)$$

$$= a^n - b^n \Rightarrow n = 2$$

The left side of the last equation is divisible by c, then the right side must also be divisible by c, which contradicts this condition.

Hence n = 2.

And now we will show geometrically that the equalities $a^n + b^n = c^n$ and

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acosβ+bcosα = c are satisfied for cosβ = $\frac{a^{n-1}}{c^{n-1}}$, cosα = $\frac{b^{n-1}}{c^{n-1}}$ only for n = 2, and at the same time the line of n-straight lines CH ⊥ AB.

We have already shown that to satisfy the equality $a^n + b^n = c^n$, where *a*, *b*, *c* and *n* are natural numbers, the following equalities must be satisfied:

$$\frac{x}{z} = \frac{a^{n-1}}{c^{n-1}} = \frac{\sin\beta_1}{\sin\beta_2}, \quad \frac{y}{z} = \frac{b^{n-1}}{c^{n-1}} = \frac{\sin\alpha_1}{\sin\alpha_2}$$

That is: $a\frac{x}{z} + b\frac{y}{z} = c \Rightarrow a\frac{\sin\beta_1}{\sin\beta_2} + b\frac{\sin\alpha_1}{\sin\alpha_2} = c \Rightarrow a\frac{a^{n-1}}{c^{n-1}} + b\frac{b^{n-1}}{c^{n-1}} = c \Rightarrow a^n + b^n = c^n$ On the other hand, all these formulas are written for the triangle ABC with sides a, *b*, *c* and corresponding angles α, β, γ, for which the equality: $a\cos\beta + b\cos\alpha = c$

Comparing the formulas, the last formulas can be written: $\cos\beta = \frac{\sin\beta_1}{\sin\beta_2}$, $\cos\alpha = \frac{\cos\alpha_1}{\cos\alpha_2}$

Let's transform these formulas. All trigonometric expressions obtained for angles α, α_1 α_2 will be true for angles β, β_1, β_2. Thus we get:

$$\cos\alpha = \cos(\alpha_1 + \alpha_2) = \frac{\sin\alpha_1}{\sin\alpha_2} \Rightarrow \cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2 = \frac{\sin\alpha_1}{\sin\alpha_2} \Rightarrow$$

$$\Rightarrow \cos\alpha_1\cos\alpha_2\sin\alpha_2 - \sin\alpha_1\sin^2\alpha_2 = \sin\alpha_1 \Rightarrow \frac{1}{2}\cos\alpha_1 \cdot 2\sin\alpha_2\cos\alpha_2 -$$

$$-\frac{1}{2}\sin\alpha_1 2\sin^2\alpha_2 = \sin\alpha_1 \Rightarrow \frac{1}{2}\cos\alpha_1\sin2\alpha_2 - \frac{1}{2}\sin\alpha_1(1 - \cos2\alpha_2) = \sin\alpha_1 \Rightarrow$$

$$\Rightarrow \cos\alpha_1\sin2\alpha_2 + \sin\alpha_1\cos2\alpha_2 = 3\sin\alpha_1$$

$$\sin(2\alpha_2 + \alpha_1) = 3\sin\alpha_1 \Rightarrow \frac{\sin\alpha_1}{\sin(2\alpha_2+\alpha_1)} = \frac{1}{3}$$

To visualize the application of the last equality, let's draw a triangle ABC' (ΔABC'=ΔABC), which is symmetrical to the triangle ABC with respect to the side AB (see Figure 5)

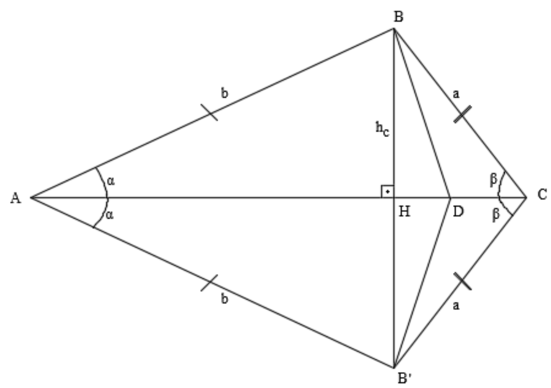

Figure 5

As we can see from the previous formula, the ratio of the sines should be $\frac{1}{3}$.

$$\sin\alpha_1 = \frac{h_1}{AM}, \quad \sin(2\alpha_2 + \alpha_1) = \frac{h_2}{AM}$$

Then $\frac{\sin\alpha_1}{\sin(2\alpha_2+\alpha_1)} = \frac{h_1}{h_2} = \frac{1}{3}$ Since $\frac{\sin\alpha_1}{\sin(2\alpha_2+\alpha_1)} = \frac{h_1}{h_2} = \frac{\frac{1}{2}bh_1}{\frac{1}{2}bh_2} = \frac{B}{M,}$ and only when BB' ⊥ AC CM=MH=HM'=M'C', then

$\frac{\sin\alpha_1}{\sin(2\alpha_2+\alpha_1)} = \frac{h_1}{h_2} = \frac{BM}{B'M'} = \frac{1}{3}$, only then the line of n straight lines is perpendicular to the base (for n = 2) (see Figure 6).

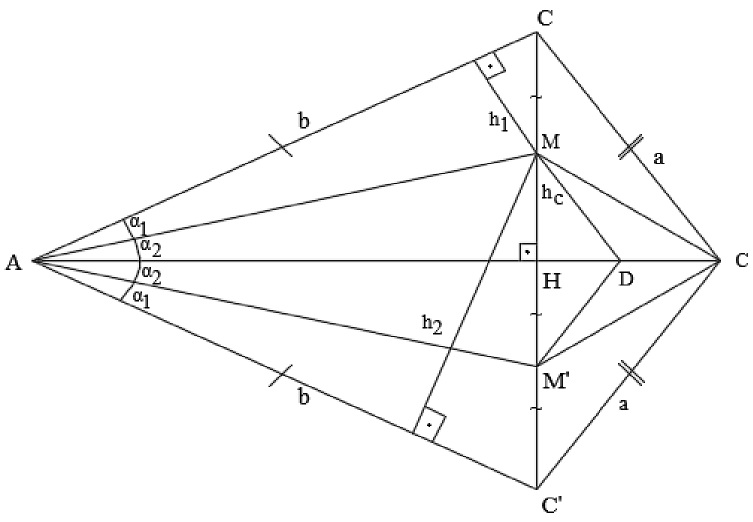

Figure 6

Let us show that for n >2 the ratio of sines is not equal to $\frac{1}{3}$.

Indeed, $\frac{\sin\alpha_1}{\sin(2\alpha_2+\alpha_1)} = \frac{\frac{h_1}{AM}}{\frac{h_2}{AM}} = \frac{h_1}{h_2} = \frac{\frac{1}{2}h_1 AC}{\frac{1}{2}h_2 AF} = \frac{CM}{MF} = \frac{\frac{1}{2}L_c}{\frac{1}{2}L_c + L_c t} = \frac{L_c}{L_c(1+2t)} = \frac{1}{1+2t} < \frac{1}{3}$.

Here $t > 1$ (Figure 7).

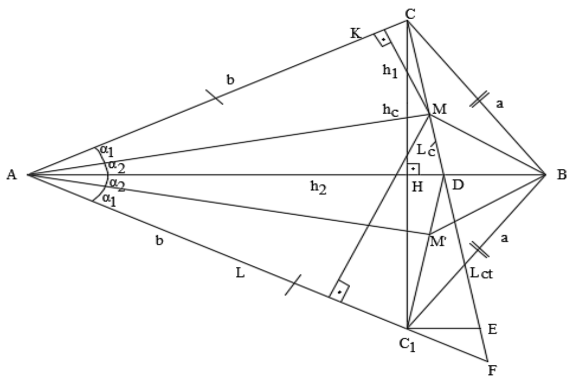

Figure 7

Thus, when n >2 $\cos\alpha \neq \frac{\sin\alpha_1}{\sin\alpha_2}$

In a triangle, side CF faces an obtuse angle, i.e. she is the biggest side.

Here CM = MD $= \frac{1}{2} L_c$.   DF=DE+EF

Since CD=DE=L_c, то DF= L_c+ EF.   That is DF $> L_c \Rightarrow t = \frac{DE}{L_c} > 1$

Thus, we get that, $\cos\beta = \frac{a^{n-1}}{c^{n-1}}$ and $\cos\alpha = \frac{b^{n-1}}{c^{n-1}}$ only when n=2. However, the cosines$(\cos\beta, \cos\alpha)$ take values close respectively to the values $\frac{a^{n-1}}{c^{n-1}}$ and $\frac{b^{n-1}}{c^{n-1}}$ Let's write the following expressions:

$$\cos\beta = \frac{a^{n-1}}{c^{n-1}} f_1, \cos\alpha = \frac{b^{n-1}}{c^{n-1}} f_2 \tag{31}$$

Here $f_1$and $f_2$are functions of positive integers a,b,c and n. Let us determine the values of these functions. To do this, from the triangles ΔACD and ΔBCD we find cosα and cosβ in the following form:

$$\cos\alpha = \frac{b^2 c^{2n-2} + b^{2n} - c^{2n-2} L_c^2}{2b^{n+1} c^{n-1}} \tag{32}$$

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$$cos\beta = \frac{a^2c^{2n-2}+a^{2n}-c^{2n-2}L_c^2}{2a^{n+1}c^{n-1}} \quad (33)$$

We substitute in (32) and (33) the value $L_c^2 = \frac{a^2b^n+a^nb^2}{c^n} - \frac{a^nb^2}{c^{2n-2}},$ derived from (26)
we get: $cos\beta = \frac{a^{n-1}}{c^{n-1}} f_1$ and $cos\alpha = \frac{b^{n-1}}{c^{n-1}} f_2$ where

$$f_1 = \frac{1}{2}\left[1 + \left(\frac{c}{a}\right)^{2n-2} - \left(\frac{b}{a}\right)^n \left(\frac{c}{a}\right)^{n-2} - \left(\frac{c}{a}\right)^{n-2} \left(\frac{b}{a}\right)^2 + \left(\frac{b}{a}\right)^n\right] \quad (34)$$

$$f_2 = \frac{1}{2}\left[1 + \left(\frac{c}{b}\right)^{2n-2} - \left(\frac{a}{b}\right)^n \left(\frac{c}{b}\right)^{n-2} - \left(\frac{c}{b}\right)^{n-2} \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^n\right] \quad (35)$$

As seen, for n=2 we have:

$$f_1 = \frac{1}{2}\left[1 + \left(\frac{c}{a}\right)^2 - \left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right)^2 + \left(\frac{b}{a}\right)^2\right] = \frac{1}{2}\left(1 + \frac{c^2 - b^2}{a^2}\right) = \frac{1}{2}\left(1 + \frac{a^2}{a^2}\right) = 1$$

$$f_2 = \frac{1}{2}\left[1 + \left(\frac{c}{b}\right)^2 - \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^2\right] = \frac{1}{2}\left(1 + \frac{c^2 - a^2}{b^2}\right) = \frac{1}{2}\left(1 + \frac{b^2}{b^2}\right) = 1$$

Then $cos\beta = \frac{a^{n-1}}{c^{n-1}} f_1 = \frac{a}{c}, cos\alpha = \frac{b^{n-1}}{c^{n-1}} f_2 = \frac{b}{c}.$

That is, for a right triangle, everything is correct. Now suppose that $n>2,\ n\epsilon N$.

In this case, substituting (31) into (15), we obtain the equality

$$a^n f_1 + b^n f_2 = c^n \quad (36)$$

Thus, for any integer triangular numbers a, b, c and natural powers of n, we can find such functions f_1 and f_2, which, when formulated, give equality (36).

If we pay attention to the functions (34) and (35), we see that for natural values of a, b, c and n, the functions f_1 and f_2 must be rational. This can also be seen from (36).

Now suppose that there are values n(n>2) under which the condition $a^n + b^n = c^n$, is satisfied, that is, such values f_1 and f_2 $(f_1 \neq 1, f_2 \neq 1),$, under which the following conditions were simultaneously met:

$\begin{cases} a^n + b^n = c^n \\ a^n f_1 + b^n f_2 = c^n \end{cases}$ Then, subtracting the first from the equation from the second, we get $a^n(1 - f_1) + b^n(1 - f_2) = 0$ or $a^n(1 - f_1) = b^n(f_2 - 1)$.

Where we get

$$\left(\frac{a}{b}\right)^n = \frac{1-f_2}{f_1-1} \quad (37)$$

Now we can get the following expression:

$$\left(\frac{a}{c}\right)^n = \frac{1-f_2}{f_1-f_2},\ \left(\frac{b}{c}\right)^n = \frac{f_1-1}{f_1-f_2} \quad (38)$$

Formulas (37) and (38) can be written as:

$$\frac{a}{b} = \sqrt[n]{\frac{1-f_2}{f_1-1}},\ \frac{a}{c} = \sqrt[n]{\frac{1-f_2}{f_1-f_2}},\ \ \frac{b}{c} = \sqrt[n]{\frac{f_1-1}{f_1-f_2}} \quad (39)$$

They can also be presented, for example, as:

$$cos\beta = \left(\frac{a}{c}\right)^{n-1} \cdot f_1 = \sqrt[n]{\left(\frac{1-f_2}{f_1-f_2}\right)^{n-1}} f_1;\ cos\alpha = \left(\frac{b}{c}\right)^{n-1} \cdot f_2 = \sqrt[n]{\left(\frac{f_1-1}{f_1-f_2}\right)^{n-1}} f_2 \quad (40)$$

Let's note:

1) For n=2, that is, when the triangle is right-angled, regardless of a, b, c, we get $f_1 = f_2 = 1$. Then the expression $a^2 + b^2 = c^2$ can have positive integer solutions for a, b, c.

2) With a=b, that is, when the triangle is isosceles, regardless of a, b, c, we get $f_1 = f_2 = 1$. However, then we have $a^n + a^n = c^n$ and $2a^n = c^n$, this case is impossible if a and c are rational.

In formula (36), for an arbitrary triangle, an equality is obtained in which a, b, c, n can receive not only integer values, but also any real positive values.

And now consider the case for which the line of n-lines is perpendicular to the base of the triangle.

As can be seen from Figure 5 the projection of "b" on side "c" is $AH = b \cdot cos\alpha$, and the projection of a on side "c" is $BH = a \cdot cos\beta$, where $c = a \cdot cos\beta + b \cdot cos\alpha$

$$\text{Let } \left(\frac{b}{a}\right)^k = \frac{b\cdot cos\alpha}{a\cdot cos\beta} \tag{41}$$

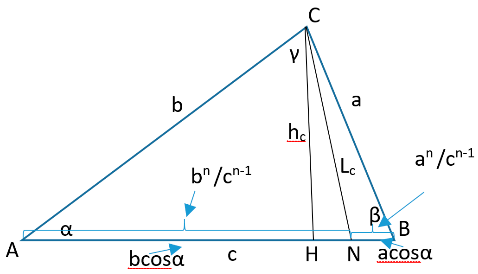

Figure 8

From formula (41) one can also obtain the formula

$$\left(\frac{b}{a}\right)^{k-1} = \frac{cos\alpha}{cos\beta} \tag{42}$$

On the other hand, from formula (34) we have: $\left(\frac{b}{c}\right)^{n-1} = \frac{1}{f_2}\cdot cos\alpha$, $\left(\frac{a}{c}\right)^{n-1} = \frac{1}{f_1}\cdot cos\beta$

If we divide the corresponding sides of these formulas into each other, then we get:

$$\left(\frac{b}{a}\right)^{n-1} = \frac{f_1}{f_2}\cdot\frac{cos\alpha}{cos\beta} \tag{43}$$

or

$$\left(\frac{b}{a}\right)^{n} = \frac{b}{a}\cdot\frac{f_1}{f_2}\cdot\frac{cos\alpha}{cos\beta} \tag{44}$$

If we divide the corresponding sides of formulas (43) and (42) into each other, then we obtain:

$$\left(\frac{b}{a}\right)^{n-k} = \frac{f_1}{f_2} \tag{45}$$

Now consider the n-line CN. As is known: $AN = \frac{b^n}{c^{n-1}}$, $BN = \frac{a^n}{c^{n-1}}$

Formula (34) can be written in the following form: $f_1 = \frac{a\cdot cos\beta}{\frac{a^n}{c^{n-1}}}$, $f_2 = \frac{b\cdot cos\alpha}{\frac{b^n}{c^{n-1}}}$

$$\text{Or } \quad a\cdot cos\beta = \frac{a^n}{c^{n-1}}\cdot f_1, \quad b\cdot cos\alpha = \frac{b^n}{c^{n-1}}\cdot f_2 \tag{46}$$

Let's find the length of NH:

$$NH = \Delta = \frac{b^n}{c^{n-1}} - b\cdot cos\alpha = \frac{b^n}{c^{n-1}} - \frac{b^n}{c^{n-1}} f_2 = \frac{b^n}{c^{n-1}}\,(1-f_2) \tag{47}$$

$$\text{Or: } \Delta = a\cdot cos\beta - \frac{a^n}{c^{n-1}} = \frac{a^n}{c^{n-1}}\cdot f_1 - \frac{a^n}{c^{n-1}} = \frac{a^n}{c^{n-1}}\,(f_1-1) \tag{48}$$

Then the length of the n-line L_c can be found as follows:

$$L_c^2 = h_c^2 + \Delta^2 \tag{49}$$

And now, using the obtained formulas, we will try to prove some statements related to Fermat's theorem.

As we have already noted, formula (36) is true for any triangle with any positive real sides *a,b,c* and real values "*n*".

As we have noted, f_1 and f_2 must be rational.

We are only interested in triangles with natural values *a, b, c* and "*n*".

Note also that for positive integer values *a,b,c*, the expressions cosα, cosβ, cosγ must be rational. This is clear from the cosine theorem.

$$cos\alpha = \frac{b^2+c^2-a^2}{2bc}; \quad cos\beta = \frac{a^2+c^2-b^2}{2ac} \tag{50}$$

It is known that the ratio of integers is rational.

Consider the case

$$f_1 = f_2 = 1 \tag{51}$$

From formula (45) we get that this is possible only for *n=k*, i.e. equality must hold

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$$a^k + b^k = c^k \tag{52}$$

On the other hand, as can be seen from formulas (47) and (48), with $f_1 = f_2 = 1$, we have Δ=0. From (49) we obtain that $L_c = h_c$. That is, by formulas (45), (47), (48) we obtain that in this case the line of n-straight lines must be perpendicular to the base. Then formula (26) is transformed into the following form

$$L_c^2 = h_c^2 = \frac{a^2b^k + b^2a^k}{c^k} - \frac{c^2a^kb^k}{c^{2k}} = \frac{a^2b^2(b^{k-2}c^{k-2} + a^{k-2}c^{k-2} - a^{k-2}b^{k-2}}{c^{2k-2}}$$

Or $h_c^2c^{2k-2} = a^2b^2(b^{k-2}\cdot c^{k-2} + a^{k-2}\cdot c^{k-2} - a^{k-2}\cdot b^{k-2})$ (53)

Here we have:

$$h_c^2 = b^2(sin\alpha)^2 = b^2(1-(cos\alpha)^2) = b^2(1-(\frac{b^2+c^2-a^2}{2bc})^2) = b^2 - \frac{(b^2+c^2-a^2)^2}{4c^2} \tag{54}$$

Substituting (54) into (53) we get:

$$c^{2k-4}(b^2c^2 - \left(\frac{b^2+c^2-a^2}{2}\right)^2) = a^2b^2(b^{k-2}c^{k-2} + a^{k-2}c^{k-2} - a^{k-2}b^{k-2}) \tag{55}$$

Here k ≥ 2.

Since the numbers a, b, c satisfy the condition $c^k = a^k + b^k$, then two of them must be odd, and one of them must be an even number.

Then the value of the expression $(\frac{b^2+c^2-a^2}{2})^2$ will be a natural number. On the other hand, the left side of equality (55) is divisible by "c". Therefore, the right side must also be divisible by "c". Since a^2b^2 and $a^{k-2}b^{k-2}$are not divisible by "c", then equality (55) is possible only when 2k – 4 = 0 or when k = 2.

Indeed, for k = 2, formula (55) transforms into the form:

$$b^2c^2 - (\frac{b^2+c^2-a^2}{2})^2) = a^2b^2$$

If we simplify this expression, we get:

$$b^4 + c^4 + a^4 + 2b^2c^2 - 2a^2b^2 - 2a^2c^2 + 4a^2b^2 - 4b^2c^2 = 0$$

$$b^4 + c^4 + a^4 - 2b^2c^2 + 2a^2b^2 - 2a^2c^2 = 0 \Rightarrow \left(c^2 - (a^2+b^2)\right)^2 = 0 \Rightarrow c^2 = a^2 + b^2 \tag{56}$$

Which is to be expected.

This can be shown in another way.

As you know, for the triangle ABC, you can write the following equalities:

$$b\cos\gamma + c\cos\beta = a \Rightarrow cos\gamma = \frac{a}{b} - \frac{c}{b}cos\beta$$

$$a\cos\gamma + c\cos\alpha = b \Rightarrow cos\gamma = \frac{b}{a} - \frac{c}{a}cos\alpha$$

If formulas (31) are taken into account in the last equalities, then we obtain the following equalities:

$$cos\gamma = \frac{a}{b}(1 - (\frac{a}{c})^{n-2}f_1), \quad cos\gamma = \frac{b}{a}(1 - (\frac{b}{c})^{n-2}f_2)$$

As can be easily seen, for $f_1 = f_2 = 1$ и $\gamma = 90^0$, we get $n = 2$, or for $n = 2$, $\gamma = 90^0$ we get $f_1 = f_2 = 1$.

And now we will try to prove that for n > k > 2 and for $f_1 \neq f_2 \neq 1$,we come to a contradiction.

Formulas (46) are transformed into the following form:

$$a^n = \frac{1}{f_1}ac^{n-1}\cos\beta = \frac{c^{n-2}}{f_1} \bullet \frac{a^2+c^2-b^2}{2} = \frac{c^{n-2}A}{f_1}$$

and

$$b^n = \frac{1}{f_2}bc^{n-1}\cos\alpha = \frac{c^{n-2}}{f_2} \bullet \frac{a^2+c^2-b^2}{2} = \frac{c^{n-2}B}{f_2}$$

i.e.,

$$a^n = \frac{c^{n-2}A}{f_1} \qquad b^n = \frac{c^{n-2}B}{f_2} \tag{57}$$

$$\text{Here } A = \frac{a^2+c^2-b^2}{2}, \ B = \frac{b^2+c^2-a^2}{2} \tag{58}$$

Assume that the equality $a^n + b^n = c^n$ holds. Substituting expressions (57) here, we obtain:

$$\frac{c^{n-2}A}{f_1} + \frac{c^{n-2}B}{f_2} = c^n \quad (59) \qquad \text{or} \qquad \frac{A}{f_1} + \frac{B}{f_2} = c^2 \quad (60)$$

The last formula is the relationship between f_1 and f_2

These formulas can also be presented in the following form:

$$f_1 = \frac{1}{1 - \frac{B}{A}(\frac{1}{f_2} - 1)} \quad \text{and} \quad f_2 = \frac{1}{1 + \frac{B}{A}(1 - \frac{1}{f_1})}$$

Without proof, we also present the following formulas:

$$f_1 = \frac{(\frac{c}{a})^n}{1+(\frac{b}{a})^k} \quad , \quad f_2 = \frac{(\frac{c}{b})^n}{1+(\frac{a}{b})^k}$$

On the other hand, dividing formulas (57) (corresponding sides) by each other, one can obtain

$$(\frac{b}{a})^n = \frac{B}{A} \bullet \frac{f_1}{f_2} \qquad (61)$$

From the last formula, taking into account (45), we obtain $(\frac{b}{a})^n = \frac{B}{A} \bullet (\frac{b}{a})^{n-k} \Rightarrow \frac{B}{A} = (\frac{b}{a})^k$ (62)

Note also that by substituting formula (37) into (61), we can obtain the equality

$$\frac{B}{A} = \frac{f_{2(f_1-1)}}{f_{1(1-f_2)}} \qquad (63)$$

From (60) it can be seen that for $f_1 = f_2$ (for n=2) we also get

$$B + A = \frac{b^2+c^2-b^2}{2} + \frac{a^2+c^2-b^2}{2} = c^2 \qquad (64)$$

However, if $n > 2$, then we get $B = \frac{b^2+c^2-b^2}{2} < b^2$ and $A = \frac{a^2+c^2-b^2}{2} < a^2$ (65)

Let's prove these inequalities.

Let n > k > 2 and $a^n + b^n = c^n$. Whence it turns out: $(\frac{a}{c})^n + (\frac{b}{c})^n = 1$. Since in this case $\frac{a}{c} < 1$, $\frac{b}{c} < 1$, we can write the following inequality:

$$(\frac{a}{c})^2 + (\frac{b}{c})^2 > (\frac{a}{c})^k + (\frac{b}{c})^k > (\frac{a}{c})^n + (\frac{b}{c})^n = 1$$

Therefore, in this case we get $a^2 + b^2 + c^2$. It follows from the latter that $c^2 - a^2 < b^2$, $c^2 - b^2 < a^2$. Then we have:

$$B = \frac{b^2+(c^2-a^2)}{2} < \frac{b^2+b^2}{2} = b^2 \quad \text{and} \quad A = \frac{a^2+(c^2-b^2)}{2} < \frac{a^2+a^2}{2} = a^2$$

Without proof, we also present the following double inequalities

$$(\frac{a}{c})^n \le \frac{A}{c^2} < \frac{1}{2} < \frac{B}{c^2} \le (\frac{b}{c})^n, \; 0 < (\frac{a}{c})^{n-1} \le cos\beta < cos\alpha \le (\frac{b}{c})^{n-1} < 1,$$

$$log_{\frac{b}{a}} \frac{B}{A} \le n \le log_{\frac{c}{b}} \frac{c^2}{B}, \; \frac{c^2}{2} < B \le b^2, \; A \le a^2 < \frac{c^2}{2}$$

Now consider the following formula (see 62, 63):

$$(\frac{b}{a})^k = \frac{B}{A} = \frac{f_{2(f_1-1)}}{f_{1(1-f_2)}}, \text{ где } f_1 \ne f_2 \ne 1.$$

As we already noted in (65), for $n > k > 2$

$$B < b^2 < b^k < b^n$$
$$A < a^2 < a^k < a^n$$

In formula (62), as can be seen from the last inequalities, $b^k > B$, $a^k > A$. Therefore, in order to obtain $\frac{B}{A}$ from $(\frac{b}{a})^k$ (i.e., the smallest of large numbers), it is necessary that $(\frac{b}{a})^k$ is reduced by an integer. In other words, it must be that $a^k = At$, $b^k = Bt$. And this is impossible, since b and a are irreducible numbers. Therefore, the number k cannot be an integer or a fractional number, at which the number $(\frac{b}{a})^k$ would become irrational.

Since $\frac{B}{A}$ is the ratio of two integers, it is therefore rational. That is, in any case, if n > k > 2, and the number k is a rational number, then the equality $\frac{B}{A} = (\frac{b}{a})^k$ leads to a contradiction. This contradiction is removed only when n = k = 2 and if k is (for k > 2) an irrational number.

5. Determination of the limits of change of the coefficients f_1 and f_2

And now we will show the intervals of change of the coefficients f_1 and f_2. In works [3,4] shows that the equality $a^n + b^n = c^n$ is possible only for an acute triangle. Let us show that in this case the coefficients f_1 and f_2 introduced by us (for c > b > a) should change within the following limits:

$$1 \le f_1 < 1 + \frac{1}{2}(\frac{b}{a})^n, \; \frac{1}{2} < f_2 \le 1 \qquad (66)$$

We have already noted that if we draw a straight line n from the vertex C (line CD), then $(\frac{AC}{BC})^n = (\frac{b}{a})^n = \frac{AD}{DB}$ where for n = 0 we get that CD is the median, i.e. AD = DB. Let's draw height AN from vertex C to side AB. Earlier we noted that in this case AH = bcosα,

$$BH = \alpha cos\beta, \; AD = \frac{b^n}{c^{n-1}}, \; BD = \frac{a^n}{c^{n-1}},$$

$\frac{BH}{BD} = f_1$, $\frac{AH}{AD} = f_2$. Then we get $\frac{1}{f_2} = \frac{AD}{AH} < \frac{AB}{AM} = 2$ or $f_2 > \frac{1}{2}$ (see Figure 9).

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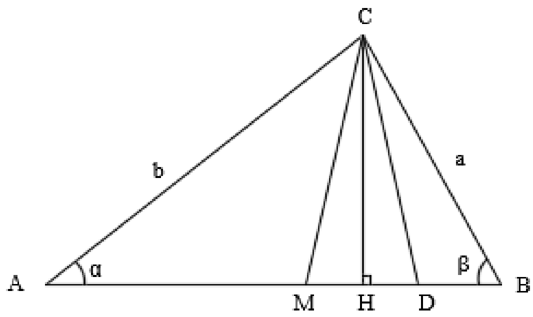

Figure 9

On the other hand, since AN < AD, then $f_2 < 1$. Therefore, $\frac{1}{2} < f_2 < 1$ (for n > 2).

Also, taking into account that BH > BD and $f_1 = \frac{BH}{BD}$, we obtain $f_1 > 1$. To determine the right side of the double inequality for f1, we use formula (36) and transform it into the following form:

$$f_1 = \frac{1}{a^n}(c^n - b^n f_2) = \frac{1}{a^n}(c^n - b^n + b^n - b^n f_2) = \frac{1}{a^n}(a^n + b^n(1 - f_2)) = 1 + (\frac{b}{a})^n(1 - f_2)$$

If we take into account the inequality $f_1 > \frac{1}{2}$ in the last expression, then we obtain

$$f_1 < 1 + \frac{1}{2}(\frac{b}{a})^n.$$

As is already known, cases of equality $f_1 = f_2 = 1$ are possible only in a right triangle. Thus, we determine the intervals of change of the coefficients (66) f_1 and f_2.

6. Determining the interval for changing the number k

Now let's set the limits for changing the number *k.*

Suppose we are given a triangle *ABC*, where *BC* = *a*, *AC* = *b*, *AB* = *c*. Draw height *CH* and median *D* from vertex *C*. Then $AD = DB = \frac{c}{2}$ (see Figure 10)

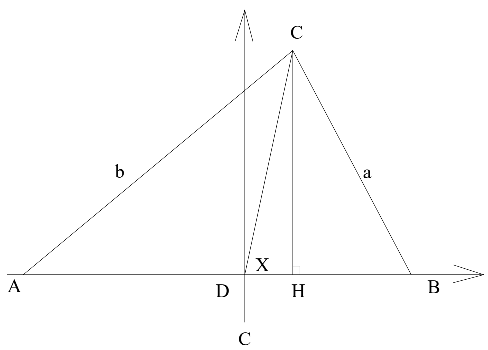

Figure 10

Let's assume DH = x. Then $AH = \frac{c}{2} + x$, $BH = \frac{c}{2} - x$,

$$b^2 - (\frac{c}{2} + x)^2 = a^2 - (\frac{c}{2} - x)^2.$$

Simplifying the last equality, we get: $b^2 = a^2 + 2cx$.

Let's transform the equality $(\frac{b}{a})^k = \frac{AH}{HB}$ into the following form: $(\frac{b^2}{a^2})^{\frac{k}{2}} = \frac{\frac{c}{2}+x}{\frac{c}{2}-x} = \frac{c+2x}{c-2x}$.

On the other side $(\frac{b^2}{a^2})^{\frac{k}{2}} = (\frac{a^2+2cx}{a^2})^{\frac{k}{2}} = (1 + (\frac{2cx}{a^2}))^{\frac{k}{2}}$.

Equating the right parts of the last expressions, we get:

$$(1+\frac{2cx}{a^2})^{\frac{k}{2}} = \frac{c+2x}{c-2x} \quad \text{or } k = \frac{2(\ell n(c+2x) - \ell n(c-2x))}{\ell n(1+\frac{2cx}{a^2})}.$$

Let's establish what the number k will tend to when n → ∞.

As is known, n → ∞, in the case when the triangle ABC is equilateral [3].

Let's calculate the following limit:

$$k = \lim_{x\to 0}\frac{2(\ell n(c+2x)-\ell n(c-2x))}{\ell n(1+\frac{2cx}{a^2})} = \left[\frac{0}{0}\right] = 2\lim_{x\to 0}\left(\frac{(\ell n(c+2x)-\ell n(c-2x))'}{(\ell n\left(1+\frac{2cx}{a^2}\right))'}\right)$$

$$= 2\lim_{x\to 0}\frac{\frac{2}{c+2x}+\frac{2}{c-2x}}{\frac{2c}{a^2+2cx}} = 2\lim_{x\to 0}\frac{\frac{4c}{c^2-4x^2}}{\frac{2c}{a^2+2cx}} = \frac{4a^2}{c^2}$$

Since in an equilateral triangle $a = c$, then $k = 4$. Thus, as $n \to\infty$ $k \to 4$.

Now let's find out what the number k will tend to if $b\to c$, $a\to 0$.

Let's calculate the following limit:

$$k = \lim_{\substack{x\to\frac{c}{2}\\ b\to c}}\frac{2(\ell n(c-2x)-\ell n(c+2x))}{\ell n\left(1-\frac{2cx}{b^2}\right)} = 2\lim_{\substack{x\to 0\\ b\to c}}\frac{\frac{-2}{c-2cx}-\frac{2}{c+2x}}{\frac{-\frac{2c}{b^2}}{1-\frac{2cx}{b^2}}} = 4\lim_{\substack{x\to\frac{c}{2}\\ b\to c}}\frac{b^2-2cx}{c^2-4x^2} = 4\lim_{x\to\frac{c}{2}}\frac{c(c-2x)}{(c-2x)(c+2x)} = 4\cdot\frac{1}{2} = 2$$

Therefore, in this case $k \to 2$.

Earlier we have already shown that for n = k = 2 we get $f_1 = f_2 = 1$ and the equality $a^2 + b^2 = c^2$. holds.

Using formulas (38) and (57) we can obtain the following equalities:

$$A = \frac{c^2 f_1\ (1-f_2\ )}{f_1 - f_2}\ ,\ B = \frac{c^2 f_2\ (f_1-1)}{f_1 - f_2} \qquad (67)$$

Here the ratios $\frac{1-f_2}{f_1-f_2} = (\frac{a}{c})^n$, $\frac{f_1-1}{f_2-f_1} = (\frac{b}{c})^n$ are not integers, since by assumption natural numbers a, b, c are pairwise coprime and the degree n is a natural number. On the other hand, the ratios $\frac{f_1}{f_1-f_2}$ and $\frac{f_2}{f_1-f_2}$ are not integers either, since if we write $\frac{f_1}{f_1-f_2} = t_1$, $\frac{f_2}{f_1-f_2} = t_2$ (where t_1, t_2 are integers), then we get respectively $\frac{f_1}{f_2} = \frac{t_1}{t_1-1}$ and $\frac{f_1}{f_2} = \frac{t_2+1}{t_2}$ For the numerator and denominator of the ratio $\frac{f_1}{f_2}$ to be an integer, we write it as $\frac{f_1\ n_1 n_2}{f_2\ \ n_1 n_2}$.

Here $f_1 = \frac{m_1}{n_1}$, $f_2 = \frac{m_2}{n_2}$, where m_1, n_1, m_2, n_2 are integers.

That is, here $f_1 n_1 n_2$ and $f_2 n_1 n_2$ must be consecutive natural numbers.

Since $\frac{f_1}{f_2} = \frac{f_1\ n_1 n_2}{f_2\ \ n_1 n_2} = (\frac{b}{a})^{n-k}$, this is impossible.

Now suppose that the ratios $\frac{f_1(1-f_2)}{f_1-f_2} = t_1$ are integers.

Then from formulas (67) we get: $A = c^2 t_1$, $B = c^2 t_2$ and $A + B = c^2(t_1 + t_2)$

On the other hand, it is known that $A + B = c^2$.

Then we have $c^2(t_1 + t_2) = c^2$ or $t_1 + t_2 = 1$, which is impossible by assumption.

Given the above, one of the assumptions may be that c2 is evenly divisible by $(f_1 - f_2)$.

Let $A = c^2\ \frac{f_1(1-f_2)n_1n_2}{(f_1-f_2)n_1n_2}$, $B = c^2\ \frac{f_2(f_1-1)n_1n_2}{(f_1-f_2)n_1n_2}$.

Then $c^2 = (f_1 - f_2)n_1n_2t$, $A = f_1(1-f_2)n_1n_2t$, $B = f_2\ (f_1 - 1)n_1n_2t$ (68)

Since $\frac{A}{B} = (\frac{a}{b})^k$, then t = 1. If the assumption is correct, then we can write:

$$a^n = (1-f_2)n_1n_2c^{n-2},\qquad b^n = (f_1-1)\ n_1n_2c^{n-2}, \qquad (69)$$

$$c^n = (f_1- f_2)\ n_1n_2c^{n-2} \qquad (70)$$

In this case, the expressions $(1-f_2)\ n_1n_2 = \frac{A}{f_1}$, $(f_1 - 1)\ n_1n_2 = \frac{B}{f_2}$ will be positive integers.

On the other hand $\frac{A}{f_1} = \frac{a^n}{c^{n-2}}$, $\frac{B}{f_2} = \frac{a^n}{c^{n-2}}$ (71)

As can be seen, the right-hand sides of these equalities can be integer only for n = 2, and for n > 2 they will be irreducible.

We come to a contradiction, that is, under this assumption, it turns out that the equality $a^n + b^n = c^n$ for natural numbers a, b, c and n is true only for n = 2.

Indeed, for n = 2, substituting expression (70) into the equality $a^n + b^n = c^n$ we get:

$$(1-f_2)\ n_1n_2c^{n-2} + (f_1 - 1)\ n_1n_2c^{n-2} = (f_1- f_2)\ n_1n_2c^{n-2} \qquad (72)$$

As we can see, in the last equality, each of the three terms has a common factor, which contradicts the condition, since by assumption the numbers a^n, b^n, c^n are coprime.

А теперь предположим, что в равенстве (59) $\frac{c^{n-2}}{f_1}$, $\frac{c^{n-2}}{f_2}$ являются целыми числами.

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Then

$$c^{n-2} = m_1m_2t \quad (73)$$

Substituting expressions (69) and (73) into (59), we get:

$$\frac{m_1 m_2\, t\, n_1}{m_1}A + \frac{m_1 m_2\, t\, n_2}{m_2}B = m_1m_2tc^2 \quad \text{or} \quad m_2n_1tA + m_1n_2tB = m_1m_2tc^2 \quad (74)$$

Let t = 1. As you can see, each of the parts of equality (74) has a common factor, since if two of the three terms are divisible by the same number, then the third term must be divisible by this number.

Now suppose that the factors m1 and m2 are both in the number $c^{n-2}$and in the numbers A and B. Then c^{n-2} has at least one of the factors of the numbers m_1 and m_2.

$$\text{Assume that } c^{n-2} = p_1t,\ m_1 = p_1k_1,\ \frac{A}{k_1} = t_1,\ \frac{B}{m_2} = t_2 \quad (75)$$

Then, substituting them into (59), we obtain: $\frac{Ap_1tn_1}{p_1k_1} + \frac{Bp_1tn_2}{m_2} = p_1tc^2$

Even for t = 1 we get:

$$t_1n_1 + t_2p_1n_2 = p_1tc^2 \quad (76)$$

As we can see, in this case, the right and left sides of the equation have a common factor p1. Since two of the three terms have a common factor, that is, we again come to a contradiction.

Thus, in all the cases presented for n > 2, we arrive at a contradiction

And now let's draw heights from point D to sides a and b. Let's designate these heights respectively through h_1 and h_2, and height CH through h_3 (see Figure 11).

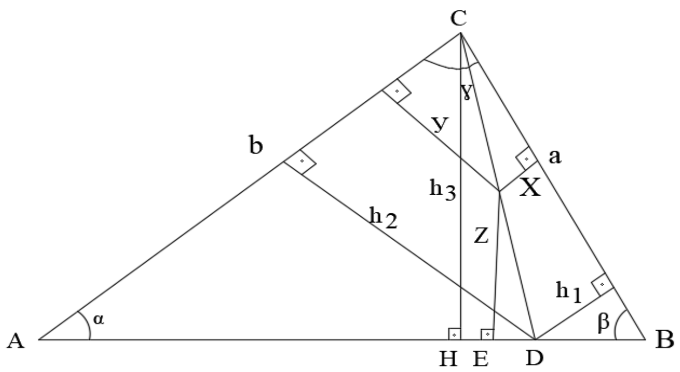

Figure 11

Since the point of intersection of n lines lies on the midline parallel to side C, then $h_1 = 2x$, $h_2 = 2y$, $h_3 = 2z$. Then, taking into account formulas (7), we can write the following relations:

$$\frac{h_1}{h_2} = \frac{x}{y} = \left(\frac{a}{b}\right)^{n-1}, \quad \frac{h_1}{h_3} = \frac{x}{z} = \left(\frac{a}{c}\right)^{n-1}, \quad \frac{h_2}{h_3} = \frac{y}{z} = \left(\frac{b}{c}\right)^{n-1} \quad (77)$$

Since by condition:

$$c > b > a, \text{ то } h_3 > h_2 > h_1 > 1 \quad (78)$$

It should be noted that for h < 1, the side a will also be less than unity.

As is known, for integer values of the sides a, b, c of the triangle ABC, according to the cosine theorem, the values of cosα, cosβ, cosγ will also be rational. In this case, the square of the area of this triangle will also be a rational number [4]. Indeed, taking into account that $S = \frac{1}{2}ab\sin\gamma$ and $\sin\gamma = \sqrt{1 - cos^2\gamma}$, we obtain the formula for the squared area:

$$S^2 = \frac{1}{4}a^2b^2(1 - cos^2\gamma).$$

It can be shown that in this case h_1^2, h_2^2, h_3^2 will also be rational numbers. Indeed, since $h_1 = BD\sin\beta = \frac{a^n}{c^{n-1}}\sin\beta$, $h_2 = AD\sin\alpha = \frac{b^n}{c^{n-1}}\sin\alpha$,

$h_3 = \frac{2S_{ABC}}{c}$, then we can write $h_1^2 = \left(\frac{a^n}{c^{n-1}}\right)^2(1 - cos^2\beta)$, $h_2^2 = \left(\frac{b^n}{c^{n-1}}\right)^2(1 - cos^2\alpha)$, $h_3^2 = \left(\frac{2S}{c}\right)^2$.

And now,

$$\text{let } a = K_1h_1,\ b = K_2h_2,\ c = K_3h_3 \quad (79)$$

Then, taking into account (77) and (78), we can obtain the following formulas:

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$$\frac{K_1^2}{K_2^2} = \left(\frac{a}{b}\right)^{2(n-2)}, \ \frac{K_1^2}{K_3^2} = \left(\frac{c}{a}\right)^{2(n-2)}, \quad \frac{K_2^2}{K_3^2} = \left(\frac{c}{b}\right)^{2(n-2)}, \quad (80)$$

It should also be noted that if we take into account in the formulas $a^2 = K_1^2 h_1^2$, $b^2 = K_2^2 h_2^2$, $c^2 = K_3^2 h_3^2$ performed inequalities (2) are satisfied, then we obtain $a^2 > K_1^2$, $b^2 = K_2^2$, $c^2 = K_3^2$

Here K_1^2, K_2^2, K_3^2 can be both integers and non-integer rational numbers. If K_1^2, K_2^2, K_3^2 are integers (or at least two of them), then we will come to a contradiction, because in order to obtain equalities (4), the ratios $\frac{b}{a}, \frac{c}{a}, \frac{c}{b}$ must be reduced. Since *a, b, c* are coprime natural numbers, then formulas (80) will be valid only for $n = 2$.

Thus, in all the cases presented for $n > 2$, we arrive at a contradiction.

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