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The (α, p)-Golden metric manifolds and their submanifolds

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Abstract: The notion of *Golden structure* was introduced 15 years ago by the present authors and know a constant interest by several geometers. Now, we propose a new generalization apart of that called *metallic structure* and considered also by the authors. By adding a compatible Riemannian metric we focus on the study of submanifolds in this setting.

1. Introduction

The Golden and the metallic structures (studied in [8],[12],[11],[10]) are particular cases of polynomial structure on a manifold, which was generally defined by Goldberg et al. in [24] and [23]).

A polynomial structure J of degree 2, on a smooth manifold $\overline{M}$, satisfying $J^2 = pJ + qId$, where Id is the identity tensor field and $p, q \in \mathbb{N}^*$ is called a *metallic structure* and $(\overline{M}, J)$ is called an *almost metallic manifold.* In particular, for $p = q = 1$ the metallic structure J becomes a *Golden structure*.

The *real metallic numbers*, $\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}$, are the positive solution of the equation $x^2 - px - q = 0$, where p, q are positive integer numbers and $p^2 + 4q > 0$ ([10]). These $\sigma_{p,q}$ numbers are members of the *metallic means family*, as generalizations of the Golden number $\phi = \frac{1 + \sqrt{5}}{2}$) ([26],[27],[28]).

The complex version of the above numbers, namely the *complex metallic numbers*, $\sigma_{p,q}^c = \frac{p + \sqrt{p^2 - 6q}}{2}$ appear as solution of the equation $x^2 - px + \frac{3}{2}q = 0$, where p, q are real numbers satisfying $q \geq 0$ and $p^2 < 6q$. Moreover, an almost complex metallic structure is defined as an (1,1)-tensor field J_M which satisfied the relation: $J_M^2 - pJ_M + \frac{3}{2}qId = 0$ ([30]). For $p = q = 1$, the almost complex metallic structure becomes complex golden structure, defined in [8].

In ([4]) the authors introduced the α-*metallic numbers* of the form $\frac{p + \sqrt{\alpha(p^2 + 4q)}}{2}$ (where p, q are positive integer numbers and $p^2 + 4q > 0$) and they classified α-metallic metric manifolds, using the α-metallic structure, defined by: $\varphi^2 = p\varphi + \frac{p^2(\alpha - 1) + 4q\alpha}{4}Id$. A classification of almost Golden Riemannian manifolds with null trace is obtained in [2] and [3].

In this paper we consider the (α, p)-metallic means family which contains the (α, p)-*Golden numbers*, obtained as the root of the polynomial $X^2 - pX - \frac{5\alpha - 1}{4}p^2$, where $\alpha \in \{-1, 1\}$, $p \in \mathbb{R}^*$ and hence we obtain a new structure of degree 2, called an *almost (α, p)-Golden structure*. Moreover, we study the properties of a Riemannian manifold endowed with a $\Phi_{\alpha,p}$ structure and a compatible Riemannian metric g, called this data as *almost (α, p) Golden Riemannian manifold*. This manifold is a generalization of an almost Hermitian Golden manifolds ([1]). Some similarly manifolds, such that: holomorphic Golden Norden Hessian Manifolds ([22]), almost Golden Riemannian manifolds ([2], [3]) and α-golden manifolds ([5]) have been studied, using a pure metric g, i.e. $g(\phi X, Y) = g(X, \phi Y)$.

2. The almost (α,p)-Golden structure

Let $\overline{M}$ be a real smooth manifold. If $\overline{M}$ is endowed by an endomorphism J_1 such that $J_1^2 = Id$ (where Id is the identity or Kronecker endomorphism) then J_1 is called an *almost product structure* and $(\overline{M}, J_1)$ is called *almost product manifold*.

An endomorphism J_{-1} on $\overline{M}$ is called an *almost complex structure* on $\overline{M}$ if $J_{-1}^2 = -Id$. The pair $(\overline{M}, J_{-1})$ is then an *almost complex manifold* and the dimension of $\overline{M}$ is even, say $2m$.

These two structures can be unified under the notions of α-structure ([4],[6]), noted by J_α; see also the definition 1.2 of [7].

Definition 1. The endomorphism J_α of type (1,1) is called α-structure, if it is satisfy:

$$J_\alpha^2 = \alpha \cdot Id, \tag{2.1}$$

on the even dimensional manifold $\overline{M}$, where $\alpha \in \{-1, 1\}$.

Let us introduce the *(α, p)-Golden numbers* given by:

$$\varphi_{\alpha,p} = p\frac{1 + \sqrt{5\alpha}}{2}, \tag{2.2}$$

as the solution of the equation $x^2 - px - \frac{5\alpha-1}{4}p^2 = 0$, where $\alpha \in \{-1, 1\}$ and $p \in \mathbb{R}^*$.

Using these numbers, we introduce a new structure on the manifold $\overline{M}$ (of even dimension), which generalizes both the almost Golden structure and the almost complex Golden structure ([8]).

Definition 2. An endomorphism $\Phi_{\alpha,p}$ (where $p \in \mathbb{R}^*$ and $\alpha \in \{-1, 1\}$) satisfying the equality:

$$\Phi_{\alpha,p}^2 = p\Phi_{\alpha,p} + \frac{5\alpha - 1}{4}p^2 \cdot Id, \tag{2.3}$$

is called an *almost (α, p)-Golden structure* and the pair $(\overline{M}, \Phi_{\alpha,p})$ is called an *almost (α, p)-Golden manifold*.

In particular, $\Phi_{\alpha,1}$ structure is named α-Golden structure, studied in ([5]).

Remark 1. The eigenvalues of the almost (α, p)-Golden structure $\Phi_{\alpha,p}$ are:

$$\varphi_{\alpha,p} = p\frac{1 + \sqrt{5\alpha}}{2}, \quad \overline{\varphi}_{\alpha,p} = p\frac{1 - \sqrt{5\alpha}}{2}. \tag{2.4}$$

In particular, for $\alpha = 1$ we obtain $\varphi_{1,p} = p\frac{1+\sqrt{5}}{2} = p\phi$ as a root of the polynomial $X^2 - pX - p^2$ and $\varphi_{1,p}$ is a member of the metallic numbers family ([10]), where $q = p^2$ and ϕ is the Golden number.

For $\alpha = -1$, we obtain the $\varphi_{-1,p} = p\frac{1+i\sqrt{5}}{2} = p\phi_c$ as a root of the polynomial $X^2 - pX + \frac{3}{2}p^2$ and $\varphi_{-1,p}$ is a member of the complex metallic numbers family ([30]), where $q = p^2$ and ϕ_c is the complex Golden number.

Moreover, if $(\alpha, p) = (1, 1)$ one obtains the *almost Golden structure* determined by an endomorphism Φ with $\Phi^2 = \Phi + Id$. In this case, $(\overline{M}, \Phi)$ is called *almost Golden manifold*, studied in ([8], [11], [2]).

If $(\alpha, p) = (-1, 1)$, one obtains the *almost complex Golden structure* determined by an endomorphism Φ_c which verifies $\Phi_c^2 = \Phi_c + \frac{3}{2}Id$. In this case, $(\overline{M}, \Phi_c)$ is called *almost complex Golden manifold*, studied in ([3], [1]).

We point out that the almost (α, p)-Golden structure $\Phi_{\alpha,p}$ and the α-structure J_α are closely related:

Proposition 1. *Every α-structure J_α on $\overline{M}$ induces two almost (α, p)-Golden structures $\Phi_{\alpha,p}$:*

$$\Phi_{\alpha,p}^1 = \frac{p}{2}(Id + \sqrt{5}J_\alpha) \quad \Phi_{\alpha,p}^2 = \frac{p}{2}(Id - \sqrt{5}J_\alpha); \tag{2.5}$$

Conversely, if the almost (α, p)-Golden structure $\Phi_{\alpha,p}$ is given then two α-structures J_α can be associated:

$$J_\alpha^1 = \frac{2}{p\sqrt{5}}\left(\Phi_{\alpha,p} - \frac{p}{2}Id\right) \quad J_\alpha^2 = -\frac{2}{p\sqrt{5}}\left(\Phi_{\alpha,p} - \frac{p}{2}Id\right) \tag{2.6}$$

Proof. First of all, we seek $a, b \in \mathbb{R}$ such that $\Phi_{\alpha,p} = aId + bJ_\alpha$. Using (2.3) we find $a = \frac{p}{2}$ and $b = \pm\frac{\sqrt{5}}{2}$ which implies (2.5). Moreover $\Phi_{\alpha,p}$ verifies (2.3).

On the other hand, if $\Phi_{\alpha,p}$ verifies (2.5) then we obtain that J_α verifies (2.6) and (2.1). Conversely, if J_α verifies (2.6) then $\Phi_{\alpha,p}$ verifies (2.5). $\square$

Example 1. (1) An almost product structure J_1 induces two almost $(1, p)$-Golden structures:

$$\Phi_{1,p}^{\pm} = p\frac{Id \pm \sqrt{5}J_1}{2}; \tag{2.7}$$

(2) An almost complex structure J_{-1} induces two almost $(-1, p)$-Golden structures:

$$\Phi_{-1,p}^{\pm} = p\frac{Id \pm \sqrt{5}J_{-1}}{2}. \tag{2.8}$$

$\square$

A straightforward computation yields:

Proposition 2. *The inverse of the almost (α, p)-Golden structure $\Phi_{\alpha,p}$ is:*

$$\Phi_{\alpha,p}^{-1} = \frac{4}{p^2(1 - 5\alpha)}\Phi_{\alpha,p} - \frac{4}{p(1 - 5\alpha)}Id. \tag{2.9}$$

For a given α-structure J_α we can naturally consider two complementary projectors P and Q:

$$P = \frac{1}{2}(Id + \sqrt{\alpha}J_\alpha), \quad Q = \frac{1}{2}(Id - \sqrt{\alpha}J_\alpha), \tag{2.10}$$

where $\alpha \in \{-1, 1\}$. Therefore, we have:

$$P + Q = Id, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0. \tag{2.11}$$

Moreover, it results immediatelly:

$$\sqrt{\alpha}J_\alpha = P - Q. \tag{2.12}$$

The eigenvalues of J_α are $\pm\sqrt{\alpha}$. Thus P and Q define two complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, where $\mathcal{D}_1$ contains the eigenvectors corresponding to the eigenvalue $\sqrt{\alpha}$ and $\mathcal{D}_2$ contains the eigenvectors corresponding to the eigenvalue $-\sqrt{\alpha}$. If J_α has the eigenvalue $\sqrt{\alpha}$ of multiplicity a and the eigenvalue $-\sqrt{\alpha}$ of multiplicity b (recall that $a + b = dim(\overline{M}) = 2m$) then the dimension of $\mathcal{D}_1$ is a while the dimension of $\mathcal{D}_2$ is b.

Conversely, if there exist in $\overline{M}$ two complementary distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ of dimension $a \geq 1$ and $b \geq 1$, respectively (where $a + b = dim(\overline{M}) = 2m$), then, we define an α-structure J_α on $\overline{M}$, which verifies (2.12).

Moreover, we obtain two complementary distributions D_P, D_Q on $\overline{M}$, corresponding to $\varphi_{\alpha,p}$ and $p - \varphi_{\alpha,p}$ respectively, given in (2.4). A straightforward computation, using (2.10) and (2.6), gives us the projection operators $P_{\alpha,p}$ and $Q_{\alpha,p}$ on the almost (α, p)-Golden manifold $(\overline{M}, \Phi_{\alpha,p})$:

$$P_{\alpha,p} = \frac{\sqrt{5\alpha}}{5p} \cdot \Phi_{\alpha,p} + \frac{5 - \sqrt{5\alpha}}{10}Id, \quad Q_{\alpha,p} = -\frac{\sqrt{5\alpha}}{5p} \cdot \Phi_{\alpha,p} + \frac{5 + \sqrt{5\alpha}}{10}Id \tag{2.13}$$

which verify:

$$P_{\alpha,p} + Q_{\alpha,p} = Id, \quad P_{\alpha,p}^2 = P_{\alpha,p}; \quad Q_{\alpha,p}^2 = Q_{\alpha,p}, \quad P_{\alpha,p} \cdot Q_{\alpha,p} = 0 \tag{2.14}$$

and

$$\Phi_{\alpha,p} = \frac{5p}{2\sqrt{5\alpha}}(P_{\alpha,p} - Q_{\alpha,p}) - \frac{\sqrt{5\alpha}}{5}. \tag{2.15}$$

3. (α, p)-Golden Riemannian geometry

Given an even dimensional manifold $\overline{M}$, endowed with an α-structure J_α, we fix a Riemannian metric $\overline{g}$, such that:

$$\overline{g}(J_\alpha X, Y) = \alpha \overline{g}(X, J_\alpha Y) \tag{3.1}$$

which is equivalent by:

$$\overline{g}(J_\alpha X, J_\alpha Y) = \overline{g}(X, Y) \tag{3.2}$$

for any vector fields $X, Y \in \Gamma(T\overline{M})$.

Definition 3. The Riemannian metric $\overline{g}$ defined on an even dimensional manifold $\overline{M}$ endowed with an α-structure J_α which verifies the equivalent relations (3.1) and (3.1) is called (α, J_α)-*compatible*.

Thus, by using (3.1) and (2.6) we obtain that the Riemannian metric $\overline{g}$ which is (α, J_α)-compatible verifies:

$$\overline{g}(\Phi_{\alpha,p} X, Y) - \alpha \overline{g}(X, \Phi_{\alpha,p} Y) = \frac{p}{2}(1 - \alpha)\overline{g}(X, Y), \tag{3.3}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Moreover, from (3.2) and (2.6) we derive:

$$\overline{g}(\Phi_{\alpha,p} X, \Phi_{\alpha,p} Y) = \frac{p}{2}(\overline{g}(\Phi_{\alpha,p} X, Y) + \overline{g}(X, \Phi_{\alpha,p} Y)) + p^2 \overline{g}(X, Y), \tag{3.4}$$

for any $X, Y \in \Gamma(T\overline{M})$.

Definition 4. An *almost* (α, p)-*Golden Riemannian manifold* is a triple $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$ where $\overline{M}$ is an even dimensional manifold, $\Phi_{\alpha,p}$ is an almost (α, p)-Golden structure and $\overline{g}$ is a Riemannian metric which verifies (3.3) and (3.4).

For $\alpha = 1$ in (3.3) and (3.4) we obtain:

$$\overline{g}(\Phi_{1,p} X, Y) = \overline{g}(X, \Phi_{1,p} Y)), \tag{3.5}$$

and:

$$\overline{g}(\Phi_{1,p} X, \Phi_{1,p} Y) = p(\overline{g}(\Phi_{1,p} X, Y) + p^2 \overline{g}(X, Y), \tag{3.6}$$

and the triple $(\overline{M}, \Phi_{1,p}, \overline{g})$ is a particular case of an almost metallic Riemannian manifold, studied in ([9],[10], [14],[15]).

For $\alpha = -1$ in (3.3) and (3.4) we have:

$$\overline{g}(\Phi_{-1,p} X, Y) + \overline{g}(X, \Phi_{-1,p} Y) = p\overline{g}(X, Y), \tag{3.7}$$

and:

$$\overline{g}(\Phi_{-1,p} X, \Phi_{-1,p} Y) = \frac{3}{2}p^2 \overline{g}(X, Y), \tag{3.8}$$

and the triple $(\overline{M}, \Phi_{-1,p}, \overline{g})$ is a particular case of almost complex metallic Riemannian manifold, studied in ([30]).

Proposition 3. *If $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$ is almost (α, p)-Golden Riemannian manifold of dimension $2m$ then the trace of the $\Phi_{\alpha,p}$ structure satisfies:*

$$trace(\Phi_{\alpha,p}^2) = p \cdot trace(\Phi_{\alpha,p}) + \frac{5\alpha-1}{2}mp^2. \tag{3.9}$$

Proof. Denoting by $\{E_1, E_2, ..., E_{2m}\}$ a local orthonormal basis of $T\overline{M}$ from (2.3) we obtain:

$$\overline{g}(\Phi_{\alpha,p}^2 E_i, E_i) = p\overline{g}(\Phi_{\alpha,p}E_i, E_i) + \frac{5\alpha-1}{4}p^2\overline{g}(E_i, E_i),$$

and summing this equality by $i \in \{1, ...2m\}$ we obtain the claimed relation. $\square$

Example 2. Using $\varphi_{\alpha,p}$ and $\overline{\varphi}_{\alpha,p}$ defined in (2.2) let us consider the endomorphism $\Phi_{\alpha,p} : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$, given by:

$$\Phi_{\alpha,p}(X^i, Y^i) := (\varphi_{\alpha,p}X^1, ..., \varphi_{\alpha,p}X^m, \overline{\varphi}_{\alpha,p}Y^1, ..., \overline{\varphi}_{\alpha,p}Y^m), \tag{3.10}$$

where $(X^i, Y^i) := (X^1, ..., X^m, Y^1, ..., Y^m)$ and $i \in \{1, ..., m\}$.

A straightforward computation yields:

$$\Phi_{\alpha,p}^2(X^i, Y^i) := (\varphi_{\alpha,p}^2 X^i, \overline{\varphi}_{\alpha,p}^2 Y^i) = p(\varphi_{\alpha,p}X^i, \overline{\varphi}_{\alpha,p}Y^i) + \frac{5\alpha-1}{4}p^2(X^i, Y^i),$$

and hence $\Phi_{\alpha,p}$ verifies (2.2) .

Let us consider:
$$J_\alpha(X^i, Y^i) := (X^1, ..., X^m, \alpha Y^1, ..., \alpha Y^m).$$

The Euclidean metric $\overline{g} := \langle,\rangle$ on $\mathbb{R}^{2m}$ is (α, J_α)-compatible:

$$\overline{g}(J_\alpha Z, Z') = \alpha\Sigma_{i=1}^m(X^i X'^i + Y^i Y'^i) = \alpha\overline{g}(Z, J_\alpha Z'),$$

for any $Z := (X^1, ..., X^m, Y^1, ..., Y^m), Z' = (X'^1, ..., X'^m, Y'^1, ..., Y'^m) \in \Gamma(\mathbb{R}^{2m})$. Using (3.1) and (2.6) we obtain that the Riemannian metric $\overline{g}$ being (α, J_α)-compatible verifies:

$$\overline{g}(\Phi_{\alpha,p}Z, \Phi_{\alpha,p}Z') = \frac{p}{2}(\overline{g}(\Phi_{\alpha,p}Z, Z') + \overline{g}(Z, \Phi_{\alpha,p}Z')) + p^2\overline{g}(Z, Z').$$

Thus, $(\mathbb{R}^{2m}, \Phi_{\alpha,p}, \overline{g})$ is an example of almost (α, p)-Golden Riemannian manifold.

If ∇ is the Levi-Civita connection on $(\overline{M}, \overline{g})$ then the covariant derivative ∇J_α is a tensor field of type $(1, 2)$ defined by:

$$(\nabla_X J_\alpha)Y := \nabla_X J_\alpha Y - J_\alpha \nabla_X Y, \tag{3.11}$$

for any $X, Y \in \Gamma(T\overline{M})$. Hence, from (2.6)(i) we obtain;

$$(\nabla_X \Phi_{\alpha,p})Y = \frac{p\sqrt{5}}{2}(\nabla_X J_\alpha)Y. \tag{3.12}$$

Let us consider now *the Nijenhuis tensor field* of J_α. Using a similar approach as in ([6]) Definition 2.8. and Proposition 2.9.) we get:

$$N_{J_\alpha}(X, Y) = J_\alpha^2[X, Y] + [J_\alpha X, J_\alpha Y] - J_\alpha[J_\alpha X, Y] - J_\alpha[X, J_\alpha Y] \tag{3.13}$$

which is equivalent with:

$$N_{J_\alpha}(X, Y) = (\nabla_{J_\alpha X}J_\alpha)Y - (\nabla_{J_\alpha Y}J_\alpha)X + (\nabla_X J_\alpha)J_\alpha Y - (\nabla_Y J_\alpha)J_\alpha X. \tag{3.14}$$

The Nijenhuis tensor of $\Phi := \Phi_{\alpha,p}$ is given by:

$$N_\Phi(X,Y) := \Phi^2[X,Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y], \tag{3.15}$$

Thus, from (3.15), we obtain :

$$N_\Phi(X,Y) = (\nabla_{\Phi X}\Phi)Y - (\nabla_{\Phi Y}\Phi)X - \Phi(\nabla_X\Phi)Y + \Phi(\nabla_Y\Phi)X. \tag{3.16}$$

Using (3.15), (3.14) and (3.12) we remark that:

$$N_\Phi(X,Y) = \frac{5p^2}{4}N_{J_\alpha}(X,Y). \tag{3.17}$$

We point out that necessary and sufficient conditions for the integrability of a polynomial structure whose characteristic polynomial has only simple roots were given in ([29]).

Recall also that the structure $\Phi_{\alpha,p}$ is *integrable* if $N_{\Phi_{\alpha,p}} = 0$. For an *integrable almost* (α, p) *- Golden structure* we drop the adjective *"almost"* and then simply call it (α, p)-*Golden structure*.

From (2.5) it results that $\Phi_{\alpha,p}$ is integrable if and only if the associated almost α-structure J_α is integrable. Moreover, the distribution D_P is integrable if $Q_{\alpha,p}\left[P_{\alpha,p}X, P_{\alpha,p}Y\right] = 0$ and analogous, $Q_{\alpha,p}$ is integrable if $P_{\alpha,p}\left[Q_{\alpha,p}X, Q_{\alpha,p}Y\right] = 0$ for any $X, Y \in \Gamma(T\overline{M})$.

Let us consider the second fundamental form Ω, which is a 2-form on $(\overline{M}, J_\alpha, \overline{g})$ where J_α is an α-structure defined in (2.1) and the metric $\overline{g}$ is (α, J_α)-compatible. The 2-form Ω is defined by:

$$\Omega(X,Y) := \overline{g}(J_\alpha X, Y), \tag{3.18}$$

for any $X, Y \in \Gamma(T\overline{M})$.

It follows directly:

Proposition 4. *If $\overline{M}$ is a Riemannian manifold, endowed by an α-structure J_α and the metric $\overline{g}$ which is (α, J_α)-compatible, then*:

$$\Omega(X,Y) = \alpha\Omega(Y,X). \tag{3.19}$$

Lemma 1. *If $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$ is an almost (α, p) - Golden Riemannian manifold, then*:

$$\Omega(X,Y) = \frac{2}{p\sqrt{5}}\overline{g}(\Phi_{\alpha,p}X, Y) - \frac{1}{\sqrt{5}}\overline{g}(X,Y), \tag{3.20}$$

$$\Omega(\Phi_{\alpha,p}X, Y) = \frac{p}{2}\Omega(X,Y) + \frac{p\alpha\sqrt{5}}{2}\overline{g}(X,Y). \tag{3.21}$$

Moreover, inverting $X \leftrightarrow Y$ in (3.21), we obtain:

$$\Omega(\Phi_{\alpha,p}Y, X) = \frac{p}{2}\Omega(Y,X) + \frac{p\alpha\sqrt{5}}{2}\overline{g}(X,Y), \tag{3.22}$$

and using (3.19) in (3.22) and multiplying by $\alpha = \pm 1$ we get:

$$\Omega(X, \Phi_{\alpha,p}Y) = \frac{p}{2}\Omega(X,Y) + \frac{p\sqrt{5}}{2}\overline{g}(X,Y). \tag{3.23}$$

Thus, we get:

Proposition 5. *Let $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$ be an almost (α, p)-Golden Riemannian manifold. Then*:

$$\Omega(\Phi_{\alpha,p}X, Y) - \Omega(X, \Phi_{\alpha,p}Y) = \frac{p(\alpha - 1)\sqrt{5}}{2}\overline{g}(X,Y), \tag{3.24}$$

and:

$$\Omega(\Phi_{\alpha,p}X,Y) + \Omega(X,\Phi_{\alpha,p}Y) = p\Omega(Y,X) + \frac{p(\alpha+1)\sqrt{5}}{2}\overline{g}(X,Y). \tag{3.25}$$

Remark 2. Let $(\overline{M},\Phi_{\alpha,p},\overline{g})$ be an almost (α,p)-Golden Riemannian manifold. In particular, we have:

i) for $\alpha = 1$:

$$\Omega(\Phi_{1,p}X,Y) = \Omega(X,\Phi_{1,p}Y) = \frac{p}{2}\Omega(X,Y) + \frac{p}{2}\sqrt{5}\overline{g}(X,Y) \tag{3.26}$$

ii) for $\alpha = -1$:

$$\Omega(\Phi_{-1,p}X,Y) + \Omega(X,\Phi_{-1,p}Y) = p\Omega(Y,X). \tag{3.27}$$

$\square$

Lemma 2. *Let $\overline{M}$ be a Riemannian manifold endowed with an α-structure J_α and the metric $\overline{g}$ which is (α,J_α)-compatible. Then:*

$$\overline{g}((\nabla_X J_\alpha)Y,Z) = \alpha\overline{g}(Y,(\nabla_X J_\alpha)Z). \tag{3.28}$$

From (3.12) and (3.28) we obtain also:

Proposition 6. *If $(\overline{M},\Phi_{\alpha,p},\overline{g})$ is an almost (α,p)-Golden Riemannian manifold then $\Phi_{\alpha,p}$ structure satisfies*:

$$\overline{g}((\nabla_X \Phi_{\alpha,p})Y,Z) = \alpha\overline{g}(Y,(\nabla_X \Phi_{\alpha,p})Z). \tag{3.29}$$

4. Submanifolds in an almost (α,p)-Golden Riemannian manifold

In ([17], [14], [15], [18], [19], [20], [21]) are presented the properties of special types of submanifolds (invariant, slant, semi-slant, hemi-slant, bi-slant submanifolds and warped product submanifolds) in locally metallic and locally Golden Riemannian manifolds. Submanifolds of Almost-Complex Metallic Manifolds was studied in ([31]).

Using similar techniques, we study some properties of submanifolds in almost (α,p)-Golden Riemannian manifold.

Let M be a $2n$-dimensional submanifold, isometrically immersed in a $2m$-dimmensional almost (α,p)-Golden Riemannian manifold $(\overline{M},\Phi_{\alpha,p},\overline{g})$.

Let us denote by T_xM (and by $T_x^\perp M$) the tangent space (and, respectively, the normal space) of $\overline{M}$ in a point $x \in \overline{M}$. Thus, the decomposition of the tangent space $T_x\overline{M}$ is given by:

$$T_x\overline{M} = T_xM \oplus T_x^\perp M,$$

for any $x \in \overline{M}$.

In the rest of the paper we shall note by X the vector field i_*X, for any $X \in \Gamma(TM)$, in order to simplify the notations. If g is the induced Riemannian metric on M, then it is given by $g(X,Y) = \overline{g}(i_*X,i_*Y)$, for any $X,Y \in \Gamma(TM)$, where i_* is the differential of the immersion $i : M \to \overline{M}$. From (3.1) and (3.1), we remark that the induced metric on the submanifold M verifies the following equalities:

$$(i)\ g(J_\alpha X,Y) = \alpha g(X,J_\alpha Y);\quad (ii)\ g(J_\alpha X,J_\alpha Y) = g(X,Y), \tag{4.1}$$

for any $X,Y \in \Gamma(TM)$.

The decomposition into the tangential and normal parts of $\Phi_{\alpha,p}X$ and $\Phi_{\alpha,p}V$, for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^\perp M)$, are given by:

$$(i)\ \Phi_{\alpha,p}X = \mathcal{T}X + \mathcal{N}X,\quad (ii)\ \Phi_{\alpha,p}U = \mathfrak{t}U + \mathfrak{n}U, \tag{4.2}$$

where $\mathcal{T} : \Gamma(TM) \to \Gamma(TM), \mathcal{N} : \Gamma(TM) \to \Gamma(T^\perp M)$ and, respectively, $\mathfrak{t} : \Gamma(T^\perp M) \to \Gamma(TM), \mathfrak{n} : \Gamma(T^\perp M) \to \Gamma(T^\perp M)$.

In the next considerations we denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections on $(\overline{M},\overline{g})$ and, respectively, on the submanifold (M,g).

The Gauss and Weingarten formulas are given by:

$$(i)\ \overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (ii)\ \overline{\nabla}_X U = -A_U X + \nabla_X^{\perp} U, \tag{4.3}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_U is the shape operator.

The second fundamental form h and the shape operator A verify:

$$\overline{g}(h(X, Y), U) = \overline{g}(A_U X, Y). \tag{4.4}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$.

For the α-structure J_α, the decompositons into tangential and normal parts of $J_\alpha X$ and $J_\alpha U$, for any $X \in \Gamma(TM)$ and $U \in \Gamma(T^{\perp}M)$, are given by:

$$(i)\ J_\alpha X = fX + \omega X, \quad (ii)\ J_\alpha U = BU + CU, \tag{4.5}$$

where $f : \Gamma(TM) \to \Gamma(TM), fX := (J_\alpha X)^T, \omega : \Gamma(TM) \to \Gamma(T^{\perp}M), \omega X := (J_\alpha X)^{\perp}$, $B : \Gamma(T^{\perp}M) \to \Gamma(TM), BU := (J_\alpha U)^T$ and $C : \Gamma(T^{\perp}M) \to \Gamma(T^{\perp}M), CU := (V)^{J_\alpha \perp}$.

A direct calculus shows that the maps f, ω, B and C satisfies the following identity:

$$(i)\ \overline{g}(fX, Y) = \alpha \overline{g}(X, fY), \quad (ii)\ \overline{g}(CU, V) = \alpha \overline{g}(U, CV) \tag{4.6}$$

and

$$\overline{g}(\omega X, V) = \alpha \overline{g}(X, BV), \tag{4.7}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$.

Using (4.2), we obtain:

Lemma 3. *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold endowed with an α-structure J_α and let $\Phi_{\alpha,p}$ be the almost (α, p)-Golden structure, related by J_α through relationships (2.5)(i). Thus, we get:*

$$(i)\ \mathcal{T}X = \frac{p}{2}X \pm \frac{\sqrt{5\alpha}}{2}fX, \quad (ii)\ \mathcal{N}X = \pm \frac{\sqrt{5\alpha}}{2}\omega X \tag{4.8}$$

$$(i)\ \mathfrak{t}V = \pm \frac{\sqrt{5\alpha}}{2}BV, \quad (ii)\ \mathfrak{n}V = \frac{p}{2}V \pm \frac{\sqrt{5\alpha}}{2}CV, \tag{4.9}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

Now, using (4.8) and (4.9) in (4.6) and (4.7), we get:

Proposition 7. *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold endowed with an almost (α, p)-Golden structure. Thus, the maps $\mathcal{T}$ and $\mathfrak{n}$ verify:*

$$\overline{g}(\mathcal{T}X, Y) = \alpha \overline{g}(X, \mathcal{T}Y) + \frac{p(1-\alpha)}{2}g(X, Y), \tag{4.10}$$

$$\overline{g}(\mathfrak{n}U, V) = \alpha \overline{g}(U, \mathfrak{n}V) + \frac{p(1-\alpha)}{2}g(U, V) \tag{4.11}$$

and $\mathcal{N}$, $\mathfrak{t}$ verifies:

$$\overline{g}(\mathcal{N}X, U) = \alpha \overline{g}(X, \mathfrak{t}U), \tag{4.12}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^{\perp}M)$.

The covariant derivatives of the tangential and normal parts of $\Phi_{\alpha,p}X$ (and $\Phi_{\alpha,p}V$), are given by:

$$(i)\ (\nabla_X \mathcal{T})Y = \nabla_X \mathcal{T}Y - \mathcal{T}(\nabla_X Y), \quad (ii)\ (\overline{\nabla}_X \mathcal{N})Y = \nabla_X^{\perp} \mathcal{N}Y - \mathcal{N}(\nabla_X Y), \tag{4.13}$$

$$(i)\ (\nabla_X \mathfrak{t})U = \nabla_X \mathfrak{t}U - \mathfrak{t}(\nabla_X^{\perp} U), \quad (ii)\ (\overline{\nabla}_X \mathfrak{n})U = \nabla_X^{\perp} \mathfrak{n}U - \mathfrak{n}(\nabla_X^{\perp} U), \tag{4.14}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(T^\perp M)$.

If M is an isometrically immersed submanifold of almost (α, p) - Golden Riemannian manifold $(\overline{M}, \Phi_{\alpha,p}, \overline{g})$, then:

$$\overline{g}((\nabla_X \mathcal{T})Y, Z) = \alpha \overline{g}(Y, (\nabla_X \mathcal{T})Z), \tag{4.15}$$

for any $X, Y, Z \in \Gamma(TM)$.

Let $2r = 2m - 2n$ be the codimension of M in $\overline{M}$. We fix a local orthonormal basis $\{N_1, ..., N_{2r}\}$ of the normal space $T_x^\perp M$. Hereafter we assume that the indices a, b, c run over the range $\{1, ..., 2r\}$.

Let $\Phi_{\alpha,p} := \Phi$ be the almost (α, p)-Golden structure. Then, similar as in ([10]), we get:

$$(i)\ \Phi X = \mathcal{T}X + \sum_{a=1}^{r} u_a(X) N_a,\ (ii)\ \Phi N_a = \xi_a + \sum_{b=1}^{r} \mathcal{A}_{ab} N_b, \tag{4.16}$$

for any $x \in M$ and $X \in T_x M$, where ξ_a are vector fields on M, u_a are 1-forms on M and $(\mathcal{A}_{ab})_{2r}$ is an $2r \times 2r$ matrix of smooth real functions on M.

Moreover, we remark that:

$$(i)\ \mathcal{N}X = \sum_{a=1}^{2r} u_a(X) N_a, \quad (ii)\ \mathfrak{t}N_a = \xi_a, \tag{4.17}$$

and

$$\mathfrak{n}N_a = \sum_{b=1}^{2r} \mathcal{A}_{ab} N_b. \tag{4.18}$$

In a similar manner as in ([9]), we obtain:

Theorem 1. The structure $\Sigma = (\mathcal{T}, g, u_a, \xi_a, (\mathcal{A}_{ab})_r)$ induced on the submanifold M by the almost (α, p)-Golden structure, $\Phi_{\alpha,p}$, on $\overline{M}$ satisfies the following equalities :

$$\mathcal{T}^2 X = p\mathcal{T}X + \frac{5\alpha - 1}{4} p^2 X - \sum_{a=1}^{2r} u_a(X)\xi_a, \tag{4.19}$$

$$u_a(\mathcal{T}X) = p \cdot u_a(X) - \sum_{b=1}^{2r} \mathcal{A}_{ab} u_b(X), \tag{4.20}$$

$$\mathcal{A}_{ab} = \alpha \mathcal{A}_{ab}, \tag{4.21}$$

$$u_b(\xi_a) = \frac{5\alpha - 1}{4} p^2 \delta_{ab} + p\mathcal{A}_{ab} - \sum_{c=1}^{2r} \mathcal{A}_{ac}\mathcal{A}_{cb}, \tag{4.22}$$

$$\mathcal{T}\xi_a = p\xi_a - \sum_{b=1}^{2r} \mathcal{A}_{ab}\xi_b, \tag{4.23}$$

and

$$u_a(X) = \frac{5\alpha - 1}{4} p^2 g(X, \xi_a) \tag{4.24}$$

for any $X \in \Gamma(TM)$, where δ_{ab} is the Kronecker delta, $p \in \mathbb{R}^*$ and $\alpha \in \{-1, 1\}$).

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