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Overlapping of Lévai's and Milson's e-Tangent-Polynomial Potentials Along Symmetric Curves

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Abstract: The paper examines common elements between Lévai's and Milson's potentials obtained by Liouville transformations of two rational canonical Sturm-Liouville equations (RCSLEs) with even density functions which are exactly solvable via Jacobi polynomials in a real or accordingly imaginary argument. We refer to the polynomial numerators of the given rational density function as 'tangent polynomial' (TP) and thereby term the aforementioned potentials as '*e*-TP'. A special attention is given to the overlap between the two potentials along symmetric curves which represent two different rational forms of the Ginocchio potential exactly quantized via Gegenbauer and Masjed-Jamei polynomials respectively. Our analysis reveals that the actual interconnection between Lévai's parameters for these two rational realizations of the Ginocchio potential is much more complicated than one could expect based on the striking resemblance between two quartic equations derived by Lévai for 'averaged' Jacobi indexes.

Keywords: rational Sturm-Liouville equation; Liouville transformation; complex Jacobi polynomials, classical Jacobi polynomials, Romanovski-Routh polynomials, Masjed-Jamei polynomials; quasi-rational solutions, almost-everywhere holomorphic solutions

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1. Introduction

Nearly 60 years ago Bose [1, 2] made a revolutionary discovery that the most solvable potentials known at that time can be obtained by the Liouville transformation [3, 4] of the some special cases of a rational Sturm-Liouville equation [4] written in its canonical (no first derivative) form and simply referred to below as 'RCSLE'. Several years later the author [5] made use of Bose's discovery to construct the family of rational potentials *exactly* solvable in terms of superpositions of two hypergeometric functions of a *real* variable as well as its confluent counter-part exactly solvable in terms of superpositions of confluent hypergeometric series and Whittaker function (see, e. g., Ch. 5 and Ch. 6 in [3]). It was shown that the corresponding eigenfunctions are expressible in terms of *classical* Jacobi and *classical* Laguerre polynomials with *degree-dependent* indexes in general. Thereby we refer to these two families of solvable rational potentials as 'Jacobi-reference' ($\mathcal{J}$Ref) and 'Laguerre-reference' ($\mathcal{L}$Ref) potentials.

The Bose approach was further advanced by Milson [6] who demonstrated that there is another family of solvable potentials obtained by the Liouville transformation of the real Fuschian SLE with two poles at complex conjugated points. It was proven by us later [7] under influence of Avarez-Castillo and Kirchbach's breakthrough paper [8] that the eigenfunctions of this RCSLE are expressible in terms of Romanovski-Routh (R-Routh) polynomials with degree-dependent indexes. (For the reasons explained in [9, 10] we prefer to refer to pseudo-Jacobi polynomials [11-13] as Routh polynomials [14] so the epithet 'R-Routh polynomials' is fully consistent with the term 'Romanovski/pseudo-Jacobi polynomials in Leski's [15, 16] classification scheme of Romanovski polynomials [17, 18]).

While making some inspirational remarks on the original draft of [19] Kirchbach drew author's attention to the reference to Milson's paper [6] in the brand-new review article [20]. It became clear that Milson has already addressed this problem to a large extent. However our study revealed some new important elements not discussed in [6]. First it was shown that hypergeometric polynomials constructed in [6] are nothing but R-Routh polynomials. Secondly the author took advantage of

Stevenson's idea [21] (also pointed to the author by Kirchbach) to express an analytically continued solution in terms of hypergeometric polynomials in a complex argument. It was just confirmed that the latter *formally complex* polynomials can be converted to real R-Routh polynomials in the real argument and that the Sturm-Liouville problem in question is indeed exactly solvable.

Our current analysis explicitly takes advantage of the interconnection [8] between the real R-Routh polynomials and Jacobi polynomials with complex conjugated indexes provided that the argument of the complex polynomials is restricted to the imaginary axis [22, 23]. To our knowledge it was Bagrov et al [24] who brought the latter polynomials into the quantum mechanics using them as polynomial components of the eigenfunctions for the *very unique trigonometric* limit of the potentials solvable in terms of the R-Routh polynomials -- the so-called 'Rosen-Morse II' potential in the Cooper-Khare-Sukhatme (CKS) [25, 26] classification scheme of solvable rational potentials. (In following [30] we refer to this trigonometric modification of the Rosen-Morse potential [28] simply as '*t*-RM'.) The cited text book [24] made no mathematical references and did not provide any arguments in support of the conjecture that the constructed eigenfunctions are real.

A few years later Dabrowska et al [29] ran into these polynomials while examining the eigenfunctions for the Gendenshtein potential [30] (the Scarf II potential in the CKS classification scheme [25, 26]). The cited authors explicitly computed first three eigenfunctions and did confirm that they are real and mutually orthogonal. Under evident influence of these authors Lévai [31], in his search for 'shape-invariant' [30] rational potentials, extended to *complex* Jacobi polynomials the systematic method suggested by Bhattacharjie and Sudarshan [32] for constructing potentials exactly solvable via classical Jacobi polynomials. He found that the list of shape-invariant potentials compiled in [29] missed the aforementioned *t*-RM potential.

Though the fact that the eigenvalues of the $\mathfrak{I}$Ref CSLE coincide with one of real roots of a quartic equation was originally recognized by Grosche [33] it was Lévai [34] who proved that the average of the Jacobi indexes of the polynomial forming a 'quasi-rational' [35] eigenfunction is determined by a root of a similar quartic equation in the particular case of an even density function. It was originally overlooked by the author [7, 10] that Lévai (apparently unaware of Milson's paper [6]) implicitly discussed both '$\mathfrak{I}$Ref' and '$\mathfrak{R}$Ref' CSLEs[1)] with even density functions. In particular the quartic equation derived by us [7] for complex conjugated indexes of the Jacobi polynomials forming eigenfunctions of the '$\mathfrak{R}$Ref' CSLE happened to be another representation for the equation obtained by Lévai [34] in a more general context. In following the terminology suggested in our papers [7, 36, 37], we refer to the numerator of the rational density function in the RCSLEs of our interest as 'tangent polynomial' (TP) so we term the Liouville potentials associated with the even density function as the '*e*-TP' potential for briefness.

In [38] Lévai presented a more thorough analysis comparing his approach [31, 34, 39] and our original technique [5] used for constructing the general 1D Schrödinger equation exactly solvable via a superposition of hypergeometric functions. An examination of his arguments illuminated in next Section revealed that, while starting from the complex Jacobi equation, Lévai implicitly converts it to the CSLE initially introduced by us [5] under the real field, with the only (though very important for *his* analysis) difference that the new CSLE was expressed in terms of the variable $2z(x)–1$ instead of the variable $z(x)$ used in [5] to convert the 1D Schrödinger equation to the hypergeometric equation in z. Obviously the conversion of the CSLE by a linear transformation of the variable may not affect the results so there is no surprise that introducing the solved-by-polynomials equation [31] and then converting it to its canonical form leads to exactly the same results as the Bose technique [1, 2] directly starting from the resultant CSLE. Contrary to Lévai's assertion in [40] it is the new variable (not the difference in the approach) 'suits better the formulation of solvable $\mathcal{PT}$-symmetric potentials than that in [5]'. The author has already took advantage of this variable introduced in Lévai's earlier works [31, 39] while discussing the exactly solvable $\mathfrak{I}$Ref and $\mathfrak{R}$Ref problems in parallel in [19]. Regrettably this re-examination of our original study [5] (as well as its extension by Milson [6]) was disregarded in Lévai's papers [38, 40, 41] (see next Section for more details).

Though the primary focus of this paper is the real rational Sturm-Liouville problems solvable by polynomials we (under influence of Lévai's cited works [31, 39]) start our analysis in next Section from a complex (non-self-adjoint} RCSLE such that its two real-field reductions result in the self-adjoint RCSLEs with ℐRef and ℜRef Liouville potentials. It is worth stressing in this context that our persistent references to the Schrödinger equation with solvable rational potentials are made solely because the RCSLEs represented in their Liouville form have broad applications in quantum mechanics and are much better known to scientists. Also note that Everitt's [4] catalogue of Sturm-Liouville differential equations includes some renowned examples of the solvable ℐRef and ℒRef Liouville potentials while disregarding their generalizations presented in our paper [5] as well as the extension of our technique to the Fuschian RCSLE with two poles on the imaginary axis [6].

In Section 3 we present a unified approach to the Liouville potentials of the ℐRef and RRef CSLEs with even density functions by treating them as two real branches of the generally complex potential of 'Lévai class'. We term these two branches as Lévai*'s* and Milson's *e*-TP potentials. The new element of our analysis of Lévai*'s* *e*-TP potential, compared with [34], is the proof [36] that the (n+1)-th eigenfunction (specified by the label **c** in our notation) for a '*nearly-symmetric*' potentials is accompanied by a triplet of 'quasi-rational' [35] solutions (q-RSs) composed of Jacobi polynomials of the same degree n which belong to three different types **a**, **b**, and **d.** Since any solution vanishing at one of the quantization ends (types **a** and **b**) and lying below the ground-energy level are necessarily nodeless [42] the q-RSs of these two types can be used as the factorization functions (FFs) for the so-called [43] 'rational Darboux transformations' (RDTs) giving rise to new exactly solvable rational potentials [37]. We also point to some important details in both Milson's [6] and our [7, 10] papers which are absent in Lévai*'s* sketch of this little known problem exactly solvable in terms of R-Routh polynomials with degree-dependent indexes.

As it has been already asserted by Lévai [34] the symmetric reduction of the ℜRef potentials is nothing but an alternative representation for the Ginocchio potential on the line [44]. Though the assertion itself happened to be correct [45] the interrelationship between the two alternative parametrizations of the Ginocchio potential examined in Section 4 turned out to be much more complicated than one would expect based on Lévai's observation that the quartic equations for the averaged Jacobi indexes in both quantization schemes may be re-written in the unified fashion.

The most important consequence of the presented proof is that, in addition to the renowned quantization scheme by classical Gegenbauer polynomials [44], the cited symmetric potential can be alternatively quantized by R-Routh polynomials of a definite parity termed by us [7, 45] 'Masjed-Jamei polynomials' to give credit to the scrupulous analysis of these polynomials in [46]. In other words the RCSLEs with even ℐRef [44] and even ℜRef [7] Bose invariants constitute the same spectral problem unambiguously defined by the common Liouville form of its two rational realizations – the main result of this study proven in Section 4.

2. Quartic equation for the average of indexes of Jacobi polynomial forming a q-RS of complex Fuschian CSLE with three singular points

Let us start our analysis from the *complex* (non-self-adjoint) Fuschian CSLE with three singular points

$$\left\{\frac{d^2}{d\eta^2}+I[\eta;\bar{\Lambda};T_2;\varepsilon]\right\}\Phi[\eta;\bar{\Lambda};T_2;\varepsilon]=0 \tag{2.1}$$

representing its Bose invariant [1, 2, 6] as

$$I[\eta;\bar{\Lambda};T_K;\varepsilon]=I^o[\eta;\bar{\Lambda}]+\rho[\eta;T_K]\varepsilon \tag{2.2}$$

with both reference polynomial fraction (RefPF)

$$I^o[\eta;\bar{\Lambda}]=-\frac{h_{o;-}}{4(\eta+1)^2}-\frac{h_{o;+}}{4(1-\eta)^2}-\frac{O_0^o}{4(1-\eta^2)} \tag{2.3}$$

and the density function

$$\rho[\eta; T_K] = \frac{T_K[\eta]}{(1-\eta^2)^2} \quad (K \le 2) \tag{2.4}$$

having second-order poles at $\eta = \pm 1$. Note that, in contrast with $\mathfrak{I}$RefPF (2.3), Bose invariant (2.2) also depends on the coefficients of the TP of degree $K \le 2$:

$$T_K[\eta; a, c_\pm] = \frac{1}{2}[c_+(1+\eta) + c_-(1-\eta)] - a(1-\eta^2) \tag{2.5}$$

$$= \frac{1}{4}[c_+(1+\eta)^2 + c_-(1-\eta)^2 - d(1-\eta^2)], \tag{2.6}$$

where

$$d \equiv 4a - c_+ - c_- . \tag{2.7}$$

The real-field self-adjoint reduction of $\mathfrak{I}$Ref CSLE (2.1) was initially introduced by us in [7] to treat $\mathfrak{I}$Ref and $\mathfrak{R}$Ref CSLEs in parallel, to a large extent under influence of Lévai's renowned papers [31, 39]. Examination of (4) in [38] shows that

$$I[\eta; \bar{\Lambda}; T_K; \varepsilon] = R[\eta; \bar{\Lambda}; T_K; \varepsilon] - \frac{1}{2}\dot{Q}[\eta; \bar{\Lambda}; T_K; \varepsilon] - \frac{1}{4}Q[\eta; \bar{\Lambda}; T_K; \varepsilon] \tag{2.8}$$

while Lévai's second-order differential equation (2) can be re-written as

$$\left\{(1-\eta^2)\frac{d^2}{d\eta^2} + \tau[\eta; \bar{\Lambda}; T_K; \varepsilon]\frac{d}{d\eta} + \varepsilon\right\} F[\eta; \bar{\Lambda}; T_K; \varepsilon] = 0 \tag{2.9}$$

with

$$\tau[\eta; \bar{\Lambda}; T_K; \varepsilon] \equiv (1-\eta^2) Q[\eta; \bar{\Lambda}; T_K; \varepsilon] \tag{2.10}$$

and

$$(1-\eta^2) R[\eta; \bar{\Lambda}; T_K; \varepsilon] = \varepsilon . \tag{2.11}$$

Note that the parameters p_I, p_{II}, and p_{III} in (9) in [38] are related to the coefficients a, $c_\pm$ of TP (2.6) in the trivial fashion:

$$p_I \equiv -a, \; p_{II} \equiv \frac{1}{2}(c_+ + c_-), \; p_{III} \equiv \frac{1}{2}(c_+ - c_-) . \tag{2.12}$$

Lévai's starting formula (13) in [38], with

$$\phi[\eta] \equiv T_K[\eta] = (1-\eta^2)^2 \rho[\eta; a, c_\pm] \tag{2.13}$$

in our notation, is nothing but the conventional representation of the given Liouville potential re-written as

$$V[\eta; h_{o;\pm}, O_0^o; a, c_\pm] + \frac{1}{2}\{\eta, x\} = -\rho^{-1}[\eta; a, c_\pm] \, I^o[\eta; h_{o;\pm}, O_0^o; a, c_\pm], \tag{2.14}$$

where the Schwarzian derivative $\{\eta, x\}$ is expressed in terms of η (cf. (2.6) in [19]), provided that the change of variable $\eta(x; T_K)$ satisfies the ordinary differential equation (ODE)

$$\frac{dx}{d\eta} = \frac{1-\eta^2}{\sqrt{T_K[\eta]}} \tag{2.15}$$

with prime denoting the derivative with respect to x. The parameters s_I, s_{II}, and s_{III} in the mentioned formula (or similarly in (1.29) in [41]) are thus related to the parameters $\bar{\Lambda}$ of $\mathfrak{I}$RefPF (2.3) above as follows:

$$\begin{aligned} s_I &\equiv \frac{1}{4}(O_0^o - h_{o;-} - h_{o;+}), \\ s_{II} &\equiv \frac{1}{2}(h_{o;+} - h_{o;-}), \\ s_{III} &\equiv \frac{1}{2}(h_{o;-} + h_{o;+}) \end{aligned} \tag{2.16}$$

Disregarding the enhancements suggested by us in [19] Lévai simply compared his approach with our initial scheme [5] using the variable z(x) satisfying the ODE

$$\frac{dx}{dz} = \frac{2z(1-z)}{\sqrt{{}_1T_K[z]}}. \tag{2.17}$$

This transformation converts the Schrödinger equation into the $\mathcal{J}$Ref CSLE with the Bose invariant

$${}_1I[z; \bar{\Lambda}; {}_1T_K; \varepsilon] \equiv 4\, I[2z-1; \bar{\Lambda}; T_K; \varepsilon], \tag{2.18}$$

where ${}_1T_K$ are the coefficients of the TP

$${}_1T_K[z] \equiv T_K[2z-1]. \tag{2.19}$$

Though our initial technique [5] dealing with hypergeometric functions of the variable z varying between 0 and 1 is necessary to prove that the given potential is indeed exactly solvable (as well as to derive close-form expressions for the scattering amplitudes [47, 48]) the use of the variable η = 2z–1 allows one to treat $\mathcal{J}$Ref and $\mathcal{R}$Ref potentials in a symmetric fashion and also makes it easier to examine $\mathcal{PT}$-symmetric reductions of the complex $\mathcal{J}$RefPF potential as it has been done by Lévai [34, 38, 40, 41]. Obviously the conversion of the CSLE by a linear transformation of the variable may not affect the results so there is no surprise that starting from the solved-by-polynomials equation [31] and then converting it to the canonical form leads to exactly the same results as the Bose technique [1, 2] directly starting from the resultant CSLE. Lévai's assertion in [40] that the discussion of this problem in [38] 'revealed that his approach suits better the formulation of solvable $\mathcal{PT}$-symmetric potentials than that in [5]' is not precisely accurate – it is the change of variable

$$\eta(x; T_K) = 2z(x; {}_1T_K) - 1 \tag{2.20}$$

that suits better his analysis [38] compared with the variable $z(x; T_K)$. Namely, if the reflection $\eta \to -\eta$ keeps the TP unchanged – the case of our current interest – then one can choose $\eta(x; T_K)$ to be an odd function of x. If all the TP coefficients are required to be real then the Schwarzian derivative $\{\eta, x\}$ becomes an even real function of x and the $\mathcal{PT}$-transformation is equivalent to the complex conjugation of RefPF (2.3) followed by the reflection of the argument η.

As another novel development inspired by Lévai's works [31, 34, 38, 39] the author can point to our recent idea [9, 49, 50] to introduce complex (non-self-adjoint) CSLE (2.1) and then examine its so-called [36, 51, 52] 'almost-everywhere holomorphic' (AEH) solutions

$$\phi_{k,m}[\eta; \bar{\Lambda}; T_K] = \prod_{\aleph=\pm} (1+\aleph\eta)^{\rho_{\aleph;k,m}} \Pi_m[\eta; \bar{\eta}^{(m)}] \tag{2.21}$$

which exist at some energies $\varepsilon_{k,m}$ inside the vertical band $|Re\,\eta| < 1$ in the complex plane. It will be proven below that the monomial product

$$\Pi_m[\eta; \bar{\eta}^{(m)}] \equiv \prod_{l=1}^{m} (\eta - \eta_l^{(m)}) \tag{2.22}$$

coincides with the monic Jacobi polynomial with generally complex indexes[2)]

$$\Pi_m[\eta; \bar{\eta}^{(m)}] = \hat{P}_m^{(\lambda_{+;k,m}, \lambda_{-;k,m})}(\eta). \tag{2.23}$$

(Here we use the term 'AEH solutions', instead of the equivalent epithet 'q-RSs' appearing in the section title, to stress that we deal with complex functions analytically continued from the real axis into the complex plane.) We then took advantage of the fact that the so-called [53, 54] 'differential polynomial system' (DPS) composed of complex Jacobi polynomials [55] allows the second real-field reduction formed by Routh polynomials [14], in addition to the one formed by conventional (real) Jacobi polynomials.

To derive the necessary and sufficient conditions for CSLE (2.1) to have an AEH solution (2.21) first note that characteristic exponents (ChExps) of these solutions for the poles of CSLE (2.1) at ±1 satisfy the indicial equations

$$(2\rho_{\pm;k,m}-1)^2 = h_{o;\pm}+1-c_{\pm}\varepsilon_{k,m} \qquad (2.24)$$

Introducing the complex exponent differences (ExpDiffs)

$$\lambda_{\pm;k,m} \equiv 2\rho_{\pm;k,m}-1 \qquad (2.25)$$

we come to the equation

$$\lambda^2_{\pm;k,m} = h_{o;\pm}+1-c_{\pm}\varepsilon_{k,m} \qquad (2.26)$$

Re-writing RefPF (2.3) as

$$I^o[\eta;h_{o;\pm},\mu_o] = -\frac{h_{o;-}+h_{o;+}+(h_{o;+}-h_{o;-})\eta}{2(1-\eta^2)^2}+\frac{\mu_o^2-1}{4(1-\eta^2)}, \qquad (2.27)$$

where the parameter

$$\mu_o \equiv \sqrt{h_{o;+}+h_{o;-}-O_0^o+1} \qquad (2.28)$$

is chosen to coincide with the ExpDiff for the pole of the $\mathfrak{I}$Ref CSLE at infinity, and examining asymptotic behavior of AEH solutions near this pole we can write the closing equation for the given system of algebraic equations in $\lambda_{\pm;k,m}$ and $\varepsilon_{k,m}$ as follows

$$a\,\varepsilon_{k,m} = {}^1\!/_4\,\mu_o^2-\rho_{\infty;k,m}(\rho_{\infty;k,m}+1), \qquad (2.29)$$

where

$$\rho_{\infty;k,m} = -\omega_{k,m}-m-1 \qquad (2.30)$$

is the ChExp for the pole of CSLE (2.1) at infinity and

$$\omega_{k,m} \equiv {}^1\!/_2\,(\lambda_{-;k,m}+\lambda_{+;k,m}). \qquad (2.31)$$

It can be shown that the derived system of algebraic equations (2.26), (2.29)-(2.31) is simply another representation for coupled equations (14)-(16) in [34].

Squaring (2.31) gives

$$2\lambda_{+;k,m}\lambda_{-;k,m} = 4\omega^2_{k,m}-\lambda^2_{+;k,m}-\lambda^2_{-;k,m} \qquad (2.32)$$

while substituting (2.30) into (2.29) brings us to the following energy dispersion formula:

$$a\,\varepsilon_{k,m} = {}^1\!/_4(\mu_o^2+1)-(\omega_{k,m}+m+{}^1\!/_2)^2 \quad (K=2,\ a\neq 0), \qquad (2.33)$$

Re-writing (2.26) as

$$\lambda^2_{\pm;k,m} = \underset{\sim}{h}_{o;\pm}+1+c_{\pm}(\omega_{k,m}+m+{}^1\!/_2)^2/a, \qquad (2.34)$$

where

$$\underset{\sim}{h}_{o;\pm} \equiv h_{o;\pm}-{}^1\!/_4\,c_{\pm}\mu_o^2/a, \qquad (2.35)$$

squaring (2.32) and making use of (2.34), we come to the following quartic equation

$$\begin{aligned}&{}^1\!/_4\,a^2[4\omega^2_{k,m}-\underset{\sim}{h}_{o;+}-\underset{\sim}{h}_{o;-}-2-(c_++c_-)(\omega_{k,m}+m+{}^1\!/_2)^2/a]^2-\\&[a(\underset{\sim}{h}_{o;+}+1)-c_+(\omega_{k,m}+m+{}^1\!/_2)^2]\times[a(\underset{\sim}{h}_{o;-}+1)-c_-(\omega_{k,m}+m+{}^1\!/_2)^2]=0\end{aligned} \qquad (2.36)$$

in $\omega_{k,m}$. Note that the leading coefficient of quartic equation (2.36) coincides with the TP discriminant

$$\Delta_T(a,c_{\pm}) = {}^1\!/_4(c_+-c_-)^2-2a(c_++c_-)+4a^2. \qquad (2.37)$$

For each of *four* generally complex roots of this equation (or for each of three roots if the TP discriminant vanishes [56]) the corresponding pair of the ExpDiffs $\lambda_{\pm;k,m}$ coincides with two roots of the quadratic equation

$$\lambda^2_{\pm;k,m} - 2\omega_{k,m}\lambda_{\pm;k,m} + \tfrac{1}{2}\omega^2_{k,m} - \tfrac{1}{2}(\underset{\sim}{h}_{o;-} + \underset{\sim}{h}_{o;+}) - 1$$
$$- \tfrac{1}{2}(c_- + c_+)(\omega_{k,m} + m + \tfrac{1}{2})^2 / a = 0 \tag{2.38}$$

Expressing again $\lambda^2_{\pm;k,m}$ in terms of $\omega^2_{k,m}$ via (2.34) one finds

$$\lambda_{\mp;k,m} = \frac{\underset{\sim}{h}_{o;\mp} - \underset{\sim}{h}_{o;\pm} - (c_\pm - c_\mp)(\omega_{k,m} + m + \tfrac{1}{2})^2 / a_2}{\omega_{k,m}} - \omega_{k,m}. \tag{2.39}$$

Keeping in mind that that AEH solutions (2.21) obey the ODE

$$\ddot{\phi}_{k,m}[\eta; h_{o;\pm}, \mu_o; T_2] + I[\eta; h_{o;\pm}; T_2; \varepsilon_{k,m}]\phi_{k,m}[\eta; h_{o;\pm}, \mu_o; T_2] = 0 \tag{2.40}$$

with the Bose invariant

$$I[\eta; h_{o;\pm}; T_2; \varepsilon_{k,m}] = -\frac{h_{o;-} - c_-\,\varepsilon_{k,m}}{4(\eta+1)^2} - \frac{h_{o;+} - c_+\,\varepsilon_{k,m}}{4(1-\eta)^2}$$
$$+ \frac{h_{o;+} + h_{o;-} + 1 - \mu_o^2 + d\,\varepsilon_{k,m}}{4(\eta^2 - 1)} \tag{2.41}$$

and substituting (2.21) and (2.23) into (2.40) then brings us to the ODE

$$(1-\eta^2)\ddot{P}_m^{(\lambda_{+;k,m},\lambda_{-;k,m})}(\eta) -$$
$$2P_1^{(\lambda_{+;k,m},\lambda_{-;k,m})}(\eta)\dot{P}_m^{(\lambda_{+;k,m},\lambda_{-;k,m})}(\eta) +$$
$$m(m + \lambda_{+;k,m} + \lambda_{-;k,m} + 1)P_m^{(\lambda_{+;k,m},\lambda_{-;k,m})}(\eta) = 0 \tag{2.42}$$

if we set

$$2\rho_{+;k,m}\,\rho_{-;k,m} + \tfrac{1}{4}(h_{o;+} + h_{o;-} - \mu_o^2 + 1 + d\varepsilon_{k,m}) = -m(2\omega_{k,m} + m + 1) \tag{2.43}$$

One can easily verify the latter relation by substituting (2.7), (2.25), (2.26), and (2.33) into the left-hand side of (2.43).

We thus proved that q-RSs (2.21) have form

$$\phi_{k,m}[\eta; \Lambda; a, c_\pm] \propto \prod_{\aleph=\pm} (1+\aleph\eta)^{\rho_{\aleph;k,m}} P_m^{(\lambda_{+;k,m},\lambda_{-;k,m})}(\eta) \tag{2.44}$$

and thereby arrived to the starting point of Lévai's approach [34, 38, 40, 41] while moving in the opposite direction.

Setting

$$\delta \equiv -\tfrac{1}{2}(c_+ + c_-)/a, \tag{2.45}$$

$$\Sigma \equiv \tfrac{1}{2}(\underset{\sim}{h}_{o;+} + \underset{\sim}{h}_{o;-} + 2) - \delta - \tfrac{1}{4}, \tag{2.46}$$

and also taking into account that, as a direct consequence of (2.34) and (2.45),

$$\tfrac{1}{2}(\lambda^2_{+;k,m} + \lambda^2_{-;k,m}) = \tfrac{1}{2}(\underset{\sim}{h}_{o;+} + \underset{\sim}{h}_{o;-} + 2) - \delta(\omega_{k,m} + m + \tfrac{1}{2})^2 \tag{2.47}$$

we come to (10) in [34] with the Jacobi indexes α and β dependent on the polynomial degree; namely,

$$\delta(\omega_{k,m} + m + \tfrac{1}{2})^2 - \delta + \tfrac{1}{2}(\lambda^2_{+;k,m} + \lambda^2_{-;k,m}) - \tfrac{1}{4} = \Sigma \tag{2.48}$$

in our notation.

The simplification utilized in Lévai's aforementioned papers takes place in the particular case:

$$c_+ = c_- = \delta / C \tag{2.49}$$

when the parameter

$$\Lambda_{k,m} \equiv \tfrac{1}{4}(\lambda^2_{-;k,m} - \lambda^2_{+;k,m}) \tag{2.50}$$

becomes independent of the polynomial degree:

$$\Lambda_{k,m} \equiv \Lambda = \tfrac{1}{4}(h_{o;-} - h_{o;+}). \tag{2.51}$$

Subtracting one of equations (2.34) from another we come to the crucial relation [34]

$$\lambda^2_{-;k,m} - \lambda^2_{+;k,m} = h_{o;-} - h_{o;+} = 4\Lambda \tag{2.52}$$

which allows one to combine two separate quartic equations in $\lambda_{\pm;k,m}$ into a single quartic equation in $\omega_{k,m}$. Namely making use of (2.31) one finds

$$\lambda_{-;k,m} - \lambda_{+;k,m} = 2\Lambda / \omega_{k,m} \tag{2.53}$$

so [34]

$$\lambda_{\pm;k,n} = \omega_{k,n} \mp \Lambda / \omega_{k,n} \tag{2.54}$$

and

$$\tfrac{1}{2}(\lambda^2_{+;k,m} + \lambda^2_{-;k,m}) = \omega^2_{k,m} + \Lambda^2 / \omega^2_{k,m}. \tag{2.55}$$

We can thus re-write (2.47) as

$$\omega^2_{k,m} + \Lambda^2 / \omega^2_{k,m} = \tfrac{1}{2}(\underset{\sim}{h}_{o;+} + \underset{\sim}{h}_{o;-} + 2) - \delta(\omega_{k,m} + m + \tfrac{1}{2})^2 \tag{2.56}$$

Substituting (2.46) into the right-hand side of this equation gives

$$\delta(\omega_{k,m} + m + \tfrac{1}{2})^2 + \omega^2_{k,m} - \Sigma - \delta - \tfrac{1}{4} + \Lambda^2 / \omega^2_{k,m} = 0 \tag{2.57}$$

which brings us to quartic equation (13) in [34]:

$$(1+\delta)\omega^4_{k,m} + \delta(2m+1)\omega^3_{k,m} + [\delta(m + \tfrac{1}{2})^2 - \delta - \Xi - \tfrac{1}{4}]\omega^2_{k,m} + \Lambda^2 = 0 \tag{2.58}$$

The trivial real-field reduction of CSLE (2.1) obtained by choosing all six parameters $\bar{\Lambda}, T_K$ to be real leads to the very special representative of the family of the $\mathfrak{J}$Ref potentials [5, 36] referred to by us as "Lévai's *e*-TP potential". However as it has been noticed by Lévai himself [34] there is another family of real CSLEs discovered by Milson [6] and thereby referred to by us [45] as 'Milson's *e*-TP potential'. It is obtained by requiring the parameters $h_{o;+}$ and $h_{o;-}$ to be complex conjugated while keeping real the parameter O^o_0. Both RefPF and density function thus become real when expressed in terms of the new variable $\eta_I = -i\eta$. Lévai's formalism outlined above made it possible to treat both potentials in the unified matter outlined in next section.

3. Two real field reductions of complex $\mathfrak{J}$Ref SLE

The important common feature of the $\mathfrak{J}$Ref and $\mathfrak{R}$Ref potentials *on the line* is that the corresponding density functions

$$\rho[\eta; T_2] = \frac{T_2[\eta]}{(1-\eta^2)^2} \quad (T_2[\pm 1] \neq 0) \tag{3.1}$$

and

$${}_i\rho[\eta_I; {}_iT_2] = \frac{{}_iT_2[\eta_I]}{(1+\eta_I^2)^2} \tag{3.2}$$

have second-order poles at the endpoints ±1 and ±∞ accordingly. As a direct consequence of this observation the algebraic Schrödinger equation

$$\left\{\frac{d}{d\xi}\,{}_\iota\rho^{-1/2}[\xi;{}_\iota T_2]\frac{d}{d\xi}-\left({}_\iota V[\xi;{}_\iota\bar{\Lambda};{}_\iota T_2]-\varepsilon\right){}_\iota\rho^{1/2}[\xi;{}_\iota T_2]\right\}{}_\iota\Psi[\xi;{}_\iota\bar{\Lambda};{}_\iota T_2;\varepsilon]=0 \quad (3.3)$$

(with $\xi\equiv\eta$ or η_I for ι = ' ' or *i* respectively) has the so-called [37] "prime" form; namely, the sum of two ChExps equal to 0 and as a result the principal solution is unambiguously determined by the Dirichlet boundary conditions (DBCs). In [37] we have proved the cited below theorem which allows one to formulate the given spectral problem as the Dirichlet problem for the prime SLE.

Theorem 3.1. If the density function ${}_\iota\rho[\xi;{}_\iota T_2]$ of the given RCSLE has second-order poles at both endpoints then the corresponding Dirichlet problem formulated for the prime SLE is fully equivalent to the requirement that the eigenfunctions of RCSLE are square integrable with the weight ${}_\iota\rho[\xi;{}_\iota T_2]$.

One of the advantages of formulating the spectral problem as the Dirichlet problem for the prime SLE is that we can automatically adopt the rigorous theorems proven in [42] for SLEs solved under the DBCs. As explained in [37] the requirement for the density function to have *second-order* poles at the endpoints automatically assures that the corresponding Liouville transformation leads to the 1D Schroedinger equation on the line. Again Gesztesy, Simon, and Teschl's meticulous proofs made it possible to extend the conventional results of the regular Sturm-Liouville theory [57] to singular Sturm-Liouville problems including the 1D Schroedinger equation on the line.

It is worth stressing in this context that the theorem does not hold if the density function has a simple pole at one of the endpoints and thereby is invalid for the radial Schroedinger equation [37, 58].

In this paper we focus solely on the $\mathfrak{I}$Ref and $\mathfrak{R}$Ref CSLEs with the numerators of density functions (3.1) and (3.2) formed by the even second-degree polynomials

$$T_e[\eta;\delta,C]=C^{-1}(1+\delta-\eta^2)\quad(\delta/C>0) \qquad (3.4)$$

and

$$_i T_e[\eta_I;{}_i a,\kappa_+]={}_i a(\eta_I^2+\kappa_+)\quad({}_i a,\ \kappa_+\equiv 1+\delta_+>0). \qquad (3.5)$$

3.1. Lévai's real e-TP potential

As initially noticed in [59] the variable $\eta_{-;s}(x;\delta,C)$ obtained by solving the ODE

$$\eta'_{-;s}(x;\delta,C)=\rho_L^{-1/2}[\eta_{-;s}(x;\delta,C);\delta,C], \qquad (3.6)$$

under the boundary condition

$$\eta_{-;s}(0;\delta,C)=0 \qquad (3.7)$$

with

$$\rho_L[\eta;\delta,C]\equiv\frac{1+\delta-\eta^2}{C(1-\eta^2)^2} \qquad (3.8)$$

coincides with variable (3.2) in [44] if one sets

$$\delta=(\lambda^2-1)^{-1},\quad C=\delta\lambda^4 \qquad (3.9)$$

in Ginocchio's notation. Namely,

$$\eta_{-;s}[y;\lambda]=\frac{\lambda y}{\sqrt{Y[y^2;\lambda^2]}}, \qquad (3.10)$$

where the function $y(x;\lambda)$ is a solution of the ODE [60, 61]

$$y'(x,\lambda) = [1 - y^2(x,\lambda)]\,Y[y^2(x,\lambda);\lambda^2] \quad \text{for } -1 < y < +1 \tag{3.11}$$

and

$$Y[y^2;\lambda^2] \equiv 1 + (\lambda^2 - 1)\,y^2. \tag{3.12}$$

To prove Levai's assertion [59] one can follow Wu's pioneering arguments [60, 61] in support of the claim that the radial Ginocchio potential [62] is nothing but another representation of the implicit radial potential exactly solvable in terms of hypergeometric functions [47, 48]. First multiplying the square of (3.10) by its reverse

$$y^2 = \frac{\eta_{-;s}^2}{N[\eta_{-;s}^2;\lambda^2]}, \tag{3.13}$$

where

$$N[\eta^2;\lambda^2] \equiv \lambda^2 + (1-\lambda^2)\eta^2, \tag{3.14}$$

gives

$$N[\eta^2;\lambda^2] = \frac{\lambda^2}{Y[y^2;\lambda^2]}. \tag{3.15}$$

Setting

$$\eta_{-;s}(x;\lambda) \equiv \eta_{-;s}[y(x,\lambda);\lambda] = \eta_{-;s}(x;\delta,\delta\lambda^4) \tag{3.16}$$

with $\delta = (\lambda^2 - 1)^{-1}$ and also taking into account that

$$N[\eta^2;1+1/\delta] = (1+\delta-\eta^2)/\delta \tag{3.17}$$

we can re-write (3.6) as

$$\eta'_{-;s}(x;\lambda) = \frac{\lambda^2[1-\eta_{-;s}^2(x;\lambda)]}{\sqrt{N[\eta_{-;s}^2(x;\lambda);\lambda^2]}}. \tag{3.18}$$

or, which is equivalent,

$$\eta'_{-;s}(x;\lambda) = [1 - y^2(x;\lambda)]\sqrt{N[\eta_{-;s}^2(x;\lambda);\lambda^2]}. \tag{3.19}$$

On other hand, we come to the same relation multiplying (3.11) by the derivative

$$\frac{d\eta_{-;s}}{dy} = \lambda^{-2} N^{3/2}[\eta_{-;s}^2;\lambda^2] \tag{3.20}$$

and making use of (3.15).

Algebraic Schrödinger equation (3.3) thus takes form

$$\left\{\frac{d}{d\eta}\rho_L^{-1/2}[\eta;\delta,C]\frac{d}{d\eta} - \rho_L^{1/2}[\eta;\delta,C]\left(V_L[\eta;\bar{\Lambda};\delta,C] - \varepsilon\right)\right\}\Psi_L[\eta;\bar{\Lambda};\delta,C;\varepsilon] = 0 \tag{3.21}$$

where $V_L[\eta_{-;s};\bar{\Lambda};C,\delta]$ is the Liouville potential converted back to the variable $\eta_{-;s}(x;\delta,C)$, i.e.,

$$V_L[\eta_{-;s}(x;\lambda);\bar{\Lambda};\delta,\delta\lambda^4] = -[\eta'_{-;s}(x;\lambda)]^2 I^0[\eta_{-;s}(x;\lambda);\bar{\Lambda}] - \tfrac{1}{2}\{\eta_{-;s}(x;\lambda),x\}, \tag{3.22}$$

where again $\delta = (\lambda^2 - 1)^{-1}$. It has been shown in [45] that the Schwarzian derivative expressed in terms of the variable $\eta_{-;s}$ can be written as

$$\{\eta_{-;s},x\} = -\frac{2C}{1+\delta-\eta_{-;s}^2} + \frac{C(1-\eta_{-;s}^2)^2}{2(1+\delta-\eta_{-;s}^2)^2}\left\{1 - \frac{4}{1-\eta_{-;s}^2} + \frac{5(\delta+1)}{1+\delta-\eta_{-;s}^2}\right\}. \tag{3.23}$$

Substituting (2.3), (3.6), and (3.23) into potential function (3.22) expressed in terms of the variable $\eta_{-;s}$ one finds

$$V_L[\eta;\bar{\Lambda};\delta,C]/C = \frac{h_{o;-}(1-\eta)^2 + h_{o;+}(1+\eta)^2 + O_0^o(1-\eta^2)+4}{4(1+\delta-\eta^2)} +$$
$$\frac{1-\eta^2}{(1+\delta-\eta^2)^2} - \frac{(1-\eta^2)^2}{4(1+\delta-\eta^2)^2}\left\{1+\frac{5(\delta+1)}{1+\delta-\eta^2}\right\} \quad (3.24)$$

or alternatively

$$V_L[\eta;\bar{\Lambda};\delta,C]/C = \tfrac{1}{4}(\mu_o^2-2) + \frac{\Xi}{1+\delta-\eta^2} +$$
$$\frac{3\delta(3\delta+2)}{4(1+\delta-\eta^2)^2} - \frac{5\delta^2(\delta+1)}{4(1+\delta-\eta^2)^3} + \frac{(h_{o;+}-h_{o;-})\eta}{2(1+\delta-\eta^2)}, \quad (3.25)$$

where

$$\Sigma \equiv \tfrac{1}{2}(h_{o;+}+h_{o;-}+2) + \tfrac{1}{4}[\delta(\mu_o^2-4)-1] \quad (3.26)$$

is simply another representation for parameter (2.46) taking into account (2.35).

It is essential that solutions of prime SLE (3.21) are related to solutions of the real ℑRef CSLE with density function (3.8) via the simple gauge transformation

$$\Psi_L[\eta;\bar{\Lambda};\delta,C;\varepsilon] \propto \rho_L^{1/4}[\eta;\delta,C]\,\Phi_L[\eta;\bar{\Lambda};\delta,C;\varepsilon] \quad (3.27)$$

and therefore the eigenfunctions of the prime SLE and the eigenfunctions of the ℑRef CSLE,

$$\phi_{\mathbf{c}n}[\eta;\Xi,\Lambda;\delta,C] = \prod_{\aleph=\pm}(1+\aleph\eta)^{\frac{1}{2}\lambda_{\aleph;\mathbf{c}n}} P_n^{(\lambda_{+;\mathbf{c}n},\lambda_{-;\mathbf{c}n})}(\eta), \quad (3.28)$$

are interrelated via the elementary formula

$$\psi_{\mathbf{c}n}[\eta;\Xi,\Lambda;\delta,C] \propto \rho_L^{1/4}[\eta;C,\delta]\,\phi_{\mathbf{c}n}[\eta;\Xi,\Lambda;\delta,C]. \quad (3.29)$$

Examination of the asymptotic behavior of eigenfunctions (3.28) near poles of the ℑRef CSLE at ±1 reveals that solutions (3.29) obey the DBCs

$$\lim_{\eta\to\pm1}\psi_{\mathbf{c}n}[\eta;\Xi,\Lambda;\delta,C;\varepsilon]=0 \quad (3.30)$$

iff

$$\lambda_{\pm;\mathbf{c}n} > 0. \quad (3.31)$$

One can then directly verify that constraints (3.31) are the necessary and sufficient conditions for q-RSs (3.28) to be square integrable with non-negative weight (3.8):

$$\int_{-1}^{+1}\phi_{\mathbf{c}n}^2[\eta;\Xi,\Lambda;\delta,C]\rho_L[\eta;\delta,C]d\eta < \infty, \quad (3.32)$$

as prescribed by Theorem 3.1. It directly follows from definition (2.31) of the sought-for roots, coupled with constraints (3.31), that the root $\omega_{\mathbf{c}n}$ associated with (n+1)-th eigenfunction must be positive.

Examination of asymptotic behavior of potential (3.24) near the singular endpoints shows that

$$V_L[\pm1;\bar{\Lambda};\delta,C]/C = (h_{o;\pm}+1)/\delta. \quad (3.33)$$

We choose the energy reference point from the requirement that the potential vanishes at η = +1 setting $h_{o;+}=-1$. We also require that the potential takes a nonnegative value at the lower endpoint by choosing

$$h_{o;-} + 1 \equiv \lambda_o^2 \equiv 4\Lambda \geq 0\,, \tag{3.34}$$

where the parameter λ_o coincides with the zero-energy ExpDiff for the pole of ℱRef CSLE (2.1) at η = –1. Taking into account (2.28) we can thus represent ℱRef PF (2.3) as

$$I^o[\eta;\lambda_o,\mu_o] = \frac{1-\lambda_o^2}{4(\eta+1)^2} + \frac{1}{4(1-\eta)^2} + \frac{\mu_o^2 + 1 - \lambda_o^2}{4(1-\eta^2)}\,. \tag{3.35}$$

With such a choice of the energy reference point two positive roots $\lambda_{\pm;\mathbf{c}n}$ of quadratic equation (2.38) specifying the (n+1)-th eigenfunction satisfy the inequality

$$\lambda_{-;\mathbf{c}n} \geq \lambda_{+;\mathbf{c}n} > 0 \tag{3.36}$$

so the sought-for roots take form

$$\lambda_{\pm;\mathbf{c}n} = \omega_{\mathbf{c}n} \mp \Lambda / \omega_{\mathbf{c}n}\,. \tag{3.37}$$

Comparing (3.37) with (12) in [34] we conclude that Lévai's definition of the *e*-TP potential differs from ours by a reflection of the argument, i.e., $z = -\eta_{-;s}$ and $\alpha = \lambda_{-;\mathbf{c}n}$, $\beta = \lambda_{+;\mathbf{c}n}$ in our terms.

In the limiting case of the Ginocchio potential on the line $(\lambda_o = \Lambda = 0)$ quartic equation (2.58) takes form:

$$\breve{\omega}^2_{\mathbf{\dagger}n}[\delta(\breve{\omega}_{\mathbf{\dagger}n} + n + {}^1\!/_2)^2 + \breve{\omega}^2_{\mathbf{\dagger}n} - \Sigma - \delta - {}^1\!/_4] = 0 \tag{3.38}$$

with the double zero root

$$\breve{\omega}_{\mathbf{\dagger}\_n} = 0 \tag{3.39}$$

and two other roots determined by the quadratic equation

$$\breve{\omega}^2_{\mathbf{\dagger}+n} + \delta(\breve{\omega}_{\mathbf{\dagger}+n} + n + {}^1\!/_2)^2 - \delta - \Xi - {}^1\!/_4 = 0\,. \tag{3.40}$$

The average of the indexes of the Jacobi polynomials in the right-hand side of (3.28) thus coincides with the positive root $\breve{\omega}_{\mathbf{c}n}$ of quadratic equation (3.40). Since the TP may not have zeros between -1 and +1 the parameter δ must be either positive or smaller than -1, i. e., the linear coefficient of the quadratic equation

$$\breve{\omega}^2_{\mathbf{\dagger}+n} + \frac{\delta}{\delta+1}(n + {}^1\!/_2)\breve{\omega}_{\mathbf{\dagger}+n} + [\delta(n + {}^1\!/_2)^2 - \delta - \Xi - {}^1\!/_4] / (\delta+1) = 0 \tag{3.41}$$

is necessarily positive and therefore the quadratic equation has a positive root iff its free term is negative, i.e., iff

$$(n + {}^1\!/_2)^2 < (\Xi + {}^1\!/_4) / \delta + 1 \tag{3.42}$$

in agreement with (23) in [34]. Setting $\Lambda = 0$ in (3.37) confirms that

$$\breve{\lambda}_{+;\mathbf{c}n}(\Xi;\delta) = \breve{\lambda}_{-;\mathbf{c}n}(\Xi;\delta) = \breve{\omega}_{\mathbf{c}n} \tag{3.43}$$

as expected. The negative root $\breve{\omega}_{\mathbf{d}n}$ of quadratic equation (3.41) specifies the q-RS composed of the Jacobi polynomial with two equal negative indexes:

$$\breve{\lambda}_{+;\mathbf{d}n}(\Xi;\delta) = \breve{\lambda}_{-;\mathbf{d}n}(\Xi;\delta) = \breve{\omega}_{\mathbf{d}n} < 0. \tag{3.44}$$

Examination of quartic equation (2.58) reveals that

$$\lim_{\Lambda\to 0} (\Lambda / \breve{\omega}_{\mathbf{\dagger}\_n}) = \sqrt{\delta + \Xi + {}^1\!/_4 - \delta(n + {}^1\!/_2)^2} \tag{3.45}$$

and therefore

$$\breve{\lambda}_{\pm;\mathbf{\dagger}\_n}(\Xi;\delta) \equiv \lim_{\Lambda\to 0} \lambda_{\pm;\mathbf{\dagger}\_n}(\Xi,\Lambda;\delta) = \mp\sqrt{\delta + \Xi + {}^1\!/_4 - \delta(n + {}^1\!/_2)^2} \tag{3.46}$$

so

$$\breve{\lambda}_{+;\breve{\mathbf{t}}\_n}(\Xi;\delta)\breve{\lambda}_{-;\breve{\mathbf{t}}\_n}(\Xi;\delta)<0 \quad (3.47)$$

as far as condition (3.42) holds and therefore both Jacobi indexes are real. We thus conclude that

$$\breve{\mathbf{t}}\_ = \mathbf{a} \text{ or } \mathbf{b}. \quad (3.48)$$

We thus confirmed that the (n+1)-th eigenfunction is indeed accompanied by a triplet of q-RSs **a**n, **b**n, and **d**n, in agreement with our more general results [36] for asymptotically levelled (AL) potentials $(\lambda_o = 0,\ c_+ \neq c_-)$.

As the asymmetry parameter $\Lambda > 0$ increases the *e*-TP ℑRef potential has at least n+1 bound energy levels as far as the positive root of quartic equation (2.58) remains larger than $\sqrt{\Lambda}$ and thereby both Jacobi indexes stay positive, as prescribed by (3.36). Indeed, the latter equation may not have a positive double root larger than $\sqrt{\Lambda}$ because this would imply existence of two eigenfunctions with exactly the same number of nodes inside the quantization interval. Therefore two positive roots may not merge giving rise to a complex conjugated pair as far as $\breve{\omega}^2_{\mathbf{c}n} > \Lambda$. The (n+1)-th bound energy level may disappear iff Λ reaches the positive root of the quadratic equation

$$(\delta+2)\Lambda+(2n+1)\sqrt{\Lambda}+(n+1/2)^2-(\Xi+1/4)/\delta-1=0 \quad (3.49)$$

in $\sqrt{\Lambda}$.

3.2. Milson's e-TP potential

As stressed above choosing all the parameters of RefPF (2.3) to be real and making the Liouville transformation on the real axis [1, 2] brings us directly to the conventional ℑRef potential [5]. However as mentioned in previous section there is another less obvious option to formulate the real Sturm-Liouville problem utilized by Milson [6]. Namely keeping real only O^o_0 and a_2 while allowing the parameters in each pair $h_{o;\pm}$ and $c_\pm$ to be complex and also restricting CSLE (2.1) to the imaginary axis $\eta = i\eta_I$ we come to the real CSLE [6]:

$$\left\{\frac{d^2}{d\eta_I^2}+I[\eta_I;h_o,{}_iO^o_0;{}_i\mathcal{T}_K]+{}_i\rho[\eta_I;{}_i\mathcal{T}_K]\tilde{\varepsilon}\right\}\Phi[\eta_I;h_o,{}_iO^o_0;{}_i\mathcal{T}_K;\tilde{\varepsilon}]=0 \quad (3.50)$$

with the Routh-reference (ℜRef) PF

$${}_iI^o[\eta_I;h_o,{}_iO^o_0]=\frac{h_o}{4(1+i\eta_I)^2}+\frac{h_o^*}{4(1-i\eta_I)^2}+\frac{{}_iO^o_0}{4(1+\eta_I^2)} \quad (3.51)$$

and the *positive* density function

$${}_i\rho_M[\eta_I;{}_i\mathcal{T}_K]=\frac{{}_i T_K[\eta_I]}{(1+\eta_I^2)^2} \quad (K=0 \text{ or } 2) \quad (3.52)$$

(assuming the TP has a positive leading coefficient and a negative discriminant ${}_i\Delta_T$ for K = 2).

The conventional Liouville transformation of CSLE (3.50) then leads to the *real* ℜRef potential in our classification scheme. Though the full credit for the discovery of the latter potential should be certainty given to Milson [6] two particular cases of RCSLE (3.50) with a constant TP (K = 0) and the second-degree TP

$${}_iT_2[\eta_I]={}_ia(1+\eta_I^2) \quad (3.53)$$

associated with the translationally shape invariant (TSI) ℜRef potentials (t-RM and Gendenshtein potentials respectively) have already implicitly appeared in Lévai's pioneering paper [31] which originally drew author's attention to this extraneous class of rational potentials solvable in terms of Jacobi polynomials with complex conjugated indexes and imaginary argument.

Let us now discuss the energy spectrum of ℜRef CSLE (3.50). Since the corresponding Liouville potential is asymptotically levelled we can choose the energy reference point via the requirement that that the potential vanishes at both quantization endpoints ±x. This can be achieved by setting [7, 10]

$$ {}_{i}O_0^o = 2h_{o;R} + 1 \tag{3.54}$$

where $h_{o;R}$ is the real part of the complex parameter

$$ h_{o;-} \equiv h_o \equiv h_{o;R} + i h_{o;I}. \tag{3.55}$$

In other words we make parameter (2.28) equal to zero so $\underset{\sim}{h}_{o;\pm} \equiv h_{o;\pm}$ and ℜRef CSLE (3.50) takes form [7, 10]

$$ \left\{ \frac{d^2}{d\eta_{+;s}^2} + {}_{i}I^o[\eta_{+;s}; h_o] + {}_{i}\rho_M[\eta_{+;s}; \kappa_+]\underset{\sim}{\varepsilon} \right\} {}_{i}\Phi[\eta_{+;s}; h_o; \kappa; \underset{\sim}{\varepsilon}] = 0, \tag{3.56}$$

where

$$ {}_{i}I^o[\eta_{+;s}; h_{o;R} + i h_{o;I}] = \frac{h_{o;R} + h_{o;I}\eta_{+;s}}{(\eta_{+;s}^2 + 1)^2} + \frac{1}{4(\eta_{+;s}^2 + 1)}, \tag{3.57}$$

$\underset{\sim}{\varepsilon} \equiv -\varepsilon$, and

$$ \rho_M[\eta_{+;s}; \kappa_+, a_+] = \frac{a_+(\eta_{+;s}^2 + \kappa_+)}{(\eta_{+;s}^2 + 1)^2} \quad (a_+ > 0). \tag{3.58}$$

(The density function with a constant TP, giving rise to the translationally form-invariant (TFI) CSLE of Group B [63] with the trigonometric Liouville potential [27], requires a special consideration.) In Lévai's notation [34]

$$ \kappa_+ - 1 = \delta_+ > -1, \; a_+ = C_+^{-1} > 0, \tag{3.59}$$

and

$$ h_{0;R} - \delta_+ + 3/4 \equiv \Sigma_+ \tag{3.60}$$

while parameter (2.51) becomes imaginary:

$$ \Lambda_+ \equiv 1/4(\lambda_{-;k,m}^2 - \lambda_{+;k,m}^2) = 1/2\, i h_{o;I}\, i \tag{3.61}$$

The change of variable defined by the ODE

$$ \eta'_{+;s}(x; \kappa_+, a_+) = \rho_M^{-1/2}[\eta_{+;s}(x; \kappa_+, a_+); \kappa_+, a_+] \tag{3.62}$$

solved under the boundary condition

$$ \eta_{+;s}(0; \kappa_+, a_+) = 0 \tag{3.63}$$

converts CSLE (3.56) into the Schrödinger equation with Milson's [6] potential

$$ V_M(x; h_o; \kappa_+, a_+) \equiv V_M[\eta_{+;s}(x; \kappa_+, a_+); h_o; \kappa_+, a_+] $$
$$ = (\eta'_{+;s})^2\, {}_{i}I^o[\eta_{+;s}(x; \kappa_+, a_+); h_o] - 1/2\{\eta_{+;s}, x\}, \tag{3.64}$$

where [45]

$$ \{\eta_{+;s}, x\} = \frac{2}{a_+(\eta_{+;s}^2 + \kappa_+)} + \frac{2(1+\eta_{+;s}^2)}{a_+(\eta_{+;s}^2 + \kappa_+)^2} - \tag{3.65}$$

$$\frac{(1+\eta_{+;s}^2)^2}{2a_+(\eta_{+;s}^2+\kappa_+)^2}\left(1+\frac{5\kappa_+}{\eta_{+;s}^2+\kappa_+}\right).$$

Setting

$$_i\omega_{k,n} \equiv Re\lambda_{\pm;k,n} \tag{3.66}$$

and again taking into account that parameter (3.61) is independent of the polynomial degree we come to the quartic equation

$$(1+\delta_+)\,_i\omega_{k,n}^4+\delta_+(2n+1)\,_i\omega_{k,n}^3+[\delta_+(n+1/2)^2-\delta_+-\Xi_+-1/4]\,_i\omega_{k,n}^2$$
$$+\Lambda_+^2=0 \tag{3.67}$$

while combining (2.54) with (3.61) gives

$$\lambda_{\pm;k,n}=\,_i\omega_{k,n}\mp i\frac{h_{o;I}}{2\,_i\omega_{k,n}}. \tag{3.68}$$

Though quartic equation (3.67) seems identical to (2.58) there is a catch: the roots specifying the eigenvalues of the corresponding real CSLEs with density functions (3.8) and (3.52) have opposite signs. Namely while $\omega_{\mathbf{c}n}$ is positive its counter-part $_i\omega_{\mathbf{c}n}$ specifying the eigenvalue of the Schrödinger equation with Milson's *e*-TP potential is restricted by the negative upper bound specified by (3.80) below. Though Lévai's assertion that the symmetric $\Re$Ref potential coincides with the Ginocchio potential on the line [44] is formally correct [45] its proof relies on rather sophisticated arguments thoroughly illuminated in next Section.

The quartic equation similar to (3.67) was independently derived by us in [19] under influence of Avarez-Castillo and Kirchbach's breakthrough study [8] on the quantization of the Gendenshtein ('Scarf II) potential [30] in terms of Romanovski polynomials [18]. The author realized that the corresponding Schrödinger equation can be obtained by the Liouville transformation of $\Re$Ref CSLE (3.50) with the density function

$$_i\rho_M[\eta_I;1]=\frac{1}{\eta_I^2+1}. \tag{3.69}$$

By searching for eigenfunctions of the given CSLE in the 'quasi-rational' [35] form

$$_i\phi_{\mathbf{c}n}[\eta_I;h_o;\kappa_+]\propto(\eta_I^2+1)^{_i\omega_{\mathbf{c}n}+1}\left(\frac{1-i\eta_I}{1+i\eta_I}\right)^{\lambda_{\mathbf{c}n;I}}\Pi_n[\eta_I;h_o;\kappa_+] \tag{3.70}$$

and examining the ExpDiff for the pole of CSLE (3.50) at infinity we found that n-degree polynomials in the right-hand side satisfy the Routh equation [14]

$$\left[(\eta_I^2+1)\frac{d^2}{d\eta_I^2}+2[(_i\omega_{\mathbf{c}n}+1)\eta_I-\lambda_{\mathbf{c}n;I}]\frac{d}{d\eta_I}-n(n+2\omega_{\mathbf{c}n}-1)\right]\times$$
$$\Re_n^{(_i\omega_{\mathbf{c}n}+i\lambda_{\mathbf{c}n;I})}[\eta_I]=0 \tag{3.71}$$

such that its polynomial solutions $\Re_n^{(_i\lambda_{\mathbf{c}n})}[\eta_I]$ after being re-written in the monic form coincide with the pseudo-Jacobi polynomials

$$\hat{\Re}_n^{(_i\lambda_{\mathbf{c}n})}[\eta]=P_n(\eta;-\lambda_{\mathbf{c}n;I},-_i\omega_{\mathbf{c}n}-1) \tag{3.72}$$

defined via (9.9.1) in [11]. It was found that that the eigenvalues of $\Re$Ref CSLE (3.50) are unambiguously determined by the real part $_i\omega_{\mathbf{c}n}$ of the complex index $_i\lambda_{\mathbf{c}n}$ of the Routh polynomial:

$$_i\varepsilon_{\mathbf{c}n}(h_o;\kappa_+)=-_i\wp_{\infty;\mathbf{c}n}^2<0, \tag{3.73}$$

with

$$ {}_i\wp_{\infty;\mathbf{c}n} = -{}_i\omega_{\mathbf{c}n} - n - \tfrac{1}{2} > 0 \tag{3.74} $$

while examination of the ExpDiffs for the poles of CSLE (3.50) brought us to the algebraic equation

$$ {}_i\lambda^2_{\mathbf{c}n} = h_{o;R} + 1 + i\,h_{o;I} + (\kappa_+ - 1)\,{}_i\varepsilon_{\mathbf{c}n}(h_o;\kappa_+) \tag{3.75} $$

similar to (2.26) except that the coefficients c_+ and c_- are real and coincide with each other. It then directly follows from (3.75) that

$$ Im\,{}_i\lambda^2_{\mathbf{c}n} = 2\,{}_i\omega_{\mathbf{c}n}\,\lambda_{\mathbf{c}n;I} = h_{o;I} \tag{3.76} $$

bringing us back to Lévai's formula (3.68).

Substituting (3.74) and (3.75) into the right-hand side of (3.76) and eliminating $\lambda_{\mathbf{c}n;I}$ from the resultant formula

$$ {}_i\omega^2_{\mathbf{c}n} - \lambda^2_{\mathbf{c}n;I} = h_{o;R} + 1 + (1-\kappa_+)({}_i\omega_{\mathbf{c}n} + n + \tfrac{1}{2})^2 \tag{3.77} $$

we [19] came to the quartic equation

$$ {}_i\omega^2_{\mathbf{c}n}[{}_i\omega^2_{\mathbf{c}n} - h_{o;R} - 1 + (\kappa_+ - 1)({}_i\omega_{\mathbf{c}n} + n + \tfrac{1}{2})^2] - \tfrac{1}{4}h^2_{o;I} = 0 \tag{3.78} $$

which is nothing but an alternative form of (3.67) with $\kappa_+ = 1 + \delta_+$ and Levai's parameters Ξ_+ and Λ_+ defined via (3.60) and (3.61) accordingly.

By requiring eigenfunctions (3.70) to be square integrable with weight (3.58):

$$ \int_{-\infty}^{+\infty} {}_i\phi^2_{\mathbf{c}n}[\eta_I;h_o;\kappa_+]\,{}_i\rho_M[\eta_I;\kappa_+]d\eta_I < \infty \tag{3.79} $$

we come to the constraint

$$ {}_i\wp_{\infty;\mathbf{c}n} = -{}_i\omega_{\mathbf{c}n} - n - \tfrac{1}{2} > 0 \tag{3.80} $$

which implies that the real part of the Routh index ${}_i\lambda_{\mathbf{c}n}$ must be restricted by the upper bound

$$ {}_i\omega_{\mathbf{c}n} < -n - \tfrac{1}{2}\,. \tag{3.81} $$

We define R-Routh polynomials (Romanovski\pseudo-Jacobi polynomials in Leski's terms [15, 16]) via (3.5) and (3.6) in [64]:

$$ R_n^{(2\alpha_I,\alpha_R+1)}(x) = (-i)^n\,P_n^{(\alpha_R+i\alpha_I,\alpha_R-i\alpha_I)}(ix) \equiv \Re_n^{(\alpha_R+i\alpha_I)}(x) \tag{3.82} $$

for

$$ n \le N_0 \equiv \left\lfloor -\alpha_R - \tfrac{1}{2} \right\rfloor. \tag{3.83} $$

Note we use the term 'pseudo-Jacobi polynomials' as a synonym for Routh polynomials and do not require for the parameter N_0 [12] to be a nonnegative integer so our use of this term is similar to the definition of these polynomials in [65]. The requirement for quartic equation

$$ ({}_i\wp_{\infty;\mathbf{c}n} + n + \tfrac{1}{2})^2[({}_i\wp_{\infty;\mathbf{c}n} + n + \tfrac{1}{2})^2 - h_{o;R} - 1 + (\kappa_+ - 1)\,{}_i\wp^2_{\infty;\mathbf{c}n}] - \tfrac{1}{4}h^2_{o;I} = 0 \tag{3.84} $$

to have an odd number of negative roots [7] results in Milson's formula for the number n_o of discrete energy levels [6]

$$n_o = \left\lfloor \sqrt{\tfrac{1}{2}(h_{o;R} + 1 + |h_o + 1|)} - \tfrac{1}{2} \right\rfloor. \tag{3.85}$$

Finally let us point to the fact that each eigenfunction is accompanied by another q-RS associated with the second real-root of the quartic equation. If the Routh polynomial forming the second q-RS has no real roots then the q-RS in question can be used as the FF for the RDT giving rise to a new solvable rational potential [10] (as originally suggested by Quesne [64] for the Scarf II potential).

It is also worth mentioning that our argumentation in [7] was restricted solely to the real (self-adjoint) CSLE and within this approach we were unable to prove that the real part of sought-for complex ExpDiff ${}_i\lambda_{cn}$ coincides with a negative real root of a quartic equation in the general case of the density function [9]

$${}_i\rho[\eta; \kappa_+ + i\,\kappa_I] = \frac{\eta^2 + \kappa_I\,\eta + \kappa_+}{(\eta^2+1)^2} \tag{3.86}$$

with $|\kappa_I| > 0$ assuming that the TP has a negative discriminant

$$\Delta_T = \tfrac{1}{4}\kappa_I^2 - \kappa_+ < 0. \tag{3.87}$$

To cover the general case [9] we (under the influence of Levai's works [31, 34, 38, 39]) had to start from complex (non-self-adjoint) CSLE (2.1) and then consider the second real-field reduction of Bochner's [55] complex Jacobi DPS to the DPS formed by Routh polynomials [14].

4. Ginocchio potential as an overlap of Lévai's and Milson's e-TP potentials

The main purpose of this Section is to prove that Lévai's [34] and Milson's [6] *e*-TP reductions of the $\mathfrak{I}$Ref and $\mathfrak{R}$Ref potentials defined via (3.22) and (3.64) overlap along the symmetric potential curves

$$V_L[\eta_{-;s}(x;\delta,C); 0, \mu_o; \delta, C] = V_M[\eta_{+;s}(x;\kappa_+, a_+); h_{o;R}; \kappa_+, a_+] \tag{4.1}$$

obtained by setting corresponding asymmetry parameters (3.34) and (3.61) to zero:

$$\Lambda = \Lambda_+ = 0\,. \tag{4.2}$$

(Here and below the $\mathfrak{I}$RefPF in the right-hand side of (3.22) is parametrized by the parameters λ_o and μ_o according to (3.35).) As explained below this assertion can be considered as the corollary of the following theorem:

Theorem 4.1. The RCSLEs with even $\mathfrak{I}$Ref and even $\mathfrak{R}$Ref Bose invariants constitute two rational realizations of the same Sturm-Liouville problem interrelated via an algebraic change of variable.

Proof of Theorem 4.1.

Let us first prove that the solutions of ODEs (3.6) and (3.62) under the boundary conditions (3.7) and (3.63) respectively are interrelated via the following algebraic formulas:

$$\eta_{+;s} = \frac{\eta_{-;s}}{\sqrt{1-\eta_{-;s}^2}} \tag{4.3}$$

or

$$\eta_{-;s} = \frac{\eta_{+;s}}{\sqrt{\eta_{+;s}^2+1}} \tag{4.4}$$

if we choose

$$1-\kappa_+ = (\kappa_- + 1)^{-1} \tag{4.5}$$

and

$$a_+ = a_-(\kappa_- + 1) \tag{4.6}$$

or, in Lévai's notation,

$$\delta_+\delta_- = 1 \tag{4.7}$$

and

$$C_{-} = \delta_{-} C_{+}, \tag{4.8}$$

so

$$\delta_{\pm} + 1 \equiv \pm\kappa_{\pm} \tag{4.9}$$

and

$$a_{\pm} = \pm C_{\pm}^{-1} \tag{4.10}$$

with $\delta_{-} \equiv \delta$, $C_{-} \equiv C$ in the notation of subsection 3.1.

If we , following [45], introduce two non-negative variables

$$z_{\pm} \equiv \eta_{\pm;s}^{2} \tag{4.11}$$

satisfying the ODEs

$$z'_{\pm}(x;\kappa_{\pm},a_{\pm}) = \frac{2\sqrt{z_{\pm}}(1 \pm z_{\pm})}{\sqrt{a_{\pm}(z_{\pm}+\kappa_{\pm})}} \tag{4.12}$$

then algebraic relation (4.3) is equivalent to the linear fractional transformation

$$z_{+} = \frac{z_{-}}{1-z_{-}} \tag{4.13}$$

assuming that the parameters $\kappa_{\pm}$, $a_{\pm}$ are chosen as specified above.

To prove (4.13) let us introduce the auxiliary PF

$$\zeta_{-}[z_{-}] \equiv \frac{z_{-}}{1-z_{-}}. \tag{4.14}$$

and demonstrate that the function

$$\zeta_{-}(x;\kappa_{-},a_{-}) \equiv \zeta_{-}[z_{-}(x;\kappa_{-},a_{-})] \tag{4.15}$$

satisfies ODE (4.12) for the variable $z_{+}(x;\kappa_{+},a_{+})$ with

$$\zeta_{-}(0;\kappa_{-},a_{-}) = z_{\pm}(0;\kappa_{+},a_{+}) = 0. \tag{4.16}$$

Indeed differentiating (4.15) with respect to x and making use of (4.12) with the lower subscript one finds

$$\zeta'_{-}(x;\kappa_{-},a_{-}) = 2[\zeta_{-}(x;\kappa_{-},a_{-})+1]\sqrt{\frac{z_{-}(x;\kappa_{-},a_{-})}{a[z_{-}(x;\kappa_{-},a_{-})+\kappa_{-}]}}, \tag{4.17}$$

where we also took into account that

$$\zeta_{-}[z_{-}]+1 = \frac{1}{1-z_{-}}. \tag{4.18}$$

Under constraints (4.5) and (4.6) the denominator of the PF in the right-hand side of (4.17) can be re-written as follows

$$a_{-}[z_{-}(x;\kappa_{-},a_{-})+\kappa_{-}] = a_{+}[1-z_{-}(x;\kappa_{-},a_{-})]\times[\zeta_{-}(x;\kappa_{-},a_{-})+\kappa_{+}] \tag{4.19}$$

which gives

$$\zeta'_{-}(x;\kappa_{-},a_{-}) = 2[\zeta_{-}(x;\kappa,a)+1]\sqrt{\frac{\zeta_{-}(x;\kappa_{-},a_{-})}{a_{+}[\zeta(x;\kappa_{-},a_{-})+\kappa_{+}]}}. \tag{4.20}$$

This confirms that

$$z_{+}(x;\kappa_{+},a_{+}) \equiv \zeta_{-}(x;\kappa_{-},a_{-}) \tag{4.21}$$

as asserted.

Our next step is to prove that that the parameters μ_{o} and $h_{o;R}$ can be interrelated in such a way that

$$I^{o}[\eta_{-;s};0,\mu_{o}]=\left(\frac{d\eta_{+;s}}{d\eta_{-;s}}\right)^{2}{}_{i}I^{o}[\eta_{+;s};h_{o;R}]+\tfrac{1}{2}\{\eta_{+;s},\eta_{-;s}\}\quad, \tag{4.22}$$

provided that the coefficients of *e*-TPs (3.4) and (3.5) obey constraints (4.5)-(4.8). As proven above the variables $z_{\pm}$ are related via linear fractional transformation (4.13) and therefore [66]

$$\{z_{+},z_{-}\}=0\,. \tag{4.23}$$

This implies that interrelation formula (4.22) is equivalent to the requirement that the RefPFs

$${}_{1}I^{o}_{-}[z_{-};\mu_{o}]=\left(\frac{d\eta_{-;s}}{d\,z_{-}}\right)^{2}\left(I^{o}[\sqrt{z_{-}};0,\mu_{o}]-\tfrac{1}{2}\{\eta^{2}_{-;s},\eta_{-;s}\}\right) \tag{4.24}$$

and

$${}_{1}I^{o}_{+}[z_{+};h_{o;R}]\equiv\left(\frac{d\eta_{+;s}}{d\,z_{+}}\right)^{2}\left({}_{i}I^{o}[\sqrt{z_{+}};h_{o;R}]-\tfrac{1}{2}\{\eta^{2}_{+;s},\eta_{+;s}\}\right)\quad, \tag{4.25}$$

where

$$\{\eta^{2},\eta\}=-\tfrac{3}{2}\eta^{-2}, \tag{4.26}$$

are interrelated as follows

$${}_{1}I^{o}_{-}[z_{-};\mu_{o}]\equiv\left(\frac{dz_{+}}{dz_{-}}\right)^{2}{}_{1}I^{o}_{+}[z_{+};h_{o;R}]\,. \tag{4.27}$$

Substituting (3.35) with $\lambda_{o}=0$ and (3.57) with $h_{o;I}=0$, into the right-hand sides of (4.24) and (4.25) accordingly, we can represent the RefPFs in question as

$${}_{1}I^{o}_{-}[z_{-};\mu_{o}]=\frac{3}{16z_{-}^{2}}+\frac{1}{4z_{-}(1-z_{-})^{2}}+\frac{\mu_{o}^{2}-1}{16z_{-}(1-z_{-})} \tag{4.28}$$

and

$${}_{1}I^{o}_{+}[z_{+};h_{o;R}]=\frac{3}{16z_{+}^{2}}+\frac{h_{o;R}}{4z_{+}(z_{+}+1)^{2}}+\frac{1}{16z_{+}(z_{+}+1)}\,. \tag{4.29}$$

Expressing the right-hand side of (4.28) in terms of the variable z_{-} we can re-write (4.27) as

$$(1-z_{-})^{4}\,{}_{1}I^{o}_{-}[z_{-};\mu_{o}]=\frac{1}{16z_{+}^{2}(z_{+}+1)^{2}}\left[3+z_{+}(\mu_{o}^{2}+3)+4z_{+}^{2}\right]. \tag{4.30}$$

and thereby confirm that the right-hand side of the latter formula turns into (4.29) if we choose [45]

$$h_{o;R}+1=\tfrac{1}{4}\mu_{o}^{2} \tag{4.31}$$

which concludes the proof. □

It seems beneficiary to also re-formulate Theorem 4.1 as the following proposition

Corollary 4.1. The Liouville transformations of the RCSLEs with even $\mathfrak{I}$Ref and even $\mathfrak{R}$Ref Bose invariants on the intervals (-1, +1) and (-∞, +∞) accordingly results in the Schrödinger equation with exactly the same potential symmetric under the reflection of its argument x.

which constitutes the main result of this paper.

As illuminated in detail in [45] the Liouville transformation of the CSLE

$$\left\{\frac{d^2}{dz_-^2}+{}_1I^o_-[z_-;\mu_o]-\frac{(z_- -\delta_- -1)\varepsilon}{4C_-z_-(1-z_-)^2}\right\}{}_1\Phi_-[z_-;\mu_o;\delta_-,C_-;\varepsilon]=0$$

(4.32)

on the finite interval (0, 1) converts it to the Schrödinger equation with the non-singular radial ∮Ref potential. By reflecting the latter potential around the origin the author [5] originally constructed the symmetric potential later re-discovered by Ginocchio [44] in the form:

$$V_G[y]=-(1-y^2)\left\{\lambda^2\nu(\nu+1)-\tfrac{1}{4}(1-\lambda^2)[5(1-\lambda^2)y^4-(7-\lambda^2)y^2+2]\right\}$$

(4.33)

where the variable y is related to the variable z_- as follows

$$z_- = \frac{\lambda^2 y^2}{1+(\lambda^2-1)y^2}. \tag{4.34}$$

It was Wu [60] who recognized the equivalence of two representations as soon as Ginocchio presented his results.

Later Lévai [59] pointed to the fact that the Ginocchio potential on the line turns into the symmetric ∮Ref potential associated with *e*-TP (3.4) if we choose

$$\mu_o = 2\nu+1,\quad \delta_- = (\lambda^2-1)^{-1},\quad C_- = \delta_-\lambda^4 \tag{4.35}$$

and then express (4.33) in terms of variable (3.10). Setting $h_{o;+} = h_{o;-} = -1$ in (3.26) and also making use of constraint (2.49) to replace $\lambda_{\pm;k,m}$ in (2.26) for

$$\lambda_{-;\mathbf{c}n} = \lambda_{+;\mathbf{c}n} \equiv \breve{\omega}_{\mathbf{c}n}. \tag{4.36}$$

thus gives

$$\Sigma_- + \delta_- + \tfrac{1}{4} = \tfrac{1}{4}\delta_-\mu_o^2 \tag{4.37}$$

and

$$\breve{\varepsilon}_{\mathbf{c}n} = -\lambda^4\breve{\omega}^2_{\mathbf{c}n} \tag{4.38}$$

respectively. As expected energy dispersion formula (4.38) and quadratic equation

$$(\lambda^2-1)\breve{\omega}^2_{\mathbf{c}n} + (\breve{\omega}_{\mathbf{c}n}+n+\tfrac{1}{2})^2 - \tfrac{1}{4}\mu_o^2 = 0 \tag{4.39}$$

match respectively (3.9) and (3.10) in [44], with μ standing for $\breve{\omega}_{\mathbf{c}n}$ in our notation. Taking into account that both leading and linear coefficients of the quadratic equation are positive the latter may have a positive root

$$\breve{\omega}_{\mathbf{c}n} = \lambda^{-2}\left[\sqrt{\tfrac{1}{4}\lambda^2\mu_o^2-(\lambda^2-1)(n+\tfrac{1}{2})^2} - n - \tfrac{1}{2}\right] > 0 \tag{4.40}$$

iff its free term is negative, i.e., iff

$$0 \le n < \tfrac{1}{2}(\mu_o - 1). \tag{4.41}$$

On other hand substituting (4.31) into (3.60) shows that

$$\Sigma_+ + \delta_+ + \tfrac{1}{4} = \tfrac{1}{4}\mu_o^2 \tag{4.42}$$

so the quadratic polynomial in the brackets in quartic equation (3.78) takes form;

$${}_i\breve{\omega}^2_{\mathbf{c}n} + (\lambda^2-1)({}_i\breve{\omega}_{\mathbf{c}n}+n+\tfrac{1}{2})^2 - \tfrac{1}{4}\mu_o^2 = 0 \quad ({}_i\breve{\omega}_{\mathbf{c}n} < -n-\tfrac{1}{2}).$$

(4.43)

Setting side by side quadratic equations (4.39) and (4.43) we conclude that the positive root of the former equation specifies ChExp (3.74) of the (n+1)-th eigenfunction for the pole of CSLE (3.56) at infinity:

$$i\breve{\wp}_{\infty;\mathbf{c}n} = \breve{\omega}_{\mathbf{c}n} = -i\breve{\omega}_{\mathbf{c}n} - n - 1/2 > 0 \tag{4.44}$$

provided that

$$0 \le n < \sqrt{h_o + 1} - 1/2, \tag{4.45}$$

as prescribed by (3.85) with $h_o = h_{o;R}$.

Finally comparing (4.42) with (4.37) we find that Lévai's parameters (3.26) and (3.60) are related via a non-trivial formula

$$\Sigma_+ = \delta_+\left(\Sigma_- - 3/4\right) + 3 \tag{4.46}$$

which confirms our assertion that, despite the *formal similarity* between quartic equations (2.58) and (3.67), the two equations are related in a rather complicated fashion in the symmetric limit $\Lambda = \Lambda_\pm = 0$.

As already pointed to in Introduction the fact that the symmetric reduction of the $\Re$Ref potential leads to a certain subclass of the $\Im$Ref potentials has been already recognized by Milson [6] who cited in this connection the quadratic transformation of the hypergeometric function using the substitution

$$1/2(1 - i\eta_{+;s}) \to \eta_{+;s}^2 + 1 \tag{4.47}$$

which is reminiscent of (4.11) for the variable z_+. However the author was unable to figure out all the details necessary for the suggested (possibly alternative) representation of the Schrödinger equation with the given symmetric potential.

As originally discovered by Ginocchio [44] the eigenfunctions of the Schrödinger equation with potential (4.33) can be expressed in terms of classical Gegenbauer polynomials with degree-dependent indexes after being converted to the variable $\eta_{-,s}$:

$$\breve{\phi}_{\mathbf{c}n}[\eta_{-,s};\mu_o;\lambda] = (1-\eta_{-;s}^2)^{1/2(\breve{\omega}_{\mathbf{c}n}+1)} C_n^{(\breve{\omega}_{\mathbf{c}n}+1/2)}(\eta_{-,s}) \tag{4.48}$$

On other hand, as demonstrated in [7, 45] the eigenfunctions of $\Re$Ref CSLE (3.56) with real h_o are expressible in terms of orthogonal Masjed-Jamei polynomials [46]

$$i\breve{\phi}_{\mathbf{c}n}[\eta_{+;s};\mu_o;\lambda] \propto (\eta_{+;s}^2 + 1)^{1/2\, i\breve{\omega}_{\mathbf{c}n} + 1/2} I_n^{(p_n)}(\eta_{+;s}) \tag{4.49}$$

with degree-dependent indexes $p_n = 1/2 - i\breve{\omega}_{\mathbf{c}n}$ larger than n+1.

5. Example: *sech*-squared potential

Lévai's [34] and Milson's [6] *e*-TP potentials for $a_- = 0$ $(a_-\kappa_- = 1)$ and $\kappa_+ = a_+ = 1$ turn into the Rosen-Morse [28] and respectively Gendenshtein [30] TSI potentials:

$$V_{RM}[\tanh(2x);\lambda_o,\mu_o] = -1/4(\mu_o^2 - 1)\,\mathrm{sech}^2(2x) + 1/2\lambda_o^2[1 - \tanh(2x)] \tag{5.1}$$

and

$$V_G[\sinh(2x); h_{o;R} + i h_{o;I}] = -\mathrm{sech}^2(2x)[h_{o;R} + 3/4 + h_{o;I}\sinh(2x)] \tag{5.2}$$

where

$$V_{RM}[\eta_{-,s};\lambda_o,\mu_o] = -1/2(1-\eta_{-,s})[1/2(\mu_o^2 - 1)(1+\eta_{-,s}) - \lambda_o^2] \tag{5.3}$$

and

$$V_G[\eta_{+,s}; h_{o;R} + i h_{o;I}] = -\frac{4h_{o;R} + 3 + 4h_{o;I}\,\eta_{+,s}}{4(\eta_{+,s}^2 + 1)} . \tag{5.4}$$

Our choice of the TP coefficients $a_{-}=0$, $\kappa_{-}=+\infty$ and $\kappa_{+}=a_{+}=1$ assures that conditions (4.5) and (4.6) automatically hold and as a result the variables

$$z_{-}(x)=tanh^{2}(2x) \tag{5.5}$$

and

$$z_{+}(x)=sinh^{2}(2x) \tag{5.6}$$

are related via linear fractional formula (4.13).

The corresponding prime SLEs take form:

$$\left\{(1-\eta^{2})\frac{d}{d\eta}(1-\eta^{2})\frac{d}{d\eta}-V_{RM}[\eta;\lambda_{o},\mu_{o}]-\varepsilon\right\}\Psi[\eta;\lambda_{o},\mu_{o};\varepsilon]=0 \tag{5.7}$$

and

$$\left\{(1+\eta^{2})\frac{d}{d\eta}\sqrt{1+\eta^{2}}\frac{d}{d\eta}-V_{G}[\eta;h_{o}]-\underset{\sim}{\varepsilon}\right\}{}_{i}\Psi[\eta;h_{o};\underset{\sim}{\varepsilon}]=0 \tag{5.8}$$

which are solved under the DBCs at ±1 and ±∞ respectively. The representation of the RCLEs of our interest in the prime form [37] makes the ChExps of two Frobenius solutions differ only by sign and therefore the DBC automatically selects the principal solution.

Setting $\kappa_{+}=1$ in (3.78) brings us to the following quadratic equation

$${}_{i}\omega^{2}_{\dagger m}({}_{i}\omega^{2}_{\dagger m}-h_{o;R})-{}^{1}\!/_{4}\,h^{2}_{o;I}=0 \tag{5.9}$$

in ${}_{i}\omega^{2}_{\dagger m}$, where we changed $\mathbf{c}n$ for $\dagger m$ to indicate that the given equation holds for the averaged indexes of the Jacobi polynomials forming not only the eigenfunctions of CSLE (3.56), but also any other q-RSs. The crucial feature of this quadratic equation is that its coefficients are independent of the polynomial degree and therefore, as a direct consequence of (3.68), this is also true for the complex-conjugated pairs of the Jacobi indexes in question. So, in contrast with RM potential (5.1), Gendenshtein potential (5.2) belongs to Group A (not B!) in Odake and Sasaki's [67] classification scheme of the rational TSI potentials. This is the direct consequence of our observation [51] that the ExpDiffs for the poles of SLE (5.8) at ±∞ are energy independent so the TFI CSLE under consideration belongs to Group A [63]. Since all the q-RSs in this case are specified by a single series of Maya diagrams [63] all the solvable rational Darboux-Crum transform of SLE (5.8) with a complex parameter h_{o} can be obtained using admissible Wronskians of Routh polynomials [68] with the common complex index. It is convenient to choose the sequence which starts from a finite orthogonal set of R-Routh polynomials forming n_{o} eigenfunctions of SLE (5.8). As a result [67, 68] the corresponding Wronskian transforms of R-Routh polynomials form finite sequences of exceptional orthogonal polynomials (EOSs) in Quesne's terms [64].

The symmetric reductions $\lambda_{o}=0$ and $h_{o;I}=0$ of potentials (5.1) and (5.2) represent exactly the same *sech*-squared potential if the parameters μ_{o} and $h_{o;R}$ are related via (4.31). This potential can be thus quantized either via classical Gegenbauer polynomials with *degree-dependent* indexes [44] or via Masjed-Jamei polynomials [46] with *degree-independent* indexes [45]. As pointed to in [63] the A or B grouping suggested by Odake and Sasaki [67] for rational TSI potentials is actually an attribute of the particular rational realization of the given potential rather than the potential itself. The h-PT and Morse potentials represent two other instances of such an atypical dualism [63, 69].

The latter quantization scheme using R-Routh polynomials of a definite parity is obviously preferable for constructing the symmetric Darboux-Crum transforms of the *sech*-squared potential [67]. Since it is the symmetric limit of the Gendenshtein potential the q-RSs of a definite parity in this limiting case are also specified by a single series of Maya diagrams and therefore all the rational

Darboux-Crum transform (RDC$\mathfrak{T}$) of SLE (5.8) with a real parameter h_o can be obtained using admissible Wronskians of Routh polynomials of a definite parity with the common real index such that the first n_o polynomials in the given infinite polynomial sequence constitute a finite orthogonal set of Masjed-Jamei polynomials. As a result the corresponding Wronskian transforms of Masjed-Jamei polynomials must form finite sequences of EOPs. A detailed study on this remarkable family of finite EOP sequences is currently under way.

6. Discussion

The presented analysis scrutinizes the striking resemblance between quartic equations (2.58) and (3.67) suggested by Lévai [34] for computing the energy spectrum of real Fuschian CSLEs (2.1) and (3.50) with two poles located symmetrically around the origin on either real or imaginary axis. In both case even density functions (3.8) and accordingly (3.52) remain positive within the corresponding quantization intervals so the change of variable $\eta_{-;s}(x;\delta,C)$ or subsequently $\eta_{+;s}(x;\delta_+,C_+)$ converting the given CSLE into the Schrödinger equation is an odd real function of x. It has been shown that the two variables are interrelated via simple algebraic formula (4.3) or its reverse (4.4).

It was also proven that the corresponding Liouville potentials referred to in the paper as Lévai's [34] and Milson's [6] *e*-TP potentials overlap along the symmetric curves which are nothing but two alternative rational representations of Ginocchio's [44] potential function (4.33).

Since the Liouville transformation converts $\mathfrak{R}$Ref CSLE (3.50) with real h_o into the Schrödinger equation with the Ginocchio potential on the line the eigenfunctions of the latter equations can be expressed in terms of Masjed-Jamei polynomials [46] with degree-dependent index [45]. In the TSI limit represented by sech-squared potential the index of the Masjed-Jamei polynomials forming the corresponding eigenfunctions becomes degree-independent as a result the Wronskians of the eigenfunctions turn into weighted Wronskians of the *orthogonal* Masjed-Jamei polynomials with the common index. This finite polynomial sequence starts an infinite sequence of Routh polynomials of a definite parity with the same *real* index. If the Wronskian formed by a subset of the latter polynomials does not have real roots than the corresponding Wronskian transforms of the Masjed-Jamei polynomials form a finite EOP sequence. The EOP sequences constructed in such a way represent the symmetric reduction of the EOP sequences generated by us in [68] for Gendenshtein potential (5.2) using the admissible Wronskians of Routh polynomials with a common complex index.

7. Conclusions

Though the main purpose of the paper was to reveal some remarkable overlapping features of two real-field reductions of the complex (non-self-adjoint) $\mathfrak{J}$Ref CSLE with an even density function the starting point for our analysis has a much broader area of application. Namely the density function does not have to be even – we only need to require that it remains positive inside the finite real interval or on the imaginary axis. The Liouville transformation on these intervals results in two families of complex Liouville potentials referred to by us as complex potentials of Lévai class.

As suggested by Lévai [34] one can significantly simplify the analysis of the resultant Schrödinger equation by choosing the even density function. Such a choice of the density function assures that the variable $\eta(x)$ used to convert the CSLE to Schrödinger equation in x is an odd function of x. The direct consequence of this constraint is that the corresponding Liouville potential is $\mathcal{PT}$-symmetric if this is true for the $\mathfrak{J}$RefPF in question.

The important advantage of the suggested complexification of the $\mathfrak{J}$Ref CSLE comes from the fact that it has a quartet of AEH solutions which can be used as FFs for RDTs whether the Liouville transformation is done on the real or imaginary axis. In particular it will be shown in a separate publication that Lévai's $\mathcal{PT}$-symmetric potential [34, 38, 40] obtained via the Liouville transformation of the $\mathcal{PT}$-symmetric $\mathfrak{J}$Ref CSLE on the real axis has AEH solutions composed of R-Routh polynomials in an imaginary argument $i\eta_I$. Since these functions may not have zeros on the real axis $-\infty < \eta_I(x) < +\infty$ each of them can be used as the FF to construct a new solvable $\mathcal{PT}$-symmetric potential. The sketched complexification of the $\mathfrak{I}$Ref CSLE thus provides a new mechanism for constructing $\mathcal{PT}$-symmetric potentials with real energy spectra specified by positive roots of quartic equations (2.58).

The presented analysis of the AEH solutions of the complex $\mathfrak{I}$Ref CSLE with the even density function thus opens new horizons for Lévai's unified approach [34] to two families of generally complex Liouville potentials obtained via the Liouville transformations of the mentioned CSLE either inside the interval (-1, +1) or on the imaginary axis.

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Notes

[1] We refer to the RCSLEs in question as '$\mathfrak{I}$Ref' and '$\mathfrak{R}$Ref' to stress that have 'quasi-rational' [35] solutions (q-RSs) expressible in terms of Jacobi and Routh polynomials accordingly and use the same epithets for the corresponding Liouville potentials.

[2] It would be more accurate to use the notation $\rho_{\pm;k,m}(\bar{\Lambda};T_K)$, $\lambda_{\pm;k,m}(\bar{\Lambda};T_K)$, and $\varepsilon_{k,m}(\bar{\Lambda};T_K)$ but we disregard the dependence of the latter quantities on the parameters $\bar{\Lambda}$ and T_K for briefness.

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