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The Quadratic Equation for the Quaternions: *The Closed Form Solution*

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Abstract: In this paper we shall arrive at a simple Closed Form Solution that resolves the two roots of following equations over the quaternions: $\vec{f} = \vec{x}^2 + \vec{b}\,\vec{x} + \vec{x}\,\vec{a} \Rightarrow -\vec{c} = \vec{x}^2 + \vec{b}\,\vec{x} + \vec{x}\,\vec{a} + \vec{a}\vec{b} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t} - \vec{v}^{\,2}$, such that; $\vec{t} = \vec{x} + \vec{u}$; $\vec{u} = \frac{1}{2}(\vec{a} + \vec{b})$; $\vec{v} = \frac{1}{2}(\vec{a} - \vec{b})$; $\vec{f} = \vec{w}^2 + \vec{w}\vec{y} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t} - \vec{v}^{\,2}$; $\vec{w} = \vec{t} + \vec{v}$; $\vec{y} = -2\vec{v}$; left-handed form. $\vec{f} = \vec{w}^2 + \vec{z}\vec{w} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t} - \vec{v}^{\,2}$; $\vec{w} = \vec{t} - \vec{v}$; $\vec{z} = +2\vec{v}$; right-handed form. $\vec{f} = \vec{y}\vec{z}\vec{y} + \vec{r}\,\vec{y} + \vec{y}\,\vec{s} \Rightarrow \vec{z}\,\vec{f} = \vec{z}(\vec{y}\vec{z}\vec{y}) + \vec{z}(\vec{r}\,\vec{y}) + \vec{z}\,(\vec{y}\,\vec{s}) = \vec{x}^2 + \vec{b}\,\vec{x} + \vec{x}\,\vec{a}$, such that: $\vec{x} = \vec{z}\vec{y}$; $\vec{b} = \vec{z}\vec{r}\ \frac{1}{\vec{z}}$; $\vec{a} = \vec{s}$; $\vec{y} = \frac{1}{\vec{z}}\vec{x}$. The entire argument centers around the expression $-\vec{t}\vec{v} + \vec{v}\vec{t}$, which would ordinarily cancel out to zero over the reals and complex numbers; however, for the quaternions, the expression $-\vec{t}\vec{v} + \vec{v}\vec{t}$ produces a vector that is orthogonal to both $\vec{t}$ and $\vec{v}$. In order to ensure to the referees that this is not a waste of their time, a calculator is provided below for $\vec{f} = \vec{x}^2 + \vec{b}\,\vec{x} + \vec{x}\,\vec{a}$. Enter the $\vec{f}$ vector in cells T2:W2; $\vec{a}$ in cells J2:M2 and $\vec{b}$ in cells O2:R2. The two roots appear in cells D2:G2 and D3:G3.

Keywords: 11Rxx Algebraic number theory; 11R11 Quadratic extensions; 11R16 Cubic and quartic extensions

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Theorem 1 The Quadratic Equation , The General Depressed Case

Quadratic Quaternionic Calculator, Closed Form Solution
https://docs.google.com/spreadsheets/d/1X8sKNNuxFq5HLq5qk-93SDVdk6yeh14Xp_OV3bVl2ZI/edit?usp=sharing

Let $\vec{f} = \vec{x}^2 + \vec{x}\,\vec{b} + \vec{a}\,\vec{x}$

$-\vec{c} = (\vec{x} + \vec{a}\,)(\vec{x} + \vec{b}\,) = \vec{x}^2 + \vec{x}\,\vec{b} + \vec{a}\,\vec{x} + \vec{a}\,\vec{b}; \quad -\vec{c} = \vec{f} + \vec{a}\vec{b}$

We know that there exists a two dimensional basis in the four dimensional space of quaternions that describes vectors $\vec{a}$ and $\vec{b}$. Namely the bisector of the roots (mistakenly known as the Axis of Symmetry) and the straight line that is between the two roots and the bisector.

We shall define the bisector as $\vec{u} = \frac{1}{2}(\vec{a} + \vec{b}\,)$ as the first vector of the basis.

We now define the locator as $\vec{v} = \frac{1}{2}(\vec{a} - \vec{b}\,)$

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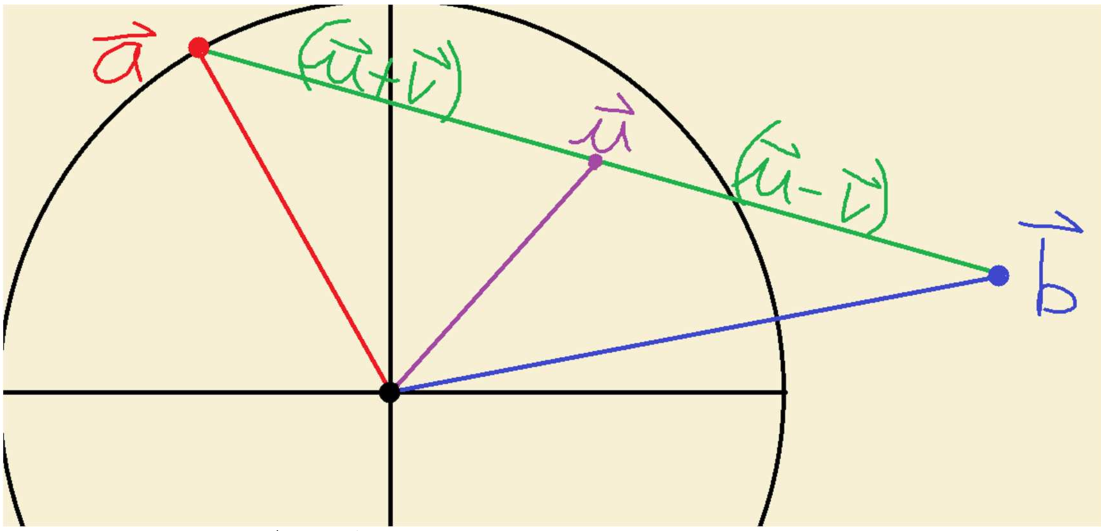

EQ.1 $\vec{u} = \frac{1}{2}(\vec{a} + \vec{b})$

EQ.2 $\vec{v} = \frac{1}{2}(\vec{a} - \vec{b})$, compelling $\vec{a} = \vec{u} + \vec{v}$ and $\vec{b} = \vec{u} - \vec{v}$

EQ.3 Let $\vec{t} = \vec{x} + \vec{u}$, therefore $\vec{x} = \vec{t} - \vec{u}$

EQ.4a $-\vec{c} = \vec{x}^2 + \vec{x}\,\vec{b} + \vec{a}\,\vec{x} + \vec{a}\vec{b}$

EQ.5 $-\vec{c} = \vec{x}^2 + \vec{x}(\vec{u} - \vec{v}) + (\vec{u} + \vec{v})\vec{x} + (\vec{u} + \vec{v})(\vec{u} - \vec{v})$

EQ.6 $-\vec{c} = (\vec{t} - \vec{u})^2 + (\vec{t} - \vec{u})(\vec{u} - \vec{v}) + (\vec{u} + \vec{v})(\vec{t} - \vec{u}) + (\vec{u} + \vec{v})(\vec{u} - \vec{v})$

EQ.7 $-\vec{c} = (\vec{t}^2 - \vec{t}\vec{u} - \vec{u}\vec{t} + \vec{u}^{\,2}) + (\vec{t}\vec{u} - \vec{t}\vec{v} - \vec{u}^2 + \vec{u}\vec{v}) + (\vec{u}\vec{t} - \vec{u}^2 + \vec{v}\vec{t} - \vec{v}\vec{u}) + (\vec{u}^2 - \vec{u}\vec{v} + \vec{v}\vec{u} - \vec{v}^2)$

EQ.8a $-\vec{c} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t} - \vec{v}^2 = (\vec{t} + \vec{v})(\vec{t} - \vec{v})$

EQ.9a $-\vec{c} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t} - \vec{v}^2$

EQ.9b $\vec{d} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t}$; $\vec{d} = -\vec{c} + \vec{v}^2$.

This is the fundamental middle handed form, from which we derive the solution.

EQ.10a Let $\vec{w} = \vec{t} + \vec{v}$, such that $\vec{t} = \vec{w} - \vec{v}$

EQ.11a $-\vec{c} = (\vec{w} - \vec{v})^2 - (\vec{w} - \vec{v})\vec{v} + \vec{v}(\vec{w} - \vec{v}) - \vec{v}^2$

EQ.12a $-\vec{c} = (\vec{w}^2 - \vec{w}\vec{v} - \vec{v}\vec{w} + \vec{v}^2) - (\vec{w}\vec{v} - \vec{v}^2) + (\vec{v}\vec{w} - \vec{v}^2) - \vec{v}^2$

EQ.13a $-\vec{c} = \vec{w}^2 - \vec{w}\vec{v} - \vec{v}\vec{w} + \vec{v}^2 - \vec{w}\vec{v} + \vec{v}^2 + \vec{v}\vec{w} - \vec{v}^2 - \vec{v}^2$

EQ.14a $-\vec{c} = \vec{w}^2 - 2\vec{w}\vec{v}$

EQ.15a $-\vec{c} = \vec{w}^2 - 2\vec{w}\vec{v} = \vec{w}(\vec{w} - 2\vec{v})$

EQ.16a Let $\vec{y} = -2\vec{v}$

EQ.17a $-\vec{c} = \vec{w}^2 + \vec{w}\vec{y} = \vec{w}(\vec{w} + \vec{y})$. This is the second fundamental form, the left-handed form.

EQ.9b $-\vec{c} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t} - \vec{v}^2$ (a restatement of EQ 9a)

EQ.10b Let $\vec{z} = \vec{t} - \vec{v}$, such that $\vec{t} = \vec{z} + \vec{v}$, thence:

$\vec{w} = \vec{z} + 2\vec{v} = \vec{z} - \vec{y}$

$\vec{z} = \vec{w} - \vec{2}\vec{v} = \vec{w} + \vec{y}$

EQ.11b $-\vec{c} = (\vec{z} + \vec{v})^2 - (\vec{z} + \vec{v})\vec{v} + \vec{v}(\vec{z} + \vec{v}) - \vec{v}^2$

EQ.12b $-\vec{c} = (\vec{z}^2 + \vec{z}\vec{v} + \vec{v}\vec{z} + \vec{v}^2) + (-\vec{z}\vec{v} - \vec{v}^2) + (+\vec{v}\vec{z} + \vec{v}^2) - \vec{v}^2$

EQ.13b $-\vec{c} = \vec{z}^2 + 2\vec{v}\vec{z} = \vec{z}^2 - \vec{y}\vec{z} = (\vec{z} - \vec{y})\vec{z}$. This is the third fundamental form, the right-handed form.

However, before we can proceed to resolve the roots of $\vec{d} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t}$, some general definitions and lemmas are in order.

Definition 1 Orthogonal Imaginary Unit Vector Bases

EQ.1 $\vec{\imath}\,\vec{\jmath} = -\vec{k}$ and $\vec{\lambda}\vec{\mu} = -\vec{\nu}$; furthermore that, $\vec{\jmath}\,\vec{\imath} = +\vec{k}$ and $\vec{\mu}\vec{\lambda} = +\vec{\nu}$.
EQ.2 $\vec{\imath}\,\vec{k} = +\vec{\jmath}$ and $\vec{\lambda}\vec{\nu} = +\vec{\mu}$; furthermore that, $\vec{k}\,\vec{\imath} = -\vec{\jmath}$ and $\vec{\nu}\vec{\lambda} = -\vec{\mu}$.
EQ.3 $\vec{\jmath}\,\vec{k} = -\vec{\imath}$ and $\vec{\mu}\vec{\nu} = -\vec{\lambda}$; furthermore that, $\vec{k}\,\vec{\jmath} = +\vec{\imath}$ and $\vec{\nu}\vec{\mu} = +\vec{\lambda}$.
If, and only if:

1. Let $\vec{z} = z_0\vec{q} + z_1\vec{\imath} + z_2\vec{\jmath} + z_3\vec{k}$; let $\alpha = ATAN2\left(\frac{z_2}{z_1}\right)$; let $\beta = ATAN2\left(\frac{z_3}{\sqrt{z_1^2+z_2^2}}\right)$; let $M = \sqrt{z_1^2 + z_2^2 + z_3^2}$
2. $\vec{\lambda} = +\vec{\imath}\,(cos\alpha)(cos\beta) + \vec{\jmath}(sin\alpha)(cos\beta) + \vec{k}(sin\beta)$. This is Lambda in respect to $\vec{z}$.
3. $\vec{\mu} = -\vec{\imath}\,(sin\alpha) + \vec{\jmath}(cos\alpha) + 0\vec{k}$. This is Mu in respect to $\vec{z}$.
4. $\vec{\nu} = -\vec{\imath}\,(cos\alpha)(sin\beta) + \vec{\jmath}(sin\alpha)(sin\beta) + \vec{k}(cos\beta)$. This is Nu in respect to $\vec{z}$.
5. This rigid body rotation of …
   a. $\vec{\imath} = 1\vec{\imath} + 0\vec{\jmath} + 0\vec{k}$ to $\vec{\lambda}$
   b. $\vec{\jmath} = 0\vec{\imath} + 1\vec{\jmath} + 0\vec{k}$ to $\vec{\mu}$
   c. $\vec{k} = 0\vec{\imath} + 0\vec{\jmath} + 1\vec{k}$ to $\vec{\nu}$

   which maintains the relative spatial property that $\vec{\nu}\,\vec{\mu} = \vec{\lambda} = -\vec{\mu}\,\vec{\nu}$; $[-\vec{\mu}]_{4L}\vec{\nu} = [+\vec{\mu}]_{4R}\vec{\nu}$, such that:
6. $\vec{z} = z_0\vec{q} + M\vec{\lambda} + 0\vec{\mu} + 0\vec{\nu} = z_0\vec{q} + z_1\vec{\imath} + z_2\vec{\jmath} + z_3\vec{k}$

Definition 6 Hypercomplex Chiral Orthogonal Basis Conversion, For the Quaternions

EQ.1f $\vec{x} = c_0\vec{q} + c_1\vec{\imath} + c_2\vec{\jmath} + c_3\vec{k} = d_0\vec{q} + d_1\vec{\lambda} + d_2\vec{\mu} + d_3\vec{\nu}$; $c_0 = d_0$
Since $c_0 = d_0$, we are only concerned with the imaginary bases, thus:
EQ.2f $\vec{y} = (\vec{x} - c_0\vec{q}\,) = c_1\vec{\imath} + c_2\vec{\jmath} + c_3\vec{k} = d_1\vec{\lambda} + d_2\vec{\mu} + d_3\vec{\nu}$
We already know that:

1. $\alpha = ATAN2\left(\frac{c_2}{c_1}\right)$; $\beta = ATAN2\left(\frac{3}{\sqrt{c_1^2+c_2^2}}\right)$
2. $\vec{\lambda} = +\vec{\imath}\,(cos\alpha)(cos\beta) + \vec{\jmath}(sin\alpha)(cos\beta) + \vec{k}sin\beta$
3. $\vec{\mu} = -\vec{\imath}\,(sin\alpha) + \vec{\jmath}(cos\beta) + 0\vec{k}$
4. $\vec{\nu} = -\vec{\imath}\,(cos\alpha)(sin\beta) + \vec{\jmath}(sin\alpha)(sin\beta) + \vec{k}cos\beta$

Let us rename the above as:
EQ.3f $\vec{\lambda} = \gamma_{1,1}\vec{\imath} + \gamma_{1,2}\vec{\jmath} + \gamma_{1,3}\vec{k}$
EQ.4f $\vec{\mu} = \gamma_{2,1}\vec{\imath} + \gamma_{2,2}\vec{\jmath} + \gamma_{2,3}\vec{k}$
EQ.5f $\vec{\nu} = \gamma_{3,1}\vec{\imath} + \gamma_{3,2}\vec{\jmath} + \gamma_{3,3}\vec{k}$
Which yields the system of three linear equations:
EQ.5f $d_1\gamma_{1,1}\vec{\imath} + d_2\gamma_{2,1}\vec{\imath} + d_3\gamma_{3,1}\vec{\imath} = c_1\vec{\imath}$
EQ.6f $d_1\gamma_{1,2}\vec{\jmath} + d_2\gamma_{2,2}\vec{\jmath} + d_3\gamma_{3,2}\vec{\jmath} = c_2\vec{\jmath}$
EQ.7f $d_1\gamma_{1,3}\vec{k} + d_2\gamma_{2,3}\vec{k} + d_3\gamma_{3,2}\vec{k} = c_3\vec{k}$
EQ8f Let $\mathbf{\Gamma}$ be a 3x3 real matrix whose pairwise entries are equal to $\gamma_{m,n}$.
EQ9f Let C be a 1x3 real column matrix whose entries are c_1, c_2 and c_3 respectively.
EQ10f Let $D = \mathbf{\Gamma}^{-1}C$, which is also a 1x3 real column matrix
Then d_1, d_2, d_3 are the respective entries of D from top to bottom.

Theorem 7 The Lambda Choice Quaternion Eraser

Statement One $\vec{0} = -\vec{t}\vec{\nu} + \vec{\nu}\vec{t}$ when both $\vec{t}$ and $\vec{\nu}$ are on the same Great Circle of $\vec{\lambda}$.

Statement Two $\alpha_1\vec{\mu} + \alpha_2\vec{\nu} = -\vec{t}\vec{\nu} + \vec{\nu}\vec{t}$ when both $\vec{t}$ and $\vec{\nu}$ are not on the same Great Circle and $\alpha_1\vec{\mu} + \alpha_2\vec{\nu}$ is orthogonal to both $\vec{t}$ and $\vec{\nu}$.

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Proof:

EQ.1 Let $\vec{v} = v_0\vec{q} + v_1\vec{\lambda}$, establishing $\vec{v}$ as the reference frame, compelling the orthogonal basis $\{\vec{\lambda}, \vec{\mu}, \vec{v}\}$, then:

EQ.2 Let $\vec{t} = t_0\vec{q} + t_1\vec{\lambda} + t_2\vec{\mu} + t_3\vec{v}$

EQ.3 $-\vec{t}\vec{v} + \vec{v}\vec{t} = -(t_0\vec{q} + t_1\vec{\lambda} + t_2\vec{\mu} + t_3\vec{v})(v_0\vec{q} + v_1\vec{\lambda}) + (v_0\vec{q} + v_1\vec{\lambda})(t_0\vec{q} + t_1\vec{\lambda} + t_2\vec{\mu} + t_3\vec{v})$

EQ.4 $\vec{0} = (-t_0\vec{q} - t_1\vec{\lambda})(v_0\vec{q} + v_1\vec{\lambda}) + (v_0\vec{q} + v_1\vec{\lambda})(+t_0\vec{q} + t_1\vec{\lambda})$, since they commute upon the same Great Circle.

EQ.5 $-\vec{t}\vec{v} + \vec{v}\vec{t} = (-t_2\vec{\mu} - t_3\vec{v})(v_0\vec{q} + v_1\vec{\lambda}) + (v_0\vec{q} + v_1\vec{\lambda})(+t_2\vec{\mu} + t_3\vec{v})$

EQ.6 $-\vec{t}\vec{v} + \vec{v}\vec{t} = (-t_2v_0\vec{\mu} - t_2v_1\vec{v} - t_3v_0\vec{v} + t_3v_1\vec{\mu}) + (t_2v_0\vec{\mu} + t_3v_0\vec{v} - t_2v_1\vec{v} + t_3v_1\vec{\mu})$

EQ.7 $-\vec{t}\vec{v} + \vec{v}\vec{t} = 2(t_3v_1\vec{\mu} - t_2v_1\vec{v})$, such that $-\vec{t}\vec{v} + \vec{v}\vec{t}$ is orthogonal to $\vec{v}$, vanishing t_0 and t_1 and v_0 .

Q.E.D.

Likewise we could establish $\vec{t}$ as the reference frame via: $\vec{t} = t_{2,0}\vec{q} + t_{2,1}\vec{\lambda}_2$, compelling the orthogonal basis $\{\vec{\lambda}_2, \vec{\mu}_2, \vec{v}_2\}$, and the expression

$-\vec{t}\vec{v} + \vec{v}\vec{t}$ will result in vector that is also orthogonal to $\vec{t}$, vanishing the real part and $\vec{\lambda}_2$ part of $\vec{y}$, leaving only $\vec{\mu}_2$ and $\vec{v}_2$ as the remaining dimensions.

Regardless of which reference frame we choose, we know that $(-\vec{t}\vec{v} + \vec{v}\vec{t})$ is orthogonal to both $\vec{t}$ and $\vec{v}$, and, by definition, anything in the form of

$M(\overrightarrow{\mu_v}cos\theta + \vec{v}_v sin\theta)$ is strictly orthogonal to $\vec{v}$; however, not everything in form of $M(\overrightarrow{\mu_v}cos\theta + \vec{v}_v sin\theta)$ is strictly orthogonal to $\vec{t}$.

However, the most important takeaway is that $-\vec{t}\vec{v} + \vec{v}\vec{t} = 2(t_3v_1\vec{\mu} - t_2v_1\vec{v})$, meaning the real part of $\vec{v}$, which is v_0, has no effect; thus, the equation $\vec{d} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t}$ remains equal to $\vec{d}$, no matter the real parts of either $\vec{t}$ or $\vec{v}$, nor the lambda part of $\vec{t}$, thus t_0, t_1 and v_0 are erased from existence, allowing us to reduce the equation to (let M and N be positive reals):

$$\vec{d} = (\vec{t}^2 - \vec{t}\vec{v} + \vec{t}\vec{v}) = \vec{t}^2 + 2N(t_3\vec{\mu} - t_2\vec{v}) ; \quad N = v_1$$

Theorem 8 The Right Triangle Theorem

Thence, the expression: $\vec{d} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t}$ geometrically compels $\vec{d}$ to be the hypotenuse of a right triangle, since $-\vec{t}\vec{v} + \vec{v}\vec{t}$ is orthogonal to both $\vec{t}$ and its square, as both $\vec{t}$ and $\vec{t}^2$ lay upon the same Great Circle.

Although there exists an entire family of right triangles that share $\vec{d}$ as the hypotenuse, there are only two **congruent** right triangles within this family that satisfy $\vec{t}$.

EQ.1 $\vec{d} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t}$; $\vec{v} = v_0\vec{q} + v_1\vec{\lambda}_v \Rightarrow \{\vec{\lambda}_v, \vec{\mu}_v, \vec{v}_v\}$, which is the orthogonal basis in respect to $\vec{v}$.

EQ.2 $-\vec{t}\vec{v} + \vec{v}\vec{t} = \alpha\vec{\mu}_v + \beta\vec{v}_v$

EQ.3 $\vec{\mu}_v cos\theta + \vec{v}_v sin\theta$ is orthogonal to $(-\vec{\mu}_v sin\theta + \vec{v}_v cos\theta)$ by definition. We shall choose this orthogonality to generate the family of solutions.

EQ.4 $\vec{\mu}_v cos\theta + \vec{v}_v sin\theta$ is orthogonal to $(+\vec{\mu}_v sin\theta - \vec{v}_v cos\theta)$ by definition. We discard this orthogonality in favor of the former.

EQ.5 $\vec{d} = d_0\vec{q} + d_1\vec{\lambda}_v + d_2\vec{\mu}_v + d_3\vec{v}_v$

EQ.6 $\vec{t} = t_0\vec{q} + t_1\vec{\lambda}_v + t_2\vec{\mu}_v + t_3\vec{v}_v$

EQ.7 $\vec{t}^2 = (t_0^2 - t_1^2 - t_2^2 - t_3^2)\vec{q} + 2t_0t_1\vec{\lambda}_v + 2t_0t_2\vec{\mu}_v + 2t_0t_3\vec{v}_v$; $d_0 = (t_0^2 - t_1^2 - t_2^2 - t_3^2)$; $d_1 = 2t_0t_1$

EQ.8 Let $\vec{f} = d_2\vec{\mu}_v + d_3\vec{v}_v$

EQ.9 Let $\vec{g} = G\,(+\vec{\mu}_v cos\beta + \vec{v}_v sin\beta) = (-\vec{t}\vec{v} + \vec{t}\vec{v}) = +2t_3v_1\vec{\mu}_v - 2t_2v_1\vec{v}_v = x\vec{\mu}_v + y\vec{v}_v$

EQ.10 Let $\vec{t}^2 = \vec{h} = H\,(-\vec{\mu}_v sin\beta + \vec{v}_v cos\beta) = +2t_0t_2\vec{\mu}_v + 2t_0t_3\vec{v}_v = w\vec{\mu}_v + z\vec{v}_v$

EQ.11 $\vec{f} = G\vec{g} + H\vec{h}$

EQ.12 $d_2\vec{\mu}_v = +G\vec{\mu}_v cos\theta - H\vec{\mu}_v sin\theta = +2t_3v_1\vec{\mu}_v + 2t_0t_2\vec{\mu}_v = x\vec{\mu}_v + w\vec{\mu}_v$

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EQ.13 $d_3\vec{\nu}_v = +G\vec{\nu}_v sin\theta + H\vec{\nu}_v cos\theta = -2t_2 v_1\vec{\nu}_v + 2t_0 t_3\vec{\nu}_v = y\vec{\nu}_v + z\vec{\nu}_v$

EQ.14 $\vec{t}^2 = \vec{d} - \vec{g} \quad \Rightarrow \quad \vec{t} = \pm\sqrt{\vec{d} - \vec{g}}$

The above relationship provides us with enough information to brute the roots of the quadratic equation by simply comparing every value of the angular argument of β against the real number magnitude of error from the return on $\vec{d}$ in a preliminary search, and then converge rapidly upon the roots via bisection.

In fact, it was by empirical observation of the roots (using the above rapidly convergent algorithm) that I was able to resolve the closed form solution. We shall first simplify the quadratic equation further in lieu of those empirical results.

Theorem 9 The Orthogonal Basis Rotation Theorem; The Relative Frame Theorem

EQ.1 Let $\vec{d} = d_0\vec{q} + d_1\vec{\lambda}_v + d_2\vec{\mu}_v + d_3\vec{\nu}_v = A(\vec{q}cos\alpha + \vec{\lambda}_v sin\alpha) + \Omega(\vec{\mu}_v cos\phi + \vec{\nu}_v sin\phi);\ \Omega = \sqrt{d_2^2 + d_3^2}$

EQ.2a $\vec{\mu_2} = +\vec{\mu}_v cos\phi + \vec{\nu}_v sin\phi$

EQ.2b $\vec{\nu_2} = -\vec{\mu}_v sin\phi + \vec{\nu}_v cos\phi$

EQ.3 $\vec{d} = d_0\vec{q} + d_1\vec{\lambda}_v + d_2\vec{\mu}_v + d_3\vec{\nu}_v = A(\vec{q}cos\alpha + \vec{\lambda}_v sin\alpha) + \Omega\vec{\mu}_2 + 0\vec{\nu}_2$

We are able to perform this basis conversion because all we did was rotate $\{\vec{\mu}_v, \vec{\nu}_v\}$ about $\vec{\lambda}_v$; hence, $(\vec{\lambda}_v, \vec{\mu}_2, \vec{\nu}_2)$ preserves the multiplicative relationships 171 expected in the original basis. In fact, there is **no preferred frame of reference** for the $\vec{\mu}$ and $\vec{\nu}$ axes for an Observer on the Great Circle of $\vec{\lambda}$, only $\vec{\lambda}$ is absolute from the Observer's perspective. The Observer is free to rotate the $\{\vec{\mu}_v, \vec{\nu}_v\}$ axes in any manner that simplifies the existing problem.

Thus, in the equation $\vec{d} = \vec{t}^2 - \vec{t}\vec{\nu} + \vec{\nu}\vec{t}$, the $\vec{\nu}$ variable establishes $\vec{\lambda}$, and the $\vec{d}$ variables establishes $\vec{\mu}_2$ and $\vec{\nu}_2$.

Theorem 10 The Fully Depressed Case of the Quadratic Equation

We now combine Theorems 14 and 15 to yield the fully depressed case of the quadratic equation.

EQ.1a $\vec{d} = \vec{t}^2 - \vec{t}\vec{\nu} + \vec{\nu}\vec{t} = \vec{t}^2 - \vec{t}(v_0\vec{q} + N\vec{\lambda}) + (v_0\vec{q} + N\vec{\lambda})\vec{t}$, where $\vec{\lambda}_v$ is in respect to $\vec{\nu}$.

EQ.1b $\vec{d} = \vec{t}^2 - \vec{t}(N\vec{\lambda}) + (N\vec{\lambda})\vec{t}$

EQ.1c $\vec{d} = d_0\vec{q} + d_1\vec{\lambda} + \omega_1\vec{\mu}_v + \omega_2\vec{\nu}_v$, where $(\vec{\mu}_v, \vec{\nu}_v)$ is the initial orthogonal basis in respect to $\vec{\nu}$.

EQ.2a Let $\Omega = \sqrt{\omega_1^2 + \omega_2^2}$

EQ.2b Let $\phi = ATAN2\left(\frac{\omega_2}{\omega_1}\right)$

EQ.2c Let $\vec{\mu}_2 = +\vec{\mu}_v cos\phi + \vec{\nu}_v sin\phi$

EQ.2d Let $\vec{\nu}_2 = -\vec{\mu}_v sin\phi + \vec{\nu}_v cos\phi$

EQ.2e Let $\vec{d} = d_0\vec{q} + d_1\vec{\lambda} + \Omega\vec{\mu}_2 + 0\vec{\nu}_2$

EQ.3a $\vec{t} = t_0\vec{q} + t_1\vec{\lambda} + t_2\vec{\mu}_2 + t_3\vec{\nu}_2$

EQ.3b $\vec{t}^2 = (t_0^2 - t_1^2 - t_2^2 - t_3^2)\vec{q} + 2t_0t_1\vec{\lambda} + 2t_0t_2\vec{\mu}_2 + 2t_0t_3\vec{\nu}_2$

EQ.3c $-\vec{t}(N\vec{\lambda}) + (N\vec{\lambda})\vec{t} = 2Nt_3\vec{\mu}_2 - 2Nt_2\vec{\nu}_2$ (Lambda Choice Quaternion Eraser)

EQ.4a $d_0 = (t_0^2 - t_1^2 - t_2^2 - t_3^2)$

EQ.4b $d_1 = 2t_0t_1$

EQ.4c $\Omega = 2t_0t_2 + 2Nt_3$

EQ.4d $0 = 2t_0t_3 - 2Nt_2$

Lemma 11 The Mu Part and Nu Part Equivalence.

EQ.5a $0 = 2t_0t_3 - 2Nt_2$

EQ.5b $0 = t_0t_3 - Nt_2$

EQ.5c $t_0 = \frac{Nt_2}{t_3}$

EQ.6a $\Omega = 2t_0t_2 + 2Nt_3$

EQ.6b $\Omega - 2Nt_3 = 2t_0t_2$

EQ.6c $t_0 = \frac{\Omega - 2Nt_3}{2t_2}$

EQ.7a $\frac{Nt_2}{t_3} = \frac{\Omega - 2Nt_3}{2t_2}$

EQ.7b $2Nt_2^2 = \Omega t_3 - 2Nt_3^2$

EQ.7c $t_2^2 = \frac{\Omega t_3 - 2Nt_3^2}{2N}$.

Lemma 12 The Real Part and Nu Part Equivalence.

EQ.1a $0 = 2t_0t_3 - 2Nt_2$

EQ.1b $0 = t_0t_3 - Nt_2$

EQ.2a $t_2 = \frac{t_0t_3}{N}$

EQ.2b $\Omega = 2t_0t_2 + 2Nt_3$

EQ.2c $\Omega - 2Nt_3 = 2t_0t_2$

EQ.2d $t_2 = \frac{\Omega - 2Nt_3}{2t_0}$

EQ.3a $\frac{t_0t_3}{N} = \frac{\Omega - 2Nt_3}{2t_0}$

EQ.3b $2t_0^2t_3 = \Omega N - 2N^2t_3$

EQ.3c $t_0^2 = \frac{\Omega N - 2N^2t_3}{2t_3}$

EQ.3d $\frac{1}{t_0^2} = \frac{2t_3}{\Omega N - 2N^2t_3}$

Lemma 13 The Lambda Part Identity

EQ.1 $d_1 = 2t_0t_1$

EQ.2 $t_1 = \frac{d_1}{2t_0}$

EQ.3 $t_1^2 = \frac{d_1^2}{4t_0^2} = \frac{d_1^2}{4}\left(\frac{2t_3}{\Omega N - 2N^2t_3}\right) = \frac{2d_1^2t_3}{4\Omega N - 8N^2t_3}$

Lemma 14 The Real Part Identity

EQ.1 $d_0 = (t_0^2 - t_1^2 - t_2^2 - t_3^2)$

EQ.2 $d_0 = \left(t_0^2 - \frac{d_1^2}{4t_0^2} - t_2^2 - t_3^2\right)$

EQ.3 $d_0 = \left(\frac{\Omega N - 2N^2t_3}{2t_3} - \frac{2d_1^2t_3}{4\Omega N - 8N^2t_3} - \frac{\Omega t_3 - 2Nt_3^2}{2N} - t_3^2\right)$

EQ.4 $d_0 = \frac{\Omega N - 2N^2t_3}{2t_3} - \frac{2d_1^2t_3}{4\Omega N - 8N^2t_3} - \left(\frac{\Omega t_3 - 2Nt_3^2}{2N} + t_3^2\right)$

EQ.5 $d_0 = \frac{\Omega N - 2N^2t_3}{2t_3} - \frac{2d_1^2t_3}{4\Omega N - 8N^2t_3} - \left(\frac{\Omega t_3}{2N}\right)$

EQ.6 $d_0 = \frac{\Omega N^2 - 2N^3t_3 - \Omega t_3^2}{2Nt_3} - \frac{2d_1^2t_3}{4\Omega N - 8N^2t_3}$

EQ.7 $d_0 = \frac{\Omega N^2 - 2N^3t_3 - \Omega t_3^2}{2Nt_3} - \frac{2Nt_3}{2Nt_3}\left(\frac{2d_1^2t_3}{4\Omega N - 8N^2t_3}\right)$

EQ.8 $d_0 = \frac{\Omega N^2 - 2N^3t_3 - \Omega t_3^2}{2Nt_3} - \frac{4Nd_1^2t_3^2}{8\Omega N^2t_3 - 16N^3t_3^2}$

EQ.9 $d_0 = \frac{(4\Omega N - 8N^2t_3)}{(4\Omega N - 8N^2t_3)}\left(\frac{\Omega N^2 - 2N^3t_3 - \Omega t_3^2}{2Nt_3}\right) - \frac{4Nd_1^2t_3^2}{8\Omega N^2t_3 - 16N^3t_3^2}$

EQ.10 $d_0 = \frac{4\Omega^2N^3 - 8\Omega N^4t_3 - 4\Omega^2Nt_3^2 - 8\Omega N^4t_3 + 16N^5t_3^2 + 8\Omega N^2t_3^3 - 4Nd_1^2t_3^2}{8\Omega N^2t_3 - 16N^3t_3^2}$

EQ.11 $d_0 = \frac{+8\Omega N^2t_3^3 + (16N^5 - 4Nd_1^2 - 4\Omega^2N)t_3^2 - 16\Omega N^4t_3 + 4\Omega^2N^3}{8\Omega N^2t_3 - 16N^3t_3^2}$

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EQ.12 $8\Omega N^2 d_0 t_3 - 16N^3 d_0 t_3^2 = +8\Omega N^2 t_3^3 + (16N^5 - 4Nd_1^2 - 4\Omega^2 N)t_3^2 - 16\Omega N^4 t_3 + 4\Omega^2 N^3$

EQ.13 $0 = 8\Omega N^2 t_3^3 + (16N^5 - 4Nd_1^2 - 4\Omega^2 N)t_3^2 - 16\Omega N^4 t_3 + 4\Omega^2 N^3 - 8\Omega N^2 d_0 t_3 + 16N^3 d_0 t_3^2$

EQ.14 $0 = 8\Omega N^2 t_3^3 + (16N^5 + 16N^3 d_0 - 4Nd_1^2 - 4\Omega^2 N)t_3^2 - (16\Omega N^4 + 8\Omega N^2 d_0)t_3 + 4\Omega^2 N^3$

EQ.15 $0 = 2\Omega N^2 t_3^3 + (4N^5 + 4N^3 d_0 - Nd_1^2 - \Omega^2 N)t_3^2 - (4\Omega N^4 + 2\Omega N^2 d_0)t_3 + \Omega^2 N^3$

EQ.16 $0 = 2\Omega N t_3^3 + (4N^4 + 4N^2 d_0 - d_1^2 - \Omega^2)t_3^2 - (4\Omega N^3 + 2\Omega N d_0)t_3 + \Omega^2 N^2$

EQ.17 $0 = At_3^3 + Bt_3^2 + Ct_3 + D$

EQ.17a $A = +2\Omega N$

EQ.17b $B = +4N^4 + 4N^2 d_0 - d_1^2 - \Omega^2$

EQ.17c $C = -4\Omega N^3 - 2\Omega N d_0$

EQ.17d $D = +\Omega^2 N^2$

Definition 15 Cardano's Theorem: The Real Cubic Identity of the Nu Part

We now use the Cardano Method to depress the Cubic of the Nu Part.

EQ.1 $0 = At_3^3 + Bt_3^2 + Ct_3 + D$

EQ.2 $0 = \zeta^3 + p\zeta + q$

EQ.3 $\zeta = t_3 + \frac{B}{3A}$

EQ.4 $p = \frac{3AC - B^2}{3A^2}$

EQ.5 $q = \frac{2B^3 - 9ABC + 27A^2 D}{27A^3}$ Completing Cardano's depression of the Cubic. We now implement Vieta's Substitution:

Definition 16 Vieta's Theorem: The Resolution of the Nu Part.

EQ.1 $\zeta = w - \frac{p}{3w}$

EQ.2 $0 = w^3 + q - \frac{p^3}{27w^3}$

EQ.3 $0 = w^6 + qw^3 - \frac{p^3}{27}$

EQ.4 $y = w^3$

EQ.5 $0 = y^2 + qy - \frac{p^3}{27}$

EQ.6 $y = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$

EQ.7 $w = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, either sign of the square root shall suffice.

EQ.8 $t_3 + \frac{B}{3A} = w - \frac{p}{3w}$

EQ.9 $t_3 = -\frac{B}{3A} + w - \frac{p}{3w}$

We now use the identities from t_3 to yield t_0, t_1 and t_2.

EQ.10 $t_0 = \pm\sqrt{\frac{\Omega N - 2N^2 t_3}{2t_3}}$

EQ.11 $t_1 = \frac{d_1}{2t_0}$

EQ.12 $t_2 = \pm\sqrt{\frac{\Omega t_3 - 2Nt_3^2}{2N}}$

Of course, we have a serious dilemma. Which of the three real roots do we accept for t_3? Which sign of the above squares do we choose in unison? Only the polar form of solution will elucidate which root of t_3 to accept, and then how to produce t_0, t_1 and t_2 .

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Theorem 17 The Offset Circle Theorem

For the moment, let us suppose we know which cubic root to select as t_3, then we now examine the following relationship: $\frac{Nt_2}{t_3} = \frac{\Omega - 2Nt_3}{2t_2}$. This equation informs us that the coordinate $t_2\vec{\mu}_2 + t_3\vec{\nu}_2$ lays upon a circle, with a radius of $\frac{\Omega}{4N}$, offset from the origin by $+\frac{\Omega}{4N}$.

Let $t_2\vec{\mu}_2 + t_3\vec{\nu}_2 = M(\vec{\mu}_2 cos\theta + \vec{\nu}_2 sin\theta)$, then the Law of Cosines reveals that:

EQ.1 $$M^2 = \left(\frac{\Omega}{4N}\right)^2 + \left(\frac{\Omega}{4N}\right)^2 - 2\left(\frac{\Omega}{4N}\right)^2 cos2\theta$$

EQ.2 $$M^2 = 2\left(\frac{\Omega}{4N}\right)^2 (1 - cos2\theta)$$

EQ.3 $$M = \left(\frac{\Omega}{2N}\right) sin\theta \ ,$$ upholding the Law of Sines.

We now examine the relationship $t_0 = \frac{Nt_2}{t_3}$.

EQ.4 $$t_0 = N\frac{Mcos\theta}{Msin\theta}$$

EQ.5 $$t_0 = Ncot\theta$$

EQ.6 $$t_1 = \frac{d_1}{2t_0} = \frac{d_1}{2N}tan\theta$$

EQ.7 $$d_0 = (t_0^2 - t_1^2 - t_2^2 - t_3^2)$$

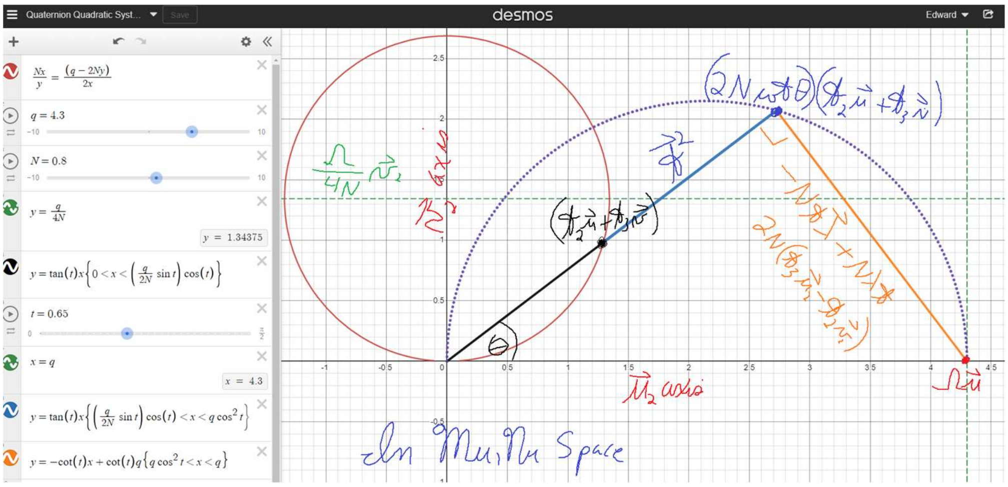
EQ.8 $$d_0 = N^2cot^2\theta - \frac{d_1^2}{4N^2}tan^2\theta - M^2cos^2\theta - M^2sin^2\theta$$

EQ.9 $$d_0 = N^2cot^2\theta - \frac{d_1^2}{4N^2}tan^2\theta - M^2(cos^2\theta + sin^2\theta) \ ; \quad M^2 = \left(\frac{\Omega^2}{4N^2}\right)sin^2\theta \ ; \ (cos^2\theta + sin^2\theta) = 1$$

EQ.10 $$d_0 = N^2cot^2\theta - \frac{1}{4N^2}(d_1^2tan^2\theta + \Omega^2 sin^2\theta)$$, which leads to a nasty degree six equation with 6 pairs of conjugate solutions for θ.

Before we proceed, the below image is the geometric appearance of the question at hand in $\vec{\mu}_2, \vec{\nu}_2$ space.

In the following url link, q is Omega, N is N, and t is theta: https://www.desmos.com/calculator/q2bfcbs7wq



However, when we yield the roots of the cubic to produce t_3 we can solve for theta without any of the hassle that the polar form introduces.

EQ.11 $$t_2\vec{\mu}_2 + t_3\vec{\nu}_2 = M(\vec{\mu}_2 cos\theta + \vec{\nu}_2 sin\theta)$$

EQ.12 $$t_3 = Msin\theta$$

EQ.13 $$t_3 = \left(\frac{\Omega}{2N}\right)sin^2\theta$$

EQ.14 $$\frac{2Nt_3}{\Omega} = sin^2\theta \quad ; \quad \sqrt{\frac{2Nt_3}{\Omega}} = sin\theta$$

EQ.15 $$\theta = Arcsine\left(+\sqrt{\frac{2Nt_3}{\Omega}}\right)$$. We know to take the positive root, since the (t_2, t_3) coordinate resides in the first quadrant, since both magnitude variables, N and Ω, are positive by definition, forcing the red circle (in the above image) in the upper two quadrants. We also both angles for the Arcsine function. This is not because we cannot resolve the ambiguity; rather, both θ solutions fulfill $\vec{d} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t}$ simultaneously. Hence:

EQ.16 $\theta_1 = Arcsine\left(+\sqrt{\frac{2Nt_3}{\Omega}}\right)$; $\theta_2 = (\pi - \theta_1)$; yielding the empirically observed form: $\vec{t} = t_3\vec{v}_2 \pm \sqrt{\Psi^2}$, where Ψ has no $\vec{v}_2$ part.

With both values of theta known, we simply use the identities above to yield t_0, t_1, t_2

EQ.17a $t_{0,1} = N cot\theta_1;\ t_{1,1} = \frac{d_1}{2t_{0,1}};\ t_{2,1} = \left(\frac{\Omega}{2N}\right) sin\theta_1 cos\theta_1$

EQ.17b $t_{0,2} = N cot\theta_1;\ t_{1,2} = \frac{d_1}{2t_{0,1}};\ t_{2,2} = \left(\frac{\Omega}{2N}\right) sin\theta_2 cos\theta_2$; $t_{0,1} = -t_{0,2}$; $t_{1,1} = -t_{1,2}$; $t_{2,1} = -t_{2,2}$; $t_{3,1} = t_{3,2}$. **Q.E.D.**

Theorem 18 Which Root Theorem

We shall use the randomly generated components of $\vec{f} = \vec{x}^2 + \vec{x}\vec{b} + \vec{a}\vec{x}$ seen below to demonstrate that all three roots of t_3 are valid by symmetry.

$$\vec{a} = -6.198\vec{q} \quad \vec{\imath} + \quad \vec{\jmath} + \quad \vec{k}$$
$$\vec{b} = +6.472\vec{q} - 7.628\vec{\imath} + 5.019\vec{\jmath} + 1.531\vec{k}$$
$$\vec{f} = -8.299\vec{q} + 5.952\vec{\imath} + 6.088\vec{\jmath} + 2.996\vec{k}$$
$$\vec{x}_1 = +1.1457138\vec{q} + 1.5397790\vec{\imath} + 1.6404340\vec{\jmath} - 0.1822954\vec{k}$$
$$\vec{x}_2 = -1.4197138\vec{q} - 4.1986025\vec{\imath} - 3.0201749\vec{\jmath} - 2.0290342\vec{k}$$

The resultant equation $\vec{d} = \vec{t}^2 - \vec{t}\vec{v} + \vec{v}\vec{t}$; $\Omega =$; $N =$

$\vec{d} = -87.5983675\vec{q} + 36.5451060\vec{\imath} + 70.9961880\vec{\jmath} - 44.7808970\vec{k} = -87.5983675\vec{q} + 7.89969368971\vec{\lambda} + \Omega\vec{\mu}_2 + 0\vec{v}_2$

$$\vec{v} = -6.33500000\vec{q} + 3.3755000\vec{\imath} + 1.0005000\vec{\jmath} + 3.4690000\vec{k}$$
$$= -6.33500000\vec{q} + N\vec{\lambda} + 0\vec{\mu}_2 + 0\vec{v}_2$$
$$\vec{\lambda} = 0\vec{q} + 0.6829448113\vec{\imath} + 0.2024252062\vec{\jmath} + 0.7018621094\vec{k}$$
$$\vec{\mu}_2 = 0\vec{q} + 0.3415270497\vec{\imath} + 0.7608650002\vec{\jmath} - 0.551764194\vec{k}$$
$$\vec{v}_2 = 0\vec{q} - 0.6457132948\vec{\imath} + 0.6165293888\vec{\jmath} + 0.4504951205\vec{k}$$

Has the roots, accepting the angular argument of $\theta_1 = 1.316874331\, radians$; $\theta_2 = \pi - \theta_1 = 1.824718323 radians$

$$\vec{t}_1 = +1.282713777\vec{q} + 3.079289328\vec{\lambda} + 2.243471181\vec{\mu}_2 + 8.644566873\vec{v}_2$$
$$\vec{t}_2 = -1.282713777\vec{q} - 3.079289328\vec{\lambda} - 2.243471181\vec{\mu}_2 + 8.644566873\vec{v}_2$$

t3,1	t3,2	t3,3	Theta 1	Theta 2	Lambda Frame	t0 q	t1 lambda	t2 mu2	t3 nu2
10.08356777	8.644566873	-2.585822472	1.316874331	1.824718323	Root 1	1.282713777	3.079289328	2.243471181	8.644566873
1.092856303	0.9368974962	-0.2802512416			Root 2	-1.282713777	-3.079289328	-2.243471181	8.644566873

The three roots for t_3 are as follows: $t_3 = +10.08356777, +8.644566873, -2.585822472$

$\theta_1 = Arcsine\left(+\sqrt{\frac{2Nt_3}{\Omega}}\right) = \left(\frac{\pi}{2} - \quad i\right)$, , $0 +$

$t_{0,1} = N cot\theta_1;\ t_{1,1} = \frac{d_1}{2N} tan\theta;\ t_{2,1} = \left(\frac{\Omega}{2N}\right) sin\theta_1 cos\theta_1;\ t_{2,1} = \left(\frac{\Omega}{2N}\right) sin\theta_1{}^2$.

$t_{0,1} = 0 +$; $t_{1,1} =$; $t_{2,1} = 0 +$; $t_{3,1} =$ for $\theta_1 = \left(\frac{\pi}{2} - \quad i\right)$

$t_{0,1} =$ $+$; $t_{1,1} =$; $t_{2,1} =$ $+ 0$; $t_{3,1} =$ for $\theta_1 =$

$t_{0,1} = 0$; $t_{1,1} =$; $t_{2,1} = 0 +$; $t_{3,1} =$ for $\theta_1 = 0 +$

$$d_0 = -87.5983675 = (1.4407i)^2 - (-2.7416i)^2 - (2.93926i)^2 - (10.08356)^2$$
$$= -2.0756 + 7.5163 + 8.6392 - 101.6781$$
$$d_0 = -87.5983675 = (1.28277)^2 - (3.07928)^2 - (2.24347)^2 - (8.64456)^2$$
$$= +1.6454 - 9.4819 - 5.0332 - 74.7284$$
$$d_0 = -87.5983675 = (-10.5639i)^2 - (0.37389i)^2 - (5.52678i)^2 - (-2.58582)^2$$
$$= -111.59 + 0.1397 + 30.545 - 6.68646$$

$$d_1 = 7.89969 = 2t_0t_1 = 2(0 + 1.4407i)(0 - 2.7416i) = 2(1.28277)(3.07928)$$
$$= 2(0 - 10.5639i)(0 + 0.37389i)$$
$$d_2 = 91.2081 = \Omega = 2t_0t_2 + 2Nt_3 = 2(1.4407i)(2.93926i) + 2(4.9425)(10.0835)$$
$$= -8.4691 \quad + 99.6753$$
$$d_2 = 91.2081 = \Omega = 2t_0t_2 + 2Nt_3 = 2(1.28277)(2.24347) + 2(4.9425)(8.64456)$$
$$= +5.7557 \quad + 85.4514$$
$$d_2 = 91.2081 = \Omega = 2t_0t_2 + 2Nt_3 = 2(-10.5639i)(5.52678i) + 2(4.9425)(-2.58582)$$
$$= +116.7687 - 25.5608$$
$$0 = 2t_0t_3 + 2Nt_2 = 2(1.4407i)(10.0835) - 2(4.9425)(2.93926i) = 29.054i - 29.054i$$
$$0 = 2t_0t_3 + 2Nt_2 = 2(1.28277)(8.64456) - 2(4.9425)(2.24347) = 22.177i - 22.177i$$
$$0 = 2t_0t_3 + 2Nt_2 = 2(-10.5639i)(-2.58582) - 2(4.9425)(5.52678i) = 54.632i - 54.632i$$

That is, all three Arcsine arguments of t_3 produce the same $\vec{d}$ vector after recombination. In other words, a Quadratic Equation over the Quaternions has one pair of roots with four real coefficients, and two pairs of roots with three purely imaginary coefficients for $\vec{q}, \vec{\lambda}, \vec{\mu}_2$ and one pure real coefficient for $\vec{v}_2$.

However, the geometric meaning of complex coefficients remains unclear. For now, we accept the guaranteed real argument for θ_1 Q.E.D.

Theorem 19 The Closed Form Solution for a General Quadratic Equation for the Quaternions.

We now combine all of the steps to solve original query:

EQ.1 $\vec{f} = \vec{x}^2 + \vec{x}\,\vec{b} + \vec{a}\,\vec{x}$

EQ.2 Let $-\vec{c} = \vec{f} + \vec{a}\,\vec{b}$

EQ.3 $-\vec{c} = \vec{x}^2 + \vec{x}\,\vec{b} + \vec{a}\,\vec{x} + \vec{a}\,\vec{b}$

EQ.4 $\vec{u} = \frac{1}{2}(\vec{a} + \vec{b})$

EQ.5 $\vec{v} = \frac{1}{2}(\vec{a} - \vec{b})$

EQ.6 Let $\vec{t} = \vec{x} + \vec{u}$, therefore $\vec{x} = \vec{t} - \vec{u}$

EQ.7 $-\vec{c} = \vec{t}^2 - \vec{t}\,\vec{v} + \vec{v}\,\vec{t} - \vec{v}^{\,2}$

EQ.8 Let $\vec{d} = -\vec{c} + \vec{v}^2 = d_{0,0}\vec{q} + d_{1,0}\vec{\imath} + d_{2,0}\vec{\jmath} + d_{3,0}\vec{k}$

EQ.9 $\vec{d} = \vec{t}^2 - \vec{t}\,\vec{v} + \vec{v}\,\vec{t}$

EQ.10 $\vec{v} = v_{0,0}\vec{q} + v_{1,0}\vec{\imath} + v_{2,0}\vec{\jmath} + v_{3,0}\vec{k}$

EQ.11 $\alpha = ATAN2\left(\frac{v_2}{v_1}\right)$

EQ.12 $\beta = ATAN2\left(\frac{v_3}{\sqrt{v_1^1 + v_2^2}}\right)$

EQ.13 $\vec{\lambda} = +\vec{\imath}\,(cos\alpha)(cos\beta) + \vec{\jmath}(sin\alpha)(cos\beta) + \vec{k}(sin\beta) = \gamma_{1,1}\vec{\imath} + \gamma_{1,2}\vec{\jmath} + \gamma_{1,3}\vec{k}$

EQ.14 $\vec{\mu}_1 = -\vec{\imath}\,(sin\alpha) + \vec{\jmath}(cos\alpha) + 0\vec{k} = \gamma_{2,1}\vec{\imath} + \gamma_{2,2}\vec{\jmath} + \gamma_{2,3}\vec{k}$

EQ.15 $\vec{v}_1 = -\vec{\imath}\,(cos\alpha)(sin\beta) + \vec{\jmath}(sin\alpha)(sin\beta) + \vec{k}(cos\beta) = \gamma_{3,1}\vec{\imath} + \gamma_{3,2}\vec{\jmath} + \gamma_{3,3}\vec{k}$

EQ.16 Let **Γ** be a 3x3 real matrix whose pairwise entries are equal to $\gamma_{m,n}$.

EQ.17 Let A be a 1x3 real column matrix whose entries are $v_{1,0}, v_{2,0}$ and $v_{3,0}$ respectively.

EQ.18 Let B be a 1x3 real column matrix whose entries are $d_{1,0}, d_{2,0}$ and $d_{3,0}$ respectively.

EQ.19 Let **V** =**Γ**A , which is also a 1x3 real column matrix, let it the results be named $N, 0, 0$.

***N* is our first primary variable.**

EQ.20 Let D =**Γ**B , which is also a 1x3 real column matrix, let it the results be named $d_{1,1}, d_{2,1}, d_{3,1}$

We do **not** require the inverse Gamma Matrix for this process.

Γ Gamma Matrix A Matrix B Matrix

ΓA = V Matrix **Γ**B = D Matrix

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$\gamma_{1,1}$	$\gamma_{1,2}$	$\gamma_{1,3}$	$v_{1,0}$	$d_{1,0}$	N	$d_{1,1}$
$\gamma_{2,1}$	$\gamma_{2,2}$	$\gamma_{2,3}$	$v_{2,0}$	$d_{2,0}$	0	$d_{2,1}$
$\gamma_{3,1}$	$\gamma_{3,2}$	$\gamma_{3,3}$	$v_{3,0}$	$d_{3,0}$	0	$d_{3,1}$

EQ.21 $\phi = ATAN2\left(\frac{d_{3,1}}{d_{2,1}}\right);\ \ \Omega = \sqrt{d_{2,1}^2 + d_{3,1}^2}$

EQ.22 $\vec{\mu}_2 = +\vec{\mu}_1 cos\phi + \vec{v}_1 sin\phi = +\vec{\imath}(-sin\alpha\ cos\phi - cos\alpha\ sin\beta\ sin\phi) + \vec{\jmath}(+cos\alpha\ cos\phi + sin\alpha\ sin\beta\ sin\phi) + \vec{k}(cos\beta\ sin\phi)$

EQ.23 $\vec{\mu}_2 = \mu_1\vec{\imath} + \mu_2\vec{\jmath} + \mu_3\vec{k}$

EQ.24 $\vec{v}_2 = -\vec{\mu}_1 sin\phi + \vec{v}_1 cos\phi = +\vec{\imath}(+sin\alpha\ sin\phi - cos\alpha\ sin\beta\ cos\phi) + \vec{\jmath}(-cos\alpha\ sin\phi + sin\alpha\ sin\beta\ cos\phi) + \vec{k}(cos\beta\ cos\phi)$

EQ.25 $\vec{v}_2 = v_1\vec{\imath} + v_2\vec{\jmath} + v_3\vec{k}$

EQ.26 $\vec{v} = \frac{1}{2}(\vec{a} - \vec{b}\,) = v_{0,0}\vec{q} + N\vec{\lambda}\ + 0\vec{\mu}_1\ + 0\vec{v}_1\ \ = v_{0,0}\vec{q} + N\vec{\lambda}\ + 0\vec{\mu}_2 + 0\vec{v}_2$

EQ.27 $\vec{d} = d_{0,0}\vec{q} + d_{1,1}\vec{\lambda} + d_{2,1}\vec{\mu}_1 + d_{3,1}\vec{v}_1 = d_{0,0}\vec{q} + d_{1,1}\vec{\lambda} + \Omega\vec{\mu}_2 + 0\vec{v}_2$

EQ.28 $A = +2\Omega N$

EQ.29 $B = +4N^4 + 4N^2 d_{0,0} - d_{1,1}^2 - \Omega^2$

EQ.30 $C = -4\Omega N^3 - 2\Omega N d_{0,0}$

EQ.31 $D = +\Omega^2 N^2$

EQ.32 $0 = At_3^3\ + Bt_3^2 + Ct_3 + D$

EQ.33 $0 = \zeta^3\ + p\zeta + q$

EQ.34 $\zeta = t_3 + \frac{B}{3A}$

EQ.35 $p = \frac{3AC - B^2}{3A^2}$

EQ.36 $q = \frac{2B^3 - 9ABC + 27A^2D}{27A^3}$

EQ.37 $\zeta = w - \frac{p}{3w}$

EQ.38 $0 = w^3 + q - \frac{p^3}{27w^3}$

EQ.39 $0 = w^6 + qw^3 - \frac{p^3}{27}$

EQ.40 $y = w^3$

EQ.41 $0 = y^2 + qy - \frac{p^3}{27}$

EQ.42 $y = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$

EQ.43 $w = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$, either sign of the square root shall suffice, and any cube root will suffice.

EQ.44 $t_3 + \frac{B}{3A} = w - \frac{p}{3w}$

EQ.45 $t_3 = -\frac{B}{3A} + w - \frac{p}{3w}$. All roots will be real.

EQ.46 $\theta = Arcsin\left(+\sqrt{\frac{2Nt_3}{\Omega}}\right)$. If θ is a complex number, then $\vec{\lambda}$ is the imaginary unit.

We evaluate θ for all three roots of t_3 and select the real-valued argument. Hopefully someone will elucidate the meaning of the complex arguments in due time, for I dare not feign knowledge of their geometric interpretation.

EQ.47 $\vec{t}_1 = +\vec{q}Ncot\theta + \vec{\lambda}\frac{d_{1,,1}}{2N}tan\theta + \vec{\mu}_2\left(\frac{\Omega}{2N}\right)sin\theta cos\theta + \vec{v}_2\left(\frac{\Omega}{2N}\right)sin^2\theta = t_{0,1}\vec{q} + t_{1,1}\vec{\lambda} + t_{2,1}\vec{\mu}_2 + t_{3,1}\vec{v}_2$

EQ.48 $\vec{t}_2 = -\vec{q}Ncot\theta - \vec{\lambda}\frac{d_{1,,1}}{2N}tan\theta - \vec{\mu}_2\left(\frac{\Omega}{2N}\right)sin\theta cos\theta + \vec{v}_2\left(\frac{\Omega}{2N}\right)sin^2\theta = t_{0,2}\vec{q} + t_{1,2}\vec{\lambda} + t_{2,2}\vec{\mu}_2 + t_{3,2}\vec{v}_2$

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The above two equations are the roots in the proper orthogonal basis of $\vec{\lambda}, \vec{\mu}_2, \vec{v}_2$, however we must now convert back to $\vec{\imath}, \vec{\jmath}, \vec{k}$.

EQ.49 $t_{0,1}\vec{q} + t_{1,1}\vec{\lambda} + t_{2,1}\vec{\mu}_2 + t_{3,1}\vec{v}_2 = t_{0,3}\vec{q} + t_{1,3}\vec{\imath} + t_{2,3}\vec{\jmath} + t_{3,3}\vec{k}$

EQ.50 $t_{0,3} = t_{0,1}$

EQ.51 $t_{1,3} = t_{1,1}\gamma_{1,1} + t_{2,1}\mu_1 + t_{3,1}v_1$

EQ.52 $t_{2,3} = t_{1,1}\gamma_{1,2} + t_{2,1}\mu_2 + t_{3,1}v_2$

EQ.53 $t_{2,3} = t_{1,1}\gamma_{1,3} + t_{2,1}\mu_3 + t_{3,1}v_3$

EQ.54 $t_{0,2}\vec{q} + t_{1,2}\vec{\lambda} + t_{2,2}\vec{\mu}_2 + t_{3,2}\vec{v}_2 = t_{0,4}\vec{q} + t_{1,4}\vec{\imath} + t_{2,4}\vec{\jmath} + t_{3,4}\vec{k}$

EQ.55 $t_{0,4} = t_{0,2}$

EQ.56 $t_{1,4} = t_{1,2}\gamma_{1,1} + t_{2,2}\mu_1 + t_{3,2}v_1$

EQ.57 $t_{2,4} = t_{1,2}\gamma_{1,2} + t_{2,2}\mu_2 + t_{3,2}v_2$

EQ.58 $t_{2,4} = t_{1,2}\gamma_{1,3} + t_{2,2}\mu_3 + t_{3,2}v_3$

Recall that $\vec{t} = \vec{x} + \vec{u}$ and therefore $\vec{x} = \vec{t} - \vec{u}$ and that $\vec{u} = \frac{1}{2}(\vec{a} + \vec{b})$.

EQ.59 $\vec{t}_1 = \vec{x}_1 + \vec{u} = t_{0,3}\vec{q} + t_{1,3}\vec{\imath} + t_{2,3}\vec{\jmath} + t_{3,3}\vec{k}$

EQ.60 $\vec{t}_2 = \vec{x}_2 + \vec{u} = t_{0,4}\vec{q} + t_{1,4}\vec{\imath} + t_{2,4}\vec{\jmath} + t_{3,4}\vec{k}$

EQ.61 $\vec{u} = u_0\vec{q} + u_1\vec{\imath} + u_2\vec{\jmath} + u_3\vec{k}$

EQ.62 $\vec{x}_1 = (t_{0,3} - u_0)\vec{q} + (t_{1,3} - u_1)\vec{\imath} + (t_{2,3} - u_2)\vec{\jmath} + (t_{3,3} - u_3)\vec{k}$

EQ.63 $\vec{x}_2 = (t_{0,4} - u_0)\vec{q} + (t_{1,4} - u_1)\vec{\imath} + (t_{2,4} - u_2)\vec{\jmath} + (t_{3,4} - u_3)\vec{k}$

The above two equations satisfy the original query $\vec{f} = \vec{x}^2 + \vec{x}\,\vec{b} + \vec{a}\,\vec{x}$, proving that all Quadratic Equations over the Quaternions adhere to the same closed form solution.

Q.E.D.

Appendix B: The M-th Root of N-Unity for Class of Algebraic Hypercomplex Numbers of Even Dimensions, The Great Circle Theorem

Assuming that we are in a Hypercomplex Space that is Cayley Algebraic, then let $\sqrt[n,m]{\vec{x}}$ be the function that returns the m^{th} principal root unity for $\vec{x}$.

EQ.11d Let $\vec{q}$ be the observation vector.

EQ.12d Let D be the set of pairwise orthogonal imaginary unit vectors, $|D| = n$ and n must be odd, such that $|D \cup \{\vec{q}\}|$ is even.

EQ.13d Let $\vec{x} = \alpha_0\vec{q} + \sum_{z=1}^{z=n} \alpha_z\vec{d}_z$ such that $\forall z, \alpha_z \in \mathbb{R}$

EQ.14d Let $\beta = +\sqrt{\sum_{z=1}^{z=n} \alpha_z^2}$, which is the real number magnitude of the imaginary part of $\vec{x}$.

EQ.15d Let $\gamma = +\sqrt{\alpha_0^2 + \sum_{z=1}^{z=n} \alpha_z^2}$, which is the real number magnitude of $\vec{x}$.

EQ.16d Let $\vec{\lambda} = \frac{1}{\beta}(\vec{x} - \alpha_0\vec{q}\,)$, compelling $\vec{\lambda}$ to be a unit vector.

EQ.17d Let $\tau = \frac{1}{\beta}\alpha_0$, giving us the ratio between the magnitudes of the real part and the imaginary part.

EQ.18d Let $\theta = ATAN2\left(\frac{1}{\tau}\right) = ACOTAN2(\tau)$, that is, the four-quadrant arccotagent of τ.

EQ.19d $\vec{x} = \gamma(\vec{q}cos\theta + \vec{\lambda}sin\theta)$

EQ.20d $\sqrt[n,m]{\vec{x}} = (\sqrt[n]{\gamma})\left(\vec{q}cos\left(\frac{\theta+2\pi m}{n}\right) + \vec{\lambda}sin\left(\frac{\theta+2\pi m}{n}\right)\right), \forall(m,n) \in \mathbb{Z}, m \leq n$

Appendix B2: Corollary: The Square Root of a Quaternion, The Well Defined Positive and Negative Square Roots

For a quaternion $\vec{x} = \alpha_0\vec{q} + \alpha_1\vec{\imath} + \alpha_2\vec{\jmath} + \alpha_3\vec{k}$, the square root is given by:
EQ.21d

$$+\sqrt{\vec{x}} = \left(+\sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2}\right)\left(\vec{q}cos\left(0 + \frac{1}{2}ATAN2\left(\frac{+\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}{\alpha_0}\right)\right)\right.$$
$$\left. + \frac{1}{+\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}(\vec{x} - \alpha_0\vec{q}\,)sin\left(0 + \frac{1}{2}ATAN2\left(\frac{+\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}{\alpha_0}\right)\right)\right)$$

$$-\sqrt{\vec{x}} = \left(+\sqrt{\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2}\right)\left(\vec{q}cos\left(\pi + \frac{1}{2}ATAN2\left(\frac{+\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}{\alpha_0}\right)\right)\right.$$
$$\left. + \frac{1}{+\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}(\vec{x} - \alpha_0\vec{q}\,)sin\left(\pi + \frac{1}{2}ATAN2\left(\frac{+\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}{\alpha_0}\right)\right)\right)$$

Appendix C: The Quadratic Equation , The Trivial Case of Symmetric Roots, For All Hypercomplex Dimensions

EQ.1e $-\vec{C} = \vec{x}^2 + \vec{x}\vec{y} + \vec{y}\vec{x}$ can be readily solved for $\vec{x}$ even if $\vec{C}$ and $\vec{y}$ are not on the same Great Circle.

EQ.2e Let $\vec{t} = (\vec{x} + \vec{y})$, such that $\vec{x} = (\vec{t} - \vec{y})$

EQ.3e $-\vec{C} = (\vec{t} - \vec{y}\,)^2 + (\vec{t} - \vec{y}\,)\vec{y} + \vec{y}(\vec{t} - \vec{y}\,)$

EQ.4e $-\vec{C} = (\vec{t}^2 - \vec{t}\vec{y} - \vec{y}\vec{t} + \vec{y}^2) + (\vec{t}\vec{y} - \vec{y}^{\,2}) + (\vec{y}\vec{t} - \vec{y}^2\,)$

EQ.5e $-\vec{C} = \vec{t}^2 - \vec{y}^2$

EQ.6e $\vec{t}^2 = \vec{y}^2 - \vec{C}$

EQ.7e $\vec{t} = \pm\sqrt{\vec{y}^2 - \vec{C}} \Leftrightarrow (\vec{x} + \vec{y}) = \pm\sqrt{\vec{y}^2 - \vec{C}}$

EQ.8e $\vec{x} = -\vec{y} \pm \sqrt{\vec{y}^2 - \vec{C}}$

EQ.9e $\vec{0} = \left(-\vec{y} \pm \sqrt{\vec{y}^2 - \vec{C}}\right)^2 + \left(-\vec{y} \pm \sqrt{\vec{y}^2 - \vec{C}}\right)\vec{y} + \vec{y}\left(-\vec{y} \pm \sqrt{\vec{y}^2 - \vec{C}}\right) + \vec{C}$

Q.E.D

It is our goal to transform the earlier equation, $-\vec{c} = \vec{w}^2 + \vec{w}\vec{y}$, into the Symmetric Case via a series of additional substitutions.

The Symmetric Case occurs when $-\vec{C} = \vec{x}^2 + ([\,\vec{y}\,]_{4L} + [\,\vec{y}\,]_{4R})\vec{x}$.

Statements and Declarations: I have nothing to declare.

Quadratic Quaternionic Calculator, Closed Form Solution

https://docs.google.com/spreadsheets/d/1X8sKNNuxFq5HLq5qk-93SDVdk6yeh14Xp_OV3bVl2ZI/edit?usp=sharing

	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W
1	Roots	q	i	j	k	Equation #	Input A	q	i	j	k	Input B	q	i	j	k	Input F	q	i	j	k
2	x[1] = Root1	0.8628096	0.0577428	-0.2100223	1.0096663	1	a	-0.6686400	-0.8734909	0.6985943	-0.7593366	b	0.3174079	0.3362282	0.6152269	0.2311811	f	0.2147400	-0.2603369	-0.3192916	0.6827207
3	x[2]= Root2	-0.5115775	0.3123006	0.1216380	-0.1741465	1															
4	Verify Tolerance	Verified				1															
5						1															

References

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