Article

On the Constant Terms of Certain Laurent Series

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Keywords: Laurent series; constant term; modular form; integer partition

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On the Constant Terms of Certain Laurent Series

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Abstract: We study the divisibility properties of the constant terms of certain meromorphic modular forms for Hecke groups and relate those properties to several O.E.I.S. sequences and several other sequences, the members of which appear in congruences of Ramanujan. At the end of the article, we construct from elementary arithmetic functions some meromorphic but not necessarily modular functions and study their constant terms. For our use in subsequent drafts, we work out a variation of the multinomial theorem convenient for application to single-variable power series.

Keywords: Laurent series; constant term; modular form; integer partition

1. Introduction

For $g(x)$ a Laurent series in x, let $C(g)$ denote its constant term. Let $f(x) = 1/x + \sum_{n=0}^{\infty} a_{n+1}x^n$ (*sic.*) In this article, we study $C(f^k)$, mostly in settings where (after substituting one of several exponential functions for x) f is a meromorphic modular form for some matrix group.

The constant $C(f^k)$ is a function of the coefficients $a_1, ..., a_k$. Furthermore the numbers $C(f), C(f^2), ...$ determine $a_1, a_2, ...$ To see this, let h_k be the coefficient of x^k in the polynomial $h(x) = (1 + \sum_{n=1}^{k} a_n x^n)^k$. It is clear that $h_k = C(f^k)$. We have $h_1 = a_1$, $h_2 = a_1^2 + 2a_2$, $h_3 = a_1^3 + 6a_1a_2 + 3a_3$, *etc.*

The numerical coefficients on the right sides of these equations may be calculated using the multinomial theorem. An entirely straightforward application of the multinomial theorem expresses $h(x)$ in terms of the monomials $a_i x^i$. But to obtain the numerical coefficients of the sums for h_k, we need to express $h(x)$ as a linear combination of the powers of x. We do this in the next section.

The occasion for our interest was a problem in the theory of quadratic forms, which led us to the empirical finding that equations (1) and (2) below and corresponding equations for other meromorphic modular forms are valid for $k \leqslant 50$ [1]. Here we test (1), (2) and several analogues for $k \leqslant 5000$.

The function $\Delta(z)$ occurring in equation (2) is defined as follows. For z in the upper half plane and $q = q(z) = \exp(2\pi i z)$, $\Delta(z)$ is the weight twelve normalized cusp form for $SL(2, \mathbb{Z})$ with Fourier expansion $\sum_{n=1}^{\infty} \tau(n)q^n$, where τ denotes Ramanujan's function. The reciprocal $1/\Delta(z)$ appears in expressions for the dimensions of certain Lie algebras ([2], page 328; [3], page 45.) It also appears in string theory, for example, in the counting of black hole microstates ([4], equation (14).) We will study the constant terms $C(1/\Delta^k)$ (k a positive integer).

The Klein invariant appearing in equation (1), $j(z) = 1/q + \sum_{n=0}^{\infty} c(n)q^n$, defined on the upper half of the complex plane with $c(0) = C(j) = 744$, is central (for example) to the classical theory of modular forms and to the moonshine phenomenon. We will also study the constant terms $C(j^k)$.

Constant terms of meromorphic modular forms came into our work on quadratic forms as follows. Siegel studied the constant terms in the Fourier expansions of a particular family of meromorphic modular forms T_h for $SL(2, \mathbb{Z})$ ("level one modular forms") in 1969 [5,6]. Siegel demonstrated that these constant terms never vanish. He used this to establish a bound on the exponent of the first non-vanishing Fourier coefficient for a level one entire modular form f of weight h such that the constant term of f is itself non-vanishing. Theta functions fit this description, so Siegel was able to give an upper bound on the least positive integer represented by a positive-definite even unimodular quadratic form in $2h$ variables. While working on an extension of Siegel's result on the non-vanishing of the T_h constant terms to higher-level modular forms, we came across the regularities described in equations (1) and (2). It is apparent that, if only we had proofs of these statements and their analogues, we would have known that the constant terms of $1/\Delta^k, j^k$, and their analogues were non-zero immediately.

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One question we study is the nature of the special role of the primes $p = 2$ and 3 in equations (1) and (2): why these primes but not others (apparently)? We searched for regularities involving other primes among the modular forms for Hecke groups (section 7.) Along the way we made observations relevant to the classical situation as well (conjecture 3.) Constant terms of meromorphic modular forms of certain kinds appear to have multiplicative structure. While seeking a level two version of Siegel's result, the present writer found numerical evidence for divisibility properties of the constant terms for several kinds of modular form, including the T_h [1]; if these properties hold, the constant terms cannot vanish. [1] Let $d_b(n)$ be the sum of the digits in the base b expansion of n. Then (apparently)

$$\mathrm{ord}_2(C(j^k)) = \mathrm{ord}_2(C(1/\Delta^k)) = 3d_2(k) \tag{1}$$

and

$$\mathrm{ord}_3(C(j^k)) = \mathrm{ord}_3(C(1/\Delta^k)) = d_3(k). \tag{2}$$

We argue (based on numerical experiments) that the $C(j^k)$ inherit the stated properties from the O.E.I.S. sequence A005148 [9], which was originally studied by Newman, Shanks and Zagier [10,11] in an article on its use in series approximations to π.

We tried to find patterns in the p-orders of constant terms of j and other modular forms for $SL(2,\mathbb{Z})$ for p larger than three. Our search within $SL(2,\mathbb{Z})$ seemed to fail, so we searched among the Hecke groups $G(\lambda_n), n = 3, 4, ...$ The matrix group $SL(2,\mathbb{Z})$ coincides with the Hecke group $G(\lambda_3)$, discussed below. It is isomorphic to the product of cyclic groups $C_2 * C_3$; while in general $G(\lambda_m) \cong C_2 * C_m$ for $m = 3, 4, ...$ We will state some conjectures about the constant terms, for example, of meromorphic forms for Hecke groups isomorphic to $C_2 * C_{p^k}$, p prime.

Recently we found apparent regularities for $p = 5, 7, 11$ in the original case of $SL(2,\mathbb{Z})$ (conjectures 2 and 13.) They are conditions equivalent to the statement that $\mathrm{ord}_p(C(f))$ vanishes (for $p = 5, 7, 11$ when $f = j^k$, and for $p = 5$ and 7 when $f = 1/\Delta^k$.) These conditions are simple restrictions on the digits in the base p expansions of k. The author's thesis advisor[2] remarked that (1) and (2) might follow from congruences of Ramanujan. We report experiments that support this suggestion in the last section.

The present article states several conjectures based on extensive computations (mainly done with *SageMath*), but no theorems. The data is available in a GitHub repository [12].

2. Constant terms

We want to write the polynomial $h(x)$ discussed in the introduction as a linear combination of powers of x. The usual multinomial theorem evaluates expressions of the form $(y_1 + y_2 + ... + y_k)^k$. The result is a linear combination over "partitions" λ of the products $\prod_t y_t^{\lambda_t}$ (but not quite the usual partitions; see below.) To get what we want, we would need to substitute $a_t x^t$ for y_t, shift the index t by one, and then sort out the powers of x. We prefer to re-derive everything from scratch. First we analyze $h(x)$ in terms of the elementary school process of multiplying polynomials (in our jargon: use of "paths".) Then we translate this into the language of integer partitions. An integer partition λ is usually defined as a finite non-increasing sequence of positive integers. We loosen this a little in order to simplify our arguments.

Definition 1. *1. A partition $(\lambda_1, \lambda_2, ..., \lambda_N)$ is a finite non-increasing sequence of non-negative integers, which may be empty.*
2. D_λ is the set of distinct parts λ_i of λ.
3. If λ is non-empty, $l(\lambda) = N$.
4. $l(\lambda) = 0$ if λ is empty.
5. $|\lambda| = \sum_{w=1}^{N} \lambda_w$ if λ is non-empty.

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[1] For example, see Serre [7], section 3.3, equation (22), or the Wikipedia page [8].
[2] Glenn Stevens

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6. $|\lambda| = 0$ *if* λ *is empty.*
7. *Setting* $|\lambda| = n$*, we say that* λ *is a partition of* n*.*
8. $\Lambda[k, n]$ *is the set of partitions* λ *such that* $|\lambda| = n$ *and* $l(\lambda) = k$*.*
9. *Let* B_λ *be the set of sequences* $\{B_{\lambda,i}\}_{i \in D_\lambda}$ *such that* $B_{\lambda,i} = (i, i, ..., i)$ *and* $l(B_{\lambda,i}) = \#\{j \text{ s.t. } i = \lambda_j\}$*, so that* $l(\lambda) = \sum_{i \in D_\lambda} l(B_{\lambda,i})$ *and* $|\lambda| =$
$\sum_{i \in D_\lambda} i \cdot |B_{\lambda,i}|$*. The* $B_{\lambda,i}$ *are the blocks of* λ *and* B_λ *is the block decomposition of* λ*.*

Remark 1. $B_{\lambda_1} = B_{\lambda_2}$ *if and only if* $\lambda_1 = \lambda_2$*.*

Definition 2.
1. *Write* $1 = a_0$ *and* $h(x) = (\sum_{n=0}^{k} a_n x^n)^k = \sum_{n=0}^{k^2} h_n x^n$ *(say.)*
2. *Let* L *be the* $(k+1) \times (k+1)$ *lattice of monomials*

$$\begin{array}{cccc} a_0x^0 & a_1x^1 & ... & a_kx^k \\ a_0x^0 & a_1x^1 & ... & a_kx^k \\ ..... & ..... & ... & ..... \\ ..... & ..... & ... & ..... \\ a_0x^0 & a_1x^1 & ... & a_kx^k \end{array} .$$

3. *Let* $\mathcal{P}_L$ *be the collection of all* $(k+1)$*-tuples ("paths in* L*")* $\pi = (w_0, w_1, .., w_k)$ *such that* $0 \leqslant w_i \leqslant k$ *for each* i*. The path* π *visits each row of* L *once, in numerical order, and visits the columns of* L *in the order of, and with the multiplicity of, the terms in the sequence* $(w_0, w_1, .., w_k)$*.*
4. *For* π *in* $\mathcal{P}_L$*, let* $v(\pi, w)$ *be the number of visits* π *makes to column* w *of* L*.*
5. *For* π *in* $\mathcal{P}_L$ *such that* $\pi = (w_0, w_1, ..., w_k)$*, let*

$$\mu(\pi) = \prod_{i=0}^{k} a_{w_i} x^{w_i} .$$

6. *For* π *in* $\mathcal{P}_L$*,* π^* *is the unique partition that belongs to the set of permutations of* π *(perhaps with multiplicity greater than one.)*
7. M_n *is the set of paths* π *in* $\mathcal{P}_L$ *such that* $|\pi^*| = n$*.*
8. λ_* *is the number of paths* π *in* $\mathcal{P}_L$ *such that* $\pi^* = \lambda$*.*
9. Λ_L *is the set of distinct partitions in* $\mathcal{P}_L$*.*

Remark 2.
1. *By the ordinary distributive law,* $h(x) = \sum_{\pi \in \mathcal{P}_L} \mu(\pi)$*.*
2. *For* π *in* $\mathcal{P}_L$ *and* $0 \leqslant w \leqslant k$*,* $v(\pi, w) = l(B_{\pi^*,w})$*.*
3. *For* π *in* $\mathcal{P}_L$*,*

$$\mu(\pi) = \prod_{w=0}^{k} (a_w x^w)^{v(\pi,w)} .$$

4. $\mathcal{P}_L$ *is closed under permutations.*
5. *If* π_1 *and* π_2 *are both in* $\mathcal{P}_L$ *then* π_2 *is a permutation of* π_1 *if and only if* $\mu(\pi_1) = \mu(\pi_2)$*.*
6. *Let* $\lambda = \pi^*$ *for any* π *in* $\mathcal{P}_L$*. Then* $l(\lambda) = k + 1$*, so the multiplicity of* λ *among the permutations of* π *is*

$$\frac{(k+1)!}{\prod_{w=0}^{k} (lB_{\lambda,w})!} .$$

This is the multiplicity of π^* *mentioned in the previous definition. It is independent of the choice of* π*. It is equal to* λ_**.*
7. *Hence* $h(x) =$

$$\sum_{\lambda \in \Lambda_L} \lambda_* \, \mu(\lambda) .$$

8. *It is clear that* Λ_L *is the disjoint union*

$$\Lambda_L = \bigcup_{n \geqslant 0} \Lambda[k+1, n] ,$$

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so $h(x) =$

$$\sum_{n\geqslant 0}\left(\sum_{\lambda\in\Lambda[k+1,n]}\lambda_*\mu(\lambda)\right).$$

9. *A partition* λ *belongs to* $\Lambda[k+1,n]$ *if and only if* $n = \deg_x \mu(\lambda)$.

Therefore

Theorem 1.

$$C(f^k) = \sum_{\lambda\in\Lambda[k+1,k]}\frac{(k+1)!}{\prod_{w=0}^{k} l(B_{\lambda,w})!}\cdot\prod_{w=0}^{k} a_w^{l(B_{\lambda,w})}\ .$$

Proof. As we remarked in our introduction, $C(f^k) = h_k$.

We will apply this result in later drafts; stated in this way, it makes manifest the numerical coefficients mentioned in the introduction.

3. Modular forms

3.1. Ramanujan's congruences.

Here are the congruences of Ramanujan mentioned above ([13], page 290 and elsewhere.)[3] [4]

$$\tau(n) \equiv \sigma_{11}(n) \ (\bmod\ 2^8) \tag{3}$$

for odd n.

$$\tau(n) \equiv n\sigma_1(n) \ (\bmod\ 3). \tag{4}$$

$$\tau(n) \equiv n^2\sigma_1(n) \ (\bmod\ 3^2). \tag{5}$$

$$\tau(n) \equiv n^2\sigma_7(n) \ (\bmod\ 3^3). \tag{6}$$

$$\tau(n) \equiv n\sigma_1(n) \ (\bmod\ 5). \tag{7}$$

$$\tau(n) \equiv n\sigma_9(n) \ (\bmod\ 5^2). \tag{8}$$

$$\tau(n) \equiv n\sigma_3(n) \ (\bmod\ 7). \tag{9}$$

$$\tau(n) \equiv \sigma_{11}(n) \ (\bmod\ 691). \tag{10}$$

$$\tau_r(n) \equiv n\sigma_{2r-1}(n) \ (\bmod\ 11) \tag{11}$$

for $r = 2, 3$ and 4.[5]

Remark 3. *Equation (3) extends to all of the positive integers as follows: let* $o = ord_2(n)$ *and* $g(n) = 8^o \cdot \sigma_{11}(n/2^o)$. *Then*

$$\tau(n) \equiv g(n) \ (\bmod\ 2^8).$$

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[3] The following congruences appear in Ramanujan's unpublished (before Berndt and Ono did publish it) manuscript [14]: (4) is Ramanujan's (11.8), (5) is Ramanujan's (12.3), (6) is also Ramanujan's (12.3), (7) is Ramanujan's (2.1), (8) is Ramanujan's (4.2), and (10) is Ramanujan's (12.7).

[4] It is well known that they have been strengthened; see the articles [14], [13], [15], [16], [17], [18], [19], [20], [21], and [22].

[5] The congruences in (11) are displayed in the table at the top of page SwD-32 (page 32 of the proceedings [15]), and, in the form shown here, as equation (13) on page Ran-6 (page 8 of the proceedings [23].)

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To see this, recall Ramanujan's conjecture (proved by Mordell [24]) that, for $n \geqslant 1$ and p prime: $\tau(p)\tau(p^n) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$.[6] *Setting $p = 2$, an easy induction argument shows that $ord_2(\tau(2^o)) = 3o$, and the claim follows from the multiplicativity of $\tau(n)$.*

3.2. Modular forms for Hecke groups.

For $m = 3, 4, ...$, let $\lambda_m = 2\cos\pi/m$ and let J_m be a certain meromorphic modular form for the Hecke group $G(\lambda_m)$, built from triangle functions, with Fourier expansion

$$J_m(z) = \sum_{n=-1}^{\infty} a_n(m) q_m^n,$$

where $q_m(z) = \exp 2\pi i z/\lambda_m$. (For further details, the reader is referred to the books by Carathéodory [25,26] and by Berndt and Knopp [27], the articles of Lehner and Raleigh [28,29], to the dissertation of Leo [30], and to a summary, including pertinent references to that material, in the 2021 article [31].)

Raleigh gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_m^{2n+2} a_n(m) = P_n(m)$ for $n = -1, 0, 1, 2$ and 3. He conjectured that similar relations hold for all positive integers n [29]. [7] Akiyama proved Raleigh's conjectures in 1992 [35].

Using the weight-raising properties of differentiation and the J_m, Hecke constructed families $\mathcal{H}$ comprising modular forms of positive weight for each $G(\lambda_m)$ sharing certain properties [27,36]. It seems apparent that Akiyama's result can be extended: there should exist polynomials $Q_{\mathcal{H},n}(x)$ interpolating the coefficient of X_m^n in the Fourier expansions of the members of Hecke families $\mathcal{H}$. [8]

In section 4 of the 2021 article, we made use of a certain uniformizing variable $X_m(z)$ for z in the upper half plane [31]. By Akiyama's theorem, we have a series of the form $\mathcal{J}_m(x) := \sum_{n=-1}^{\infty} \tilde{P}_n(x) X_m^n$ for polynomials $\tilde{P}_n(x)$ in $\mathbb{Q}[x]$ with the property that $J_m = \mathcal{J}_m(m)$. We will make use of the change of variables $X_m \mapsto 2^6 m^3 X_m$ for a $G(\lambda_m)$-modular form (originally employed, apparently, by Leo ([30], page 31). It has the effect when $m = 3$ of recovering the Fourier series of a variety of standard modular forms. This is set up as a

Definition 3. *For z in the upper half plane and $k_a \neq 0$, let*

$$f(z) := \sum_{n=a}^{\infty} k_n X_m(z)^n$$

and

$$g(z) = \sum_{n=a}^{\infty} k_n 2^{6n} m^{3n} X_m(z)^n.$$

If the last expansion is written as $g(z) = \sum_{n=a}^{\infty} \tilde{k}_n X_m(z)^n$, then let

$$\overline{f}(z) := g(z)/\tilde{k}_a.$$

Also, for $m = 3, 4, ...$, let $j_m(z) := \overline{J_m}(z)$.

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[6] See equation (53) of proposition 14 in section 5.5 of Serre's book [7].

[7] For more on expansions over polynomial fields, see, for example, the book of Boas and Buck [32] and the articles by Buckholtz and Byrd ([33], [34].)

[8] See the paper [31].

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The Fourier expansion of j_3 is [9]

$$j_3(z) = 1/X_3(z) + 744 + 196884X_3(z) + 21493760X_3(z)^2 + ...,$$

which matches the standard expansion $j(z) =$

$$1/\exp(2\pi iz) + 744 + 196884\exp(2\pi i \cdot z) + 21493760\exp(2\pi i \cdot 2 \cdot z) + ...$$

Definition 4. *Let $\mathcal{F} = \{f_3, ..., f_m, ...\}$ where f_m is modular for $G(\lambda_m)$. Then let the Fourier expansion of f_m^k in powers of X_m be written*

$$f_m(z)^k = \sum_n a(f_m^k, n)X_m^n.$$

(Thus $C(f_m^k) = a(f_m^k, 0)$.)

Proposition 1. *Let $\mathcal{K} = \{J_3, , J_4, ...\}$ and $\overline{\mathcal{K}} = \{j_3, j_4, ...\}$ Then there exist polynomials $Q_{\mathcal{K},k,n}(x)$ and $Q_{\overline{\mathcal{K}},k,n}(x)$ in $\mathbb{Q}[x]$ such that $a(J_m^k, n) = Q_{\mathcal{K},k,n}(m)$ and $a(j_m^k, n) = Q_{\overline{\mathcal{K}},k,n}(m)$ for $k = 1, 2, ..., m = 3, 4, ...,$ and $n = -k, 1-k, ...$*

For k equal to one, the first claim is just Akiyama's theorem and the claim for k not equal to one is then obvious. The second statement follows immediately.

3.3. Polynomial interpolation of Fourier coefficients.

When, given a sequence of functions f_m modular for $G(\lambda_m)$ in a family $\mathcal{F}$, we looked for polynomials $Q_{\mathcal{F},n}(x)$ such that each f_m with Fourier expansion

$$f_m(\tau) = \sum_n a_{m,n}X_m^n(\tau),$$

satisfied $Q_{\mathcal{F},n}(m) = a_{m,n}$, We evaluated finite sequences $\{a_{m,n}\}_{m=1,2,3,4,...,M_n}$ (with n held constant) and generated candidates $g_n(x)$ for $Q_{\mathcal{F},n}(x)$ by Lagrange interpolation. The bounds M_n were linear in n and chosen large enough that the degrees of the $g_n(x)$ produced in this way also appeared to be linear in n. Over the course of experiments described in the article [31], this linearity seemed to be associated with systematic behavior. For example, if a polynomial $g_n(x)$ was factored as $g_n(x) = r_n \cdot p_1(x) \cdot p_2(x)... \cdot p_a(x)$ where each of the p_i was monic, r_n was rational, and the degree of $g_n(x)$ was linear in n, then often the sequence $\{r_3, r_4, ...\}$ was readily identifiable (sometimes after resorting to Sloane's encyclopedia.) We take such regularities as evidence that $g_n(m) = a_{m,n}$ for all m. Thus, when formulating conjectures about the $C(J_m^k)$ and $C(j_m^k)$[10], we did not always use tables of the $C(J_m^k)$ and $C(j_m^k)$ directly. Instead (for example), we used Lagrange interpolation to identify polynomials $h_k(x)$ and $\overline{h}_k(x)$ such that $C(J_m^k) = h_k(m)$ and $C(j_m^k) = \overline{h}_k(m)$ by letting m run through a small set of values sufficient to produce the linearity behavior mentioned above; so we assumed (in this example) that $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ and $\overline{h}_k(x) \equiv Q_{\overline{\mathcal{K}},k,0}(x)$ identically. We made tables of p orders of the $h_k(m)$ and the $\overline{h}_k(m)$. In this way we checked larger sets of m values than would have been practicable if we had checked the constant terms themselves.

Unlike the later conjectures, conjecture 1 is not a way of summarizing patterns in experimental data. Rather it codifies our assumption that the linearity behavior is a reliable signal.

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[9] See equation (23) of Serre's book [7], section 3, and the *SageMath* notebook "jpower constant term NewmanShanks 26oct22.ipynb" in [12].

[10] See the *SageMath* notebooks in the repository [12], in the folder "conjectures".

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Conjecture 1. *1. $h_k(x) \equiv Q_{\mathcal{K},k,0}(x)$ identically; consequently, $h_k(m) = C(J_m^k)$ identically.*
2. $\overline{h}_k(x) \equiv Q_{\overline{\mathcal{K}},k,0}(x)$ identically; consequently, $\overline{h}_k(m) = C(j_m^k)$ identically.

4. The reciprocals of cusp forms for $SL(2,\mathbb{Z})$

Let E_{2r} denote the weight $2r$ Eisenstein series with q-series

$$1 + \gamma_r \sum_{r=1}^{\infty} \sigma_{r-1}(n) q^n$$

for certain rational numbers γ_r; this is Rankin's notation. In our experiments, including the case $r = 1$, which is not in Rankin's list, we rely on *SageMath* to pick out the unique normalized cusp form of weight $12 + 2r$, so there is no need to specify γ_r by hand. Recall several facts:[11] Setting $E_0(z) = 1, \tau_0(n) = \tau(n)$, and $r = 0, 2, 3, 4, 5$ or 7:

1. $\Delta(z)E_{2r}(z)$ generates the space of weight $12 + 2r$ cusp forms for $SL(2,\mathbb{Z})$.
2. Writing $\Delta_r = \Delta(z)E_{2r}(z)$ and $\Delta_r = \sum_{n=1}^{\infty} \tau_r(n) q^n$: the functions $n \mapsto \tau_r(n)$ are multiplicative.

Conjecture 2. *Suppressing the dependence upon k and r, let $d_p = d_p(k), C = C(1/\Delta_r^k)$ and $o_p = ord_p(C)$.*

1. *Let $r = 0$.*
   - *(a) $o_2 = 3d_2$ and $o_3 = d_3$.*
   - *(b) $C/3^{o_3} \equiv 1 \pmod 3$ if and only if k is even.*
   - *(c) $C/3^{o_3} \equiv 2 \pmod 3$ if and only if k is odd.*
   - *(d) i. $C \equiv 0, 1$, or $4 \pmod 5$.*
     ii. $o_5 = 0$ if and only if the set of digits in the base 5 expansion of k is a subset of $\{0, 1, 2\}$.[12]
   - *(e) $o_7 = 0$ if and only if the set of digits in the base 7 expansion of k is a subset of $\{0, 1, 2, 3, 4\}$.*
2. *Let $r = 2$.*[13]
   - *(a) $o_2 = 3d_2$.*
   - *(b) i. $o_3 = d_3$ if and only if $k \equiv 0 \pmod 3$.*
     ii. If D is a positive integer such that $D \equiv 2 \pmod 3$, $k > D$, and, for some positive $n, k = D + 3^n$, then o_3 is constant for large n. Let $L_D = \lim_{n\to\infty} o_3$ and N_D be the smallest value of n such that $n \geqslant N_D \Rightarrow o_3 = L_D$. Below is a table for small D. More extensive tables are posted on GitHub [12].

D	2	5	8	11	14	17	20	23
L_D	4	5	8	5	6	8	7	8
N_D	1	2	3	3	3	3	4	3

   - *(c) If k is even and $k \equiv 0 \pmod 3$, then $C/3^{o_3} \equiv 1 \pmod 3$.*
   - *(d) If k is odd and $k \equiv 0 \pmod 3$, then $C/3^{o_3} \equiv 2 \pmod 3$.*
   - *(e) $o_5 = 0$ if and only if the set of digits in the base 5 expansion of k is a subset of $\{0, 1, 2\}$.*
3. *Let $r = 3$.*[14]
   - *(a) $o_2 = 3d_2$ if and only if k is even.*
   - *(b) i. $o_3 = d_3$.*
     ii. $C/3^{o_3} \equiv 1 \pmod 3$ if and only if k is even.
     iii. $C/3^{o_3} \equiv 2 \pmod 3$ if and only if k is odd.
   - *(c) If $o_5 = 0$, then the set of digits in the base 5 expansion of k is a subset of $\{0, 1, 2, 3\}$.*
   - *(d) If $o_7 = 0$, then the set of digits in the base 7 expansion of k is a subset of $\{0, 1, 2, 3, 4\}$.*
4. *Let $r = 4$.*

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[11] See page ran-4 (page six in the proceedings volume) of Rankin's article [23].
[12] See O.E.I.S. page [37].
[13] The converses of clauses (c) and (d) are false.
[14] Again, the converses of clauses (c) and (d) are false.

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(a) *For all positive* $k, o_2 = 3d_2$.
(b) i. *For all positive* k, $C \equiv 0 \pmod 3$.
ii. *If* $k \equiv 0 \pmod 3$, *then* $o_3 = d_3$.
iii. *If* $k \equiv 1 \pmod 3$, *then* $o_3 = d_3$ *if and only if* k *belongs to O.E.I.S sequence A191107 [38]* [15] $\{1, 4, 10, ...\}$,
iv. *If* $k \equiv 2 \pmod 3$ *and* d_3 *divides* o_3, *then* $o_3/d_3 = 2$.
(c) i. $c \equiv 0, 1,$ *or* $4 \pmod 5$.
ii. $o_5 = 0$ *if and only if the digits in the base* 5 *expansion of* k *is a subset of* $\{0, 1, 2\}$.
iii. *If* $o_5 = 0$, *then* $C/5^{o_5} \equiv 1$ *or* $4 \pmod 5$. [16]

5. *Let* $r = 5$.

(a) *If* k *is even, then* $o_2 = 3d_2$.
(b) *If* D *is a positive odd integer,* $k > D$ *and* $k = D + 2^n$, *then* o_2 *is constant for large* n. *Let* $L_D = \lim_{n\to\infty} o_2$ *and* N_D *be the smallest value of* n *such that* $n \geqslant N_D \Rightarrow o_2 = L$. *Below is a table for small* D. *More extensive tables are posted on GitHub [12].*

D	1	3	5	7	9	11	13	15	17	19
L_D	10	13	17	19	15	17	23	27	17	17
N_D	3	3	6	5	6	5	8	9	6	6

Remark 4. 1. *A p-adic geometric view of conjectures 2.2.b (ii) and 2.5.b is that the function* $k \mapsto C(1/\Delta^k)$ *takes certain units* k *in sufficiently small disks around certain* other *units to circles around zero.*
2. *Conjectures 2.2.b (ii) and 2.5.b have only limited empirical support because the mentioned p-adic units* k *grow exponentially with* n *and, on account of drastic slowdowns for large* k, *our experiments tested only* $k \leqslant 5000$. *Thus for* $p = 2$ *and* 3, *we could only check* $n \leqslant 12$ *and* 7, *respectively. We will include tables of what empirical data we do have in the appendix.*

5. Constant terms for $j^k, k = 1, 2, ...$

Recall that $h_k(x)$ and $\overline{h}_k(x)$ are polynomials identified from numerical data by Lagrange interpolation conjectured to satisfy $C(J_m^k) = h_k(m)$ and $C(j_m^k) = \overline{h}_k(m)$. In this section, we illustrate connections between the divisibility patterns (described in the introduction) for the constant terms of the $j(\tau)^k = j_3(\tau)^k$ Fourier expansions on one side, and the $h_k(x)$ on the other. Let $\overline{h}_k(x)$ factor as $\overline{h}_k(x) = \nu_k \cdot p_{k,1}(x) \times p_{k,2}(x) \times ... \times p_{k,\alpha}(x) =$ (say) $\nu_k \cdot \tilde{p}_k(x)$ where each of the $p_{k,n}(n = 1, 2, ..., \alpha)$ is monic and ν_k is rational. We represent O.E.I.S. sequence A005148 [9] $\{0, 1, 47, 2488, 138799, ...\}$ as $\{a_0, a_1, ...\}$.

Conjecture 3. 1. $\nu_k = 24a_k$.
2. $\tilde{p}_k(3)$ *is always odd.*
3. $ord_2(a_k) = 3d_2(k) - 3$.
4. $ord_3(\tilde{p}_k(3)) = d_3(k) - 1$.
5. *From the introduction:* $ord_2(C(j_3^k)) = 3d_2(k)$ *and* $ord_3(C(j_3^k)) = d_3(k)$.
6. *We restate another observation from the article [1]. Let* $o_k = ord_3(C(j_3^k))$, $\kappa = C(j_3^k)/3^{o_k}$, *and* $\rho_k = mod(\kappa, 3)$. *Then* $\rho_k = 1$ *or* 2, *according as* k *is even or odd, respectively.*
7. (a) *Let* $p = 5$ *or* 7 *and let* $o = ord_p(C(j_3^k))$. *Then* $o = 0$ *if and only if the set of digits in the base* p *expansion of* k *is a subset of* $\{0, 1, 2\}$.
(b) *Let* $p = 11$. *With notation as above,* $o = 0$ *if and only if the set of digits in the base* p *expansion of* k *is a subset of* $\{0, 1, 2, 3, 4\}$.

Remark 5. *Clause 5 of the conjecture follows from the earlier clauses. First claim:* $ord_2(C(j_3^k)) = ord_2(\overline{h}_k(3)) = ord_2(\nu_k \cdot \tilde{p}_k(3)) = ord_2(24a_k) \cdot \tilde{p}_k(3)) = ord_2(24) + ord_2(a_k) + ord_2(\tilde{p}_k(3)) = 3 + 3d_2(k) - 3 + 0 = 3d_2(k)$.

---

[15] Description: "Increasing sequence generated by these rules: $a(1) = 1$, and if x is in a then $3x - 2$ and $3x + 1$ are in a." *Mathematica* code: `h = 3; i = -2; j = 3; k = 1; f = 1; g = 7; a = Union[Flatten[NestList[{h # + i, j # + k} &, f, g]]]`.
[16] The converse is false.

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Second claim: In their 1984 article [10], Newman, Shanks and Zagier demonstrated that $ord_3(a_k) = 0$ *for all k. Therefore (under the previous clauses)* $ord_3(C(j_3^k)) = ord_3(\bar{h}_k(3)) = ord_3(\nu_k) + ord_3(\bar{p}_k(3)) = 1 + ord_3(a_k) + d_3(k) - 1 = d_3(k)$.

6. Sufficient conditions for equations (1), (2)

We construct some Laurent series (not necessarily modular, even after an appropriate substitution) such that their constant terms satisfy analogues of equation (1) or equation (2). Some conjectures in this section were tested with Monte Carlo methods.

Conjecture 4. [17]

1. *Let* $A_n = lcm\left(\{2 \cdot 8^{d_2(k)}\}_{k=1,...,n+1}\right)$. *If*

$$f(x) = \sum_{k=1}^{\infty} a_k x^k$$

is in $\mathbb{Z}[[x]]$, $a_k \equiv \tau(k) \ (mod \ A_n)$ *for* $k = 1, 2, ..., n+1$, *then*

$$ord_2(C(1/f(x)^n)) = 3d_2(n).$$

2. *Let* $B_n = lcm\left(\{3 \cdot 3^{d_3(k)}\}_{k=1,...,n+1}\right)$. *If*

$$f(x) = \sum_{k=1}^{\infty} a_k x^k$$

is in $\mathbb{Z}[[x]]$, $a_k \equiv \tau(k) \ (mod \ B_n)$ *for* $k = 1, 2, ..., n+1$, *then*

$$ord_3(C(1/f(x)^n)) = d_3(n).$$

3. *Let* $C_n = lcm\left(\{6 \cdot 8^{d_2(k)} \cdot 3^{d_3(k)}\}_{k=1,..,n+1}\right)$. *If*

$$f(x) = \sum_{k=1}^{\infty} a_k x^k$$

is in $\mathbb{Z}[[x]]$, $a_k \equiv \tau(k) \ (mod \ C_n)$ *for* $k = 1, 2, ..., n+1$, *then*

$$ord_2(C(1/f(x)^n)) = 3d_2(n)$$

and

$$ord_3(C(1/f(x)^n)) = d_3(n).$$

In the following conjectures, analogues to the series expansion of $\Delta(z)$ from the right sides of Ramanujan's congruences (3) – (11) are constructed. Graphical tests indicate that they are not modular forms, but they each appear to have some of the behaviors conjectured for Δ.

Conjecture 5. 1. *Let* $o_k = ord_2(k), g_k = 8^{o_k} \cdot \sigma_{11}(k/2^{o_k})$, *and*

$$f(x) = \sum_{k=1}^{\infty} g_k x^k.$$

[17] See the folder "conjectures" in the repository [12].

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Then

(a) $ord_2 C(1/f(x)^n)) = 3d_2(n)$.
(b) $C(1/f(x)^n) \equiv 1 \pmod 3$.

2. *Let A_n be as in the previous conjecture, g_k be as above, and let*

$$f(x) = \sum_{k=1}^{\infty} a_k x^k,$$

where $a_k \equiv g_k \pmod{A_n}$. Then $ord_2(C(1/f(x)^n)) = 3d_2(n)$.

Conjecture 6. *Let $o_2 = ord_2(k), o_3 = ord_3(k), g_k = k \cdot \sigma_1(k)$, and*

$$f(x) = \sum_{k=1}^{\infty} g_k x^k.$$

1. *If n is divisible by 4, then $ord_2(C(1/f(x)^n)) = 3d_2(n)$.*
2. *If n is divisible by 3, then $ord_3(C(1/f(x)^n)) = d_3(n)$.*
3. *If $n-1$ is divisible by 3 and $n-2$ is a power of 3 or twice a power of 3, then once again $ord_3(C(1/f(x)^n)) = d_3(n)$.*[18]

Conjecture 7. *Let $g_k = k^2 \cdot \sigma_1(k)$ and*

$$f(x) = \sum_{k=1}^{\infty} g_k x^k.$$

1. *If n is even, then $ord_2(C(1/f(x)^n)) = 3d_2(n)$.*
2. *For $n = 1, 2, ..., ord_3(C(1/f(x)^n)) = d_3(n)$.*

7. Powers of reciprocals of generating functions of certain other arithmetic functions

The functions studied in this section are constructed from certain multiplicative or additive (in the sense that $f(ab) = f(a) + f(b)$ when $\gcd(a, b) = 1$) arithmetic functions. They are not necessarily modular or consistent with analogues of equations (1) and (2).

Conjecture 8. *Let*

$$f_r(x) = \sum_{k=1}^{\infty} \sigma_r(k) x^k.$$

1. *$C(1/f_0(x)^n))$ is odd if and only if n is divisible by three.*
2. *For all positive integers n, $C(1/f_1(x)^n))$ is odd.*

In the following conjecture we study divisor sums with multiplicity.

Definition 5. 1. *For $n = \prod_i p_i^{n_i}$, $d = \prod_i p_i^{d_i}$ with $0 \leqslant d_i \leqslant n_i$, and $\binom{a}{b}$ the usual binomial coefficient, the multiplicity of d in n is*

$$\mu(d, n) := \prod_{d|n} \binom{n_i}{d_i}.$$

2.

$$\sigma_r^{\mu}(n) := \sum_{d|n} \mu(d, n)\, d^r.$$

[18] For this sequence, see the O.E.I.S. page [39] of K. Brockhaus.

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Conjecture 9. [19] *Let*

$$f_r^\mu(x) = \sum_{k=1}^{\infty} \sigma_r^\mu(k) x^k$$

and $C_{r,n} = C(1/f_r^\mu(x)^n)$.

1. (a) $C_{r,n}$ *is odd for all positive integers* r *and* n.
   (b) *If* r *is odd and* n *is even, then* $C_{r,n} \equiv 1 (\mathrm{mod}\ 3)$.
2. $C_{0,n}$ *is even for all positive integers* n.
3. (a) $C_{1,n}$ *is odd for all positive integers* n.
   (b) $C_{1,n} \equiv 0 (\mathrm{mod}\ 3)$ *if and only if* n *is even.*

8. Constant terms for $j_m^k, k = 1, 2, ...$

By imposing restrictions on k and m, we found several narrow conjectures about constant term p orders for various primes p.

8.1. m a prime power.

[20]

Conjecture 10. *If* p *is prime and* a *is an integer that is larger than* 2, *then*

$$ord_p(C(j_{p^a}^k)) = (a-3)k + ord_p(C(j_{p^3}^k)).$$

Conjecture 11. *Let* $a \geqslant 2$. *Then* $ord_2(C(j_{2^a}^2)) = 2a + 7$.

Conjecture 12. *Let* p *be a prime number larger than* 2 *and let* a *be a positive integer. Then* $ord_p(C(j_{p^a}^p)) = ap - 2$.

8.2. Other m.

Conjecture 13. *If* $d_2(k) = 1$, $a = ord_2(m), a \geqslant 2$, *and* $o = ord_2(C(j_m^k))$, *then* $o = k(a+2) + 3$.

Conjecture 14. *Let* $d_2(k) = 1$, $m \equiv 2\,(mod\,4)$, *and* $a = ord_2(m) (= 1$, *of course.) Then* $ord_2(C(j_m^k)) = k(a+6) + 1 = 7k + 1$.

Now let $K_n, n = 0, 1, 2, ...$ be the n^{th} Catalan number. (We depart from the standard notation because we have been using the letter "c" in so many other contexts.) One of several explicit formulas for K_n is

$$K_n = \frac{(2n)!}{(n+1)!n!}.$$

For n positive let $K_{1,n}$ denote the n^{th} Catalan number K such that $K \neq K_0$ and $\mathrm{ord}_2(K) = 1$.[21]

Conjecture 15. *Let* k *be the* n^{th} *positive integer such that* $d_2(k) = 2$; *also,* $m = 4j, (j = 1, 2, ...)$, *and* $a = ord_2(m)$. *Furthermore, let* $o = ord_2(C(j_m^k))$ *and* $t = ((a+6)k + 2 - o)/4$. *Then* $t = K_{1,n}$.

Conjecture 16. *Let* $d_2(k) = 2$, $m = 4j + 2, j = 1, 2, ...$, *and* $a = ord_2(m)$ *(again,* $a = 1$.*) Then* $ord_2(C(j_m^k)) = (a+6)k + 2 = 7k + 2$.

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[19] Clause 1 is based on substantially less data than the clauses that specify particular values of r.
[20] Again, see the *SageMath* notebooks in the folder "conjectures" in the repository [12]. Also see O.E.I.S. pages [40],[41], [42],[43].
[21] See Bottomley's O.E.I.S. page [44].

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Conjecture 17. *If* $m \equiv 0 \,(mod\, 3)$, *then* $ord_3(C(j_m^k)) = k \cdot ord_3(m) + d_3(k) - k$.

8.3. The constant terms $c(J_m^k)$.

The Fourier coefficients of the J_m are rational numbers, but typically they are not integers.

Conjecture 18. [22] *Let* p *be a prime number greater than two and let* $c(J_p^p) = a/b$ *(*a, b *relatively prime integers,* b *positive.) Then* $b = 2^{6p-3d_2(p)}p^{2p+2}$.

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[22] See [45] and other O.E.I.S. pages cited within it.

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