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Optimal Control of Discrete Time-varying System with Multiple Delays and Multiplicative Noises

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Abstract: This paper is concerned with the optimal linear quadratic Gaussian (LQG) control problem for discrete time-varying system with multiple input delays and multiplicative noises. The main contributions are two-fold. Firstly, when the state variables can be observed exactly, we obtain a necessary and sufficient condition for the multiple-delays system in terms of the non-homogeneous relationship between the state and costate, which is the solution to the coupled forward and backward stochastic difference equations. Secondly, when the state variables are partially observed, we derive a suboptimal linear output feedback controller for the discrete-time system based on the obtained results of the optimal LQG control. Numerical examples are shown to illustrate the proposed algorithm.

Keywords: LQG control; multiple delays; discrete time-varying system; multiplicative noises

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1. Introduction

Linear quadratic Gaussian (LQG) control problem stems from the optimal stochastic control theory of the systems with additive Gaussian white noises and state/control-dependent, which combines the concept of linear quadratic regulators for full state feedback and Kalman filters for state estimation [1–3]. Recently, the optimal LQG control has been applied in various fields, such as the robots of power substation, all-electric vehicles, electrical safety engineering networked control systems (NCSs) [5–7,9]. Specificly, for mobile monitoring robot in a ultrahigh-voltage power substation, the LQG conotrol is proposed to minimize the difference between the actual SNIR and its expectation and the change in transmitting power[4]. These motivate us to study the more complicated LQG control systems with multiple input delays and multiplicative noises.

It is generally known that random time delay and packet dropout always occur in the data transmission of NCSs. Many literatures have been investigated on LQG control problems with input delays and packet loss [8,10–14]. Basin [8] presented an optimal linear regulator (LQR) with input delay by using the duality principle. Cacace[10] studied the LQG problems for linear system with single input delay. Matni[11] presented an explicit solution to a two-player distributed LQG problem in which communication between controllers occurs across a communication link with varying delay. Basin[12] further established a necessary and sufficient condition of the optimal LQR control for the linear system with multiple input delays. Zhang[14] studied the classical LQR problem with multiple input delays for both continuous-time and discrete-time cases.

On the other hand, packet dropout is generally described as the multiplicative noises. Many references have focused on the LQG system with multiplicative noises [15–17]. Gupta[15] solved the optimal LQG problem with packet-dropping links by decomposing the problem into a standard LQR state-feedback controller designing. Liang[16] studied the optimal control and stabilization problems for NCSs with remote controller and local controller subject to packet dropout. For systems with both input delay and packet dropout, Liang[18] presented the optimal LQR controller, and derived the necessary and sufficient condition for the mean-square stabilization. Liang[19] considered the

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discrete-time LQG system with input delay and multiplicative noises, and obtained both optimal state feedback controller and suboptimal output feedback controller.

The aforementioned literatures are mainly focused on single delay and packet dropout. To our best knowledge, little progress has been made on the optimal LQG control for time-varying systems with multiple input delays and multiplicative noises.

Motivated by the work of [18–20], this paper studies the optimal LQG control for discrete time-varying system involving with multiple delays and multiplicative noises. The main contributions of this paper are summarized as follows: 1) When the state variables can be observed exactly, by introducing the stochastic maximum principle for system with multiple delays and multiplicative noises, a solution to the forward backward differential equations (FBSDEs) is obtained based on the coupled Riccati equations. 2) In terms of the solution to the FBSDEs, a necessary and sufficient condition is given for the optimal LQG control. 3) When the state variables are partially observed, we derive a suboptimal linear output feedback controller by linearizing the optimal estimator and neglecting higher order terms.

The rest of the paper is organized as follows. In Section 2, we give the results of optimal state feedback control problem. In Section 3, we derive a suboptimal linear output feedback controller for the LQG systems involving multiple input delays and multiplicative noises. Numerical examples are provided in Section 4. Conclusions are given in Section 5.

Notation: R^n denotes the n-dimensional real Euclidean space. I presents the unit matrix of appropriate dimension. The superscript $'$ denotes the transpose of the matrix. $\{\Omega, \mathcal{F}, \mathcal{P}, \{\mathcal{F}_k\}_{k\geq 0}\}$ denotes a complete probability space on which random variable w_k are defined such that $\{\mathcal{F}_k\}_{k\geq 0}$ is the natural filtration generated by w_k and ν_k, i.e., $\mathcal{F}_k = \sigma\{w_0, \ldots, w_k, \nu_0, \ldots, \nu_k\}$, augmented by all the $\mathcal{P}-$null sets in $\mathcal{F}$. A symmetric $A > 0 (\geq 0)$ means that it is a positive definite (positive semi-definite) matrix. $Tr(A)$ represents the trace of matrix A.

2. State Feedback Controller

When the state variable x_k can be observed exactly, we consider the following discrete time-varying LQG system with multiple input delays and multiplicative noises

$$\begin{aligned} x_{k+1} = & \left[C(k) + \nu_k \bar{C}(k)\right] x_k + \left[D_0(k) + \nu_k \bar{D}_0(k)\right] u_k \\ & + \left[D_d(k) + \nu_k \bar{D}_d(k)\right] u_{k-d} + w_k, \end{aligned} \tag{1}$$

where $x_k \in R^n$ is the state, $u_k \in R^m$ is the input control with the delay $d > 0$, ν_k is the scalar white noise with zero mean and variance ϕ^2, $w_k \in R^n$ is the random variables satisfying $E[w_k|\mathcal{F}_{k-1}] = \bar{w}_k$ and $E[w_k w_k'] = Q_{w_k}$. $C(k), \bar{C}(k), D_i(k)$ and $\bar{D}_i(k)$ with $i = 0, d$ are coefficient matrices with compatible dimensions. ν_k and w_k are correlated with $E[v_k w_k' | \mathcal{F}_{k-1}] = \rho$, $E[v_k w_l'] = 0$, $k \neq l$. The initial state x_0, u_i for $i = -d, \ldots, -1$ are known.

The associated cost function for system (1) is given by

$$J_N = E\left\{\sum_{k=0}^{N} x_k' Q_k x_k + u_k' R_k u_k + x_{N+1}' \mathcal{P}_{N+1} x_{N+1}\right\}, \tag{2}$$

where Q_k and $\mathcal{P}_{N+1}$ are positive semi-definite constant matrices with appropriate dimensions, control cost matrix R_k should be positive definite matrix, and N is the horizon length.

Problem 1. *Find the unique $\mathcal{F}_{k-1}$-measurable state feedback controller u_k, for $k = 0, \ldots, N$, to minimize (2) subject to (1).*

For simplicity, we make the following definitions

$$C_k(k) = C(k) + \nu_k \bar{C}(k), \quad D_k^i(k) = D_i(k) + \nu_k \bar{D}_i(k),$$

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for $i = 0, d$. Then the system (1) becomes

$$x_{k+1} = C_k(k)x_k + D_k^0(k)u_k + D_k^d(k)u_{k-d} + w_k. \tag{3}$$

Following the similar discussion of [19], in virtue of the Pontryagin's maximum principle for (3) and (2), we have

$$\zeta_N = \mathcal{P}_{N+1}x_{N+1}, \tag{4}$$
$$\zeta_{k-1} = E[C_k'(k)\zeta_k|\mathcal{F}_{k-1}] + Q_k x_k, \tag{5}$$
$$0 = E[(D_k^0(k))'\zeta_k + (D_{k+d}^d(k+d))'\zeta_{k+d}|\mathcal{F}_{k-1}] + R_k u_k, \tag{6}$$

for $k = 0, \dots, N$, where ζ_k is the costate with $\zeta_k = 0$ for $k > N$.

For further study, the following coupled Riccati difference equations are given:

$$\begin{aligned}\mathcal{P}_k =& C'(k)\mathcal{P}_{k+1}C(k) + \phi^2\bar{C}'(k)\mathcal{P}_{k+1}\bar{C}(k)\\ &- M_k'\Omega_k^{-1}M_k + Q_k,\end{aligned} \tag{7}$$

where

$$\begin{aligned}\Omega_k =& R_k + D_0'(k)\mathcal{P}_{k+1}D_0(k) + \phi^2\bar{D}_0'(k)\mathcal{P}_{k+1}\bar{D}_0(k)\\ &+ D_d'(k+d)\mathcal{P}_{k+d+1}D_d(k+d) + \phi^2\bar{D}_d'(k+d)\\ &\times \mathcal{P}_{k+d+1}\bar{D}_d(k+d) + D_0'(k)\mathcal{P}_{k+1}^{d-1} + (\mathcal{P}_{k+1}^{d-1})'\\ &\times D_0(k) - \sum_{i=1}^{d}(M_{k+i}^{d-i})'\Omega_{k+i}^{-1}M_{k+i}^{d-i},\end{aligned} \tag{8}$$

$$\begin{aligned}M_k =& D_0'(k)\mathcal{P}_{k+1}C(k) + \phi^2\bar{D}_0'(k)\mathcal{P}_{k+1}\bar{C}(k)\\ &+ (\mathcal{P}_{k+1}^{d-1})'C(k),\end{aligned} \tag{9}$$

with

$$\begin{aligned}M_k^0 =& D_0'(k)\mathcal{P}_{k+1}D_d(k) + \phi^2\bar{D}_0'(k)\mathcal{P}_{k+1}\bar{D}_d(k)\\ &+ (\mathcal{P}_{k+1}^{d-1})'D_d(k),\end{aligned} \tag{10}$$

$$\begin{aligned}M_k^j =& D_0'(k)\mathcal{P}_{k+1}^{j-1} + (\mathcal{P}_{k+j+1}^{d-j-1})'D_d(k+j)\\ &- \sum_{i=1}^{j}(M_{k+i}^{d-i})'\Omega_{k+i}^{-1}M_{k+i}^{j-i},\end{aligned} \tag{11}$$

$$\begin{aligned}\mathcal{P}_k^0 =& C'(k)\mathcal{P}_{k+1}D_d(k) + \phi^2\bar{C}'(k)\mathcal{P}_{k+1}\bar{D}_d(k)\\ &- M_k'\Omega_k^{-1}M_k^0,\end{aligned} \tag{12}$$

$$\mathcal{P}_k^j = C'(k)\mathcal{P}_{k+1}^{j-1} - M_k'\Omega_k^{-1}M_k^j, j = 1, \dots, d-1. \tag{13}$$

The terminal values are given by

$$\begin{aligned}&\mathcal{P}_{N+1}, \quad \mathcal{P}_{N+i+1} = 0, \quad \mathcal{P}_{N+i}^j = 0,\\ &M_{N+i}^j = 0, \quad \Omega_{N+i} = I, \quad i \geq 1, j = 0, \dots, d-1.\end{aligned} \tag{14}$$

Remark 1. *As can be seen that the costate equations (4)-(6) are quite different from those of Liang [19] and Zhang [20]. What's more, the coupled Riccati equations (7)-(13) are more complicated than those in Liang [19] and Zhang [20].*

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It is stressed that the key to solve the optimal LQG control problem is to obtain the solution to the FBSDEs (3) and (4)-(6). We now show the solution to the FBSDEs in the following lemma.

Lemma 1. *Supposing that Ω_k are positive definite for $k = 0, \ldots, N$, the following equation*

$$\zeta_{k-1} = \mathcal{P}_k x_k + \sum_{j=0}^{d-1} \mathcal{P}_k^j u_{j+k-d} + \Phi_k, \tag{15}$$

is the solution to FBSDEs (3) and (4)-(6), with

$$\Phi_k = C'(k)(\mathcal{P}_{k+1}\bar{w}_k - M_k'\Omega_k^{-1}\Sigma_k + \Phi_{k+1}) + \bar{C}'(k)\mathcal{P}_{k+1}\rho, \tag{16}$$

$$\begin{aligned}\Sigma_k =& D_0'(k)(\mathcal{P}_{k+1}\bar{w}_k + \Phi_{k+1}) + \bar{D}_0'(k)\mathcal{P}_{k+1}\rho + D_d'(k)\\ &\times (\mathcal{P}_{k+d+1}\bar{w}_{k+d} + \Phi_{k+d+1}) + \bar{D}_d'(k)\mathcal{P}_{k+d+1}\rho\\ &+\sum_{j=0}^{d-1}(\mathcal{P}_{k+j+1}^{d-j-1})'\bar{w}_{k+j} - \sum_{i=0}^{d}(M_{k+i}^{d-i})'\Omega_{k+i}^{-1}\Sigma_{k+i},\end{aligned} \tag{17}$$

where $\Phi_{k+1} = 0$ and $\Sigma_{k+1} = 0$ for $k \geq N$. Besides, $\mathcal{P}_k$, $\mathcal{P}_k^j$ satisfy the coupled equations (7), (12), (13).

Proof. The proof of Lemma 1 is put into Appendix A. □

Now we are ready to present the solution to Problem 1.

Theorem 1. *There exists the unique $\mathcal{F}_{k-1}$-measurable u_k for Problem 1 if and only if Ω_k, for $k = 0, \ldots, N$, are positive definite. In this case, the optimal controller u_k is given by*

$$u_k = -\Omega_k^{-1} M_k x_k - \Omega_k^{-1}\sum_{j=0}^{d-1} M_k^j u_{j+k-d} - \Omega_k^{-1}\Sigma_k. \tag{18}$$

The associated optimal performance index is as

$$\begin{aligned}J_N^* =& x_0'\mathcal{P}_0 x_0 + 2x_0'\sum_{j=0}^{d-1}\mathcal{P}_0^j u_{j-d} + \sum_{j=0}^{d-1} u_{j-d}'(D_d'(j)\mathcal{P}_{j+1}D_d(j)\\ &+\phi^2\bar{D}_d'(j)\mathcal{P}_{j+1}\bar{D}_d(j))u_{j-d} + 2\sum_{j=0}^{d-1}\sum_{i=0}^{d-1} u_{j-d}'D_d'(j)\\ &\times \mathcal{P}_{j+1}^{i-j-1}u_{i-d} - \sum_{j=0}^{d-1}\sum_{i=0}^{d-1}\sum_{m=0}^{d-1} u_{j-d}'(M_m^{j-m})'\Omega_m^{-1}M_m^{i-m}\\ &\times u_{i-d} + 2x_0'\Phi_0 + 2\sum_{k=0}^{N}\bar{w}_k'\Phi_{k+1} - \sum_{k=0}^{N}\Sigma_k'\Omega_k^{-1}\Sigma_k\\ &+\sum_{k=0}^{N} Tr[\mathcal{P}_{k+1}Q_{w_k}],\end{aligned} \tag{19}$$

where Ω_k, M_k, M_k^j, $\mathcal{P}_k$, $\mathcal{P}_k^j$, Φ_k, Σ_k satisfy the coupled equations (7)-(13),(16),(17) and $\mathcal{P}_k^j = 0$, $M_k^j = 0$ for $j < 0$.

Proof. The proof of Theorem 1 is put into Appendix B. □

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Remark 2. *We make the coefficients of the system (3) and the cost function (2) to be time-invariant. When there is no time delay in system (3), i.e., $d = 0$, we have that $D_d = \bar{D}_d = 0$. Considering the noise-uncorrelated case with $\varpi_k = 0$, it is obviously obtained that the coupled equations (10) and (12) can be rewritten as*

$$M_k^0 = 0, \qquad \mathcal{P}_k^0 = -M_k' \Omega_k^{-1} M_k^0 = 0.$$

Substituting M_k^0 and $\mathcal{P}_k^0$ into (11) and (13), it can be derived that $M_k^j = 0, \mathcal{P}_k^j = 0$ for $j = 0, \ldots, d-1$. Then the difference equations (8) and (9) yield to

$$\Omega_k = R + D_0' \mathcal{P}_{k+1} D_0 + \phi^2 \bar{D}_0' \mathcal{P}_{k+1} \bar{D}_0,$$
$$M_k = D_0' \mathcal{P}_{k+1} C + \phi^2 \bar{D}_0' \mathcal{P}_{k+1} \bar{C}.$$

The optimal controller reduces to

$$u_k = -\Omega_k^{-1} M_k x_k,$$

which is exactly the result of Moore[2].

Remark 3. *When the system (3) is a time-invariant system, (3) can be rewritten as*

$$x_{k+1} = C(k)x_k + D_0(k)u_k + D_d(k)u_{k-d} + w_k$$

with $C(k) = C + \nu_k \bar{C}$, $D_0(k) = D_0 + \nu_k \bar{D}_0$, $D_d(k) = D_d + \nu_k \bar{D}_d$. The performance index becomes

$$J_N = E\left\{\sum_{k=0}^{N} x_k' Q x_k + u_k' R u_k + x_{N+1}' \mathcal{P}_{N+1} x_{N+1}\right\}.$$

By using the results of Theorem 1, the optimal time-invariant LQG controller yields that

$$u_k = -\Omega_k^{-1} M_k x_k - \Omega_k^{-1} \sum_{j=0}^{d-1} M_k^j u_{j+k-d} - \Omega_k^{-1} \Sigma_k,$$

and the minimal cost function is as (19) where the coefficient matrices in Ω_k, M_k, M_k^j, P_k, P_k^j are time-invariant.

In view of obtaining the special case of optimal LQG control for system (3), now we shall show the results for the general system with multiple delays and multiplicative noises.

Consider the following general discrete time-varying system

$$x_{k+1} = C_k(k)x_k + \sum_{i=0}^{d} D_k^i(k) u_{k-i} + w_k, \tag{20}$$

and the cost function is as (2).

Problem 2. *Find the unique $\mathcal{F}_{k-1}$-measurable state feedback controller u_k, for $k = 0, \ldots, N$, to minimize the cost function (2) subject to the system (20).*

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Combining the system (20) and the cost function (2), we apply the Pontryagin's maximum principle to yield the following costate equations:

$$\zeta_N = \mathcal{P}_{N+1} x_{N+1}, \tag{21}$$
$$\zeta_{k-1} = E[C_k'(k)\zeta_k | \mathcal{F}_{k-1}] + Q_k x_k, \tag{22}$$
$$0 = E[\sum_{i=0}^{d} (D_{k+i}^i(k+i))'\zeta_{k+i} | \mathcal{F}_{k-1}] + R_k u_k, \tag{23}$$

with $i = 0, \ldots, d$ for $k = 0, \ldots, N$, and $\zeta_k = 0$ for $k > N$.

We introduce the following coupled Riccati equations subject to the system with multiple deleys:

$$\begin{aligned}\Omega_k =& R_k + \sum_{i=0}^{d} \big(D_i'(k+i)\mathcal{P}_{k+i+1}D_i(k+i) + \phi^2 \bar{D}_i'(k+i) \\ & \times \mathcal{P}_{k+i+1}\bar{D}_i(k+i)\big) + \sum_{i=0}^{d-1} D_i'(k+i)\mathcal{P}_{k+i+1}^{d-i-1} \\ & + \sum_{i=0}^{d-1} (\mathcal{P}_{k+i+1}^{d-i-1})' D_i(k+i) - \sum_{i=1}^{d} (M_{k+i}^{d-i})'\Omega_{k+i}^{-1} M_{k+i}^{d-i},\end{aligned} \tag{24}$$

$$\begin{aligned}M_k^j =& \sum_{i=0}^{j} \big(D_i'(k+i)\mathcal{P}_{k+i+1}D_{i-j+d}(k+i) + \phi^2 \bar{D}_i'(k+i) \\ & \times \mathcal{P}_{k+i+1}\bar{D}_{i-j+d}(k+i)\big) + \sum_{i=0}^{j-1} D_i'(k+i)\mathcal{P}_{k+i+1}^{j-i-1} \\ & + \sum_{i=0}^{j} (\mathcal{P}_{k+i+1}^{d-i-1})' D_{i-j+d}(k+i) - \sum_{i=1}^{j} (M_{k+i}^{d-i})'\Omega_{k+i}^{-1} M_{k+i}^{j-i},\end{aligned} \tag{25}$$

$$\begin{aligned}\mathcal{P}_k^j =& C'(k)\mathcal{P}_{k+1}D_{d-j}(k) + \phi^2 \bar{C}'(k)\mathcal{P}_{k+1}\bar{D}_{d-j}(k) \\ & + C'(k)\mathcal{P}_{k+1}^{j-1} - M_k'\Omega_k^{-1}M_k^j,\end{aligned} \tag{26}$$

for $j = 0, \ldots, d-1$, where the terminal value is as (14).

Now we give the main results for Problem 2 in the following theorem.

Theorem 2. *There exists the unique $\mathcal{F}_{k-1}$-measurable u_k for Problem 2 if and only if Ω_k, for $k = 0, \ldots, N$, are positive definite. In this case, the optimal controller u_k is calculated by*

$$u_k = -\Omega_k^{-1} M_k x_k - \Omega_k^{-1} \sum_{j=0}^{d-1} M_k^j u_{j+k-d} - \Omega_k^{-1}\Sigma_k, \tag{27}$$

where

$$\begin{aligned}\Sigma_k =& \sum_{i=0}^{d} [D_i'(k+i)(\mathcal{P}_{k+i+1}\varpi_{k+i} + \Phi_{k+i+1}) + \bar{D}_i'(k+i)\mathcal{P}_{k+i+1} \\ & \times \rho] + \sum_{j=0}^{d-1} (\mathcal{P}_{k+j+1}^{d-j-1})'\varpi_{k+j} - \sum_{i=1}^{d} (M_{k+i}^{d-i})'\Omega_{k+i}^{-1}\Sigma_{k+i}\end{aligned}$$

and the optimal cost is as

$$
\begin{aligned}
J_N^* =& x_0'\mathcal{P}_0 x_0 + 2x_0' \sum_{j=0}^{d-1} \mathcal{P}_0^j u_{j-d} + \sum_{j=0}^{d-1}\sum_{i=0}^{d-1}\sum_{m=0}^{d-1} u_{j-d}'\big[D_{m+d-j}'(m) \\
&\times \mathcal{P}_{m+1} D_{m+d-i}(m) + \phi^2 \bar{D}_{m+d-j}'(m)\mathcal{P}_{m+1}\bar{D}_{m+d-i}(m) \\
&+ D_{m+d-j}'(m)\mathcal{P}_{m+1}^{i-m-1} + (\mathcal{P}_{m+1}^{i-m-1})' D_{m+d-i}(m) - (M_m^{j-m})' \\
&\times \Omega_m^{-1} M_m^{i-m}\big] u_{i-d} + \sum_{k=0}^{d} Tr[\mathcal{P}_{k+1} Q_{w_k}] 2x_0'\Phi_0 \\
&+ 2\sum_{k=0}^{N} \bar{w}_k'\Phi_{k+1} - \sum_{k=0}^{N} \Sigma_k'\Omega_k^{-1}\Sigma_k
\end{aligned} \tag{28}
$$

where $D_i = 0$ for $i > d$.

In addition, the relationship of the optimal costate ζ_{k-1} and state x_k is as (15) in Lemma 1.

Proof. The proof is similar to that of Theorem 1, and to save the space of the paper, we omit it here. □

3. Output Feedback Controller

When the state variable x_k are partially observed, we study the following discrete-time stochastic system:

$$
\begin{aligned}
x_{k+1} =& [C(k) + \nu_k \bar{C}(k)]x_k + [D_0(k) + \nu_k \bar{D}_0(k)]u_k + [D_d(k) \\
&+ \nu_k \bar{D}_d(k)]u_{k-d} + w_k \\
z_k =& [H(k) + g_k \bar{H}(k)]x_k + e_k
\end{aligned} \tag{29}
$$

where $z_k \in R^q$ is the measurement, g_k is the scalar white noise with zero mean and variance Q_{g_k}, w_k and e_k are Gaussian zero-mean white noises with covariance Q_{w_k} and Q_{e_k}. $H(k)$ and $\bar{H}(k)$ are deterministic matrices with compatible dimensions. In this case, the initial value x_0 is known, ν_k, w_k, g_k and e_k are independent of each other.

Obviously, there exist multiplicative noises ν_k and g_k in system (29). As we can not obtain the exact information of the state by (29), we introduce the state estimation to design the controller instead. We first obtain the linear optimal state estimator for by applying standard filtering results in [3]. Then, we will derive the suboptimal linear state estimate feedback controller through the following linearizations.

The aim of this section is to find the suboptimal linear state estimate feedback controller for system (29) in order to minimize the cost function (2).

First, we introduce the linear optimal state estimator in Lemma 2.

Lemma 2. *Based on the system (29) with input delays and multiplicative noises, the linear optimal estimator is given by*

$$
\begin{aligned}
\hat{x}_{k+1|k} =& E[x_{k+1}|z_0,\ldots,z_k] \\
=& C(k)\hat{x}_{k|k-1} + D_0(k)u_k + D_d(k)u_{k-d} + K_k\tilde{z}_k,
\end{aligned} \tag{30}
$$

where

$$
\begin{aligned}
\tilde{z}_k =& z_k - C(k)\hat{x}_{k|k-1}, \\
K_k =& C(k)\Sigma_{k|k-1}H'(k)\big(H(k)\Sigma_{k|k-1}H'(k) + Q_{g_k}\bar{H}(k) \\
&\times (\hat{x}_{k|k-1}\hat{x}_{k|k-1}' + \Sigma_{k|k-1})\bar{H}'(k) + Q_{e_k}\big)^{-1}
\end{aligned}
$$

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Besides, the estimator error covariance matrix is

$$\begin{aligned}\Sigma_{k+1|k} =&E[(x_{k+1}-\hat{x}_{k+1|k})(x_{k+1}-\hat{x}_{k+1|k})'|z_0,\ldots,z_k]\\ =&C(k)\Sigma_{k|k-1}C'(k)-K_k(H(k)\Sigma'_{k|k-1}C'(k))+\phi^2\\ &\times[\bar{C}(k)(\Sigma_{k|k-1}+\hat{x}_{k|k-1}\hat{x}'_{k|k-1})\bar{C}'(k)+\bar{D}_0(k)u_k\\ &\times u'_k\bar{D}'_0(k)+\bar{D}_d(k)u_{k-d}u'_{k-d}\bar{D}'_d(k)]+Q_{w_k}.\end{aligned}$$

The initial values $\hat{x}_{0|-1}=\bar{x}_0$ *and* $\Sigma_{0|-1}=\bar{\mathcal{P}}^e_0$.

Proof. The proof of Lemma 2 is put into Appendix C.

Now, the state estimation is obtained, and we can consider (30) as the state instead of the unavailable exact state information. Observing (3) and (30), we know that the filter gain K_k on $\hat{x}_{k|k-1}$ should be affine, so that we can apply the results of Theorem 1 in this section. Then, we will linearize the filter gain K_k.

Applying first order of Taylor expansion on K_k through the fixed point $\hat{x}_{k|k-1}=\bar{x}_0$, the linearization of K_k yields

$$K_k= K^0_k+K^1_k(\hat{x}_{k|k-1}-\bar{x}_0)+o(\left\|\hat{x}_{k|k-1}\right\|). \tag{31}$$

Ignoring the quadratic and higher order terms in (31), and plug (30) into it, the approximation of $\hat{x}_{k+1|k}$ becomes

$$\begin{aligned}\hat{x}_{k+1|k}\approx&C(k)\hat{x}_{k|k-1}+D_0(k)u_k+D_d(k)u_{k-d}+(K^0_k\\ &+K^1_k(\hat{x}_{k|k-1}-\bar{x}_0))\tilde{z}_k\\ =&(C(k)+K^1_k\tilde{z}_k)\hat{x}_{k|k-1}+D_0(k)u_k+D_d(k)u_{k-d}\\ &+(K^0_k-K^1_k\bar{x}_0)\tilde{z}_k.\end{aligned} \tag{32}$$

With (30)-(32), the cost function (2) can be reorganized as

$$\begin{aligned}J^e_N\approx&E\Big[\sum_{k=0}^{N}\big(\hat{x}'_{k|k-1}Q_k\hat{x}_{k|k-1}+u'_kR_ku_k+Tr[Q_k\Sigma_{k|k-1}]\big)\\ &+\hat{x}'_{N+1|N}\mathcal{P}^e_{N+1}\hat{x}_{N+1|N}+Tr[\mathcal{P}^e_{N+1}\Sigma_{N+1|N}]\Big].\end{aligned} \tag{33}$$

In this case, the coupled Riccati equations can be derived as

$$\mathcal{P}^e_k=C'(k)\mathcal{P}^e_{k+1}C(k)+\eta_k(K^1_k)'\mathcal{P}^e_{k+1}(K^1_k)-(M^e_k)'(\Omega^e_k)^{-1}M^e_k+Q_k,$$

where

$$\begin{aligned}\Omega^e_k =& R_k+D'_0(k)\mathcal{P}^e_{k+1}D_0(k)+D'_d(k+d)\mathcal{P}^e_{k+d+1}D_d(k+d)\\ &+D'_0(k)\mathcal{P}^{d-1}_{k+1}+(\mathcal{P}^{d-1}_{k+1})'D_0(k)-\sum_{i=1}^{d}(M^{d-i}_{k+i})'\Omega^{-1}_{k+i}M^{d-i}_{k+i},\\ M^e_k =& D'_0(k)\mathcal{P}^e_{k+1}C(k)+(\mathcal{P}^{d-1}_{k+1})'C(k),\end{aligned}$$

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with

$$\begin{aligned}(M_k^0)^e &= D_0'(k)\mathcal{P}_{k+1}^e D_d(k) + ((\mathcal{P}_{k+1}^{d-1})^e)' D_d(k),\\ (M_k^j)^e &= D_0'(k)(\mathcal{P}_{k+1}^{j-1})^e + ((\mathcal{P}_{k+j+1}^{d-j-1})^e)' D_d(k+j)\\ &\quad - \sum_{i=1}^{j}((M_{k+i}^{d-i})^e)'(\Omega_{k+i}^e)^{-1}(M_{k+i}^{j-i})^e,\\ (\mathcal{P}_k^0)^e &= C'(k)\mathcal{P}_{k+1}^e D_d(k) - (M_k^e)'(\Omega_k^e)^{-1}(M_k^0)^e,\\ (\mathcal{P}_k^j)^e &= C'(k)(\mathcal{P}_{k+1}^{j-1})^e - (M_k^e)'(\Omega_k^e)^{-1}(M_k^j)^e,\end{aligned}$$

$j = 1, \ldots, d-1$, with the terminal values

$$\begin{aligned}&\mathcal{P}_{N+1}^e, \quad \mathcal{P}_{N+i+1}^e = 0, \quad (\mathcal{P}_{N+i}^j)^e = 0,\\ &(M_{N+i}^j)^e = 0, \quad \Omega_{N+i}^e = I, \quad i \geq 1, \quad j = 0, \ldots, d-1,\end{aligned}$$

and

$$\begin{aligned}\eta_k =& E[\tilde{z}_k\tilde{z}_k'|z_0, \ldots, z_{k-1}]\\ =& H(k)\Sigma_{k|k-1}H'(k) + Q_{g_k}\bar{H}'(k)E[x_kx_k'|z_0, \ldots, z_{k-1}]\\ & \times \bar{H}'(k) + Q_{e_k},\end{aligned}$$

where

$$\begin{aligned}&E[x_{k+1}x_{k+1}'|z_0, \ldots, z_k]\\ =& C(k)E[x_kx_k'|z_0, \ldots, z_{k-1}]C'(k) + \phi^2\bar{C}'(k)E[x_kx_k'|z_0, \ldots, z_{k-1}]\\ &\times \bar{C}(k) + D_0(k)u_ku_k'D_0'(k) + \bar{D}_0(k)u_ku_k'\bar{D}_0'(k) + D_d(k)u_{k-d}\\ &\times u_{k-d}'D_d'(k) + \bar{D}_d(k)u_{k-d}u_{k-d}'\bar{D}_d'(k)Q_{w_k} + C(k)\hat{x}_{k|k-1}u_k'D_0'(k)\\ &+ \phi^2\bar{C}(k)\hat{x}_{k|k-1}u_k'\bar{D}_0'(k) + C(k)\hat{x}_{k|k-1}u_{k-d}'D_d'(k) + \phi^2\bar{C}(k)\\ &\times \hat{x}_{k|k-1}u_{k-d}'\bar{D}_d'(k) + D_0(k)u_k\hat{x}_{k|k-1}'C'(k) + \phi^2\bar{D}_0(k)u_k\hat{x}_{k|k-1}'\\ &\times \bar{C}'(k) + D_0(k)u_ku_{k-d}'D_d'(k) + \phi^2\bar{D}_0(k)u_ku_{k-d}'\bar{D}_d'(k) + D_d(k)\\ &\times u_{k-d}\hat{x}_{k|k-1}'C'(k) + \phi^2\bar{D}_d(k)u_{k-d}\hat{x}_{k|k-1}'\bar{C}'(k) + D_d(k)u_{k-d}\\ &\times u_k'D_0'(k) + \phi^2\bar{D}_d(k)u_{k-d}u_k'\bar{D}_0'(k).\end{aligned}$$

with the initial value $E[x_0x_0'] = \bar{x}_0\bar{x}_0' + \bar{\mathcal{P}}_0^e$.

Now, we can find the suboptimal controller to minimize the cost function (33) subject to (29), by the results of Theorem 1.

Theorem 3. *The suboptimal linear state estimate feedback controller for system (29) that minimizes the cost function (33) is given by*

$$u_k^e = -(\Omega_k^e)^{-1}M_k^e\hat{x}_{k|k-1} - (\Omega_k^e)^{-1}\sum_{j=0}^{d-1}(M_k^j)^e u_{j+k-d}^e. \tag{34}$$

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The minimized cost function is given by

$$
\begin{aligned}
(J_N^e)^* =& \bar{x}_0' \mathcal{P}_0^e \bar{x}_0 + 2\bar{x}_0' \sum_{j=0}^{d-1} (\mathcal{P}_0^j)^e u_{j-d} + \sum_{j=0}^{d-1} u_{j-d}' D_d'(j) \mathcal{P}_{j+1}^e \\
& \times D_d(j) u_{j-d} + 2\sum_{j=0}^{d-1} \sum_{i=0}^{d-1} u_{j-d}' D_d'(j) (\mathcal{P}_{j+1}^{i-j-1})^e u_{i-d}^e \\
& - \sum_{j=0}^{d-1} \sum_{i=0}^{d-1} \sum_{m=0}^{d-1} (u_{j-d}^e)' ((M_m^{j-m})^e)' (\Omega_m^e)^{-1} (M_m^{i-m})^e \\
& \times u_{i-d}^e + \sum_{k=0}^{N} Tr[\mathcal{P}_{k+1}^e Q_{w_k}],
\end{aligned} \tag{35}
$$

4. Numerical examples

Example 1 Consider the scalar case of time-invariant LQG control system (3) in Remark 3. We consider the case that the additive noise w_k is the zero-mean white noise. The associate parameters are as

$$
\begin{aligned}
& C = 2, \quad \bar{C} = 1, \quad D_0 = 3, \quad \bar{D}_0 = 1, \quad D_d = 2, \quad \bar{D}_d = 2, \\
& d = 5, \quad \phi = 1, \quad Q_w = 1, \quad \rho = 0.6, \quad \bar{\varpi} = 0.2,
\end{aligned}
$$

with the initial values

$$
\begin{aligned}
& x_0 = 1, \quad u_{-5} = -0.5, \quad u_{-4} = 0.8, \quad u_{-3} = -1.2, \\
& u_{-2} = -1, \quad u_{-1} = -0.6,
\end{aligned}
$$

and the cost function (2) with $Q = 1, R = 1, \mathcal{P}_{N+1} = 1$. When the delay $d = 5$, and $N = 30$, by applying Theorem 1 and the equations (7)-(13), direct calculations yield that $\mathcal{P}_k, \Omega_k, M_k, \mathcal{P}_k^j, M_k^j$ for $k = 0, \ldots, N$. It can be obviously known that Ω_k is positive definite for $k = 0, \ldots, N$. Thus, there exists a unique u_k from Theorem 1, and the optimal controller can be calculated with (18), which is shown in Figure 1.

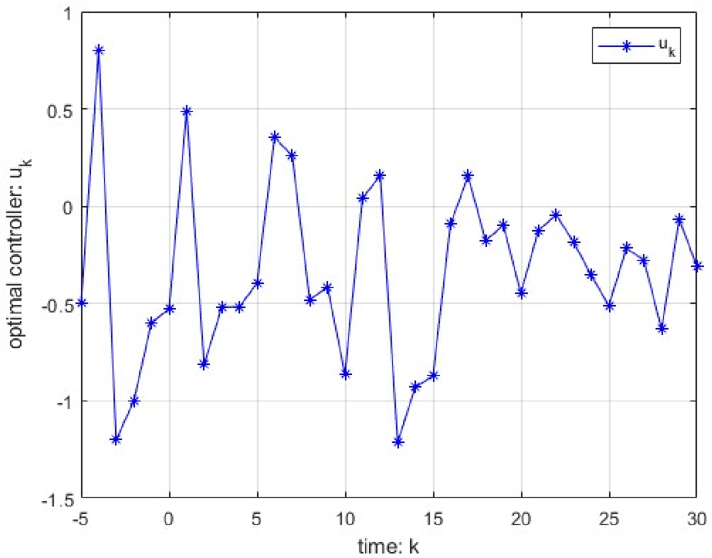

Figure 1. The Optimal Controller u_k.

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Accordingly, the associated optimal value of (17) is J_N^* =107.5150.

In order to illustrate that the proposed LQG controller can minimize performance index, let us consider the time-invariant standard state feedback controller $u_k = -\Omega_k^{-1} M_k x_k$. Based on the above parameters and by substituting into cost function, the controller u_k are shown in Figure 2, and the associated value is J_N^* =255.0603, which confirmed the effectiveness of the algorithm.

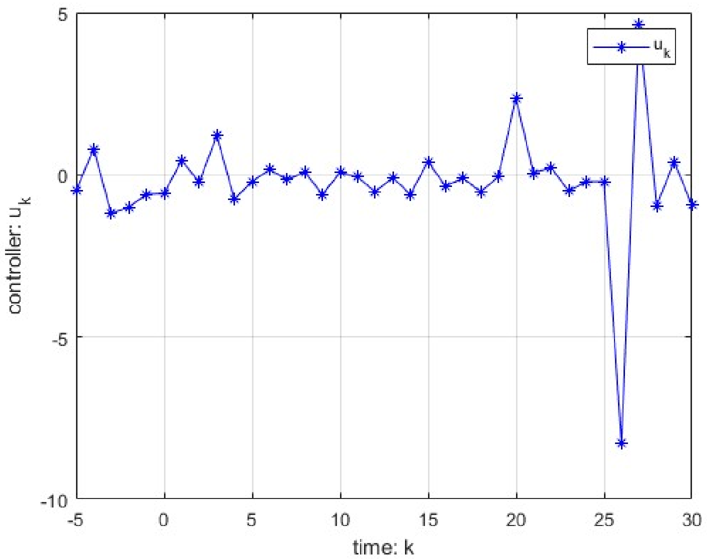

Figure 2. The Non-optimal Controller u_k.

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Example 2 Consider the discrete time-varying LQG control system with multiple delays and multiplicative noises with $x_k \in R^2$, $u_k \in R^2$, and the cost function (2). The associate coefficients are:

$$C(1) = \begin{bmatrix} -0.2 & -0.1 \\ -0.8 & 1.1 \end{bmatrix}, C(2) = \begin{bmatrix} -1.9 & 1.1 \\ -1.4 & -1.9 \end{bmatrix},$$
$$C(3) = \begin{bmatrix} -1.4 & -0.3 \\ 0 & 0.4 \end{bmatrix}, C(4) = \begin{bmatrix} -1.6 & 0.6 \\ 1.9 & -0.8 \end{bmatrix},$$
$$\bar{C}(1) = \begin{bmatrix} -1.3 & 0.8 \\ -1.9 & 0.4 \end{bmatrix}, \quad \bar{C}(2) = \begin{bmatrix} -1.4 & 1.6 \\ -0.7 & 0.4 \end{bmatrix},$$
$$\bar{C}(3) = \begin{bmatrix} -0.9 & 0.3 \\ 0.3 & -1.5 \end{bmatrix}, \bar{C}(4) = \begin{bmatrix} 1.4 & 1 \\ -1.8 & 1.5 \end{bmatrix},$$
$$D_0(1) = \begin{bmatrix} -0.3 & 0 \\ 1.6 & 1.6 \end{bmatrix}, D_0(2) = \begin{bmatrix} -1.9 & -1.2 \\ -0.7 & -0.9 \end{bmatrix},$$
$$D_0(3) = \begin{bmatrix} -1.7 & -1.3 \\ -1.8 & 0.1 \end{bmatrix}, D_0(4) = \begin{bmatrix} 0.5 & -0.2 \\ 0.8 & 1 \end{bmatrix},$$
$$\bar{D}_0(1) = \begin{bmatrix} 0 & 0.8 \\ 1.7 & 0.7 \end{bmatrix}, \quad \bar{D}_0(2) = \begin{bmatrix} -1.9 & 1 \\ 1.3 & 0.8 \end{bmatrix},$$
$$\bar{D}_0(3) = \begin{bmatrix} 1.3 & -1.2 \\ -0.8 & -1.7 \end{bmatrix}, \bar{D}_0(4) = \begin{bmatrix} -1.9 & -1.6 \\ -1.2 & -1.3 \end{bmatrix},$$
$$D_d(1) = \begin{bmatrix} -1.9 & -0.3 \\ 0.1 & 0.5 \end{bmatrix}, D_d(2) = \begin{bmatrix} 1.3 & -1.5 \\ 1 & -0.5 \end{bmatrix},$$
$$D_d(3) = \begin{bmatrix} 1.7 & 0.5 \\ -1 & -1.1 \end{bmatrix}, D_d(4) = \begin{bmatrix} 1.7 & -1.5 \\ 1.4 & 0.6 \end{bmatrix},$$
$$D_d(5) = \begin{bmatrix} 1.8 & -1.9 \\ 0 & -0.3 \end{bmatrix}, D_d(6) = \begin{bmatrix} -0.3 & -0.2 \\ 0.4 & -1.9 \end{bmatrix},$$
$$D_d(7) = \begin{bmatrix} -0.9 & -1 \\ 0.3 & 1.8 \end{bmatrix}, \bar{D}_d(1) = \begin{bmatrix} 0.2 & 0.2 \\ -1.1 & 1.4 \end{bmatrix},$$
$$\bar{D}_d(2) = \begin{bmatrix} -1.9 & -2 \\ -2 & -1.3 \end{bmatrix}, \bar{D}_d(3) = \begin{bmatrix} -1 & -1.4 \\ 1 & 1.3 \end{bmatrix},$$
$$\bar{D}_d(4) = \begin{bmatrix} -1.8 & -1.2 \\ -1.4 & 0.6 \end{bmatrix}, \bar{D}_d(5) = \begin{bmatrix} -1.5 & -0.9 \\ 1.8 & 1.5 \end{bmatrix},$$
$$\bar{D}_d(6) = \begin{bmatrix} -1.8 & 1.9 \\ -0.4 & 0.1 \end{bmatrix}, \bar{D}_d(7) = \begin{bmatrix} -1.7 & 0.5 \\ -1.2 & -0.8 \end{bmatrix},$$
$$\mathcal{P}_{N+1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$
$$Q_3 = \begin{bmatrix} 0.8 & 1 \\ 1 & 0.8 \end{bmatrix}, Q_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$
$$R_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, R_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

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When the delay time is $d = 3$ and the final time is $N = 4$, given the initial value

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u_{-3} = \begin{bmatrix} 1.7 \\ -1.9 \end{bmatrix}, u_{-2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u_{-1} = \begin{bmatrix} 0 \\ 1.6 \end{bmatrix},$$
$$C(0) = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}, \bar{C}(0) = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, D_0(0) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$
$$\bar{D}_0(0) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, D_d(0) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \bar{D}_d(0) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix},$$

by applying Theorem 1 and (8)-(13), it yields that

$$\mathcal{P}_1 = \begin{bmatrix} 25.33 & -2.03 \\ -2.03 & 2.60 \end{bmatrix}, \mathcal{P}_2 = \begin{bmatrix} 9.10 & 1.45 \\ 1.45 & 12.47 \end{bmatrix},$$
$$\mathcal{P}_3 = \begin{bmatrix} 2.83 & -0.76 \\ -0.76 & 2.90 \end{bmatrix}, \mathcal{P}_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\Omega_1 = \begin{bmatrix} 69.41 & 48.05 \\ 48.05 & 47.49 \end{bmatrix}, \Omega_2 = \begin{bmatrix} 28.53 & 5.16 \\ 5.16 & 8.26 \end{bmatrix},$$
$$\Omega_3 = \begin{bmatrix} 9.46 & 1.83 \\ 1.83 & 7.03 \end{bmatrix}, \quad \Omega_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$M_1 = \begin{bmatrix} -59.03 & 31.96 \\ -46.01 & 32.31 \end{bmatrix}, M_2 = \begin{bmatrix} 15.30 & -12.32 \\ 3.33 & 4.43 \end{bmatrix},$$
$$M_3 = \begin{bmatrix} 0.97 & 0.36 \\ 2.39 & 1.84 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For $i = 1, 2, 3, 4$, $\Omega_i > 0$, thus, there is an optimal solution to the LQG system with multiple delays and state/control noises from Theorem 1. Based on the above data, the optimal controller can be calculated as

$$u_0 = \begin{bmatrix} 0.59 \\ -1.35 \end{bmatrix}, u_1 = \begin{bmatrix} -0.96 \\ -0.46 \end{bmatrix}, u_2 = \begin{bmatrix} 1.38 \\ -2.34 \end{bmatrix},$$
$$u_3 = \begin{bmatrix} -0.10 \\ 1.34 \end{bmatrix}, u_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

According to (17), the optimal performance index of system (3) is $J_N^* = 51.1915$.

5. Conclusions

In this paper, the discrete time-varying optimal linear quadratic Gaussian (LQG) control problem involving multiple delays and state/control- dependent noises has been studied. A necessary and sufficient condition for the existence of unique optimal controller to the problem is given, which is based on the obtained maximum principle and the relationship between the state and costate. Under this context, the optimal controller and the minimized performance index are represented. What's more, as the state variables observed partially, the suboptimal linear state estimate feedback controllers for the LQG models with input delays and multiplicative noises are derived.

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Sheng, Xiao Lu and Haixia Wang; data curation, Qiyan Zhang, Chunyang Sheng, Xiao Lu and Haixia Wang; writing—original draft preparation, Qiyan Zhang and Chunyang Sheng; writing—review and editing, Qiyan Zhang, Chunyang Sheng, Xiao Lu and Haixia Wang; funding acquisition, Chunyang Sheng, Xiao Lu and Haixia Wang. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement: The data of this study is included within the article.

Conflicts of Interest: The authors declare no conflict of interest with respect to the research, authorship, and/or publication of this article.

Appendix A The proof of Lemma 1

Utilizing the maximum principle (4)-(6) to system (3) with cost function (2). We can obtain for $k = N$,

$$\begin{aligned}0 =&(D_0(N)'\mathcal{P}_{N+1}C(N) + \phi^2\bar{D}_0'(N)\mathcal{P}_{N+1}\bar{C}(N))x_N + (D_0'(N)\\ &\times \mathcal{P}_{N+1}D_0(N) + \phi^2\bar{D}_0'(N)\mathcal{P}_{N+1}\bar{D}_0(N) + R_N)u_N\\ &+ (D_0'(N)\mathcal{P}_{N+1}D_d(N) + \phi^2\bar{D}_0'(N)\mathcal{P}_{N+1}\bar{D}_d(N))u_{N-d}\\ &+ D_0'(N)\mathcal{P}_{N+1}\varpi_N + \bar{D}_0'(N)\mathcal{P}_{N+1}\rho.\end{aligned}$$

With (9)-(11), the optimal controller u_N is as

$$u_N = -\Omega_N^{-1}M_Nx_N - \Omega_N^{-1}\sum_{j=0}^{d-1}M_N^ju_{j+N-d} - \Omega_N^{-1}\Sigma_N.$$

From (4)-(6), we also have

$$\begin{aligned}\zeta_{N-1}=&(C'(N)\mathcal{P}_{N+1}C(N) + \phi^2\bar{C}'(N)\mathcal{P}_{N+1}C(N) + Q_N)x_N\\ &+ (C'(N)\mathcal{P}_{N+1}D_0(N) + \phi^2\bar{C}(N)\mathcal{P}_{N+1}\bar{D}_0(N))u_N\\ &+ (C'(N)\mathcal{P}_{N+1}D_d(N) + \phi^2\bar{C}(N)\mathcal{P}_{N+1}\bar{D}_d(N))u_{N-d}\\ &+ C'(N)\mathcal{P}_{N+1}\varpi_N + \bar{C}'(N)\mathcal{P}_{N+1}\rho\end{aligned}$$

Substituting (7), (12) and (13), ζ_{N-1} yields

$$\zeta_{N-1} =\mathcal{P}_Nx_N + \sum_{j=0}^{d-1}\mathcal{P}_N^ju_{j+N-d} + \Phi_N.$$

We have verified (15) for $k = N$. Assuming that ζ_{k-1} are as (15) for all $k \geq n+1$ with $n > N-d$, then we will show that (15) also holds for $k = n$. Set u_k to be optimal for all $k \geq n+1$, with equations (3) and (15), ζ_n can be calculated as

$$\begin{aligned}\zeta_n=&\mathcal{P}_{n+1}(C_n(n)x_n + D_n^0(n)u_n + D_n^d(n)u_{n-d} + w_n)\\ &+ \sum_{j=0}^{d-1}\mathcal{P}_{n+1}^ju_{j+n+1-d} + \Phi_{n+1}.\end{aligned} \tag{A1}$$

Insert ζ_n to (5), (5) will become

$$0 = M_nx_n + \Omega_nu_n + \sum_{j=0}^{d-1}M_n^ju_{j+n-d} + \Sigma_n$$

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Thus, the optimal controller is given by

$$u_n = -\Omega_n^{-1} M_n x_n - \Omega_n^{-1} \sum_{j=0}^{d-1} M_n^j u_{j+n-d} - \Omega_n^{-1} \Sigma_n,$$

for $n = N, \ldots, N-d+1$. Using the equations (3),(5) and (A1), ζ_{n-1} yields that

$$\begin{aligned}
&\zeta_{n-1} \\
&= \big(C'(n)\mathcal{P}_{n+1}C(n) + \phi^2 \bar{C}'(n)\mathcal{P}_{n+1}\bar{C}(n) + Q_n - M_n'\Omega_n^{-1}M_n\big) \\
&\quad \times x_n + \big(C'(n)\mathcal{P}_{n+1}D_d(n) + \phi^2 \bar{C}'(n)\mathcal{P}_{n+1}\bar{D}_d(n) - M_n' \\
&\quad \times \Omega_n^{-1} M_n^0\big) u_{n-d} + \sum_{j=1}^{d-1} \big(C'(n)\mathcal{P}_n^{j-1} - M_n'\Omega_n^{-1}M_n^j\big) u_{j+n-d} \\
&\quad - M_n'\Omega_n^{-1}\Sigma_n + C'(n)(\mathcal{P}_{n+1}\varpi_n + \Phi_{n+1}) + \bar{C}'(n)\mathcal{P}_{n+1}\rho \\
&= \mathcal{P}_n x_n + \sum_{j=0}^{d-1} \mathcal{P}_n^j u_{j+n-d} + \Phi_n,
\end{aligned}$$

which implies that (15) holds for $k = n, N-d < n \leq N$.

Then we obtained

$$\begin{aligned}
\zeta_{N-d} &= \mathcal{P}_{N-d+1} x_{N-d+1} + \sum_{j=0}^{d-1} \mathcal{P}_{N-d+1}^j u_{j+N-2d+1} + \Phi_{N-d+1}, \\
u_{N-d+1} &= -\Omega_{N-d+1}^{-1} M_{N-d+1} x_{N-d+1} - \Omega_{N-d+1}^{-1} \sum_{j=0}^{d-1} M_{N-d+1}^j \\
&\quad \times u_{j+k-d} - \Omega_{N-d+1}^{-1} \Sigma_{N-d+1}.
\end{aligned} \tag{A2}$$

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Analogy with the method, assuming that ζ_{k-1} are as (15) for all $k \geq n+1, n = 0, \ldots, N-d$, and we will verify that (15) also holds for $k = n$. As ζ_n is calculated as (A1), then for $n = 0, \ldots, N-d$, (6) will be obtained

$$
\begin{aligned}
0 = {} & \Psi + \big(D_d'(n)\mathcal{P}_{n+d+1}C(n) + \phi^2 \bar{D}_d'(n)\mathcal{P}_{n+d+1}\bar{C}(n)\big)x_{n+d} \\
& + (M_{n+d}^0)'\big(-\Omega_{n+d}^{-1}x_{n+d} - \Omega_{n+d}^{-1}\sum_{j=0}^{d-1} M_{n+d}^j u_{j+n} - \Omega_{n+d}^{-1} \\
& \times \Sigma_{n+d}\big) + D_0'(n)\sum_{j=0}^{d-2}\mathcal{P}_{n+1}^j u_{j+n-d+1} + D_d'(n+d)\sum_{j=0}^{d-2}\mathcal{P}_{n+d+1}^j u_{j+n+1} \\
= {} & \Psi + \big((\mathcal{P}_{n+d-1}^1)'C(n+d) - (M_{n+d-2}^2)'\Omega_{n+d-2}^{-1}M_{n+d-2}\big) \\
& \times x_{n+d-2} - (M_{n+d-2}^2)'\Omega_{n+d-2}^{-1}\sum_{j=0}^{d-1} M_{n+d-2}^j u_{j+n-2} \\
& + \big((\mathcal{P}_{n+d-1}^1)'D_d(n+d-2) + D_0'(n)\mathcal{P}_{n+1}^{d-3}\big)u_{n-2} \\
& + \big((\mathcal{P}_{n+d}^0)'D_d(n+d-1) + D_0'(n)\mathcal{P}_{n+1}^{d-2} - (M_{n+d-1}^1)' \\
& \times \Omega_{n+d-1}^{-1}M_{n+d-1}^0\big)u_{n-1} - (M_{n+d}^0)'\Omega_{n+d}^{-1}\sum_{j=0}^{d-3} M_{n+d}^j u_{j+n} \\
& - (M_{n+d-1}^1)'\Omega_{n+d-1}^{-1}\sum_{j=1}^{d-2} M_{n+d-1}^j u_{j+n-1} + D_0'(n)\sum_{j=0}^{d-4}\mathcal{P}_{n+1}^j \\
& \times u_{j+n-d+1} + D_d'(n+d)\sum_{j=0}^{d-4}\mathcal{P}_{n+d+1}^j u_{j+n+1} \\
& - \sum_{i=d-1}^{d}(M_{n+i}^{d-i})'\Omega_{n+i}^{-1}\Sigma_{n+i} + \sum_{i=d-2}^{d-1}\mathcal{P}_{n+1+i}^{d-1-i}\varpi_{n+i}
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi = {} & \big(D_0'(n)\mathcal{P}_{n+1}C(n) + \phi^2 \bar{D}_0'(n)\mathcal{P}_{n+1}\bar{C}(n)\big)x_n + \big(D_0'(n) \\
& \times \mathcal{P}_{n+1}D_0(n) + \phi^2 \bar{D}_0'(n)\mathcal{P}_{n+1}\bar{D}_0(n) + R_n + D_0'(n)\mathcal{P}_{n+1}^{d-1} \\
& + D_d'(n+d)\mathcal{P}_{n+d+1}D_d(n+d) + \phi^2 \bar{D}_d'(n+d)\mathcal{P}_{n+d+1} \\
& \times \bar{D}_d(n+d)\big)u_n + \big(D_0'(n)\mathcal{P}_{n+1}D_d(n) + \phi^2 \bar{D}_0'(n)\mathcal{P}_{n+1} \\
& \times \bar{D}_d(n)\big)u_{n-d} + D_0'(n)(\mathcal{P}_{n+1}\varpi_n + \Phi_{n+1}) + \bar{D}_0'(n)\mathcal{P}_{n+1}\rho \\
& + D_d'(n+d)(\mathcal{P}_{n+d+1}\varpi_{n+d} + \Phi_{n+1}) + \bar{D}_d'(n+d)\mathcal{P}_{n+d+1}\rho
\end{aligned}
$$

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After inserting (3) and (A2), and combing like terms, we can summarize that

$$\begin{aligned}0 =& \Psi + (\mathcal{P}_{n+1}^{d-1})' x_{n+1} + ((\mathcal{P}_{n+2}^{d-2})' D_d(n+1) + D_0'(n)\mathcal{P}_{n+1}^0 \\ &- (M_{n+1}^{d-1})'\Omega_{n+1}^{-1} M_{n+1}^0) u_{n-d+1} + ((\mathcal{P}_{n+3}^{d-3})' D_d(n+2) \\ &+ D_0(n)\mathcal{P}_{n+1}^1 - \sum_{i=1}^{2} (M_{n+i}^{d-i})'\Omega_{n+i}^{-1} M_{n+i}^{2-i}) u_{n-d+2} + \dots \\ &+ ((\mathcal{P}_{n+d-1}^1)' D_d(n+d-2) + D_0'(n)\mathcal{P}_{n+1}^{d-3} - \sum_{i=1}^{d-2} (M_{n+i}^{d-i})' \\ &\times \Omega_{n+i}^{-1} M_{n+i}^{d-2-i}) u_{n-2} + ((\mathcal{P}_{n+d}^0)' D_d(n+d-1) + D_0'\mathcal{P}_{n+1}^{d-2} \\ &- \sum_{i=1}^{d-1} (M_{n+i}^{d-i})'\Omega_{n+i}^{-1} M_{n+i}^{d-1-i}) u_{n-1} + \sum_{i=1}^{d} (M_{n+i}^{d-i})'\Omega_{n+i}^{-1} \\ &\times M_{n+i}^{d-i} u_n + \sum_{i=1}^{d-1} \mathcal{P}_{n+1+i}^{d-1-i} \varpi_{n+i} - \sum_{i=1}^{d} (M_{n+i}^{d-i})'\Omega_{n+i}^{-1} \Sigma_{n+i} \\ =& M_n x_n + \Omega_n u_n + \sum_{j=0}^{d-1} M_n^j u_{j+n-d} + \Sigma_n.\end{aligned}$$

Now, the optimal controller for $n = 0, \dots, N-d$ is obtained as

$$u_n = -\Omega_n^{-1} M_n x_n - \Omega_n^{-1} \sum_{j=0}^{d-1} M_n^j u_{j+n-d} - \Omega_n^{-1}\Sigma_n.$$

In the same way, substituting u_n into (5), we can also prove that

$$\zeta_{n-1} = \mathcal{P}_n x_n + \sum_{j=0}^{d-1} \mathcal{P}_n^j u_{j+n-d} + \Phi_n, \quad n = 0, \dots, N-d.$$

This completes the proof of the lemma.

Appendix B The proof of Theorem 1

"Necessity": Suppose there exists the unique $\mathcal{F}_{k-1}$-measurable u_k to make the cost function (2) minimized. We will show by induction that $\Omega_k, k = d, \dots, N$ are positive definite and the optimal controller can be designed as (15). Define

$$J(k) = \sum_{i=k}^{N} E\left[x_i' Q_i x_i + u_i' R_i u_i + x_{N+1}' \mathcal{P}_{N+1} x_{N+1}\right],$$

for $k = 0, \dots, N$, and when $k = N$ the above equation becomes

$$\begin{aligned}J(N) =& E\left[x_N' Q_N x_N + u_N' R_N u_N + \left(C_N(N) x_N + D_N^0(N) u_N\right.\right. \\ &\left. + D_N^d(N) u_{N-d} + w_N\right)' \mathcal{P}_{N+1} \left(C_N(N) x_N + D_N^0(N)\right. \\ &\left.\left. \times u_N + D_N^d(N) u_{N-d} + w_N\right)\right].\end{aligned}$$

Using (3), we can obviously know that the uniqueness of the optimal controller only depends on whether $u_N > 0$. Then setting $x_N = 0$, and $u_{N-d} = 0$, $J(N)$ can be presented as

$$\begin{aligned}J(N) =& u_N' \Omega_N u_N + 2u_N' (D_0'(N)\mathcal{P}_{N+1} \varpi_N + \bar{D}_0'(N)\mathcal{P}_{N+1}\rho) \\ &+ Tr[\mathcal{P}_{N+1} Q_{w_N}]. \end{aligned} \quad \text{(A3)}$$

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We know that $J(N)$ is expressed as a quadratic function of u_N, and as there is a unique solution for system (3), then $J(N) > 0$, it follows that $\Omega_N > 0$, i.e. Ω_k is positive definite for $k = N$. In order to accomplish the proof, we assume $\Omega_k > 0$ for all $k \geq n+1$. Then we will prove that $\Omega_n > 0$. With (3), (5) and (6), for $k \geq n+1$, we construct that

$$\begin{aligned}
&E\left[x_k'\zeta_{k-1} - x_{k+1}'\zeta_k\right] \\
=&E\left[x_k'Q_kx_k + u_k'R_ku_k\right] + E\left[u_k'(D_{k+d}^d(k+d))'\zeta_{k+d} \right.\\
&\left. - u_{k-d}'(D_k^d(k))'\zeta_k\right] - E\left[w_k'\zeta_k\right].
\end{aligned}$$

Adding from $k = n+1$ to $k = N$ on both sides of the above equation in order to get the form of $J(N)$, we have

$$\begin{aligned}
E[x_{n+1}'\zeta_n - x_{N+1}'\zeta_N] =& \sum_{k=n+1}^{N} E\left[x_k'Q_kx_k + u_k'R_ku_k\right] + \sum_{k=n+1}^{N} \\
E\left[u_k'(D_{k+d}^d(k+d))'\zeta_{k+d} - u_{k-d}'(D_k^d(k))'\zeta_k\right] -& \sum_{k=n+1}^{N}\left[w_k'\zeta_k\right].
\end{aligned}$$

Then

$$\begin{aligned}
&E\Big[\sum_{k=n+1}^{N}(x_k'Q_kx_k + u_k'R_ku_k) + x_{N+1}'\mathcal{P}_{N+1}x_{N+1}\Big] \\
=&E\Big[x_{n+1}'\zeta_n + \sum_{k=n+1}^{n+d} u_{k-d}'(D_k^d(k))'\zeta_k + \sum_{k=n+1}^{N} w_k'\zeta_k\Big].
\end{aligned}$$

Using (2), it yields that

$$\begin{aligned}
J(n) =& E\Big[x_{n+1}'\zeta_n + x_n'Q_nx_n + u_n'R_nu_n + \sum_{k=n+1}^{n+d} u_{k-d}'(D_k^d(k))'\zeta_k \\
&+ \sum_{k=n+1}^{N} w_k'\zeta_k\Big].
\end{aligned} \tag{A4}$$

Setting $x_n = 0$, $u_{n-i} = 0$ as same as the condition $k = N$. And plugging (15) into (A4), we obtain

$$\begin{aligned}
J(n) =& u_n'\big(D_0'(n)\mathcal{P}_{n+1}D_0(n) + \phi^2\bar{D}_0'(n)\mathcal{P}_{n+1}\bar{D}_0(n) + R_n \\
&+ D_0'(n)\mathcal{P}_{n+1}^{d-1} + D_d'(n+d)\mathcal{P}_{n+d+1}D_d(n+d) + \phi^2 \\
&\times \bar{D}_d'(n+d)\mathcal{P}_{n+d+1}\bar{D}_d(n+d) + (\mathcal{P}_{n+1}^{d-1})'D_0(n) \\
&- \sum_{i=1}^{d}(M_{n+i}^{d-i})'\Omega_{n+i}^{-1}M_{n+i}^{d-i}\big)u_n + \sum_{k=n}^{N} w_k'\zeta_k + u_n'(D_0'(n) \\
&\times \mathcal{P}_{n+1}\varpi_n + \bar{D}_0'(n)\mathcal{P}_{n+1}\rho) + u_n'(D_d'(n+d)\mathcal{P}_{n+d+1} \\
&\times \varpi_n + \bar{D}_d'(n+d)\mathcal{P}_{n+d+1}\rho) + u_n'\sum_{i=0}^{d-1}(\mathcal{P}_{n+d-i}^i)' \\
&\times \varpi_{n+d-1-i} + D_0'(n)\Phi_{n+1} + D_d'(n+d)\Phi_{n+d+1}.
\end{aligned}$$

Similarly to the case $\Omega_N > 0$ above, we obviously get $\Omega_n > 0$ for all $k = 0, \ldots, N$. This ends the proof of necessity.

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"Sufficiency": Suppose $\Omega_k > 0$ for $k \geq 0$ is true, we will show the uniqueness of the $\mathcal{F}_{k-1}$-measurable u_k to minimize (2). Denoted by

$$\begin{aligned}
&\mathcal{V}_k(x_k)\\
=&E\Big[x_k'\mathcal{P}_k x_k+2x_k'\sum_{j=0}^{d-1}\mathcal{P}_k^j u_{j-d+k}+\sum_{j=0}^{d-1} u_{j-d+k}'\big((D_{k+j}^d(k+j)\big)'\\
&\times\mathcal{P}_{k+j+1}D_{k+j}^d(k+j)\big)u_{j-d+k}+2\sum_{j=0}^{d-1}\sum_{i=0}^{d-1}u_{j-d+k}'\\
&\times(D_{k+j}^d(k+j))'\mathcal{P}_{k+j+1}^{i-j-1}u_{i-d+k}-\sum_{j=0}^{d-1}\sum_{i=0}^{d-1}\sum_{m=0}^{d-1}u_{j-d+k}'\\
&\times(M_{k+m}^{j-m})'\Omega_{k+m}^{-1}M_{k+m}^{i-m}u_{i-d+k}\Big]+2x_k'\Phi_k.
\end{aligned}$$

First, for $\mathcal{V}_{k+1}(x_{k+1})$, using the equivalent substitution $j=j+1$, $i=i+1$, and $m=m+1$, we calculate as follows

$$\begin{aligned}
&\mathcal{V}_{k+1}(x_{k+1})\\
=&E\Big[x_{k+1}'\mathcal{P}_{k+1}x_{k+1}+2x_{k+1}'\sum_{j=0}^{d-1}\mathcal{P}_{k+1}^{j-1}u_{j-d+k}+\sum_{j=0}^{d-1}u_{j-d+k}'\\
&\times(D_{k+j}^d(k+j))'\mathcal{P}_{k+j+1}D_{k+j}^d(k+j)u_{j-d+k}+2\sum_{j=0}^{d-1}\sum_{i=0}^{d-1}\\
&u_{j-d+k}'(D_{k+j}^d(k+j))'\mathcal{P}_{k+j+1}^{i-j-1}u_{i-d+k}-\sum_{j=0}^{d-1}\sum_{i=0}^{d-1}\sum_{m=0}^{d-1}u_{j-d+k}'\\
&\times(M_{k+m}^{j-m})'\Omega_{k+m}^{-1}M_{k+m}^{i-m}u_{i-d+k}\Big]+E\big[2x_{k+1}'\mathcal{P}_{k+1}^{d-1}u_k+u_k'\\
&\times(D_{k+d}^d(k+d))'\mathcal{P}_{k+d+1}D_{k+d}^d(k+d)u_k-u_{k-d}'(D_k^d(k))'\\
&\times\mathcal{P}_{k+1}D_k^d(k)u_{k-d}+2\sum_{j=0}^{d-1}u_{j-d+k}'(D_{k+j}^d(k+j))'\mathcal{P}_{k+1+j}^{d-1-j}u_k\\
&-2\sum_{i=0}^{d-1}u_{k-d}'(D_k^d(k))'\mathcal{P}_{k+1}^{i-1}u_{i-d+k}-2u_{k-d}(D_k^d(k))'\mathcal{P}_{k+1}^{d-1}u_k\\
&+\sum_{j=0}^{d-1}\sum_{i=0}^{d-1}u_{j-d+k}'(M_k^j)'\Omega_k^{-1}M_k^i u_{k-d+i}-\sum_{m=0}^{d-1}u_k'(M_{k+m}^{d-m})'\\
&\times\Omega_{k+m}^{-1}M_{k+m}^{d-m}u_k-u_k'(M_{k+d}^0)'\Omega_{k+d}^{-1}M_{k+d}^0u_k+u_k'(M_k^d)'\\
&\times\Omega_k^{-1}M_k^d u_k-\sum_{j=0}^{d-1}\sum_{m=0}^{d-1}u_{j-d+k}'(M_{k+m}^{j-m})'\Omega_{k+m}^{-1}M_{k+m}^{d-m}u_k\\
&-\sum_{i=0}^{d-1}\sum_{m=0}^{d-1}u_k'(M_{k+m}^{d-m})'\Omega_{k+m}^{-1}M_{k+m}^{i-m}u_{i-d+k}\Big]+2x_{k+1}'\Phi_{k+1}.
\end{aligned}$$

Construct the equation $\mathcal{V}_k(x_k) - \mathcal{V}_{k+1}(x_{k+1})$, then we have

$$\begin{aligned}
&\mathcal{V}_k(x_k) - \mathcal{V}_{k+1}(x_{k+1}) \\
=&x_k' Q_k x_k + u_k' R_k u_k + (u_k + \Omega_k^{-1} M_k x_k + \Omega_k^{-1} \sum_{j=0}^{d-1} M_k^j u_{j+k-d} \\
&+\Omega_k^{-1}\Sigma)' \Omega_k (u_k + \Omega_k^{-1} M_k x_k + \Omega_k^{-1} \sum_{j=0}^{d-1} M_k^j u_{j+k-d} \Omega_k^{-1} \\
&\times \Sigma_k) + \Sigma_k' \Omega_k^{-1} \Sigma_k - Tr[\mathcal{P}_{k+1} Q_{w_k}] - 2\bar{w}_k' \Phi_{k+1}.
\end{aligned} \tag{A5}$$

Denote

$$\Delta_k = u_k + \Omega_k^{-1} M_k x_k + \Omega_k^{-1} \sum_{j=0}^{d-1} M_k^j u_{j+k-d} + \Omega_k^{-1} \Sigma_k, \tag{A6}$$

and by virtue of (A6) and (7)-(13), and adding from $k = 0$ to $k = N$ on both sides of (A5), then we get

$$\begin{aligned}
&\mathcal{V}_0(x_0) - \mathcal{V}_{N+1}(x_{N+1}) \\
=&\sum_{k=0}^{N} [x_k' Q_k x_k + u_k' R_k u_k - \Delta_k' \Omega_k \Delta_k + \Sigma_k' \Omega_k^{-1} \Sigma_k \\
&- 2\bar{w}_k' \Phi_{k+1} - Tr[\mathcal{P}_{k+1} Q_{w_k}]]
\end{aligned}$$

Then the cost function (2) becomes

$$\begin{aligned}
J_N = &\mathcal{V}_0(x_0) + \sum_{k=0}^{N} (\Delta_k' \Omega_k \Delta_k - \Sigma_k' \Omega_k^{-1} \Sigma_k + 2\bar{w}_k' \Phi_{k+1}) \\
&+ \sum_{k=0}^{N} Tr[\mathcal{P}_{k+1} Q_{w_k}].
\end{aligned}$$

As $\Omega_k > 0$, the unique optimal controller must match the condition $\Delta_k = 0$. In this case, the cost function (2) will be the minimum, i.e., the optimal controller is

$$u_k^* = -\Omega_k^{-1} M_k x_k - \Omega_k^{-1} \sum_{j=0}^{d-1} M_k^j u_{j+k-d} - \Omega_k^{-1} \Sigma_k. \tag{A7}$$

and the optimal cost is as (19).

Above all, the proof of sufficiency is completed.

Appendix C The proof of Lemma 2

By applying standard filtering results in [?], we can obtain the linear optimal estimator for system (29) as follows.

$$\begin{aligned}
\hat{x}_{k+1|k} =& E[x_{k+1} | z_0, \ldots, z_k] = E[x_{k+1} | \tilde{z}_0, \ldots, \tilde{z}_k] \\
=& E[x_{k+1} | \tilde{z}_k] + E[x_{k+1} | \tilde{z}_0, \ldots, \tilde{z}_{k-1}] - E[x_{k+1}]
\end{aligned}$$

In view of the jointly gaussian nature of x_{k+1} and $\tilde{z}_k$, we know

$$E[x_{k+1} | \tilde{z}_k] = E[x_{k+1}] + cov(x_{k+1}, \tilde{z}_k)[cov(\tilde{z}_k, \tilde{z}_k)]^{-1} \tilde{z}_k. \tag{A8}$$

Using (29) and the orthogonality of $\hat{x}_{k|k-1}$ and $x_k - \hat{x}_{k|k-1}$, the covariance matrixes yield

$$\begin{aligned} cov(x_{k+1}, \tilde{z}_k) &= C(k)\Sigma_{k|k-1}H'(k), \\ cov(\tilde{z}_k, \tilde{z}_k) &= H(k)\Sigma_{k|k-1}H'(k) + Q_{g_k}\bar{H}(k)(\hat{x}_{k|k-1}\hat{x}'_{k|k-1} \\ &\quad + \Sigma_{k|k-1})\bar{H}'(k) + Q_{e_k}, \end{aligned}$$

where $x_k - \hat{x}_{k|k-1}$ is independent of e_k with zero mean,and the error covariance matrix

$$\begin{aligned} \Sigma_{k+1|k} =& C(k)\Sigma_{k|k-1}C'(k) - K_k(H(k)\Sigma'_{k|k-1}C'(k)) + \phi^2 \\ & \times [\bar{C}(k)(\Sigma_{k|k-1} + \hat{x}_{k|k+1}\hat{x}'_{k|k+1})\bar{C}'(k) + \bar{D}_0(k) \\ & \times u_k u'_k \bar{D}'_0(k) + \bar{D}_d(k) u_{k-d} u'_{k-d} \bar{D}'_d(k)] + Q_{f_k}. \end{aligned}$$

Substituting above equations into (A8), it becomes

$$\hat{x}_{k+1|k} = C(k)\hat{x}_{k|k-1} + D_0(k)u_k + D_d(k)u_{k-d} + K_k\tilde{z}_k$$

with

$$\begin{aligned} K_k =& C(k)\Sigma_{k|k-1}H'(k)\big(H(k)\Sigma_{k|k-1}H'(k) + Q_{g_k}\bar{H}(k) \\ & \times (\hat{x}_{k|k-1}\hat{x}'_{k|k-1} + \Sigma_{k|k-1})\bar{H}'(k) + Q_{e_k}\big)^{-1}. \end{aligned}$$

The proof of Lemma 2 is completed.

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