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Not peer-reviewed version

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Posted Date: 15 June 2023

doi: 10.20944/preprints202306.1147.v1

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Article

On the Sub Convexlike Optimization Problems

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Abstract: In this paper, we show that the sub convexlikeness introduced by V. Jeyakumar [1], and the subconvexlikeness defined in V. Jeyakumar [2] are equivalent in locally convex topological spaces. And then, we deal with set-valued vector optimization problems and obtains some vector saddle-point theorems and vector Lagrangian theorems.

Keywords: locally convex topological space; subconvexlikeness; sub convexlikeness; vector Lagrangian multiplier theorems; vector saddle-point theorems

AMS Classifications: 90C26; 90C48

0. Introduction

Generalized convex optimization are very well studied branches of mathematics. There are many very meaningful and useful definitions of generalized convexities. Let X be a normed space, and X_+ a convex cone of X . K. Fan [3] introduced the definition of X_+ -convexlike function. Jeyakumar [1] introduced the definition of X_+ -sub convexlike function. And, Jeyakumar defined X_+ -subconvexlike function in [2]. There are plenty of research articles discussing subconvexlike optimization problems, e.g., see [4–12]. In this paper, we show that the sub convexlikeness introduced in [1] and the subconvexlikeness in [2] are actually equivalent in locally convex topological spaces (including normed linear spaces).

Most of papers in set-valued optimization studied the problem with inequality constraint and the abstract constraint. In this paper we consider the set-valued optimization problem with not only inequality, abstract but also equality constraints. The explicit statement of the equality constraint would be very convenient in applications. For example, recently, the mathematical programs with equilibrium constraints has received a lot of attentions from the optimization community. The mathematical programs with equilibrium constraints are a class of optimization problem with variational inequality constraints. By representing the variational inequality as a generalized equation, e.g., [8, 13, 14], a mathematical program with equilibrium constraints can be reformulated as an optimization problem with an equality constraint. This paper deals with the set-valued optimization problem with inequality, equality as well as abstract constraints, and obtains some vector saddle-point theorems and vector Lagrangian theorems.

1. Preliminary

Let X be a real topological vector space, a subset X_+ of X is said to be a convex cone if

$$\alpha x^1 + \beta x^2 \in X_+, \forall x^1, x^2 \in X_+, \forall \alpha, \beta \geq 0.$$

We denote by 0_X the zero element in the topological space X and simply by 0 if there is no confusion.

A convex cone X_+ of X is called a pointed cone if $X_+ \cap (-X_+) = \{0\}$.

A real topological vector space X with a pointed cone is said to be an ordered topological linear space. We denote $\text{int}X_+$ the topological interior of X_+ . The partial order on X is defined by

$$x^1 \prec_{X_+} x^2, \text{ if } x^1 - x^2 \in X_+,$$

$$x^1 \prec\prec_{X_+} x^2, \text{ if } x^1 - x^2 \in \text{int } X_+.$$

Or, if there is no confusion, they may just be denoted by

$$x^1 \prec x^2, \text{ if } x^1 - x^2 \in X_+,$$

$$x^1 \prec\prec x^2, \text{ if } x^1 - x^2 \in \text{int } X_+.$$

If $A, B \subseteq X$, we denoted by

$$A \prec_{X_+} B, \text{ if } x \prec_{X_+} y \text{ for } \forall x \in A, \forall y \in B,$$

$$A \prec\prec_{X_+} B, \text{ if } x \prec\prec_{X_+} y \text{ for } \forall x \in A, \forall y \in B.$$

Or,

$$A \prec B, \text{ if } x \prec y \text{ for } \forall x \in A, \forall y \in B,$$

$$A \prec\prec B, \text{ if } x \prec\prec y \text{ for } \forall x \in A, \forall y \in B.$$

A linear functional on X is a continuous linear function from X to \mathbb{R} (1-dimensional Euclidean space). The set X^* of all linear functionals on X is the dual space of X . The subset

$$X_+^* = \{\xi \in X^* : \langle x, \xi \rangle \geq 0, \forall x \in X_+\}$$

of X^* is said to be the dual cone of the cone X_+ , where $\langle x, \xi \rangle = \xi(x)$.

Suppose that X and Y are two real topological vector spaces. Let $f: X \rightarrow 2^Y$ be a set-valued function, where 2^Y denotes the power set of Y .

Let D be a nonempty subset of X . Setting $f(D) = \bigcup_{x \in D} f(x)$, and

$$\langle f(x), \eta \rangle = \{\langle y, \eta \rangle : y \in f(x)\},$$

$$\langle f(D), \eta \rangle = \bigcup_{x \in D} \langle f(x), \eta \rangle.$$

For $x \in D, \eta \in Y^*$, we write

$$\langle f(x), \eta \rangle \geq 0, \text{ if } \langle y, \eta \rangle \geq 0, \forall y \in f(x),$$

$$\langle f(D), \eta \rangle \geq 0, \text{ if } \langle f(x), \eta \rangle \geq 0, \forall x \in D.$$

The following Definitions 1.1 and 1.2 can be found in [15].

Definition 1.1 (convex, bounded, and absorbing) A subset M of X is said to be convex, if $x_1, x_2 \in M$ and $0 < \alpha < 1$ implies $\alpha x_1 + (1 - \alpha)x_2 \in M$; M is said to be balanced if $x \in M$ and $|\alpha| \leq 1$ implies $\alpha x \in M$; M is said to be absorbing if for any given neighbourhood U of 0 , there exists a positive scalar β such that $\beta^{-1}M \subseteq U$, where $\beta^{-1}M = \{x \in X; x = \beta^{-1}v; v \in M\}$.

Definition 1.2 (locally convex topological space) A topological vector space X is called a locally convex topological space if any neighborhood of 0_X contains a convex, balanced, and absorbing open set.

From [15, pp.26 Theorem, pp.33 Definition 1], a normed linear space is a locally convex topological space.

2. The Sub Convexlikeness

This section shows that the definitions of sub convexlikeness and subconvexlikeness given by Jeyakumar [1, 2] are actually one.

A set-valued function $f: X \rightarrow 2^Y$ is said to be Y_+ -convex on D if $\forall x^1, x^2 \in D, \forall \alpha \in [0, 1]$, one has

$$\alpha f(x^1) + (1 - \alpha)f(x^2) \prec_{Y_+} f(\alpha x^1 + (1 - \alpha)x^2).$$

The following definition of convexlikeness was introduced by Ky Fan [8].

A set-valued function $f: X \rightarrow 2^Y$ is said to be Y_+ -convexlike on D if $\forall x^1, x^2 \in D, \forall \alpha \in [0, 1], \exists x^3 \in D$ such that

$$\alpha f(x^1) + (1-\alpha)f(x^2) \prec_{Y_+} f(x^3).$$

Jeyakumar [2] introduced the following subconvexlikeness.

Definition 2.1 (subconvexlike) Let Y be a topological vector space and $D \subseteq X$ be a nonempty set and Y_+ be a convex cone in Y . A set-valued map $f: D \rightarrow 2^Y$ is said to be Y_+ -subconvexlike on D if $\exists \theta \in \text{int } Y_+$ such that $\forall x_1, x_2 \in D, \forall \varepsilon > 0, \forall \alpha \in [0, 1], \exists x_3 \in D$ here holds

$$\varepsilon\theta + \alpha f(x_1) + (1-\alpha)f(x_2) \prec_{Y_+} f(x_3).$$

Lemma 2.1 is [16, Lemma 2.3].

Lemma 2.1 Let Y be a topological vector space and $D \subseteq X$ be a nonempty set and Y_+ be a convex cone in Y . A set-valued map $f: D \rightarrow 2^Y$ is Y_+ -subconvexlike on D if and only if $\forall \theta \in \text{int } Y_+, \forall x_1, x_2 \in D, \forall \alpha \in [0, 1], \exists x_3 \in D$ such that

$$\theta + \alpha f(x_1) + (1-\alpha)f(x_2) \prec_{Y_+} f(x_3).$$

A bounded function in a topological space can be defined as following Definition 2.2 (e.g., see Yosida [15]).

Definition 2.2 (bounded set-valued map) A subset M of a real topological vector space Y is said to be a bounded subset if for any given neighbourhood U of 0, there exists a positive scalar β such that $\beta^{-1}M \subseteq U$, where $\beta^{-1}M = \{y \in Y; y = \beta^{-1}v; v \in M\}$. A set-valued map $f: D \rightarrow Y$ is said to be bounded map if $f(D)$ is a bounded subset of Y .

Jeyakumar [1] introduced the following sub convexlikeness.

Definition 2.3 (sub convexlike) Let Y be a topological vector space and $D \subseteq X$ be a nonempty set. A set-valued map $f: D \rightarrow 2^Y$ is said to be Y_+ -sub convexlike on D if \exists bounded set-valued map $u: D \rightarrow Y, \forall x_1, x_2 \in D, \forall \varepsilon > 0, \forall \alpha \in [0, 1], \exists x_3 \in D$ such that

$$\varepsilon u + \alpha f(x_1) + (1-\alpha)f(x_2) \prec_{Y_+} f(x_3).$$

The following Lemma 2.1 is from Li and Wang [16, Lemma 2.3].

Lemma 2.2 Let Y be a locally convex topological space and $D \subseteq X$ be a nonempty set Y . A set-valued map $f: D \rightarrow 2^Y$ is Y_+ -subconvexlike on D if and only if $f(D) + \text{int } Y_+$ is Y_+ -convex.

Theorem 2.1 Let Y be a locally convex topological space, $D \subseteq X$ a nonempty set, and Y_+ a convex cone in Y . A set-valued map $f: D \rightarrow 2^Y$ is Y_+ -sub convexlike on D if and only if $f(D) + \text{int } Y_+$ is Y_+ -convex.

Proof. The necessity.

Suppose that f is Y_+ -sub convexlike..

$$\forall z_1 = y_1 + y_0^1, z_2 = y_2 + y_0^2 \in f(D) + \text{int } Y_+, \exists x_1, x_2 \in D \text{ such that } y_1 \in f(x_1), y_2 \in f(x_2).$$

Let

$$y_0 = \alpha y_0^1 + (1-\alpha)y_0^2,$$

then $y_0 \in \text{int } Y_+$. Therefore, \exists neighbourhood U of 0 such that $y_0^0 + U$ is a neighbourhood of y_0^0 and

$$y_0^0 + U \subseteq \text{int } Y_+.$$

By Definition 1.2, we may assume that U is convex, balanced, and absorbing.

From the assumption of sub convexlikeness, i.e., \exists bounded set-valued map $u: x_1, x_2 \in D$, $\varepsilon > 0$, $\alpha \in [0, 1]$, $\exists x_3 \in D$ such that

$$\varepsilon u + \alpha f(x_1) + (1 - \alpha)f(x_2) \subseteq f(x_3) + Y_+.$$

Therefore

$$\begin{aligned} & \alpha z_1 + (1 - \alpha)z_2 \\ &= \alpha y_1 + (1 - \alpha)y_2 + \alpha y_+^1 + (1 - \alpha)y_+^2 \\ &\subseteq f(x_3) - \varepsilon u + Y_+ + y_+^0 \end{aligned}$$

Since U is convex, balanced, and absorbing, we may take $\varepsilon > 0$ small enough such that $-\varepsilon u \subseteq U$.

Therefore

$$-\varepsilon u + y_+^0 \subseteq y_+^0 + U \subseteq \text{int } Y_+.$$

And then

$$\begin{aligned} & \alpha z_1 + (1 - \alpha)z_2 \\ &= \alpha y_1 + (1 - \alpha)y_2 + \alpha y_+^1 + (1 - \alpha)y_+^2 \\ &\subseteq f(x_3) + \text{int } Y_+ \\ &\subseteq f(D) + \text{int } Y_+. \end{aligned}$$

Hence, $f(D) + \text{int } Y_+$ is a Y_+ -convex set.

The sufficiency.

If $f(D) + \text{int } Y_+$ is Y_+ -convex, then, by Lemma 2.1, f is Y_+ -subconvexlike. And, it is clear that Y_+ -subconvexlikeness implies Y_+ -sub convexlikeness.

From Lemma 2.2 and Theorem 2.1 one has Theorem 2.2.

Theorem 2.2 Let Y be a locally convex topological space and $D \subseteq X$ a nonempty set, and Y_+ a convex cone in Y . A set-valued map $f: D \rightarrow 2^Y$ is Y_+ -subconvexlike on D if and only if f is Y_+ -sub convexlike on D .

3. Vector Saddle-Point Theorems

This section works on vector saddle-point theorems for set-valued optimization problems.

A set-valued map $f: D \rightarrow 2^Y$ is said to be affine on D if $\forall x_1, x_2 \in D, \forall \beta \in R$, there holds

$$\beta f(x_1) + (1 - \beta)f(x_2) = f(\beta x_1 + (1 - \beta)x_2).$$

We introduce the notion of sub affineline functions as follows.

Definition 3.1 (sub affineline) A set-valued map $f: D \rightarrow 2^Y$ is said to be Y_+ -sub affineline on D if $\forall x_1, x_2 \in D, \forall \alpha \in (0, 1), \exists v \in \text{int } Y_+, \exists x_3 \in D$ there holds

$$v + \alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3).$$

Theorem 3.1 Let X, Y, Z and W be real topological vector spaces, $D \subseteq X$. Y_+, Z_+, W_+ pointed convex cones of Y, Z and W , respectively. Assume that functions $f: D \rightarrow Y, g: D \rightarrow Z, h: D \rightarrow W$ satisfy that

(a) f and g are sub convexlike maps on D , i.e., $\forall u_1 \in \text{int } Y_+, \forall u_2 \in \text{int } Z_+, \forall \alpha \in (0,1), \forall x^1, x^2 \in D, \exists x', x'' \in D$ such that

$$\begin{aligned} u_1 + \alpha f(x^1) + (1-\alpha)f(x^2) &< f(x'), \\ u_2 + \alpha g(x^1) + (1-\alpha)g(x^2) &< g(x''); \end{aligned}$$

(b) h is a sub affineline map on D , i.e., $\forall \alpha \in (0,1), \forall x^1, x^2 \in D, \exists x''' \in D, v \in \text{int } W_+$ such that

$$v + \alpha h(x^1) + (1-\alpha)h(x^2) = h(x''');$$

(c) $\text{int } h(D) \neq \emptyset$;

and (i) and (ii) denote the systems

- (i) $\exists x \in D, s.t., f(x) < 0, g(x) < 0, h(x) = 0$;
(ii) $\exists (\xi, \eta, \zeta) \in (Y_+^* \times Z_+^* \times W_+^*) \setminus \{(0_Y, 0_Z, 0_W)\}$ such that

$$\xi(f(x)) + \eta(g(x)) + \zeta(h(x)) \geq 0, \forall x \in D.$$

If (i) has no solution then (ii) has solutions.

Moreover if (ii) has a solution (ξ, η, ζ) with $\xi \neq 0_{Y^*}$ then (i) has no solutions.

Proof. $\forall w_1, w_2 \in \bigcup_{t>0} t h(D) + \text{int } W_+, \forall \alpha \in (0,1), \exists x_1, x_2 \in D, \exists b_1, b_2 \in \text{int } W_+, \exists t_1, t_2 > 0$ such that

$$\begin{aligned} &\alpha w_1 + (1-\alpha)w_2 \\ &= \alpha t_1 h(x_1) + (1-\alpha)t_2 h(x_2) + \alpha b_1 + (1-\alpha)b_2 \\ &= (\alpha t_1 + (1-\alpha)t_2) \left[\frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} h(x_1) + \frac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} h(x_2) \right] + \alpha b_1 + (1-\alpha)b_2. \end{aligned}$$

By the assumption (b), $\exists x_3 \in D, \exists v \in \text{int } W_+, \forall \varepsilon > 0$ such that

$$\frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} h(x_1) + \frac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} h(x_2) = h(x_3) - \varepsilon v.$$

Since $v \in \text{int } W_+, \exists$ neighbourhood U of 0 in W for which $V = \alpha b_1 + (1-\alpha)b_2 + U$ is a neighbourhood of $\alpha b_1 + (1-\alpha)b_2$.

By Definition 1.2, we may take $\varepsilon > 0$ small enough such that

$$-\varepsilon(\alpha t_1 + (1-\alpha)t_2)v \subseteq U.$$

Then,

$$\alpha b_1 + (1-\alpha)b_2 - \varepsilon(\alpha t_1 + (1-\alpha)t_2)v \subseteq V \subseteq \text{int } W_+.$$

Therefore,

$$\begin{aligned}
& \alpha w_1 + (1-\alpha)w_2 \\
& \subseteq \alpha t_1 h(x_1) + (1-\alpha)t_2 h(x_2) + \alpha b_1 + (1-\alpha)b_2 \\
& = (\alpha t_1 + (1-\alpha)t_2) \left[\frac{\alpha t_1}{\alpha t_1 + (1-\alpha)t_2} h(x_1) + \frac{(1-\alpha)t_2}{\alpha t_1 + (1-\alpha)t_2} h(x_2) \right] + \alpha b_1 + (1-\alpha)b_2 \\
& = (\alpha t_1 + (1-\alpha)t_2) h(x_3) + \alpha b_1 + (1-\alpha)b_2 - \varepsilon (\alpha t_1 + (1-\alpha)t_2) v \\
& \subseteq \bigcup_{t>0} th(D) + \text{int } W_+.
\end{aligned}$$

So, $\bigcup_{t>0} th(D) + \text{int } Y_+$, is a convex set.

Similarly, $\bigcup_{t>0} tf(D) + \text{int } Y_+$, and $\bigcup_{t>0} tg(D) + \text{int } Z_+$ are also convex. Therefore, the set

$$C = \left(\bigcup_{t>0} tf(D) + \text{int } Y_+ \right) \times \left(\bigcup_{t>0} tg(D) + \text{int } Z_+ \right) \times \left(\bigcup_{t>0} th(D) + \text{int } W_+ \right)$$

is convex.

From the assumption (c), $\text{int } C \neq \emptyset$. We also have $(0_Y, 0_Z, 0_W) \notin B$ since (i) has no solution. Therefore, according to the separation theorem of convex sets of topological vector space, \exists nonzero vector $(\xi, \eta, \zeta) \in Y^* \times Z^* \times W^*$ such that

$$\xi(t_1 f(x) + y^0) + \eta(gt_2(x) + z^0) + \zeta(t_3 h(x) + w^0) \geq 0,$$

for $\forall t_1, t_2, t_3 > 0, \forall x \in D, \forall y^0 \in \text{int } Y_+, \forall z^0 \in \text{int } Z_+, \forall w^0 \in B$.

Since $\text{int } Y_+, \text{int } Z_+$ are convex cones, and B is a linear space, one gets

$$\xi(t_1 f(x) + \lambda_1 y^0) + \eta(t_2 g(x) + \lambda_2 z^0) + \zeta(t_3 h(x) + \lambda_3 w^0) \geq 0$$

$\forall x \in D, \forall y^0 \in \text{int } Y_+, \forall z^0 \in \text{int } Z_+, \forall w^0 \in B, \forall \lambda_i > 0, (i=1,2,3), \forall t_i > 0, (i=1,2,3)$.

Let $\lambda_i \rightarrow 0 (i=2,3), t_i \rightarrow 0 (i=1,2,3)$ one has

$$\xi(y^0) \geq 0, \forall y^0 \in \text{int } Y_+.$$

Therefore $\xi(y) \geq 0, \forall y \in Y_+$. Hence $\xi \in Y_+^*$. Similarly, $\eta \in \text{int } Z_+, \zeta \in \text{int } W_+$.

Thus

$$(\xi, \eta, \zeta) \in Y_+^* \times Z_+^* \times W_+^*.$$

Therefore,

$$\xi(f(x)) + \eta(g(x)) + \zeta(h(x)) \geq 0, \quad x \in D.$$

Which means that (ii) has solutions.

On the other hand, suppose that (ii) has a solution (ξ, η, ζ) with $\xi \neq 0_{Y_+^*}$, i.e.,

$$\xi(f(x)) + \eta(g(x)) + \zeta(h(x)) \geq 0, \quad x \in D.$$

We are going to prove that (i) has no solution.

Otherwise, if (i) has a solution $\tilde{x} \in D$, then $f(\tilde{x}) \prec \prec 0, g(\tilde{x}) \leq 0, h(\tilde{x}) = 0$. Hence, one would have

$$\xi(f(\tilde{x})) + \eta(g(\tilde{x})) + \zeta(h(\tilde{x})) < 0.$$

Which is a contradiction. The proof is completed.

We consider the following optimization problem with set-valued maps:

$$\begin{aligned}
 (VP) \quad & Y_+ \text{-min} \quad f(x) \\
 & \text{s.t. } g_i(x) \cap (-Z_{i+}) \neq \emptyset, \quad i = 1, 2, \dots, m, \\
 & 0 \in h_j(x), \quad j = 1, 2, \dots, n, \\
 & x \in D,
 \end{aligned}$$

where $f: X \rightarrow 2^Y$, $g_i: X \rightarrow 2^{Z_i}$, $h_j: X \rightarrow 2^{W_j}$ are set-valued maps, Z_{i+} is a closed convex cone in Z_i and D is a nonempty subset of X .

Definition 3.2 (weakly efficient solution) A point $\bar{x} \in F$ is said to be a weakly efficient solution of (VP) if there exists no $x \in D$ satisfying $f(\bar{x}) \succ f(x)$, where $F := \{x \in D : g(x) < 0, h(x) = 0\}$.

Let

$$\begin{aligned}
 P \min[A, Y_+] &= \{y \in A : (y - A) \cap \text{int } Y_+ = \emptyset\}, \\
 P \max[A, Y_+] &= \{y \in A : (A - y) \cap \text{int } Y_+ = \emptyset\}.
 \end{aligned}$$

In the sequel, $B(W, Y)$ denotes the set of all continuous linear mappings T from W to Y ; $B^+(Z, Y)$ denotes the set of all non-negative and continuous linear mappings S from Z to Y , where non-negative mapping S means that $S(z) \in Y_+, \forall z \in Z$.

$$\text{Let } L(\bar{x}, \bar{S}, \bar{T}) = f(\bar{x}) + \bar{S}(g(\bar{x})) + \bar{T}(h(\bar{x})).$$

Definition 3.3 (vector saddle-point) $(\bar{x}, \bar{S}, \bar{T}) \in X \times B^+(Z, Y) \times B(W, Y)$ is said to be a vector saddle-point of $L(\bar{x}, \bar{S}, \bar{T})$ if

$$L(\bar{x}, \bar{S}, \bar{T}) \in P \min[L(X, \bar{S}, \bar{T}), Y_+] \cap P \max[L(\bar{x}, B^+(Z, Y), B(W, Y)), Y_+].$$

Where

$$\begin{aligned}
 & P \max[L(\bar{x}, B^+(Z, Y), B(W, Y)), Y_+] \\
 &= \{\mu : \mu = P \max[L(\bar{x}, S, T), Y_+], (S, T) \in B^+(Z, Y) \times B(W, Y)\}.
 \end{aligned}$$

Theorem 3.2 $(\bar{x}, \bar{S}, \bar{T}) \in X \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L(\bar{x}, \bar{S}, \bar{T})$, if and only if $\exists \bar{y} \in f(\bar{x}), \bar{z} \in g(\bar{x})$, such that

- (i) $\bar{y} \in P \min[L(X, \bar{S}, \bar{T}), Y_+]$,
- (ii) $g(\bar{x}) \subseteq -Z_+, h(\bar{x}) = \{0\}$,
- (iii) $(f(\bar{x}) - \bar{y} - \bar{S}(\bar{z})) \cap \text{int } Y_+ = \emptyset$.

Proof. The sufficiency. Suppose that the conditions (i)-(iii) are satisfied. Note that $-g(\bar{x}) \subseteq Z_+, h(\bar{x}) = \{0\}$ imply

$$-S(g(\bar{x})) \subseteq Y_+, T(h(\bar{x})) = \{0\}, \quad \forall (S, T) \in B^+(Z, Y) \times B(W, Y), \quad (3.1)$$

and the condition (i) states that

$$\{\bar{y} - [f(X) + \bar{S}(g(X)) + \bar{T}(h(X))]\} \cap \text{int } Y_+ = \emptyset,$$

So $Y_+ + \text{int } Y_+ \subseteq Y_+$ and $-S(\bar{z}) \in Y_+$ together imply

$$\{\bar{y} + \bar{S}(\bar{z}) + \bar{T}(\bar{w}) - [f(X) + \bar{S}(g(X)) + \bar{T}(h(X))]\} \cap \text{int } Y_+ = \emptyset.$$

Hence

$$\bar{y} + \bar{S}(\bar{z}) + \bar{T}(\bar{w}) \in P \min[L(X, \bar{S}, \bar{T}), Y_+].$$

On the other hand, since $(f(\bar{x}) - [\bar{y} + S(\bar{z})]) \cap \text{int } Y_+ = \emptyset$, from (3.1), and from $\text{int } Y_+ + Y_+ \subseteq \text{int } Y_+$ we conclude that

$$\left\{ \bigcup_{(S,T) \in B^+(Z,Y) \times B(W,Y)} [f(\bar{x}) + S(g(\bar{x})) + T(h(\bar{x}))] - [\bar{y} + \bar{S}(\bar{z}) + \bar{T}(\bar{w})] \right\} \cap \text{int } Y_+ = \emptyset.$$

Hence

$$\bar{y} + \bar{S}(\bar{z}) + \bar{T}(\bar{w}) \in P \max[L(\bar{x}, B^+(Z, Y), B(W, Y)), Y_+]$$

Consequently,

$$L(\bar{x}, \bar{S}, \bar{T}) \cap P \min[L(X, \bar{S}, \bar{T}), Y_+] \cap P \max[L(\bar{x}, B^+(Z, Y), B(W, Y)), Y_+] \neq \emptyset.$$

Therefore $(\bar{x}, \bar{S}, \bar{T}) \in X \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L(\bar{x}, \bar{S}, \bar{T})$.

The necessity. Assume that $(\bar{x}, \bar{S}, \bar{T}) \in X \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L(\bar{x}, \bar{S}, \bar{T})$. From Definition 3.3 one has

$$L(\bar{x}, \bar{S}, \bar{T}) \cap P \min[L(X, \bar{S}, \bar{T}), Y_+] \cap P \max[L(\bar{x}, B^+(Z, Y) \times B(W, Y)), Y_+] \neq \emptyset.$$

So, $\exists \bar{y} \in f(\bar{x}), \bar{z} \in g(\bar{x}), \bar{w} \in h(\bar{x})$, i.e.,

$$\bar{y} + S(\bar{z}) + T(\bar{w}) \in L(\bar{x}, \bar{S}, \bar{T}) = f(\bar{x}) + S(g(\bar{x})) + T(h(\bar{x})),$$

such that

$$\left\{ f(\bar{x}) + S(g(\bar{x})) + T(h(\bar{x})) - [\bar{y} + \bar{S}(\bar{z}) + \bar{T}(\bar{w})] \right\} \cap \text{int } Y_+ = \emptyset, \quad (3.2)$$

$$\forall (S, T) \in B^+(Z, Y) \times B(W, Y),$$

and

$$(\bar{y} + \bar{S}(\bar{z}) + \bar{T}(\bar{w}) - [f(X) + \bar{S}(g(X)) + \bar{T}(h(X))]) \cap \text{int } Y_+ = \emptyset. \quad (3.3)$$

Taking $T = \bar{T}$ in (3.2) we get

$$S(z) - \bar{S}(\bar{z}) \notin \text{int } Y_+, \forall z \in g(\bar{x}), \forall S \in B^+(Z, Y). \quad (3.4)$$

Aim to show that $-\bar{z} \in Z_+$.

Otherwise, since $0 \in -Z_+$, if $-\bar{z} \notin Z_+$, we would have $-\bar{z} \neq 0$,

Because Z_+ is a closed convex set, by the separate theorem $\exists \eta \in Z^* \setminus \{0\}$

$$\eta(tz_+) > \eta(-\bar{z}), \forall z \in Z_+, \forall t > 0. \quad (3.5)$$

i.e.,

$$\eta(z_+) > \frac{1}{t} \eta(-\bar{z}), \forall z \in Z_+, \forall t > 0.$$

Let $t \rightarrow \infty$ we obtain $\eta(z_+) \geq 0, \forall z \in Z_+$. Which means that $\eta \in Z^* \setminus \{0\}$. Meanwhile, $0 \in Z_+$ and (3.5) yield that $\eta(\bar{z}) > 0$. Given $\tilde{z} \in \text{int } Z_+$ and let

$$S(z) = \frac{\eta(z)}{\eta(\tilde{z})} \tilde{z} + \bar{S}(z).$$

Then $\bar{S} \in B^+(Z, Y)$ and

$$S(\bar{z}) - \bar{S}(\bar{z}) = \tilde{z} \in \text{int } Y_+.$$

Contradicting to (3.4). Therefore

$$-\bar{z} \in Z_+.$$

Now, aim to prove that $-g(\bar{x}) \subseteq Z_+$.

Otherwise, if $-g(\bar{x}) \not\subseteq Z_+$, then $\exists z_0 \in g(\bar{x})$ such that $0 \neq -z_0 \notin Z_+$. Similar to the above $\exists \eta_0 \in Z^* \setminus \{0\}$ such that $\eta_0 \in Z_+^* \setminus \{0\}$, $\eta_0(z_0) > 0$. Given $\tilde{z} \in \text{int } Z_+$ and let

$$S_0(z) = \frac{\eta_0(z)}{\eta_0(z_0)} \tilde{z}.$$

Then $S_0 \in B^+(Z, Y)$ and $S_0(z_0) = \tilde{z} \in \text{int } Y_+$. And we have proved that $-\bar{z} \in Z_+$, so $-\bar{S}(\bar{z}) \in Y_+$. Therefore

$$S_0(z_0) - \bar{S}(\bar{z}) \in \text{int } Y_+ + Y_+ \subseteq \text{int } Y_+.$$

Again, contradicting to (3.4).

Therefore $-g(\bar{x}) \subseteq Z_+$. Similarly, one has $-h(\bar{x}) \subseteq W_+$. From (3.2) we get

$$[T(h(\bar{x})) - \bar{T}(\bar{w})] \cap \text{int } Y_+ = \emptyset.$$

Hence

$$T(\bar{w}) - \bar{T}(\bar{w}) \notin \text{int } Y_+, \forall T \in B(W, Y). \quad (3.6)$$

Similarly, from (3.2) again we have

$$T(w) - \bar{T}(\bar{w}) \notin \text{int } Y_+, \forall w \in h(\bar{x}), \forall T \in B(W, Y). \quad (3.7)$$

If $\bar{w} \neq 0$, since $-h(\bar{x}) \subseteq W_+$ and W_+ is a pointed cone, we have $\bar{w} \notin W_+$. Because Y_+ is a closed convex set, by the separation theorem $\exists \zeta \in W^*$, such that

$$\zeta(w) < \zeta(\bar{w}), \forall w \in W_+. \quad (3.8)$$

So $\zeta(\bar{w}) \neq 0$ since $0 \in W_+$. Taking $y^0 \in \text{int } Y_+$ and define $T^0 \in B^+(W, Y)$ by

$$T^0(w) = \frac{\zeta(w)}{\zeta(\bar{w})} y^0 + \bar{T}(w).$$

Then

$$T^0(\bar{w}) - \bar{T}(\bar{w}) = y^0 \in \text{int } Y_+,$$

Contradicting to (3.6). Therefore $\bar{w} = 0$. Thus

$$0 \in h(\bar{x}).$$

Now, we'd like to prove $h(\bar{x}) = \{0\}$.

Otherwise, if $w^0 \in h(\bar{x}) : w^0 \neq 0$, similar to (3.8) $\exists \zeta^0 \in W^*$, such that $\zeta^0(w) < \zeta^0(w^0), \forall w \in W_+$. So $\zeta^0(w^0) \neq 0$. Given $y_0 \in \text{int } Y_+$ and define $T_0 \in B(W, Y)$, by

$$T_0(w) = \frac{\zeta^0(w)}{\zeta^0(w^0)} y_0.$$

Then $T_0(w^0) = y_0 \in \text{int } Y_+$, i.e., $\bar{w} = O$) $T^0(w^0) - \bar{T}(\bar{w}) \in \text{int } Y_+$. Contradicting to (3.7). Therefore we must have

$$h(\bar{x}) = \{0\}. \quad (3.9)$$

Combining (3.2), (3.3), (3.9), and we conclude that

$$\bar{y} \in P \min[L(X, \bar{S}, \bar{T}), Y_+], \quad (3.10)$$

and

$$(f(\bar{x}) - \bar{y} - \bar{S}(\bar{z})) \cap \text{int } Y_+ = \emptyset.$$

We have proved that, if $(\bar{x}, \bar{S}, \bar{T}) \in X \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L(\bar{x}, \bar{S}, \bar{T})$, then the conditions (i)-(iii) hold.

Theorem 3.3 If $(\bar{x}, \bar{S}, \bar{T}) \in X \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L(\bar{x}, \bar{S}, \bar{T})$, and if $0 \in \bar{S}(g(\bar{x}))$, then \bar{x} is a weak efficient solution of (VP).

Proof. Assume that $(\bar{x}, \bar{S}, \bar{T}) \in D \times B^+(Z, Y) \times B(W, Y)$ is a vector saddle-point of $L(\bar{x}, \bar{S}, \bar{T})$, from Theorem 3.2 we have

$$-S(g(\bar{x})) \subseteq Y_+, h(\bar{x}) = \{0\}. \quad (3.11)$$

So $\bar{x} \in D$ (the feasible solution of (VP). And $\exists \bar{y} \in f(\bar{x})$ such that $\bar{y} \in P \min[L(X, \bar{S}, \bar{T}), Y_+]$, i.e.

$$(\bar{y} - [f(X) + \bar{S}(g(X)) + \bar{T}(h(X))] \cap \text{int } Y_+ = \emptyset.$$

Thus

$$(\bar{y} - [f(D) + \bar{S}(g(\bar{x})) + \bar{T}(h(\bar{x}))]) \cap \text{int } Y_+ = \emptyset.$$

Since $0 \in \bar{S}(g(\bar{x}))$, by (3.11), one has

$$(\bar{y} - f(D)) \cap \text{int } Y_+ = \emptyset.$$

Therefore, \bar{x} is a weakly efficient solution of (VP).

4. Vector Lagrangian Theorems

Definition 4.1 (vector Lagrangian map) The vector Lagrangian map $L : X \times B^+(Z, Y) \times B(W, Y) \rightarrow 2^Y$ of (VP) is defined by the set-valued map

$$L(x, S, T) = f(x) + S(g(x)) + T(h(x)).$$

Given $(S, T) \in B^+(Z, Y) \times B(W, Y)$, we consider the minimization problem induced by (VP):

$$\begin{aligned} \text{(VPST)} \quad & Y_+ - \min L(x, S, T), \\ & \text{s.t.}, x \in D. \end{aligned}$$

Definition 4.2 (slater constrained qualification (SC)) Let $\bar{x} \in F$. We say that (VP) satisfies the Slater Constrained Qualification at \bar{x} if the following conditions hold:

- (1) $\exists x \in D$, s.t. $h_j(x) = 0, g_i(x) \ll 0$;

(2) $0 \in \text{int } h_j(D)$ for all j .

According to the following Theorem 4.1, (VPST) can also be considered as a dual problem of (VP).

Theorem 4.1 Let $\bar{x} \in D$. Assume that $f(x) - f(\bar{x}), g(x), h(x)$ satisfy the generalized convexity condition (a), the generalized affinity condition (b), as well as the inner point condition (c), and (VP) satisfies the Slater Constrained Qualification (SC). Then, $\bar{x} \in D$ is a weakly efficient solution of (VP) if and only if $\exists(S, T) \in B^+(Z, Y) \times B(W, Y)$ such that $\bar{x} \in D$ is a weakly efficient solution of (VPST).

Proof. Assume $\exists(S, T) \in B^+(Z, Y) \times B(W, Y)$ such that $\bar{x} \in D$ is a weakly efficient solution of (VPST). Then there exist $\bar{y} \in f(\bar{x}), \bar{z} \in g(\bar{x}), \bar{w} \in h(\bar{x})$, such that

$$(\bar{y} + S(\bar{z}) + T(\bar{w}) - [f(D) + S(g(D)) + T(h(D))]) \cap \text{int } Y_+ = \emptyset,$$

If $(\bar{y} - f(D)) \cap \text{int } Y_+ \neq \emptyset$, then $\exists y \in f(D)$ such that $\bar{y} - y \in \text{int } Y_+$, i.e.,

$$(\bar{y} + S(\bar{z}) + T(\bar{w}) - [y + S(\bar{z}) + T(\bar{w})]) \in \text{int } Y_+.$$

Which means that

$$(\bar{y} + S(\bar{z}) + T(\bar{w}) - [f(D) + S(g(D)) + T(h(D))]) \cap \text{int } Y_+ \neq \emptyset.$$

Which is a contradiction.

Therefore

$$(\bar{y} - f(D)) \cap \text{int } Y_+ = \emptyset.$$

Hence, $\bar{x} \in D$ is a weakly efficient solution of (VP).

Conversely, suppose that $\bar{x} \in D$ is a weakly efficient solution of (VP). So $\exists \bar{y} \in f(\bar{x})$ such that there is not any $x \in D$ for which $f(x) - \bar{y} \in -\text{int } Y_+$. That is to say, there is not any $x \in X$ such that

$$f(x) - \bar{y} \in -\text{int } Y_+, g(x) \in -Z_+, 0_W \in h(x).$$

By Theorem 3.1, $\exists(\xi, \eta, \zeta) \in Y_+^* \times Z_+^* \times W^* \setminus \{(0_{Y^*}, 0_{Z^*}, 0_{W^*})\}$ such that

$$\xi(f(x) - \bar{y}) + \eta(g(x)) + \zeta(h(x)) \geq 0, \forall x \in D. \quad (4.1)$$

Since $\bar{y} \in f(\bar{x})$ and $0_W \in h(\bar{x})$, take $x = \bar{x}$ in (1) we obtain

$$\eta(g(\bar{x})) \geq 0.$$

But $\bar{x} \in D$ and $\eta \in Z_+^*$ imply that $\exists \bar{z} \in g(\bar{x}) \cap (-Z_+)$ for which

$$\eta(\bar{z}) \leq 0.$$

Hence $\eta(\bar{z}) = 0$, which means

$$0 \in \eta(g(\bar{x})). \quad (4.2)$$

Since $x \in D$ implies $0_W \in h(x)$, and $g(x) \cap (-Z_+) \neq \emptyset$ implies $\exists z \in g(x) \cap (-Z_+)$ such that $\eta(z) \leq 0$, we have

$$\xi(f(x) - \bar{y}) \geq 0, \forall x \in D.$$

Because the Slater Constraint Qualification is satisfied, similar to the proof of Theorem 3.2, we have $\xi \neq 0_{Y^*}$. So we may take $y_0 \in \text{int } Y_+$ such that

$$\xi(y_0) = 1.$$

Define the operator $S : Z \rightarrow Y$ and $T : W \rightarrow Y$ by

$$S(z) = \eta(z)y_0, T(w) = \zeta(w)y_0. \quad (4.3)$$

It is easy to see that

$$\begin{aligned} S &\in B^+(Z, Y), S(Z_+) = \eta(Z_+)y_0 \subseteq Y_+, \\ T &\in B(W, Y). \end{aligned}$$

And (4.2) implies

$$S(g(\bar{x})) = \eta(g(\bar{x}))y_0 \in 0 \cdot Y_+ = 0_Y. \quad (4.4)$$

Since $\bar{x} \in D$, we have $0_W \in h(\bar{x})$. Hence

$$0_Y \in T(h(\bar{x})). \quad (4.5)$$

Therefore, by (4.4) and (4.5) one gets

$$\bar{y} \in f(\bar{x}) \subseteq f(\bar{x}) + S(g(\bar{x})) + T(h(\bar{x})).$$

From (4.1) and (4.3)

$$\begin{aligned} &\xi[f(x) + S(g(x)) + T(h(x))] \\ &= \xi(f(x)) + \eta(g(x))\xi(y_0) + \zeta(h(x))\xi(y_0) \\ &= \xi(f(x)) + \eta(g(x)) + \zeta(h(x)) \\ &\geq \xi(\bar{y}), \forall x \in D. \end{aligned}$$

i.e.,

$$\xi[f(x) - \bar{y} + S(g(x)) + T(h(x))] \geq 0, \forall x \in D. \quad (4.6)$$

Taking $F(x) = f(x) + S(g(x)) + T(h(x))$, $G(x) = \{0_Z\}$ and $H(x) = \{0_W\}$, applying Theorem 3.2 to the functions $F(x) - \bar{y}, G(x), H(x)$, then (4.6) deduces that

$$(\bar{y} - [f(D) + S(g(D)) + T(h(D))]) \cap \text{int } Y_+ = \emptyset,$$

and

$$\bar{y} \in F(\bar{x}) = f(\bar{x}) + S(g(\bar{x})) + T(h(\bar{x})),$$

since $0_Y \in S(g(\bar{x})), 0_Y \in T(h(\bar{x}))$.

Consequently, $\bar{x} \in D$ is a weakly efficient solution of (VPST).

We complete the proof.

Definition 4.3 (NNAMCQ) Let $\bar{x} \in F$. We say that (VP) satisfies the No Nonzero Abnormal Multiplier Constraint Qualification (NNAMCQ) at \bar{x} if there is no nonzero vector $(\eta, \zeta) \in \Pi_{i=1}^m Z_i^* \times \Pi_{j=1}^n W_j^*$ satisfying the system

$$\begin{aligned} &\min_{x \in D \cap U(\bar{x})} [\sum_{i=1}^m \eta_i g_i(x) + \sum_{j=1}^n \zeta_j h_j(x)] = 0 \\ &\sum_{i=1}^m \eta_i g_i(\bar{x}) = 0, \end{aligned}$$

where $U(\bar{x})$ is some neighborhood of \bar{x} .

Similar to the proof of Theorem 4.1, one has Theorem 4.2.

Theorem 4.2 Let $\bar{x} \in D$. Assume that $f(x) - f(\bar{x}), g(x), h(x)$ satisfy the generalized convexity condition (a), the generalized affinity condition (b), as well as the inner point condition (c). If \bar{x} is a weakly efficient solution of (VP), then \exists vector Lagrangian multiplier $(S, T) \in B^+(Z, Y) \times B(W, Y)$ such that $\bar{x} \in D$ is a weakly efficient solution of (VPST). Inversely, if (NNAMCQ) holds at $\bar{x} \in D$, and if \exists vector Lagrangian multiplier $(S, T) \in B^+(Z, Y) \times B(W, Y)$ such that \bar{x} is a weakly efficient solution of (VPST), then \bar{x} is a weakly efficient solution of (VP).

5. Conclusions.

Jeyakumar [1] introduced the following definition of sub convexlike functions for single-valued functions.

Let Y be a topological vector space and $D \subseteq X$ be a nonempty set. A set-valued map $f : D \rightarrow 2^Y$ is said to be Y_+ -sub convexlike on D if \exists bounded set-valued map $u : D \rightarrow Y$, $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$, $\forall \alpha \in [0, 1]$, $\exists x_3 \in D$ such that

$$\varepsilon u + \alpha f(x_1) + (1 - \alpha)f(x_2) \prec_{Y_+} f(x_3),$$

where the partial order is induced by a convex cone Y_+ of Y .

Jeyakumar [2] introduced the following subconvexlikeness.

A set-valued map $f : D \rightarrow 2^Y$ is said to be Y_+ -subconvexlike on D if $\exists \theta \in \text{int } Y_+$ such that $\forall x_1, x_2 \in D$, $\forall \varepsilon > 0$, $\forall \alpha \in [0, 1]$, $\exists x_3 \in D$ here holds

$$\varepsilon \theta + \alpha f(x_1) + (1 - \alpha)f(x_2) \prec_{Y_+} f(x_3).$$

In this paper, we prove that the above two generalized convexities are equivalent.

A set-valued map $f : D \rightarrow 2^Y$ is said to be affine on D if $\forall x_1, x_2 \in D$, $\forall \beta \in R$, there holds

$$\beta f(x_1) + (1 - \beta)f(x_2) = f(\beta x_1 + (1 - \beta)x_2).$$

We define the following sub affinitylike maps, in order to weaken the condition of the "equality constraints" for optimization problems.

A set-valued map $f : D \rightarrow 2^Y$ is said to be Y_+ -sub affinitylike on D if $\forall x_1, x_2 \in D$, $\forall \alpha \in (0, 1)$, $\forall v \in \text{int } W_+$, $\exists x_3 \in D$ there holds

$$v + \alpha f(x_1) + (1 - \alpha)f(x_2) = f(x_3).$$

And then, we consider the following optimization problem with set-valued maps:

$$\begin{aligned} (VP) \quad & Y_+ \text{-min} \quad f(x) \\ & \text{s.t. } g_i(x) \cap (-Z_{i+}) \neq \emptyset, \quad i = 1, 2, \dots, m, \\ & 0 \in h_j(x), \quad j = 1, 2, \dots, n, \\ & x \in D, \end{aligned}$$

where $f : X \rightarrow 2^Y$ and $g_i : X \rightarrow 2^{Z_i}$ are sub convexlike, and $h_j : X \rightarrow 2^{W_j}$ are sub affinitylike.

For a single-valued situation, above optimization problem (VP) may be written as follows.

$$\begin{aligned} & Y_+ \text{-min} \quad f(x) \\ & \text{s.t. } g_i(x) \prec 0, \quad i = 1, 2, \dots, m, \\ & h_j(x) = 0, \quad j = 1, 2, \dots, n, \\ & x \in D. \end{aligned}$$

We obtain some vector saddle-point theorems and some vector Lagrangian theorems for the set-valued optimization problem (VP). Our Theorem 3.1 is a generalization of theorems of alternatives in [1, 2], a modification of theorems of alternatives in [5, 10, 11, 17, 18]. Our saddle-point theorems (Theorems 3.2 and 3.3) are generalizations of the saddle-point theorem in [16, 20], and modifications of saddle-point theorems in [4]. Our Lagrangian theorems (Theorems 4.1 and 4.2) are generalizations of Lagrangian theorems in [16] and modifications of those in [19].

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