Article

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General Quantum Gravity

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Abstract: General Quantum Gravity (GQG) is a formalization of quantized gravity that emerges from General Relativity through Quantum Mechanics. GQG is formalized in three different aspects, such as: Semi-quantum Minkowski GQG, Quantum Minkowski GQG, and Quantum Non-Minkowski GQG. Every observable $(3+1)D$ spacetime in GQG must have internal hidden unobservable $(n+1)D$ spacetime inside it which yields extra hidden dimensions by a closed continuous mapping, so the overall system must acquire Supersymmetry. Strings are natural and universal but forever hidden inside every $(3+1)D$ observable spacetime. Strings have eleven-dimensions by nature in GQG. If we replace Minkowski spacetime with an internal hidden spacetime, then M-theory acquires strings inside of this observable $(3+1)D$ spacetime. Obtaining type IIA string theory from M-theory by dimensional reduction is now non-restricted for the 11^{th} direction but universally for any n^{th} direction. Instead of introducing the cosmological constant in Einstein field equation, we yield Dark Energy as well as Dark Matter from GQG quite naturally. We have developed here Gravitational Electroweak Dark Energy interactions (GED), where gravity and Dark Energy are combined with electroweak symmetry. Likewise, in Gravitational Chromodynamic Dark Energy interactions (GCD), we have combined QCD with gravitational and Dark Energy symmetries. Further in a Dark Matter gauge symmetry model, we have combined Dark matter along with GED and GCD. Finally, from GED, GCD and Dark Matter gauge symmetries, we have developed a Universal Model, $SU(3)_{GED} \otimes SU(4)_{GCD} \otimes SU(5)_{DM} \subset SU(7)_{UM}$, where it is clear that Dark energy field is homogeneous, as well as non-decaying, in all kind of matter fields.

Keywords: quantum gravity; M-theory; dark energy; dark matter

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This work is dedicated to those innocent children, women and infants, who were either beheaded or brutally killed by the invading militants during and the aftermath week of 7^{th} October, 2023.

1. Introduction

Only at Planck scales ($\sim 10^{19}$ GeV), the quantum effects of gravity is believed to be showed up. The authors of Ref. [1] proposed that the quantum effects of gravity should be testable at laboratory scales without regarding Planck scales, and in this context, they also proposed in their paper that the possibility of looking for the effects of Dark Energy at atomic (i.e., laboratory) scales.

Cosmological observations are inconsistent with Einstein's equations of General Relativity in the absence of Dark Energy (or Λ) and Dark Matter. If we extend the idea of Ref. [2] by manipulating the original text of J. Frieman and O. Lahav in the following way as: "It is also theoretically possible that the cosmological constant problem could be resolved by replacing General Relativity with an alternative theory of gravity, with no dark components being imposed separately but comprised within the explanation of this alternative theory of gravity", then we can able to develop a formalism of General Quantum Gravity (GQG), where, the above idea of Ref. [1] is automatically being included.

In GQG, we describe gravity through Quantum Mechanics without considering Planck scales in general. But, if we consider Planck scales in this quantum gravity formalism, our proposed scenarios immediately develop a sense of bosonic and fermionic fields for both Dark Energy and Dark Matter quite naturally without presuming any additional conditions, such as supersymmetry, superstrings, etc.

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During the development of GQG, we have found that every $(3+1)D$ observable system must contain forever hidden string and extra dimensions, whether an external observer considers any strings in these systems or not. This string has eleven-dimensions by nature in GQG, that is why eleven is the maxinium spacetime dimension in which one can formulate a consistent supersymmetric theory.

Unification of gravity with Standard Model was tried previously by numerous authors, but no one ever thought to unify gravity, Dark Energy, Dark Matter and all fundamental interactions of particles or fields under the same Lagrangian. We are not only unified them all under a single formalism, but successfully presented here the solutions of two open problems, viz.

1. *Why Dark Matter contents* 26.8% *of the critical density in the Universe against* 4.9% *of the critical density of baryonic matters?*, or,
2. *Why is the energy density of matter nearly equal to the Dark Energy density today?*

We have developed here the Gravitational Electroweak Dark (GED) interaction, where gravity and Dark Energy are unified with Electroweak interaction. In non-abelian Dark Energy gauge symmetry, Casimir energy is considered to associate with an abelian gauge group to complete the Dark Energy scenario properly. Likewise, in the second scheme of unification, gravity and Dark Energy are unified with Quantum Chromodynamics and we have developed the Gravitational Chromodynamic Dark (GCD) interaction, where we get another set of Dark particles, which are quite different from the particles for non-abelian Dark gauge group of GED.

Finally, unifying both GED and GCD in a restricted manner, we have developed a Lagrangian to explain Dark Matter gauge symmetry.

2. General Quantum Gravity

General Quantum Gravity (GQG) is a formalization of quantized gravity that emerges from General Relativity through Quantum Mechanics. In the basic formalisms of GQG, we are going to develop three different aspects of GQG, such as:

1. Semi-quantum Minkowski GQG,
2. Quantum Minkowski GQG, and
3. Quantum Non-Minkowski GQG.

In the first one, GQG Einstein field equation is a classical-like Einstein field equation in a semi-quantum Minkowski spacetime, whereas, the second one gives us a purely Quantum Mechanical Einstein field equation in a quantum Minkowski spacetime. But in both cases, we always get the classical Schrödinger equation as a byproduct, though it is now in a $(3+1)D$ quantum spacetime. The last one is the most important one, which yields the Einstein field equation in a quantum Non-Minkowski spacetime and helps us to explain Superstring/M-theory from a different angle.

No one ever ask whether Klein-Gordon and Dirac equations are the part of any large scenario. In GQG, we generalizes that the bosonic and fermionic fields argue that the Klein-Gordon equation is a subset of the second order equations of our quantum gravity, whereas, the Dirac equation is a subset of the first order equations of GQG. Interestingly, these bosonic and fermionic fields are emerged either from Einstein field equations or from the line elements of Minkowski spacetime, which is an impossible thing in conventional physics.

Basics of these three different aspects of GQG are discussed as follows:

2.1. Semi-quantum Minkowski GQG

Let the line element of Minkowski spacetime,

$$ds^2 = c^2\,dt^2 - \sum dx^i\,dx^j = g_{\mu\nu}\,dx^\mu\,dx^\nu \equiv \frac{dt^2}{m}\,g_{\mu\nu}\,P^\mu\,\mathrm{v}^\nu, \tag{1}$$

where $P^{\mu} = m\,\mathrm{v}^{\mu}$ is the 'Four-momentum', hence $P^{\mu}\,\mathrm{v}^{\nu} \equiv p^{0}\,\mathrm{v}^{0} + p^{i}\,\mathrm{v}^{j}$ for $i,j = 1,2,3$, and $\mu,\nu = 0,1,2,3$, whereas v^{μ} is the 'four-velocity'. Be careful that it is $P^{\mu}\,\mathrm{v}^{\nu} \neq p^{0}\,\mathrm{v}^{0} - p^{i}\,\mathrm{v}^{j}$, because $g_{\mu\nu}$ takes the '$-$' sign. Also note that $m \neq m_0$ in Eq. (1) for the rest mass m_0. Thus, Eq. (1) gives us an energy-momentum invariant line element as,

$$\begin{aligned} m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= mc^2 - m\sum \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t} = E - \sum p^i\,\mathrm{v}^j = p^0\,\mathrm{v}^0 - \sum p^i\,\mathrm{v}^j \\ &= g_{\mu\nu}\,P^{\mu}\,\mathrm{v}^{\nu}, \end{aligned} \tag{2}$$

when $T = \frac{1}{2}\,m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2$ is the kinetic energy [3], that is, we have a new line element as,

$$\mathrm{d}\,\mathcal{S}_{\mathrm{v}}^2 = m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = p^0\,\mathrm{v}^0 - \sum p^i\,\mathrm{v}^j = g_{\mu\nu}\,P^{\mu}\,\mathrm{v}^{\nu}, \tag{3}$$

then, the rearrangement of Eq. (2) gives the following equation by using Eq. (1) as,

$$E = p^i\,\mathrm{v}^j + m\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = p^i\,\mathrm{v}^j + P^{\mu}\,\mathrm{v}^{\nu}\,\frac{\mathrm{d}s}{\mathrm{d}x^{\mu}}\,\frac{\mathrm{d}s}{\mathrm{d}x^{\nu}} = p^i\,\mathrm{v}^j + P^{\mu}\,\mathrm{v}^{\nu}\,g_{\mu\nu}\,. \tag{4}$$

Let us consider the representation of a wave field $\psi(\vec{r},t)$ by superposition of a free particle (de Broglie wave) for Eq. (4) as follows,

$$\begin{aligned} \psi(\vec{r},t) &= \frac{1}{(2\pi\,\hbar)^2}\,\exp\left[\frac{i}{\hbar}\left\{\left(\vec{p}\cdot\vec{r} + g_{\mu\nu}\,\vec{P}\cdot\vec{R}\right) - Et\right\}\right] \\ &\equiv \frac{1}{(2\pi\,\hbar)^2}\,\exp\left[\frac{i}{\hbar}\left\{\left(\vec{p}\cdot\vec{r} + m\,t\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\right) - Et\right\}\right], \end{aligned} \tag{5}$$

where $\vec{P} \rightarrow \vec{P}^{\,\mu}$ and $\vec{R} \rightarrow \vec{r}^{\,\mu}$, as $\vec{r}^{\,\mu} = (\mathrm{v}^{\mu}t)$, in addition, note it that we have taken here E as total energy for Eq. (4). Thus, from Eq. (5), we can get the (total) energy operator $\hat{E} \rightarrow i\,\hbar\,\partial_t$ (it is analogous with, but not exactly the same as, the Classical Quantum Mechanics, as it is now the total energy and related to $(3+1)D$ instead of $3D$ due to the presence of $g_{\mu\nu}$ in Eq. (5) in either ways), the three momentum operator $\hat{\boldsymbol{p}} \rightarrow -\,i\,\hbar\,\vec{\nabla}_i$, the 'Four-momentum' operator,

$$\begin{aligned} \hat{\boldsymbol{\mathcal{P}}}^{\mu} &\rightarrow i\,\hbar\,\vec{\nabla}_{\mu} = \left\{i\,\hbar\,\frac{\partial}{\partial(ct)},\,-\,i\,\hbar\,\frac{\partial}{\partial x^i}\right\} = \left(\frac{1}{c}\,\hat{E},\,\hat{\boldsymbol{p}}\right), \\ \text{thus,}\qquad \hat{\boldsymbol{\mathcal{P}}}_{\mu} &\rightarrow -\,i\,\hbar\,\vec{\nabla}_{\mu} = \left\{-\,i\,\hbar\,\frac{\partial}{\partial(i\,ct)},\,-\,i\,\hbar\,\frac{\partial}{\partial x^i}\right\} = \left(\frac{i}{c}\,\hat{E},\,\hat{\boldsymbol{p}}\right), \end{aligned} \tag{6}$$

and the mass operator,

$$\hat{m} \rightarrow -\,i\,\hbar\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2\frac{\partial}{\partial t} \equiv -\,i\,\hbar\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)\frac{\partial}{\partial s} = -\,i\,\hbar\,\frac{1}{(g_{\mu\nu}\,\mathrm{v}^{\mu}\,\mathrm{v}^{\nu})^{1/2}}\,\frac{\partial}{\partial s}, \tag{7}$$

where, $\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)$ is evidently relativistic, but $\frac{\partial}{\partial s}$ is no doubt Quantum Mechanical, so as,

$$s = \int_a^u \left(g_{\mu\nu}\,\mathrm{d}x^{\mu}\,\mathrm{d}x^{\nu}\right)^{1/2}\mathrm{d}u\,, \qquad \text{and} \qquad \begin{aligned} \frac{\partial}{\partial s} &= \left\{\frac{\partial^2}{\partial(ct)^2} - \frac{\partial^2}{\partial x^i\partial x^j}\right\}^{1/2} \\ &= \left\{g^{\mu\nu}\,\partial_{\mu}\partial_{\nu}\right\}^{1/2} = \Box^{1/2}. \end{aligned} \tag{8}$$

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For constant velocity, we can develop an uncertainty principle describing the intrinsic indeterminacy with which m and s can be determined as,

$$\Delta m \, \Delta s \geq \frac{\hbar}{2} \, .$$

The mass-energy relation, i.e., $E = m\,c^2$, of Eq. (2) yields for the mass operator $\hat{m}$ from Eq. (7) as,

$$i\,\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t) \;=\; -\,i\,\hbar\,\frac{\mathrm{v}^{0\,2}}{\left(g_{\mu\nu}\,\mathrm{v}^{\mu}\,\mathrm{v}^{\nu}\right)^{1/2}}\,\frac{\partial}{\partial s}\,\psi(\vec{r},t)\,, \tag{9}$$

for $c = \mathrm{v}^0$. Thus, Eq. (9) tells us that the total energy of a system directly relates to its geometry, more precisely, to $\partial/\partial s$, but not $\mathrm{v}^{0\,2} \,/\, \left(\left(g_{\mu\nu}\,\mathrm{v}^{\mu}\,\mathrm{v}^{\nu}\right)^{1/2}\right)$, since its dimensionality vanishes for v^0 and if we consider then $c = \mathrm{v}^0 = 1$. For the second term of $\hat{m}$ in Eq. (7), we can rewrite Eq. (9) by squaring its both sides after considering $c = \mathrm{v}^0 = 1$ as,

$$\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \cdot \left(i\,\hbar\,\frac{\partial}{\partial t}\right)^2 \,\psi(\vec{r},t) \;=\; -\,\hbar^2\,\frac{\partial^2}{\partial s^2}\,\psi(\vec{r},t)\,, \tag{10}$$

which clarifies more precisely that $\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2$ is evidently relativistic, but $\frac{\partial^2}{\partial s^2}$ is definitely Quantum Mechanical. Additionally, Eq. (10) yields the following if the total energy (so as the time) is fixed,

$$\mathrm{d}s \propto \; -\,i\,\hbar\,\frac{\partial}{\partial s}\,. \tag{11}$$

Thus, Eq. (10) is a very peculiar equation where the LHS spacetime is classical relativistic but the RHS spacetime is Quantum Mechanical, and the total energy is directly related to the RHS spacetime. If the LHS spacetime of Eq. (10) changes (not more than $(3+1)D$ and not less than $(1+1)D$, unless it is a vacuum state) and the total energy remains fixed (so as time), then the spacetime of RHS should not remain as same as before, but changes inversely against the LHS spacetime. Though, the increment of RHS spacetime should not be observable, i.e., all extra dimensions would have to remain hidden inside the overall system, in other words, inside the LHS observable spacetime of Eq. (10). We will discuss it below in more details very soon.

For Eq. (8), we can rewrite the mass-energy relation Eq. (10) as follows,

$$\begin{aligned} \hbar^2\,\Box\,\psi(\vec{r},t) + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 \cdot \left(i\,\hbar\,\frac{\partial}{\partial t}\right)^2 \,\psi(\vec{r},t) \;&=\; 0\,, \\ \therefore \quad \hbar^2\,\Box\,\psi(\vec{r},t) + (\mathrm{v}^{\mu})^2 \cdot \left(i\,\hbar\,\frac{\partial}{\partial t}\right)^2 \,\psi(\vec{r},t) \;&=\; 0\,, \end{aligned}$$

where we have used Eq. (1) in the last line. Thus, it is somehow a kind of Klein-Gordon like, but not an exactly similar, equation of the relativistic waves due to the above quantum scenario derived from Eq. (5). (We will see at the very end of this Subsection that Klein-Gordon equation is a subset of the Second Order Equation of Semi-quantum Minkowski GQG). This equation may yield,

$$\left[i\,\hbar\,\gamma^{\mu}\,\vec{\nabla}_{\mu} - \mathrm{v}^{\mu}\,\left(i\,\hbar\,\frac{\partial}{\partial t}\right)\right]\,\psi(\vec{r},t) = 0\,,$$

where, γ^{μ} are Dirac's gamma matrices.

Remark 1. *Definitely, the above quantum scenario derived from Eq. (5) is analogous to, but not exactly similar to, the Classical Quantum Mechanics since E is taken as total energy (and m is not rest mass) in Eq. (4). So, an expectation of the exactness between Classical Quantum Mechanics and the present quantum scenario must lead*

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a confusion and may yield wrong or faulty conclusions in a large scale. Readers are requested to be careful about it.

The wave field $\psi(\vec{r},t)$ in Eq. (5) must satisfy the eigenfunctions for a discrete Lorentz transformation as,

$$\begin{aligned}
\Psi &= \frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\sum_{i=1}^{3}\psi_i = \frac{1}{\sqrt{2}}\psi_0 + \frac{1}{\sqrt{2}}\sum_{i=1}^{3}\psi_i^\dagger = -\Psi^\dagger\,, \\
\Psi^\dagger &= \frac{1}{\sqrt{2}}\psi_0 + \frac{1}{\sqrt{2}}\sum_{i=1}^{3}\psi_i = \frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\sum_{i=1}^{3}\psi_i^\dagger = -\Psi\,,
\end{aligned} \tag{12}$$

when $\psi^\dagger$ is the complex conjugate of ψ. Then, using summation convention and Eq. (8), we can write the joint state of both spacetimes as,

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}\Psi\bigg|_{s_0} &= \left\{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^i\partial x^j}\right\}\left\{\frac{1}{\sqrt{2}}\psi_0 - \frac{1}{\sqrt{2}}\psi_i\right\}\bigg|_{x_0^\mu} \\
&= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[\psi_0-\psi_i] - \frac{\partial^2}{\partial x^i\partial x^j}[\psi_0-\psi_i]\right\}\bigg|_{x_0^\mu} \\
&= \left(\frac{\partial^2}{\partial t^2}\Psi - \frac{\partial^2}{\partial x^i\,\partial x^j}\Psi\right)\bigg|_{x_0^\mu} = g^{\mu\nu}(s)\,\partial_\mu\partial_\nu\Psi\Big|_{x_0^\mu}.
\end{aligned} \tag{13}$$

But, Eq. (13) also intends to,

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}\Psi\bigg|_{s_0} &= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[\psi_0-\psi_i] - \frac{\partial^2}{\partial x^i\partial x^j}[\psi_0-\psi_i]\right\}\bigg|_{x_0^\mu} \\
&= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[\psi_0] - \frac{\partial^2}{\partial x^i\,\partial x^j}\,[-\psi_i]\right\}\bigg|_{x_0^\mu} + \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\,[-\psi_i] - \frac{\partial^2}{\partial x^i\,\partial x^j}[\psi_0]\right\}\bigg|_{x_0^\mu} \\
&= \frac{1}{\sqrt{2}}\left\{g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu\,[\psi_0-\psi_i]\right\}\Big|_{x_0^\mu} + \frac{1}{\sqrt{2}}\left\{\widehat{g}^{\mu\nu}(\Gamma)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\,[\psi_0-\psi_i]\right\}\Big|_{x_0^\mu} \\
&= g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu\,\Psi\Big|_{x_0^\mu} + \widehat{g}^{\mu\nu}(\Gamma)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\,\Psi\Big|_{x_0^\mu} = \rho\,\Psi\Big|_{x_0^\mu} + \Gamma\,\Psi\Big|_{x_0^\mu},
\end{aligned} \tag{14}$$

for $\partial_\mu^2 = \widehat{\partial}_\mu^2 = \left[\frac{\partial^2}{\partial t^2}, \frac{\partial^2}{\partial x^i\,\partial x^j}\right]$ and $g_{\mu\nu} = \widehat{g}_{\mu\nu} = \mathrm{diag}[1,-1,-1,-1]$, where $\widehat{\partial}$ implies ∂_t's dependency on ψ_i and ∂_x's dependency on ψ_0, respectively. Here, ρ and Γ have been chosen arbitrarily. Hence, the complex conjugate of Eq. (14) is,

$$\begin{aligned}
\frac{\partial^2}{\partial s^2}\Psi^\dagger\bigg|_{s_0} &= \left\{\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^i\partial x^j}\right\}\left\{\frac{1}{\sqrt{2}}\psi_0 + \frac{1}{\sqrt{2}}\psi_i\right\}\bigg|_{x_0^\mu} \\
&= g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu\,\Psi^\dagger\Big|_{x_0^\mu} + \widehat{g}^{\mu\nu}(\Gamma)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\,\Psi^\dagger\Big|_{x_0^\mu} = \rho\,\Psi^\dagger\Big|_{x_0^\mu} + \Gamma\,\Psi^\dagger\Big|_{x_0^\mu}.
\end{aligned} \tag{15}$$

But, if we take, $\mathrm{d}s^{-2}\varphi = (\mathrm{d}t^2 - \sum \mathrm{d}x^i\,\mathrm{d}x^j)^{-1}\varphi$, where φ is an operator, we can say that, $(\mathrm{d}t^2 - \sum \mathrm{d}x^i\,\mathrm{d}x^j)^{-1} \neq \mathrm{d}t^{-2} - \sum(\mathrm{d}x^i\,\mathrm{d}x^j)^{-1}$, so, we may assume without any objection that, $(\mathrm{d}t^2 - \sum \mathrm{d}x^i\,\mathrm{d}x^j)^{-1} \equiv \frac{\partial^2}{\partial t^2} - \sum \frac{\partial^2}{\partial x^i\,\partial x^j} - \Pi$, for some value of Π. Thus,

$$\begin{aligned}\mathrm{d}s^{-2}\varphi &= (\mathrm{d}t^2 - \mathrm{d}x^i\,\mathrm{d}x^j)^{-1}\varphi = \left\{\frac{\partial^2\varphi}{\partial t^2} - \left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}\right)\right\} - \Pi\varphi \\ &= \frac{\partial^2}{\partial s^2}\varphi - \Pi\varphi = g^{\mu\nu}\,\partial_\mu\partial_\nu\varphi - \Pi\varphi\,. \end{aligned} \tag{16}$$

Mathematically, Eq. (16) contradicts Eq. (8), unless otherwise Π is an unobservable property. Let us check it.

Proposition 1. *Eq. (16) implies Eq. (8), if Γ (so as Π) is an unobservable property.*

Proof. No more combinations are possible from Eq. (14) apart from ρ, Γ and Eq. (13) itself. The arrangement of ρ and Γ implies that $g_{\mu\nu}(s) = \frac{1}{2}\big(g_{\mu\nu}(\rho) + \widehat{g}_{\mu\nu}(\Gamma)\big)$. Thus, Eq. (14) should be rewritten by using Eq. (13), Eq. (14) and additionally replacing $g_{\mu\nu}(s)$ with $g_{\mu\nu}(s) = \frac{1}{2}\big(g_{\mu\nu}(\rho) + \widehat{g}_{\mu\nu}(\Gamma)\big)$ as follows,

$$\frac{1}{2}\big(g^{\mu\nu}(\rho) + \widehat{g}^{\mu\nu}(\Gamma)\big)\,\partial_\mu\partial_\nu\,\Psi\Big|_{x_0^\mu} = g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu\Psi\Big|_{x_0^\mu} + \widehat{g}^{\mu\nu}(\Gamma)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\Psi\Big|_{x_0^\mu}. \tag{17}$$

Note that, Π exclusively has to depend upon spacetime. Comparing Eq. (16) with Eq. (14), let us say that,

$$\frac{\partial^2}{\partial s^2}\Psi\bigg|_{s_0} = \rho\Psi\Big|_{x_0^\mu} - \Pi\Psi\Big|_{x_0^\mu}, \tag{18}$$

where $\Gamma \rightarrow -\Pi$, suppose. Since $g_{\mu\nu}(s) \neq \frac{1}{2}\big(g_{\mu\nu}(\rho) - \widehat{g}_{\mu\nu}(\Pi)\big) \equiv 0$ as long as $g_{\mu\nu} = \widehat{g}_{\mu\nu} = \mathrm{diag}[1,-1,-1,-1]$, then,

$$\frac{1}{2}\big(g^{\mu\nu}(\rho) + \widehat{g}^{\mu\nu}(\Pi)\big)\,\partial_\mu\partial_\nu\,\Psi\Big|_{x_0^\mu} = g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu\Psi\Big|_{x_0^\mu} + \widehat{g}^{\mu\nu}(\Pi)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\,[-\Psi]\Big|_{x_0^\mu}. \tag{19}$$

It is impossible to decompose both the LHS of Eq. (17) and Eq. (19) as they are only depended upon ∂, thus,

1. The spacetimes of ρ and Γ (so as Π) are not easily dissociative even upto a very high energy scale.
2. The spacetime of Γ (so as Π) must be an internal hidden property of the overall system (in other words, inside the observable spacetime of ρ) since $g_{\mu\nu}(s)$ is independent of $\widehat{\partial}$. The observable spacetime is always ∂-dependent.

Thus, Eq. (16) implies Eq. (8), since Γ (so as Π) is an unobservable property. □

But, the RHS of Eq. (19) gives us,

$$\{|\Psi\rangle \in V \otimes V : F|\Psi\rangle = |\Psi\rangle\} = \mathrm{Sym}^2\,V\,,$$
$$\{|\Psi\rangle \in V \otimes V : F|\Psi\rangle = -|\Psi\rangle\} = \mathrm{Anti}^2\,V\,,$$

where, the swap operator $F|\Psi\rangle = \exp[i\,\theta]$ for some phase $\exp[i\,\theta]$, whereas V is a vector space. Then the corresponding eigenspaces are called the symmetric and antisymmetric subspaces and are denoted by the state spaces $\mathrm{Sym}^2\,V$ and $\mathrm{Anti}^2\,V$, respectively. Note that, we are not intended here ρ and Π are the two indistinguishable particles for the state spaces $\mathrm{Sym}^2\,V$ and $\mathrm{Anti}^2\,V$; the above equations are just the generalization forms of their kinds, because ρ and Π do not have distinguished (opposite) spins until otherwise they are dissociated as free particles; so, the observable spin is always the spin of ρ, since Π is an internal hidden property of the overall system. Thus, Eq. (19) tells us that, if we allow Π to be dissociated as a free particle at very high energy, the internal hidden spacetime of Π must be transformed into a fermionic particle, whereas, the overall system remains bosonic, since, the observable spacetime is always ∂-dependent.

Similarly, Eq. (15) yields,

$$\begin{aligned} \left.\frac{\partial^2}{\partial s^2}\Psi^\dagger\right|_{s_0} &= \rho\,\Psi^\dagger\Big|_{x_0^\mu} - \Pi\,\Psi^\dagger\Big|_{x_0^\mu}, \\ \therefore\quad \frac{1}{2}\left(g^{\mu\nu}(\rho) + \widehat{g}^{\mu\nu}(\Pi)\right)\partial_\mu\partial_\nu\left[-\Psi\right]\Big|_{x_0^\mu} &= g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu\left[-\Psi\right]\Big|_{x_0^\mu} + \widehat{g}^{\mu\nu}(\Pi)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\Psi\Big|_{x_0^\mu}, \end{aligned} \tag{20}$$

this tells us that the internal hidden spacetime of Π must be now bosonic, whereas, the overall system is fermionic, since, the observable spacetime is always ∂-dependent. So, whatever Eq. (19) and Eq. (20) want to tell us is that the overall system has Supersymmetry and since the spacetimes of ρ and its supersymmetric partner Π are not easily dissociative even upto a very high energy scale; thus, Π must require extremely high energy to dissociate itself from the overall system as a free particle. Instead of being a free supersymmetric partner, Π actually works quite differently inside of the observable spacetime ρ, though, at the same time, Π is still satisfying all the properties of Supersymmetry, and we will show you Π's actual purpose very soon in the below. But Supersymmetry needs extra dimensions and we will discuss it below.

By the way, we can also develop a $\widehat{\partial}$-dependent scenario as follows,

$$\begin{aligned} \left.\frac{\widehat{\partial}^2}{\widehat{\partial} s^2}\Psi\right|_{s_0} &= \Pi\Psi\Big|_{x_0^\mu} - \rho\Psi\Big|_{x_0^\mu}, \\ \therefore\quad \frac{1}{2}\left(\widehat{g}^{\mu\nu}(\Pi) + g^{\mu\nu}(\rho)\right)\widehat{\partial}_\mu\widehat{\partial}_\nu\,\Psi\Big|_{x_0^\mu} &= \widehat{g}^{\mu\nu}(\Pi)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\Psi\Big|_{x_0^\mu} + g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu\left[-\Psi\right]\Big|_{x_0^\mu}, \end{aligned} \tag{21}$$

and the complex conjugate of Ψ is,

$$\begin{aligned} \left.\frac{\widehat{\partial}^2}{\widehat{\partial} s^2}\Psi^\dagger\right|_{s_0} &= \Pi\,\Psi^\dagger\Big|_{x_0^\mu} - \rho\,\Psi^\dagger\Big|_{x_0^\mu}, \\ \therefore\quad \frac{1}{2}\left(\widehat{g}^{\mu\nu}(\Pi) + g^{\mu\nu}(\rho)\right)\widehat{\partial}_\mu\widehat{\partial}_\nu\left[-\Psi\right]\Big|_{x_0^\mu} &= \widehat{g}^{\mu\nu}(\Pi)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\left[-\Psi\right]\Big|_{x_0^\mu} + g^{\mu\nu}(\rho)\,\partial_\mu\partial_\nu\Psi\Big|_{x_0^\mu}. \end{aligned} \tag{22}$$

It is not important which state spaces satisfy such bosonic or fermionic representations of Eq. (21) and Eq. (22), here, the most important thing is that the overall system as a free observable particle at very high energy is must not be baryonic because now only the internal hidden spacetime of ρ has 'proper' spacetime arrangement for its ∂-dependency, whereas, the overall (observable) system's spacetime arrangement is quite 'improper' as it is $\widehat{\partial}$-dependent. Despite of ρ's ∂-dependency, here, being a supersymmetric partner, if it is allowed to be free at very high energy, it must not be baryonic either. We should not be confused with it. We will discuss about its property in Section 4 below.

The internal hidden spacetime of Π in Eq. (19) and Eq. (20) also provides us some additional geometry for its $\widehat{g}^{\mu\nu}(\Pi)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\left[\pm\Psi\right]$ structures. Suppose, for $\widehat{g}^{\mu\nu}(\Gamma)\,\widehat{\partial}_\mu\widehat{\partial}_\nu\left[\Psi\right]$, we have, $\frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}\left[-\psi_i\right] - \frac{\partial^2}{\partial x^i\,\partial x^j}\left[\psi_0\right]\right\}$, where, neither spacetime arrangements are matched with one another in either combinations within the curvey brackets. These 'wrong' arrangements must have a noticeable effect on the acceptable spacetime, i.e., its temporal part must influence over the spatially depended ψ_i, or its spatial part must influence over the temporally depended ψ_0, or vice versa. In other words, the acceptable spacetime should not have to be four-dimensional in this case. Let us check it. Suppose, for $\frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[-\psi_i] - \frac{\partial^2}{\partial x^i \partial x^j}[\psi_0]\right\}$, we can consider a dimension function (see Ref. [4]; we also strongly follow Nagata's work throughout this paragraph and omit all the proofs hereabout as they are well explained in his book),

$$\dim\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)\ \forall\, i \in \{1,2,3\}\ , \tag{23}$$

and the space $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ satisfies a normal T_1-space. Let $\mathcal{U}$ be a collection in a $(3+1)D$ topological spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$, i.e., $\dim\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) = (3+1)$ (which is actually hidden inside the $(3+1)D$ observable spacetime $\left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right)$, i.e., $\dim\left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right) = (3+1)$, too) and p a point of $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$, then the order of $\mathcal{U}$ at p should be denoted by, $\operatorname{ord}\mathcal{U} = \sup\left\{\operatorname{ord}_p \mathcal{U} \mid p \in \left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)\right\}$, where, $\operatorname{ord}_p \mathcal{U}$ is the number of members of $\mathcal{U}$ which contain p. If for any finite open covering $\mathcal{U}$ of the topological spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ there exists an open covering $\mathfrak{B}$ such that $\mathfrak{B} < \mathcal{U}$, $\operatorname{ord}\mathfrak{B} \leq n+1$, then $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ has covering dimension $\leq n$, i.e., $\dim\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \leq n$. If $\mathcal{U}$ can be decomposed as $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}_i$ for locally finite (star-finite, discrete, etc.) collections $\mathcal{U}_i$, then $\mathcal{U}$ is called a σ-locally finite (σ-star-finite, σ-discrete, etc.) collection. The topological spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ has strong inductive dimension -1, i.e., $\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) = -1$, if $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) = \emptyset$. If for any disjoint closed sets F and G of the topological spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ there exists an open set U such that $F \subset U \subset \left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - G$, and $\operatorname{Ind} B(U) \leq n-1$, where $B(U)$ denotes the boundary of U, then $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ has strong inductive dimension $\leq n$, i.e., $\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \leq n$. Let V is an open set and $\bar{V}$ is a closed set of $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$. If $\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \leq n$, then there exists a σ-locally finite open basis $\mathfrak{B}$ of $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ such that, $\operatorname{Ind} B(V) \leq n-1$ for every $V \in \mathfrak{B}$. If a spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ has a σ-locally finite open basis $\mathfrak{B}$ such that, $B(V) = \emptyset$ for every $V \in \mathfrak{B}$, then $\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \leq 0$. Again, $\operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \leq n$ if and only if there exists a σ-locally finite open basis $\mathfrak{B}$ such that $\operatorname{Ind} B(V) \leq n-1$ for every $V \in \mathfrak{B}$. For every subset $A = \bigcup\{B(V) \mid V \in \mathfrak{B}\}$, for any integer $n \geq 0$, of a spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$, we have, $\operatorname{Ind} A \leq \operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$. Hence, if and only if $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) = \bigcup_{i=1}^{n+1} A_i$ for some $n+1$ subsets A_i with $\operatorname{Ind} A_i \leq 0$, $i = 1, \ldots, n+l$. For the spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$, we have then $\dim\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) = \operatorname{Ind}\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$. Let A be a subset of a spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ and I the unit segment. If U is an open set of the topological product $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \times I$ such that $U \supset A \times I$, then there exists an open set V of $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ such that $A \subset V$, and $V \times I \subset U$. Let F be a closed set of $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ with $\dim F \leq n$. Let F_α and U_α, $\alpha < \tau$, be closed and open sets, respectively such that $F_\alpha \subset U_\alpha$, and $\{U_\alpha \mid \alpha < \tau\}$ is locally finite. Then there exist open sets V_α satisfying $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, and $\dim B_k \leq n-k$, $k = 1, \ldots, n+1$, where, $B_k = \{p \mid p \in F, \operatorname{ord}_p B(\mathfrak{B}) \geq k\}$, and $\mathfrak{B} = \{V_\alpha \mid \alpha < \tau\}$. Let F, F_α and U_α satisfy the same condition as above, then there exist open sets V_α, W_α, $\alpha < \tau$ satisfying, $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset W_\alpha \subset U_\alpha$, and $\operatorname{ord}_p \{W_\alpha - \bar{V}_\alpha \mid \alpha < \tau\} \leq n$ for every $p \in F$. Let G_k, $k = 0, \ldots, n$, be closed sets with $\dim G_k \leq n-k$ of the spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$. Let $\{F_\alpha \mid \alpha < \tau\}$ be a closed collection and $\{U_\alpha \mid \alpha < \tau\}$ a locally finite open collection such that $F_\alpha \subset U_\alpha$. Then there exists an open collection, $\mathfrak{B} = \{V_\alpha \mid \alpha < \tau\}$, such that, $F_\alpha \subset V_\alpha \subset \bar{V}_\alpha \subset U_\alpha$, and $\operatorname{ord}_p B(\mathfrak{B}) \leq n-k$ for every $p \in G_k$. A mapping f of the spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ into a spacetime S is a closed (open) mapping if the image of every closed (open) set of

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$\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ is closed (open) in S. Then the continuous mappings which lower dimensions of the spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ should be defined as follows,

Theorem 1. *Let f be a closed continuous mapping of the $(3+1)D$ spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ onto the spacetime S such that $\dim f^{-1}(q) \le k$ for every $q \in S$. Then,*

$$dim \left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \le dim\, S + dim\, K\,, \tag{24}$$

where $\dim K \le k$ for the space K, when $0 < \dim S \le 2$, since i should not be zero in Eq. (23).

Proof. Using Theorem III.6 of Ref. [4], we can easily prove this theorem. □

Since, the temporal axis is unaltered in Lorentz transformation, as we have already seen it in Eq. (12), we can express the maximal continuous mapping of the $(3+1)D$ spacetime $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ onto the spacetime S of Eq. (24) as,

$$\begin{aligned} X^\mu(\tau,\sigma) &\le \left\{\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \mapsto S \;\middle|\; S \le \left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right)\right\}\,, \\ \text{thus,}\quad \mathfrak{X}^\mu &\le S \cup K \text{ inside the } (3+1)D \text{ observable spacetime } \left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right)\,, \end{aligned}$$

since i should not be zero in Eq. (23), if the considered state is not vacuum; then the spacetime S definitely intends the basic structure of a 2-dimensional worldsheet $X^\mu(\tau,\sigma)$ with the joint states, $\frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial\tau^2}[-\psi_\sigma] - \frac{\partial^2}{\partial\sigma^2}[\psi_\tau]\right\}$ for the spacetime $\mathfrak{X}^\mu$, where S is a $(1+1)D$ spacetime, but for the space K, we will like to discuss it below in more details in the Theorem 2. Obviously, a string can sweep out the 2-dimensional worldsheet $X^\mu(\tau,\sigma)$ for the spacetime $\mathfrak{X}^\mu$.

If the internal hidden spacetime of Π is considered as the LHS spacetime of Eq. (10) and let it to be changed from $(3+1)D$ to $(1+1)D$ when its total energy remains fixed (so as its time), then the spacetime $\mathfrak{X}^\mu$ of RHS of Eq. (10) changes inversely against the spacetime of Π. Since the spacetime of Π is hidden inside the overall system of Eq. (19), i.e., in other words, inside the observable spacetime of ρ, then the increment of RHS spacetime $\mathfrak{X}^\mu$ of Eq. (10) should not be observable by any means, i.e., the extra dimensions of $\mathfrak{X}^\mu$ remain hidden forever inside the observable spacetime of ρ. As these internal hidden extra dimensions inside the observable spacetime of ρ are considered as the representation of the spacetime S and the space K, thus, we can conclude,

1. Strings (for the hidden spacetime $\mathfrak{X}^\mu$) are natural and universal but forever hidden inside every $(3+1)D$ observable system (i.e., the spacetime $\left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right)$) in Quantum Mechanics.
2. Every $(3+1)D$ observable system in Quantum Mechanics must contain forever hidden extra dimensions (i.e., the space K) whether an external observer considers any strings in these systems or not (for more details, see Eq. (25) below and its following text therein).

But the space K should raise more extra hidden dimensions by a closed continuous mapping beyond $\dim K \le k$ by adopting the following,

Theorem 2. *Let f be a closed continuous mapping of a space R onto a space K such that for each point q of K, $B(f^{-1}(q))$ contains at most $m+1$ points $(m \ge 0)$; then $\dim K \le \dim R + \dim M$, when $\dim R \le \left[\dim \left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - \dim S\right]$ and $\dim M \le m$, where $\dim \left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \le (3+1)$.*

Proof. Using Theorem III.7 of Ref. [4], we can easily prove this theorem. □

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Then, we can say for the overall spacetime $\mathfrak{X}^{\mu}_{\text{OVERALL}}$ that,

$$\mathfrak{X}^{\mu}_{\text{OVERALL}} \leq \left\{ \left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right) \cup S \cup K \;\middle|\; S \leq \left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right) \text{ and } K \leq \left[\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - S\right] \cup M \;\forall\; \dim M \leq m \;\exists\; m \geq 0 \right\}, \tag{25}$$

for which,

$$\mathfrak{X}^{\mu} \leq S \cup K \leq \left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \cup M \quad \forall \quad \dim M \leq m \;\exists\; m \geq 0 \text{ inside the } (3+1)D \text{ observable spacetime } \left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right). \tag{26}$$

Note here that stringy spacetime S vanishes in the overall spacetime $\mathfrak{X}^{\mu}_{\text{OVERALL}}$ of Eq. (25) for the space K leaving behind the forever hidden extra dimensions m in $\mathfrak{X}^{\mu}_{\text{OVERALL}}$. Thus, in other words, strings are experimentally unobservable forever, whereas, their actions are mandatory in the purpose of particle interactions. Also notice that Supersymmetry (now having extra dimensions m for $\mathfrak{X}^{\mu}$ due to Eq. (26)) remains unaffected in $\mathfrak{X}^{\mu}_{\text{OVERALL}}$ of Eq. (25). Thus, with these extra dimensions, the above scenario is now perfect for Supersymmetry and String Theory without any further objections.

Along with Theorem 1, what Eq. (26) actually wants to say us is,

$$\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \leq \mathfrak{X}^{\mu} \leq \left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) \cup M,$$

when $\mathfrak{X}^{\mu} \leq S \cup K$, which yields,

$$\left[\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - S\right] \leq K \leq \left[\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - S\right] \cup M. \tag{27}$$

Since $S \leq \left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right)$ in Eq. (25), let the LHS of Eq. (27) gives,

$$\left[\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) - \left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right)\right] \leq \left\{ \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) \cup \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \right\}. \tag{28}$$

The most disturbing thing here is that the temporal axis is a part of S spacetime but not the part of K space, but both $\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ spaces are influenced by the (mutual) temporal axis, despite neither of them have contained any temporal axis within themselves. On the other hand, it is evidence that only an influence should not sufficient to emerge a temporal axis within M (or K) space. Moreover, Theorem 2 yields no temporal axis for M (or K) space either. But the influenced of the temporal axis should not ease to be avoided in Eq. (28).

From Theorem 2, if we think that the dimension of M space depends only on $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right) = \left\{\left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right), \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right), \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)\right\}$, then we should be mistaken, M is not independent from either elements of the set $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$. Thinking otherwise, let $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$ are related to new quantities Q_i and T_i, differently, which are the curvilinear coordinates of $\left(\widehat{\partial}^2_{[0,i]} \otimes \Psi_{[i,0]}\right)$. Let the corresponding members $\{T_1, T_2, T_3\} \subset T$ are determining γ, then if each pair of members from the either sides of these curvilinear coordinates joining the pairs of points Q_i and T_i ($i = 1,2,3$) meet in points $\mathfrak{m}_i \in M$ separately, then the three points of intersection U_i of the pairs of coordinates q_i and t_i ($i = 1,2,3$) lie on a line. Let each of the pairs of coordinates Q_i, T_i ($i = 1,2,3$) consists of two distinct coordinates and in which $q_i \neq t_i \;\forall\; i$. Let the coordinate vectors of $\mathfrak{m}_i$ be denoted by $\mathfrak{z}_i$, that of Q_i by $\mathfrak{r}_i$

$(i = 1, 2, 3)$ and that of T_i by $\mathfrak{y}_i$ $(i = 1, 2, 3)$. Then $\mathfrak{z}_i$ can be represented by a linear combination of the $\mathfrak{r}_i$ and $\mathfrak{y}_i$ for each $i = 1, 2, 3$, say,

$$\mathfrak{z}_i = \mathfrak{r}_1 + \mathfrak{y}_1 = \mathfrak{r}_2 + \mathfrak{y}_2 = \mathfrak{r}_3 + \mathfrak{y}_3 \,.$$

Hence,

$$\begin{aligned} \mathfrak{r}_1 - \mathfrak{r}_2 &= \mathfrak{y}_2 - \mathfrak{y}_1 \,, \\ \mathfrak{r}_2 - \mathfrak{r}_3 &= \mathfrak{y}_3 - \mathfrak{y}_2 \,, \\ \mathfrak{r}_3 - \mathfrak{r}_1 &= \mathfrak{y}_1 - \mathfrak{y}_3 \,. \end{aligned}$$

Let us choose two set of coordinates,

$$\begin{aligned} a_i &= (a_1, a_2, a_3) = \frac{1}{\sqrt{2}} \left\{ \frac{\partial^2}{\partial t^2} [-\varphi_i] - \frac{\partial^2}{\partial x^i \, \partial x^j} [\varphi_0] \right\} = \left(\widehat{\partial}^2_{[0,i]} \otimes \boldsymbol{\varphi}_{[i,0]} \right) \in \Lambda \,, \\ b_i &= (b_1, b_2, b_3) = \frac{1}{\sqrt{2}} \left\{ \frac{\partial^2}{\partial t^2} [-\varrho_i] - \frac{\partial^2}{\partial x^i \, \partial x^j} [\varrho_0] \right\} = \left(\widehat{\partial}^2_{[0,i]} \otimes \boldsymbol{\varrho}_{[i,0]} \right) \in \Lambda \,, \end{aligned} \tag{29}$$

for $i, j = 1, 2, 3$, such that $\{a_i\} \in \mathfrak{r}_i$, $\{b_i\} \in \mathfrak{y}_i$ and $\{a, b\}$ is a basis of Λ, whereas $\Lambda \cap Q_i = \{O\}$, where Q_i is the interior of $\boldsymbol{Q}$ and O is the origin, i.e., Λ is admissible for $\boldsymbol{Q}$. Let the quadratic form,

$$\begin{aligned} &\mathcal{Q} \left(\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]} \right), \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]} \right) \right) \\ &= \sum_{1 \le i,j \le 3} \frac{1}{2} \left[(a_i - a_j) \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]} \right) + (b_i - b_j) \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]} \right) \right] \\ &= \frac{1}{2} \left[A \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]} \right)^2 + 2B \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]} \right) \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]} \right) + C \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]} \right)^2 \right] , \end{aligned}$$

say, is reduced. The last fact means that $2\,|B| \le A \le C$, so that $3A^2 \le 4\left(A^2 - B^2\right) \le 4\left(AC - B^2\right)$. Since Λ is admissible for $\boldsymbol{Q}$, the coordinates $a + mc$ (m an integer) do not belong to int $\boldsymbol{Q}$. Thus,

$$\{ |(m + a_1)(m + a_2)(m + a_3)| \ge 1 \ \forall \text{ integers } m \} \,,$$

this implies that,

$$\begin{aligned} A &= (a_1 - a_2)^2 + (a_2 - a_3)^2 + (a_3 - a_1)^2 \\ &= 2\left(a_1^2 + a_2^2 + a_3^2\right) - 2\left(a_1\, a_2 + a_2\, a_3 + a_3\, a_1\right) \,. \end{aligned}$$

Note it here that $a_i\, a_j \neq 0$ if $i = j$ and $a_i\, a_j = 0$ if $i \neq j$ do not hold due to Eq. (29). So as,

$$\begin{aligned} C &= (b_1 - b_2)^2 + (b_2 - b_3)^2 + (b_3 - b_1)^2 \\ &= 2\left(b_1^2 + b_2^2 + b_3^2\right) - 2\left(b_1\, b_2 + b_2\, b_3 + b_3\, b_1\right) \,. \end{aligned}$$

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and we can easily find that $B = 0$. Here,

$$\begin{aligned}\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)^2 &= \left[\frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[-\psi_2] - \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0]\right\}\right]^2 \\ &= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[-\psi_2] - \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0]\right\} \times \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[-\psi_2] - \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0]\right\} \\ &= \frac{1}{2}\left\{\frac{\partial^2}{\partial t^2}[-\psi_2] \times \frac{\partial^2}{\partial t^2}[-\psi_2] - \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0] \times \frac{\partial^2}{\partial t^2}[-\psi_2] - \right. \\ &\qquad \left. - \frac{\partial^2}{\partial t^2}[-\psi_2] \times \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0] + \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0] \times \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0]\right\} \\ &= \frac{1}{2}\left\{\left(\frac{\partial^2}{\partial t^2}[-\psi_2]\right)^2 + \left(\frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0]\right)^2\right\} - \frac{\partial^2}{\partial t^2}[\psi_0]\,\frac{\partial^2}{\partial x^2\,\partial x^2}[-\psi_2] \\ &= \frac{1}{2}\left(\widehat{\partial}_0^4\,\Psi_2^2 + \widehat{\partial}_2^4\,\Psi_0^2\right) - \partial_0^2\,\Psi_0\,\partial_2^2\,\Psi_2\,. \end{aligned} \tag{30}$$

Similarly,

$$\begin{aligned}\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)^2 &= \left[\frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[-\psi_3] - \frac{\partial^2}{\partial x^3\,\partial x^3}[\psi_0]\right\}\right]^2 \\ &= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[-\psi_3] - \frac{\partial^2}{\partial x^3\,\partial x^3}[\psi_0]\right\} \times \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[-\psi_3] - \frac{\partial^2}{\partial x^3\,\partial x^3}[\psi_0]\right\} \\ &= \frac{1}{2}\left(\widehat{\partial}_0^4\,\Psi_3^2 + \widehat{\partial}_3^4\,\Psi_0^2\right) - \partial_0^2\,\Psi_0\,\partial_3^2\,\Psi_3\,. \end{aligned} \tag{31}$$

In the same way,

$$\begin{aligned} &\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \\ &= \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[-\psi_2] - \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0]\right\} \times \frac{1}{\sqrt{2}}\left\{\frac{\partial^2}{\partial t^2}[-\psi_3] - \frac{\partial^2}{\partial x^3\,\partial x^3}[\psi_0]\right\} \\ &= \frac{1}{2}\left\{\frac{\partial^2}{\partial t^2}[-\psi_2] \times \frac{\partial^2}{\partial t^2}[-\psi_3] - \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0] \times \frac{\partial^2}{\partial t^2}[-\psi_3] - \right. \\ &\qquad \left. - \frac{\partial^2}{\partial t^2}[-\psi_2] \times \frac{\partial^2}{\partial x^3\,\partial x^3}[\psi_0] + \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0] \times \frac{\partial^2}{\partial x^3\,\partial x^3}[\psi_0]\right\} \\ &= \frac{1}{2}\left\{\frac{\partial^2}{\partial t^2}[-\psi_2] \times \frac{\partial^2}{\partial t^2}[-\psi_3] + \frac{\partial^2}{\partial x^2\,\partial x^2}[\psi_0] \times \frac{\partial^2}{\partial x^3\,\partial x^3}[\psi_0]\right\} - \\ &\qquad - \frac{1}{2}\frac{\partial^2}{\partial t^2}[\psi_0]\left\{\frac{\partial^2}{\partial x^2\,\partial x^2}[-\psi_3] + \frac{\partial^2}{\partial x^3\,\partial x^3}[-\psi_2]\right\} \\ &= \frac{1}{2}\left(\widehat{\partial}_0^4\,\Psi^2_{\{2,3\}} + \widehat{\partial}^4_{\{2,3\}}\,\Psi_0^2\right) - \frac{1}{2}\,\partial_0^2\,\Psi_0\left(\partial^2_{\{2,3\}} \otimes \Psi_{\{3,2\}}\right)\,. \end{aligned} \tag{32}$$

In the last line we have used subscripts { }, which are quite different from the subscripts [ ] we have used earlier and their purposes are quite obvious here. Since, the temporal axis is a part of S spacetime but not the part of K space, so both $\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ spaces, as well as a_i and b_i spaces of Eq. (29), are influenced by the (mutual) temporal axis and neither of them have contained any temporal axis within themselves, then we can say that all axes of $a_i\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $b_i\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ (for $i, j = 1, 2, 3$) in K space are interrelated with the (mutual) temporal axis of string spacetime S, since the temporal axis is a part of $S = \left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right)$ but not the part of K space, thus, $a_i\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $b_i\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ (for $i, j = 1, 2, 3$) in K space have individual existences as independent axes $x^{(1+i)}$ and $x^{(1+(i+\ell))}$ (for $\ell = \max i$) influenced by the (mutual) temporal axis x^0. Let us assume that $a_i \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $b_i \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ (for $i, j = 1, 2, 3$) in K space have maximal weight as 1 of each dimension as an independent axis for $x^{(1+i)}$ and $x^{(1+(i+\ell))}$, which yields,

$$\begin{aligned} \dim\left[(a_1 + a_2 + a_3)\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)\right] &\leq 3\,, \\ \dim\left[(b_1 + b_2 + b_3)\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)\right] &\leq 3\,. \end{aligned} \tag{33}$$

Hence, they have the "proper" dimensions. Comparing the last line of Eq. (32) with Eq. (30) and Eq. (31), we can determine that if Eq. (30) and Eq. (31) give us some "proper" dimensions, then Eq. (32) definitely gives us an "improper" dimension, as both $\left(\widehat{\partial}^4_0 \Psi^2_{\{2,3\}} + \widehat{\partial}^4_{\{2,3\}} \Psi^2_0\right)$ and $\left(\partial^2_{\{2,3\}} \otimes \Psi_{\{3,2\}}\right)$ are depended on x^2 and x^3 axes, simultaneously. Since a and b are satisfying Eq. (29), then $a_i a_j$ and $b_i b_j$ (for $i, j = 1, 2, 3$, $i \neq j$) must give us "improper" dimensions, too. If we consider these "improper" dimensions $(a_i\, a_j)^{1/2} \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)$ and $(b_i\, b_j)^{1/2} \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)$ (for $i, j = 1, 2, 3, i \neq j$) in K space have individual existences as independent axes $x_a^{(1+(i+\ell)+\frac{1}{2}j)}$ and $x_b^{(1+(i+\ell)+\frac{1}{2}j)}$ (since they are depended on x^2 and x^3 axes, simultaneously) influenced by the (mutual) temporal axis x^0, then, on the contrary of Eq. (33), let us assume that they have maximal weight as 0.5 of each dimension for $x_a^{(1+(i+\ell)+\frac{1}{2}j)}$ and $x_b^{(1+(i+\ell)+\frac{1}{2}j)}$, so as they can give $x^{(1+(i+\ell)+j)} = x_a^{(1+(i+\ell)+\frac{1}{2}j)} + x_b^{(1+(i+\ell)+\frac{1}{2}j)}$, thus, we can say that,

$$\begin{aligned} \dim\left[\left\{(a_1\, a_2)^{1/2} + (a_2\, a_3)^{1/2} + (a_3\, a_1)^{1/2}\right\}\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)\right] &\leq 1.5\,, \\ \dim\left[\left\{(b_1\, b_2)^{1/2} + (b_2\, b_3)^{1/2} + (b_3\, b_1)^{1/2}\right\}\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)\right] &\leq 1.5\,. \end{aligned}$$

Hence, altogether they have,

$$\begin{aligned} \dim\Bigg[&\left\{(a_1\, a_2)^{1/2} + (a_2\, a_3)^{1/2} + (a_3\, a_1)^{1/2}\right\}\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) \\ &\bigcup (a_1 + a_2 + a_3)\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right)\Bigg] \quad \leq \quad 4.5\,, \\ \dim\Bigg[&\left\{(b_1\, b_2)^{1/2} + (b_2\, b_3)^{1/2} + (b_3\, b_1)^{1/2}\right\}\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \\ &\bigcup (b_1 + b_2 + b_3)\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)\Bigg] \quad \leq \quad 4.5\,. \end{aligned}$$

Since $B = 0$, the K space yields,

$$\begin{aligned} K \quad = \quad \Bigg[&(a_1 + a_2 + a_3)\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) \\ &\bigcup\left\{(a_1\, a_2)^{1/2} + (a_2\, a_3)^{1/2} + (a_3\, a_1)^{1/2}\right\}\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) \\ &\bigcup (b_1 + b_2 + b_3)\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \\ &\bigcup\left\{(b_1\, b_2)^{1/2} + (b_2\, b_3)^{1/2} + (b_3\, b_1)^{1/2}\right\}\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)\Bigg]\,, \end{aligned}$$

i.e.,

$$\dim K \leq (4.5 + 4.5) = 9\,.$$

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Thus, $\mathfrak{X}^{\mu} \leq S \cup K$ has the spacetime axes as (using summation convention),

$$\begin{aligned} &\Big[\left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right), a_i \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right), b_i \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right), \\ &\qquad \left\{ \left(a_i a_j\right)^{1/2} \left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) + \left(b_i b_j\right)^{1/2} \left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right) \right\} \Big] \\ &\mapsto \quad \left(x^0, x^1, x^{(1+i)}, x^{(1+(i+\ell))}, x^{(1+(i+\ell)+j)} \right) \in \mathfrak{X}^{\mu}, \end{aligned} \tag{34}$$

for $i, j = 1, 2, 3$, $i \neq j$ and $\ell = \max i$. So, Eq. (34) achieves,

$$\dim \mathfrak{X}^{\mu} \leq \dim (S \cup K) \leq (2+9) = 11,$$

i.e., string has eleven-dimensions by nature, that is why eleven is the maxinium spacetime dimension in which one can formulate a consistent supersymmetric theory.

Now, returning to our main purpose and using the first two terms of Eq. (5), we can generate the following wave equation for Eq. (4) as,

$$i\hbar\, \mathrm{v}^0\, \vec{\nabla}_0\, \psi(\vec{r},t) + i\hbar\, \mathrm{v}^j\, \vec{\nabla}_i\, \psi(\vec{r},t) - i\hbar\, g_{\mu\nu}\, \mathrm{v}^{\nu}\, \vec{\nabla}_{\mu}\, \psi(\vec{r},t) = 0, \tag{35}$$

where $\vec{\nabla}_0 = \left(\partial / \partial\, (\mathrm{v}^0\, t)\right)$ for $x^0 = (\mathrm{v}^0\, t)$, while the 'Four-momentum' operator is $\hat{\boldsymbol{\mathcal{P}}}^{\mu} \rightarrow i\hbar\, \vec{\nabla}_{\mu}$, and the three momentum operator is $\hat{\boldsymbol{p}} \rightarrow -i\hbar\, \vec{\nabla}_i$.

Remark 2. *The signature of the metric $g_{\mu\nu}$, i.e., $(+,-,-,-)$, has been absorbed and retained unaltered by the last term of Eq. (35), as long as it satisfies Eq. (3) and Eq. (4). Thus, readers are requested to be careful not to presume space and time separately in Eq. (35), what we usually assume in the conventional Quantum Mechanics.*

Again rearranging Eq. (35) by using Eq. (6), we may get,

$$i\hbar\, \mathrm{v}^0 \left(1 - g_{00}\right) \vec{\nabla}_0\, \psi(\vec{r},t) + i\hbar\, \mathrm{v}^j \left(1 + g_{ij}\right) \vec{\nabla}_i\, \psi(\vec{r},t) = 0,$$

or, simply discarding $\left(1 - g_{00}\right) = \left(1 + g_{ij}\right) = 0$, we can have the First Variance of the First Order Equation of Semi-quantum Minkowski GQG as,

$$\begin{aligned} i\hbar\, \mathrm{v}^0\, \vec{\nabla}_0\, \psi(\vec{r},t) + i\hbar\, \mathrm{v}^j\, \vec{\nabla}_i\, \psi(\vec{r},t) &= 0, \\ \therefore \quad i\hbar\, \frac{\partial}{\partial t}\, \psi(\vec{r},t) + i\hbar\, \mathrm{v}^j\, \vec{\nabla}_i\, \psi(\vec{r},t) &= 0. \end{aligned} \tag{36}$$

Evidently, Eq. (36) may take the form $\hat{E}\, \psi(\vec{r},t) = -i\hbar\, \mathrm{v}^j\, \vec{\nabla}_i\, \psi(\vec{r},t)$ for the energy operator $\hat{E} \rightarrow i\hbar\, \partial_t$. Setting the Hamiltonian operator as $\hat{H}\, \psi(\vec{r},t) = -i\hbar\, \mathrm{v}^j\, \vec{\nabla}_i\, \psi(\vec{r},t) \equiv \hat{\boldsymbol{p}}\, \mathrm{v}^j\, \psi(\vec{r},t)$, where the three momentum operator $\hat{\boldsymbol{p}} \rightarrow -i\hbar\, \vec{\nabla}_i$, we can therefore have, $\hat{E}\, \psi(\vec{r},t) = \hat{H}\, \psi(\vec{r},t)$. Interested readers can easily check it that Eq. (36) is nothing but the gravitational form of the Classical Schrödinger equation, where E is total energy, and now the equation has been rewritten with v^j along with the signature of the metric $(+,-,-,-)$.

It is also possible to develop a Second Variance of the First Order Equation of Semi-quantum Minkowski GQG from Eq. (35) as follows,

$$i\hbar\, \mathrm{v}^{\nu}\, \vec{\Delta}_{\mu}\, \psi(\vec{r},t) - i\hbar\, g_{\mu\nu}\, \mathrm{v}^{\nu}\, \vec{\nabla}_{\mu}\, \psi(\vec{r},t) = 0,$$

where $i\hbar\, \vec{\Delta}_{\mu} \rightarrow \left[\hat{\boldsymbol{p}}_0,\, -\hat{\boldsymbol{p}}\right]^{\mathrm{T}} \rightarrow \left[i\hbar\, \vec{\nabla}_0,\, i\hbar\, \vec{\nabla}_i\right]^{\mathrm{T}}$.

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Now, let us multiply both sides of Eq. (35) by $(\mathrm{d}t^2/m)$, so as,

$$\begin{aligned} i\hbar\,\frac{\mathrm{d}t^2}{m}\,\mathrm{v}^0\,\vec{\nabla}_0\,\psi(\vec{r},t) + i\hbar\,\frac{\mathrm{d}t^2}{m}\,\mathrm{v}^j\,\vec{\nabla}_i\,\psi(\vec{r},t) &= i\hbar\,\frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t) \\ &= \frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\hat{\boldsymbol{\mathcal{P}}}^\mu\,\psi(\vec{r},t) \end{aligned} \tag{37}$$

which has the form of a general inhomogeneous Lorentz transformation (or Poincaré transformation).

Note it that Eq. (37) is exactly equivalent to $\mathrm{d}s^2 \equiv (\mathrm{d}t^2/m)\,g_{\mu\nu}\,P^\mu\,\mathrm{v}^\nu$ of Eq. (1), i.e., $\mathrm{d}s^2 \equiv (\mathrm{d}t^2/m)\,g_{\mu\nu}\,P^\mu\,\mathrm{v}^\nu \mapsto \mathrm{d}s^2\,\psi(\vec{r},t) \equiv (\mathrm{d}t^2/m)\,g_{\mu\nu}\,\mathrm{v}^\nu\,\hat{\boldsymbol{\mathcal{P}}}^\mu\,\psi(\vec{r},t)$, for the 'Four-momentum' operator $\hat{\boldsymbol{\mathcal{P}}}^\mu \to i\hbar\,\vec{\nabla}_\mu$. In other words, we can say that the quantum line element is,

$$\begin{aligned} \mathrm{d}s^2\,\psi(\vec{r},t) &\equiv i\hbar\,\frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t) \\ &= \frac{\mathrm{v}^\mu}{\mathrm{v}^\mu}\left(i\hbar\,\frac{\mathrm{d}t^2}{m}\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\right)\psi(\vec{r},t) \\ &= \frac{1}{\mathrm{v}^\mu}\left(i\hbar\,\frac{\mathrm{d}x^\mu}{m}\,g_{\mu\nu}\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\right)\psi(\vec{r},t) \\ &= \frac{1}{(\mathrm{v}^\mu\,m)}\left(i\hbar\,g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\right)\psi(\vec{r},t), \end{aligned}$$

hence, by considering $\mathcal{E} = \left(\mathrm{v}^\mu\,m\right)^{-1}$, we have,

$$\mathrm{d}s^2\,\psi(\vec{r},t) = i\hbar\,\mathcal{E}\,g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t)\,. \tag{38}$$

Proposition 2. *In Eq. (11), relativistic spacetime is showing a relation with the quantum spacetime, if the energy of the system remains fixed (so as its time). The wave field $\psi(\vec{r},t)$ itself in Eq. (5) is relativistic due to $g_{\mu\nu}$ in its either terms. Equally, Eq. (35) also assures us that Quantum Mechanics and Relativity are must be correlated for the presence of $g_{\mu\nu}$ in the last term of wave equation Eq. (35). Lastly, the exact equivalency of Eq. (37) and Eq. (1), i.e.,*

$$ds^2 \equiv (dt^2/m)\,g_{\mu\nu}\,v^\nu\,P^\mu \quad\longmapsto\quad ds^2\,\psi(\vec{r},t) \equiv (dt^2/m)\,g_{\mu\nu}\,v^\nu\,\hat{\boldsymbol{\mathcal{P}}}^\mu\,\psi(\vec{r},t),$$

for the 'Four-momentum' operator $\hat{\boldsymbol{\mathcal{P}}}^\mu \to i\hbar\,\vec{\nabla}_\mu$, can say us that Quantum Mechanics and Relativity are correlated in the wave field $\psi(\vec{r},t)$. So, we have a sufficient reason to accept the transformation of relativistic to quantum relations, and vice versa, as,

$$\begin{aligned} P^\mu &\iff \hat{\boldsymbol{\mathcal{P}}}^\mu \\ \text{i.e.,}\quad m\,v^\mu &\iff i\hbar\,\vec{\nabla}_\mu\,. \end{aligned}$$

We will use Proposition 2 throughout our work. This Proposition is quite straightforward than some commonly used textbook procedures, for example, Ref. [5].

Remark 3. *Technically, $\mathcal{E}$ and $i\hbar\,\vec{\nabla}_\mu$ should cancel each other in Eq. (38) for the Proposition 2, leaving behind a classical-like line element. More explicitly we can say that the quantum line element, $ds^2\,\psi(\vec{r},t) = i\hbar\,\mathcal{E}\,g_{\mu\nu}\,dx^\mu\,dx^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t)$, can transform into the classical line element, $ds^2\,\psi(\vec{r},t) = g_{\mu\nu}\,dx^\mu\,dx^\nu\,\psi(\vec{r},t)$, in a quantum spacetime, as $\mathcal{E}$ and $i\hbar\,\vec{\nabla}_\mu$ are canceling each other.*

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Let us consider $\vec{\nabla}'_\mu = (\delta/\delta x^\mu)$, etc., and let us also consider that $g_{\mu\nu}$ would transform as,

$$g_{\mu\nu} = g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\right) \longmapsto g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\right)\left(\frac{i\hbar\vec{\nabla}'_\alpha}{i\hbar\vec{\nabla}'_\mu}\right) = g^{(\mu)}_{\mu\nu} \longmapsto f^{(\mu)}_{\mu\nu}, \tag{39}$$

where $f^{(\mu)}_{\mu\nu}$ is a 'semi-quantum Lorentzian tensor' in a semi-quantum Minkowski spacetime, i.e.,

$$f^{(\mu)}_{\mu\nu} = f^{(\alpha)}_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\frac{i\hbar\vec{\nabla}'_\alpha}{i\hbar\vec{\nabla}'_\mu}\right),$$

so as the 'semi-quantum Lorentzian tensor' $f^{(\mu)}_{\mu\nu}$ and the pure Lorentzian metric tensor $g_{\mu\nu}$ should establish a relation as follows,

$$\begin{aligned} i\hbar\vec{\nabla}_\mu \otimes f^{(\mu)}_{\mu\nu} &= i\hbar\vec{\nabla}_\mu \otimes f^{(\alpha)}_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\frac{i\hbar\vec{\nabla}'_\alpha}{i\hbar\vec{\nabla}'_\mu}\right) \\ &= i\hbar\vec{\nabla}_\mu \otimes g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\right)\left(\frac{i\hbar\vec{\nabla}'_\alpha}{i\hbar\vec{\nabla}'_\mu}\right) \\ &= i\hbar\, g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\right)\vec{\nabla}_\alpha = i\hbar\, g_{\mu\nu}\,\vec{\nabla}_\alpha, \end{aligned} \tag{40}$$

thus, for Eq. (39),

$$f^{(\mu)}_{\mu\nu}\left(\frac{i\hbar\vec{\nabla}'_\mu}{i\hbar\vec{\nabla}'_\alpha}\right) = g_{\mu\nu}\,. \tag{41}$$

Without any loss of generality, we may assume that the 'quantum metric tensor' is symmetric: $f^{(\mu)}_{\mu\nu} = f^{(\mu)}_{\nu\mu}$, and *det* $(f^{(\mu)}_{\mu\nu}) \neq 0$. It has an inverse matrix f^{-1} whose components are themselves the components of matrix f, as their product gives: $f^{-1}f =$ identity matrix, i.e., in terms of components, $f^{(\mu)}_{\mu\nu}\, f^{\mu\gamma}_{(\mu)} = f^{\gamma\mu}_{(\mu)}\, f^{(\mu)}_{\mu\nu} = \delta^\gamma_\nu$, where, δ^γ_ν is the Kronecker delta.

Hence, Eq. (38) should be rewritten as,

$$\begin{aligned} ds^2\,\psi(\vec{r},t) &= \mathcal{E}\, f^{(\alpha)}_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\,dx^\mu\right)\left(\frac{\partial x^\beta}{\partial x^\nu}\,dx^\nu\right)\left(\frac{i\hbar\vec{\nabla}'_\alpha}{i\hbar\vec{\nabla}'_\mu}\,i\hbar\vec{\nabla}_\mu\right)\left(\frac{i\hbar\vec{\nabla}'_\mu}{i\hbar\vec{\nabla}'_\alpha}\right)\psi(\vec{r},t) \\ &= i\hbar\,\mathcal{E}\, f^{(\alpha)}_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}\frac{i\hbar\vec{\nabla}'_\alpha}{i\hbar\vec{\nabla}'_\mu}\right)\left(\frac{i\hbar\vec{\nabla}'_\mu}{i\hbar\vec{\nabla}'_\alpha}\right)dx^\mu\, dx^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t) \\ &= i\hbar\,\mathcal{E}\, f^{(\mu)}_{\mu\nu}\left(\frac{i\hbar\vec{\nabla}'_\mu}{i\hbar\vec{\nabla}'_\alpha}\right)dx^\mu\, dx^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t) \\ &\equiv i\hbar\,\mathcal{E}\, g_{\mu\nu}\,dx^\mu\, dx^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t)\,. \end{aligned}$$

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Let us vary the length of a curve [6–9] as,

$$\begin{aligned}\delta\, L|\gamma|\,\psi(\vec{r},t) &\equiv \int \delta \left\{ i\hbar\,\mathcal{E}\, f^{(\mu)}_{\mu\nu} \left(\frac{i\hbar\,\vec{\nabla}'_{\mu}}{i\hbar\,\vec{\nabla}'_{\alpha}}\right) \dot{x}^{\mu}\,\dot{x}^{\nu}\,\vec{\nabla}_{\mu} \right\}^{1/2} \mathrm{d}\tau\,\psi(\vec{r},t) \\ &= \frac{1}{2}\int \Bigg\{ i\hbar\,\mathcal{E}\,\partial_{\epsilon}\, f^{(\mu)}_{\mu\nu} \left(\frac{i\hbar\,\vec{\nabla}'_{\mu}}{i\hbar\,\vec{\nabla}'_{\alpha}}\right) \dot{x}^{\mu}\,\dot{x}^{\nu}\,\vec{\nabla}_{\mu} - \\ &\qquad - 2\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(i\hbar\,\mathcal{E}\, f^{(\mu)}_{\epsilon\nu} \left(\frac{i\hbar\,\vec{\nabla}'_{\mu}}{i\hbar\,\vec{\nabla}'_{\alpha}}\right) \dot{x}^{\nu}\,\vec{\nabla}_{\mu} \right) \Bigg\}\, \delta x^{\epsilon}\,\mathrm{d}\tau\,\psi(\vec{r},t)\,.\end{aligned}$$

This gives,

$$\begin{aligned}&\left\{ i\hbar\,\mathcal{E}\,\partial_{\epsilon}\, f^{(\mu)}_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}\,\vec{\nabla}_{\mu} - i\hbar\,\mathcal{E}\,\partial_{\mu}\, f^{(\mu)}_{\epsilon\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}\,\vec{\nabla}_{\mu} - i\hbar\,\mathcal{E}\,\partial_{\nu}\, f^{(\mu)}_{\epsilon\mu}\,\dot{x}^{\mu}\,\dot{x}^{\nu}\,\vec{\nabla}_{\mu} \right\} \psi(\vec{r},t) \\ &\qquad = 2\,i\hbar\,\mathcal{E}\, f^{(\mu)}_{\epsilon\delta}\,\ddot{x}^{\delta}\,\vec{\nabla}_{\mu}\,\psi(\vec{r},t),\end{aligned}$$

then the Christoffel symbol $\Gamma^{\delta}_{\mu\nu}$ should be defined by,

$$\Gamma^{\delta}_{\mu\nu}\,\psi(\vec{r},t) = \frac{1}{2}\, f^{\delta\epsilon}_{(\mu)} \left(\partial_{\mu}\, f^{(\mu)}_{\epsilon\nu} + \partial_{\nu}\, f^{(\mu)}_{\epsilon\mu} - \partial_{\epsilon}\, f^{(\mu)}_{\mu\nu} \right) \psi(\vec{r},t),$$

such that the Christoffel symbols are symmetric in the lower indices: $\Gamma^{\delta}_{\mu\nu} = \Gamma^{\delta}_{\nu\mu}$.

After a little exercise, we can yield the curvature tensor,

$$\mathcal{R}^{\sigma}_{\ \nu\gamma\delta}\,\psi(\vec{r},t) = \left(\frac{\partial \Gamma^{\sigma}_{\nu\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\sigma}_{\nu\gamma}}{\partial x^{\delta}} + \Gamma^{\sigma}_{\gamma\epsilon}\,\Gamma^{\epsilon}_{\nu\delta} - \Gamma^{\sigma}_{\delta\epsilon}\,\Gamma^{\epsilon}_{\nu\gamma} \right) \psi(\vec{r},t),$$

thus we find,

$$\mathcal{R}_{\lambda\nu\gamma\delta}\,\psi(\vec{r},t) = \frac{1}{2}\left(f^{(\lambda)}_{\lambda\delta,\nu\gamma} + f^{(\lambda)}_{\nu\gamma,\lambda\delta} - f^{(\lambda)}_{\delta\nu,\lambda\gamma} - f^{(\lambda)}_{\lambda\gamma,\nu\delta} \right) \psi(\vec{r},t),$$

which satisfies the properties like symmetry, antisymmetry and cyclicity as usual. Without much ado, we can easily obtain the Semi-quantum Minkowski GQG Einstein field equations as,

$$\left[\mathcal{R}_{\zeta\eta} - \frac{1}{2}\, f^{(\zeta)}_{\zeta\eta}\,\mathcal{R} \right] \psi(\vec{r},t) = 8\pi\, G\, \mathcal{T}_{\zeta\eta}\,\psi(\vec{r},t), \tag{42}$$

where $\mathcal{T}_{\zeta\eta}$ is the quantum energy momentum tensor, it is what the graviton field couples to, and G is the gravitational coupling. Let us develop an unusual gravitational coupling G in Planck scale using Eq. (41) as follows by using Ref. [10] and by accepting $\ell_p \mapsto \ell^{\mu}_P$,

$$\begin{aligned} G &= \frac{(\mathrm{d}\ell_P)^2}{m_P}\,\frac{\mathrm{d}^2\ell_P}{(\mathrm{d}t_P)^2} = \frac{m_P}{m_P^2}\,\frac{\mathrm{d}^2\ell_P}{(\mathrm{d}t_P)^2}\,(\mathrm{d}\ell_P)^2 = \frac{F_P}{m_P^2}\,(\mathrm{d}\ell_P)^2 = \frac{F_P}{m_P^2}\, g_{\zeta\eta}\,\mathrm{d}\ell^{\zeta}_P\,\mathrm{d}\ell^{\eta}_P \\ &= \frac{F_P}{m_P^2}\, f^{(\zeta)}_{\zeta\eta} \left(\frac{i\hbar\,\vec{\nabla}'_{\zeta}}{i\hbar\,\vec{\nabla}'_{\alpha}}\right) \mathrm{d}\ell^{\zeta}_P\,\mathrm{d}\ell^{\eta}_P, \end{aligned} \tag{43}$$

where $F_P = m_P \left\{ \mathrm{d}^2\ell^{\zeta}_P/(\mathrm{d}t_P)^2 \right\}$.

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Since $\mathcal{R}_{\zeta\eta} = f^{\lambda\nu}_{(\lambda)}\, \mathcal{R}_{\lambda\zeta\eta\nu}$, $\mathcal{R} = f^{\zeta\eta}_{(\zeta)}\, \mathcal{R}_{\zeta\eta}$ and $f^{(\lambda)}_{\lambda\nu}\, f^{\lambda\nu}_{(\lambda)} = g_{\lambda\nu}\, g^{\lambda\nu}$, the LHS of Eq. (42) may give us the purely Einsteinian form, i.e., $\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\, g_{\zeta\eta}\, \mathrm{R}\right]$, if we use Eq. (40) as follows,

$$\left(i\hbar\,\vec{\nabla}_{\zeta}\right) \otimes \left(i\hbar\,\vec{\nabla}^{\zeta}\right) \otimes \left[\mathcal{R}_{\zeta\eta} - \frac{1}{2}\, f^{(\zeta)}_{\zeta\eta}\, \mathcal{R}\right] \quad \Longleftrightarrow \quad -\hbar^{2}\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\, g_{\zeta\eta}\, \mathrm{R}\right] \vec{\nabla}_{\mu}\, \vec{\nabla}^{\mu}\,.$$

Similarly, we can get the Einsteinian energy momentum tensor $\mathrm{T}_{\zeta\eta}$ from the quantum energy momentum tensor $\mathcal{T}_{\zeta\eta}$ of Eq. (42), if we assume $\mathcal{T}_{\zeta\eta}$ depends on (quantum) metric tensor (for example, $T_{\mu\nu} = -\kappa^{-2}\left(\partial^{\lambda}\partial_{\lambda}\, g_{\mu\nu} - \partial^{\lambda}\partial_{\nu}\, g_{\mu\lambda} - \partial^{\lambda}\partial_{\mu}\, g_{\nu\lambda} + \partial_{\mu}\partial_{\nu}\, g^{\sigma}_{\sigma} - \eta_{\mu\nu}\, \partial_{\lambda}\partial^{\lambda}\, g^{\sigma}_{\sigma} + \eta_{\mu\nu}\, \partial^{\lambda}\partial^{\sigma}\, g_{\lambda\sigma}\right)$, or the electrodynamic $T_{\zeta\eta} = -F_{\zeta\mu}\, F^{\mu}_{\eta} + \frac{1}{4}\, g_{\zeta\eta}\, F^{\mu\nu}\, F_{\mu\nu} = -g_{\zeta\eta}\, F_{\zeta\mu}\, F^{\zeta\mu} + \frac{1}{4}\, g_{\zeta\eta}\, F^{\mu\nu}\, F_{\mu\nu}$, etc.). By using Eq. (43) in Eq. (42), we can get the modified GQG field equation as,

$$\begin{aligned} &\left(i\hbar\,\vec{\nabla}_{\zeta}\right) \otimes \left(i\hbar\,\vec{\nabla}^{\zeta}\right) \otimes \left[\mathcal{R}_{\zeta\eta} - \frac{1}{2}\, f^{(\zeta)}_{\zeta\eta}\, \mathcal{R}\right] \psi(\vec{r},t) \\ &\qquad\qquad = \left(i\hbar\,\vec{\nabla}_{\zeta}\right) \otimes \left(i\hbar\,\vec{\nabla}^{\zeta}\right) \otimes \left(8\pi\, G\, \mathcal{T}_{\zeta\eta}\right) \psi(\vec{r},t), \end{aligned}$$

$$\therefore \quad -\hbar^{2}\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\, g_{\zeta\eta}\, \mathrm{R}\right] \vec{\nabla}_{\mu}\, \vec{\nabla}^{\mu}\, \psi(\vec{r},t) = -\hbar^{2}\, 8\pi\, G\, \mathrm{T}_{\zeta\eta}\, \vec{\nabla}_{\mu}\, \vec{\nabla}^{\mu}\, \psi(\vec{r},t)\,. \tag{44}$$

Note it that Eq. (42) and Eq. (44) are exactly the same thing despite their different appearances.

Remark 4. *It is necessary to remember that,*

1. *We should not introduce the cosmological constant Λ in Eq. (44), because we can get Dark Energy from Einstein field equation quite naturally (see the last equation of Eq. (68) in Section 2.2 below for more details). Introduction of the cosmological constant Λ in Eq. (44) should intend to double entry of Dark Energy in the same gravitational field of GQG, which should obviously be wrong. Though, in Eq. (70) below, we will develop a field equation containing Λ, which is slightly different from the Classical Einstein field equation.*
2. *Eq. (44) tells us that gravitation is an interaction in orthogonal curvilinear coordinates x^{ζ} (i.e., outer surface) of point $P(x^{\mu})$ rather than at that very spacetime x^{μ} (i.e., core). Since $i\hbar\,\vec{\nabla}_{\mu} \nrightarrow 0$ for a particle field, then the Einstein tensor must be unity, i.e., $\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\, g_{\zeta\eta}\, \mathrm{R}\right] \rightarrow 1$, in Eq. (44) at the core x^{μ}, and there, spacetime x^{μ} behaves strongly Quantum Mechanical so as the other particle interactions (i.e., Quantum Chromodynamics and Electroweak) are prioritized there locally. On the other hand, gravitational effects only start effective beyond the core x^{μ}, i.e., in orthogonal curvilinear coordinates x^{ζ} outside the core x^{μ}, in other words, the outer surface of x^{μ}.*
3. *If we consider a cutoff energy $E_{\star}$, then we can say that gravitons only appear in an energy zone $E_{\mathrm{IR}} \leq E_{\star}$ as $E_{\star} \ll E_{\mathrm{UV}}$, and beyond that state, i.e., $E_{\mathrm{UV}} > E_{\star}$, other particle interactions (i.e., Quantum Chromodynamics and Electroweak) are prioritized, where E_{IR} is the infra red energy zone, whereas E_{UV} is the ultra violet energy zone. Thus, for gravity in Eq. (44), ultra violet zone is automatically ignored, i.e., the sum over E_{IR} intends the Feynman graphs to be finite. In the energy zone $E_{\mathrm{IR}} \leq E_{\star}$, all gravitons behave as real particles. Then we can assume the energy states of a particle from the kernel energy of the core spacetime x^{μ} to its outer surface energy for spacetime x^{ζ} as:*

$$\left(E_{\mathrm{UV}}\right)_{\text{CORE},\, i.e.,\, x^{\mu}} \;\rightarrow\; \left(E_{\star}\right)_{\text{OUTER SURFACE},\, i.e.,\, x^{\zeta}} \;\rightarrow\; \left(E_{\mathrm{IR}}\right)_{\text{BEYOND THE PARTICLE}}\,. \tag{45}$$

   Between core and outer surface energies, i.e., at $E_{\mathrm{UV}} > E_{\star}$, Electroweak and Quantum Chromodynamic interactions take place, whereas, outside these states (i.e., at $E_{\mathrm{IR}} \leq E_{\star}$) gravity starts being prioritized.
4. *For the RHS factor $\hbar^{2}\, G$ of Eq. (44), the gravitational coupling G, which has the dimension of a negative power of mass, now has lost its mass dimension due to $\hbar^{2}$. Consequently, if divergences are to be present, they could now be disposed of by the technique of renormalization (though, this will not play a role in*

our present discussion, but we can develop a renormalizable scenario by using a purely quantum form of gravity, which will be discussed in Section 2.3 below).

Remark 5. *Eq. (43) is true for the outer surface x^{ζ}, but it is seemingly true for the core x^{μ}, too, due to the universality of gravitational coupling G. So, we can choose the indexes either ways.*

Now, considering d'Alembertian operator $\Box = \vec{\nabla}_{\mu}\, \vec{\nabla}^{\mu}$, as well as $\mathcal{U} = \left(8\,\pi\, G\, \mathrm{T}_{\zeta\eta}\right)$, and inputting the 'Four-momentum' operator $\hat{\mathcal{P}} \rightarrow i\,\hbar\, \vec{\nabla}_{\mu}$ into Eq. (44), we can get the Second Order Equation of Semi-quantum Minkowski GQG as,

$$\hbar^2 \left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\, g_{\zeta\eta}\, \mathrm{R}\right] \Box\, \psi(\vec{r},t) + \mathcal{U}\, \mathcal{P}^2\, \psi(\vec{r},t) = 0\,. \tag{46}$$

The wavefunction $\psi(\vec{r},t)$ in Eq. (46) is emphatically defining a bosonic field. Thus, we can immediately develop a fermionic field (or the Third Variance of the First Order Equation of Semi-quantum Minkowski GQG) out of Eq. (46) as,

$$i\,\hbar\,\gamma^{\mu} \left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\, g_{\zeta\eta}\, \mathrm{R}\right]^{1/2} \vec{\nabla}_{\mu}\, \psi(\vec{r},t) - \mathcal{U}^{1/2}\, \mathcal{P}\, \psi(\vec{r},t) = 0, \tag{47}$$

where, γ^{μ} are Dirac's gamma matrices.

Dividing Eq. (47) either by $\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\, g_{\zeta\eta}\, \mathrm{R}\right]^{1/2}$ or by $\mathcal{U}^{1/2}$ gives us $\left(i\,\hbar\,\gamma^{\mu}\, \vec{\nabla}_{\mu} - \mathcal{P}\right) \psi(\vec{r},t) = 0$, from which the classical Dirac's equation should be derivable, but here, instead of $\partial/(\partial\, t)$, we have considered $\partial/(\partial x^0)$ by absorbing v^0 and similarly, $\mathcal{P}$ is not intended here to have a factor of rest mass, since $m \neq m_0$ in Eq. (1). Thus, we can say that Dirac's equation is a subset of the Third Variance of the First Order Equation of Semi-quantum Minkowski GQG, i.e., Eq. (47). Similarly, we can also say that the Klein-Gordon equation is a subset of the Second Order Equation of Semi-quantum Minkowski GQG, i.e., Eq. (46), and it should be derivable from $\left(\hbar^2\, \Box + \mathcal{P}^2\right) \psi(\vec{r},t) = 0$. An analogous formalism is equally applicable for the following Subsection 2.2.

2.2. *Quantum Minkowski GQG*

Let the line element of Minkowski spacetime,

$$\begin{aligned}
\mathrm{d}s^2 &= g_{\mu\nu}\, \mathrm{d}x^{\mu}\, \mathrm{d}x^{\nu} \equiv \left(\frac{\mathrm{d}t}{m}\right)^2 g_{\mu\nu}\, P^{\mu}\, P^{\nu}\,, \\
\therefore \quad m^2 \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= m^2 c^2 - m^2 \sum \frac{\mathrm{d}x^i}{\mathrm{d}t}\, \frac{\mathrm{d}x^j}{\mathrm{d}t} = m\, E - \sum p^i\, p^j = p^0\, p^0 - \sum p^i\, p^j \\
&= g_{\mu\nu}\, P^{\mu}\, P^{\nu}\,, \\
\text{and,} \quad m^2 \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 &= m^2 \left(1 - \frac{\mathrm{v}^2}{c^2}\right) c^2 = m_0^2\, c^2\,,
\end{aligned} \tag{48}$$

for the rest mass m_0, when,

$$\mathrm{d}\,\mathcal{S}_P^2 = m^2 \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = p^0\, p^0 - \sum p^i\, p^j = g_{\mu\nu}\, P^{\mu}\, P^{\nu}\,, \tag{49}$$

then, rearrangement of Eq. (48) gives,

$$m\, E = p^i\, p^j + m^2 \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 = p^i\, p^j + P^{\mu}\, P^{\nu}\, \frac{\mathrm{d}s}{\mathrm{d}x^{\mu}}\, \frac{\mathrm{d}s}{\mathrm{d}x^{\nu}} = p^i\, p^j + P^{\mu}\, P^{\nu}\, g_{\mu\nu}\,. \tag{50}$$

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Then, considering the representation of a wave field $\psi(\vec{r},t)$ by superposition of a free particle (de Broglie wave) for Eq. (50) as follows,

$$\begin{aligned}\psi(\vec{r},t) &= \frac{1}{(2\pi\hbar)^2}\exp\left[\frac{i}{\hbar\, m}\left\{m\left(\vec{p}\cdot\vec{r}+g_{\mu\nu}\,\vec{P}\cdot\vec{R}\right)-mEt\right\}\right] \\ &\equiv \frac{1}{(2\pi\hbar)^2}\exp\left[\frac{i}{\hbar\, m}\left\{m\left(\vec{p}\cdot\vec{r}+m\,t\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2\right)-mEt\right\}\right], \end{aligned} \tag{51}$$

we can generate the following wave equation using Eq. (50) combining with Eq. (48) as,

$$-\hbar^2\,\vec{\nabla}_0\,\vec{\nabla}_0\,\psi(\vec{r},t)+\hbar^2\,\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)+\hbar^2\,g_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t)=0, \tag{52}$$

which may give,

$$-\hbar^2\left(1-g_{00}\right)\vec{\nabla}_0\,\vec{\nabla}_0\,\psi(\vec{r},t)+\hbar^2\left(1+g_{ij}\right)\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)=0,$$

or, simply discarding $\left(1-g_{00}\right)=\left(1+g_{ij}\right)=0$, we can get the First Variance of the Second Order Equation of Quantum Minkowski GQG as,

$$\begin{aligned} &-\hbar^2\,\vec{\nabla}_0\,\vec{\nabla}_0\,\psi(\vec{r},t)+\hbar^2\,\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)=0, \\ \therefore\quad &-\hbar^2\,\vec{\nabla}_0\,\vec{\nabla}^0\,\psi(\vec{r},t)+\hbar^2\,\vec{\nabla}_i\,\vec{\nabla}^i\,\psi(\vec{r},t)=0. \end{aligned} \tag{53}$$

Here, $\psi(\vec{r},t)$ is definitely a bosonic field. But the uppermost equation of Eq. (53) may give us the gravitational form of the Classical Schrödinger equation by using the Proposition 2 (the exercise is left for the readers) as follows,

$$i\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t)+\frac{\hbar^2}{m}\,\vec{\nabla}_i\,\vec{\nabla}_j\,\psi(\vec{r},t)=0. \tag{54}$$

Putting differently,

$$i\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t)+\frac{\hbar^2}{m}\,g_{ij}\,\vec{\nabla}_i\,\vec{\nabla}^i\,\psi(\vec{r},t)=0.$$

Hence, we get the gravitational form of the Classical Schrödinger equation for the total energy E, and now it is in a $(3+1)D$ quantum spacetime.

Applying the representation of wave field $\psi(\vec{r},t)$ either of Eq. (51), or Eq. (5), into Eq. (50), we can get,

$$\begin{aligned}\hat{E}\,\psi(\vec{r},t) &= \frac{1}{m}\left(\hat{\boldsymbol{p}}^i\,\hat{\boldsymbol{p}}^j+g_{\mu\nu}\,\hat{\boldsymbol{\mathcal{P}}}^\mu\hat{\boldsymbol{\mathcal{P}}}^\nu\right)\psi(\vec{r},t), \\ \text{i.e.,}\quad i\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t) &= -\frac{\hbar^2}{m}\left(\vec{\nabla}_i\,\vec{\nabla}_j+g_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\right)\psi(\vec{r},t), \end{aligned} \tag{55}$$

where $g_{\mu\nu}$ is actually satisfying the last two terms of Eq. (58), see below. Note that, Eq. (55) should be used as an alternate of the gravitational form of the Classical Schrödinger equation, i.e., Eq. (54).

After using the first term of mass operator $\hat{m}$ from Eq. (7), the Eq. (55) yields,

$$\begin{aligned} -\hbar^2\,\frac{\partial^2}{\partial s^2}\,\psi(\vec{r},t) &= \hbar^2\left(\vec{\nabla}_i\,\vec{\nabla}_j+g_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\right)\psi(\vec{r},t), \\ \therefore\quad \hbar^2\left(\frac{\partial^2}{\partial s^2}+\frac{\partial^2}{\partial t^2}\right)\psi(\vec{r},t) &= 0. \end{aligned} \tag{56}$$

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Compare the first equation of Eq. (56) with Eq. (10). We will use the second value of $\hat{m}\,\hat{E}$ in Eq. (56) to construct Quantum Non-Minkowski GQG in Section 2.3 below.

Now, the Second Variance of the Second Order Equation of Quantum Minkowski GQG from Eq. (52) should be,

$$-\hbar^2\,\vec{\Delta}_\mu\,\vec{\Delta}_\nu\,\psi(\vec{r},t)+\hbar^2\,g_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t)=0\,,$$

where $i\hbar\,\vec{\Delta}_\mu\to\left[\hat{\boldsymbol{p}}_0,\,-\hat{\boldsymbol{p}}\right]^{\mathrm{T}}\to\left[i\hbar\,\vec{\nabla}_0,\,i\hbar\,\vec{\nabla}_i\right]^{\mathrm{T}}$.

Multiplying both sides of Eq. (52) by $(\mathrm{d}t/m)^2$ and comparing it with Eq. (48), we have the quantum line element for the 'Four-momentum' operator $\hat{\boldsymbol{\mathcal{P}}}^{\mu}\to i\hbar\,\vec{\nabla}_\mu$ as follows,

$$\mathrm{d}s^2\,\psi(\vec{r},t)=-\hbar^2\left(\frac{\mathrm{d}t}{m}\right)^2 g_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t)\,. \tag{57}$$

Let us now prescribe $g_{\mu\nu}$ as follows by using Proposition 2 as,

$$\begin{aligned} g_{\mu\nu} &= g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^\mu}\,\frac{\partial x^\beta}{\partial x^\nu}\right)=g_{\alpha\beta}\left(\frac{m}{\partial t}\right)^2\left(\frac{\partial t}{m}\right)^2\left(\frac{\partial x^\alpha}{\partial x^\mu}\,\frac{\partial x^\beta}{\partial x^\nu}\right)=g_{\alpha\beta}\left(\frac{P^\alpha}{P^\mu}\,\frac{P^\beta}{P^\nu}\right)\\ &\to g_{\alpha\beta}\left(\frac{\hat{\boldsymbol{\mathcal{P}}}^\alpha}{\hat{\boldsymbol{\mathcal{P}}}^\mu}\,\frac{\hat{\boldsymbol{\mathcal{P}}}^\beta}{\hat{\boldsymbol{\mathcal{P}}}^\nu}\right)\to g_{\alpha\beta}\left(\frac{i\hbar\,\vec{\nabla}_\alpha}{i\hbar\,\vec{\nabla}_\mu}\,\frac{i\hbar\,\vec{\nabla}_\beta}{i\hbar\,\vec{\nabla}_\nu}\right). \end{aligned} \tag{58}$$

To avoid any confusion between the pure Lorentzian metric tensor $g_{\mu\nu}$ and the quantum Lorentzian tensor of Eq. (58), let us assume that,

$$\mathrm{g}_{\mu\nu}=g_{\alpha\beta}\left(\frac{i\hbar\,\vec{\nabla}_\alpha}{i\hbar\,\vec{\nabla}_\mu}\,\frac{i\hbar\,\vec{\nabla}_\beta}{i\hbar\,\vec{\nabla}_\nu}\right)=g_{\mu\nu}\,. \tag{59}$$

This approach is quantizing gravity. The 'quantum metric tensor' $\mathrm{g}_{\mu\nu}$ is symmetric, i.e., $\mathrm{g}_{\mu\nu}=\mathrm{g}_{\nu\mu}$, and *det* $(\mathrm{g}_{\mu\nu})\neq 0$. Components of its inverse matrix g^{-1} are themselves the components of matrix g, i.e., $\mathrm{g}_{\mu\nu}\,\mathrm{g}^{\mu\gamma}=\mathrm{g}^{\gamma\mu}\,\mathrm{g}_{\mu\nu}=\delta^\gamma_\nu$, where, δ^γ_ν is the Kronecker delta.

Then, Eq. (57) becomes as,

$$\mathrm{d}s^2\,\psi(\vec{r},t)=-\hbar^2\left(\frac{\mathrm{d}t}{m}\right)^2 \mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t)\,. \tag{60}$$

Let us vary the length of a curve [6–9] as,

$$\begin{aligned} \delta\,L|\gamma|\,\psi(\vec{r},t) &\equiv \int\delta\left\{-\hbar^2\left(\frac{\mathrm{d}t}{m}\right)^2 \mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\right\}^{1/2}\mathrm{d}\tau\,\psi(\vec{r},t)\\ &= \frac{1}{2}\int\Bigg\{-\hbar^2\left(\frac{\mathrm{d}t}{m}\right)^2\partial_\epsilon\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu-\\ &\qquad -2\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\hbar^2\left(\frac{\mathrm{d}t}{m}\right)^2\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\nu\right)\Bigg\}\,\delta\,\vec{\nabla}_\epsilon\,\mathrm{d}\tau\,\psi(\vec{r},t)\,. \end{aligned}$$

Similar to the Subsection 2.1, after a little exercise, we can develop,

$$\mathcal{R}_{\lambda\nu\gamma\delta}\,\psi(\vec{r},t)=\frac{1}{2}\left(\mathrm{g}_{\lambda\delta,\nu\gamma}+\mathrm{g}_{\nu\gamma,\lambda\delta}-\mathrm{g}_{\delta\nu,\lambda\gamma}-\mathrm{g}_{\lambda\gamma,\nu\delta}\right)\psi(\vec{r},t),$$

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and then, we can obtain the Quantum Minkowski GQG Einstein field equations as,

$$\left[\mathcal{R}_{\zeta\eta} - \frac{1}{2}\,\mathrm{g}_{\zeta\eta}\,\mathcal{R}\right]\psi(\vec{r},t) = 8\pi\,G\,\mathcal{T}_{\zeta\eta}\,\psi(\vec{r},t)\,. \tag{61}$$

Let us develop another unusual gravitational coupling G in Planck scale using Proposition 2 and adopting Remark 5 as follows by accepting $\ell_p \mapsto \ell_P^{\mu}$,

$$\begin{aligned} G \;&=\; \frac{(\mathrm{d}\ell_P)^2}{m_P}\,\frac{\mathrm{d}^2\ell_P}{(\mathrm{d}t_P)^2} = \frac{m_P}{m_P^2}\,\frac{\mathrm{d}^2\ell_P}{(\mathrm{d}t_P)^2}\,(\mathrm{d}\ell_P)^2 = \frac{F_P}{m_P^2}\,g_{\mu\nu}\,\mathrm{d}\ell_P^{\mu}\,\mathrm{d}\ell_P^{\nu} = \frac{F_P}{m_P^2}\,\mathrm{d}\ell_P^{\mu}\,\mathrm{d}\ell_{P\,\mu} \\ \to \;&-\hbar^2\,F_P\left(\frac{\mathrm{d}t_P}{m_P^2}\right)^2\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu} \;=\; -\,\hbar^2\,\frac{\ell_P^{\mu}}{m_P^3}\,\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu}, \end{aligned} \tag{62}$$

for $F_P = m_P\left\{\mathrm{d}^2\ell_P^{\mu}/(\mathrm{d}t_P)^2\right\}$.

Interested readers can easily check that Eq. (42) and Eq. (61) are exactly the same thing but comprised with different components: the earlier one with 'Four-velocity' components and the later one with 'Four-momentum' components. Another noticeable difference between them is that Eq. (42) has a mixed expression of classical and quantum geometric expressions for $f_{\mu\nu}^{(\mu)}$, whereas, Eq. (61) has a purely quantum geometric expression for $\mathrm{g}_{\mu\nu}$. In other words, we can say that Eq. (42) is in a quantum Minkowski spacetime, whereas, Eq. (61) is in a semi-quantum Minkowski spacetime.

If we transform our spacetime into Planck scale, i.e., $x^{\mu} \to \ell_P^{\mu}$ and $t \to t_P$, and consider $m_P = \sum \mathfrak{m}_{(\varrho)} = N_{(\varrho)}\mathfrak{m}_{(\varrho)}$, where $\mathfrak{m}_{(\varrho)}$ is the mass of a certain particle and $N_{(\varrho)}$ is a very large constant number since Planck mass is a very big number, i.e., m_P is not considered here as the mass of a particular particle but the amount of $N_{(\varrho)}$ number of certain particle with mass $\mathfrak{m}_{(\varrho)}$, then we can rewrite Eq. (61) using Eq. (62) as,

$$\hbar^2\,\frac{\ell_P^{\mu}}{\left(N_{(\varrho)}\mathfrak{m}_{(\varrho)}\right)^3}\,\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu}\,\psi(\vec{r}_P,t_P) + \frac{1}{8\pi\,\mathcal{T}_{P\,\zeta\eta}}\left[\mathcal{R}_{P\,\zeta\eta} - \frac{1}{2}\,\mathrm{g}_{\zeta\eta}\,\mathcal{R}_P\right]\psi(\vec{r}_P,t_P) = 0\,.$$

Let,

$$\mathscr{C}^2 = \frac{1}{8\pi\,\mathcal{T}_{P\,\zeta\eta}}\left[\mathcal{R}_{P\,\zeta\eta} - \frac{1}{2}\,\mathrm{g}_{\zeta\eta}\,\mathcal{R}_P\right] = G, \tag{63}$$

then we have,

$$\hbar^2\,\frac{\ell_P^{\mu}}{\left(N_{(\varrho)}\mathfrak{m}_{(\varrho)}\right)^3}\,\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu}\,\psi(\vec{r}_P,t_P) + \mathscr{C}^2\,\psi(\vec{r}_P,t_P) = 0\,.$$

Considering d'Alembertian operator $\Box_P = \vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\mu}$, we can get the Third Variance of the Second Order Equation of Quantum Minkowski GQG as,

$$\hbar^2\,\frac{\ell_P^{\mu}}{\left(N_{(\varrho)}\mathfrak{m}_{(\varrho)}\right)^3}\,\Box_P\,\psi(\vec{r}_P,t_P) + \mathscr{C}^2\,\psi(\vec{r}_P,t_P) = 0\,. \tag{64}$$

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Here, $\psi(\vec{r}_P, t_P)$ is definitely a bosonic field. Thus, we can immediately develop a fermionic field (or the First Order Equation of Quantum Minkowski GQG) out of Eq. (64) as,

$$i\hbar \left(\frac{\ell_P^{\mu}}{\left(N_{(\varrho)}\mathfrak{m}_{(\varrho)}\right)^3} \right)^{1/2} \gamma^{\mu}\, \vec{\nabla}_{P\mu}\, \psi(\vec{r}_P, t_P) - \mathscr{C}\, \psi(\vec{r}_P, t_P) = 0, \tag{65}$$

where, γ^{μ} are Dirac's gamma matrices. Considering $\left(\ell_P^{\mu}\right)^{-1} = m_P = N_{(\varrho)}\mathfrak{m}_{(\varrho)}$ [11], let Eq. (65) be for Eq. (63) as,

$$\begin{aligned} i\hbar\, \gamma^{\mu}\, \vec{\nabla}_{P\mu}\, \psi(\vec{r}_P, t_P) - \left(N_{(\varrho)}\mathfrak{m}_{(\varrho)}\right)^2 \mathscr{C}\, \psi(\vec{r}_P, t_P) &= 0 \\ \therefore \quad i\hbar\, \gamma^{\mu}\, \vec{\nabla}_{P\mu}\, \psi(\vec{r}_P, t_P) - \left(N_{(\varrho)}\mathfrak{m}_{(\varrho)}\right)^2 G^{1/2}\, \psi(\vec{r}_P, t_P) &= 0 \\ \Rightarrow \quad (8\pi)^{1/2}\, i\hbar\, \gamma^{\mu}\, \vec{\nabla}_{P\mu}\, \psi(\vec{r}_P, t_P) - N_{(\varrho)}\mathfrak{m}_{(\varrho)}\, \psi(\vec{r}_P, t_P) &= 0, \end{aligned} \tag{66}$$

since $8\pi\, G = m_P^{-2} = \left(N_{(\varrho)}\mathfrak{m}_{(\varrho)}\right)^{-2}$. Now, Eq. (66) is quite handy to use. But Eq. (66) also suggests us that whatever matter satisfies such a fermionic relation is definitely originated (clustered) as matter from fundamentally different physics at the Planck scale, maybe at very different cosmological epochs. Moreover, the first term of the last equation of Eq. (66) is almost five-times larger than any Dirac-like term for baryonic matters, which is quite unusual. At this characteristic Planck scale, the matter that satisfies Eq. (66) must not provide a natural mechanism of the electroweak symmetry breaking, thus the matter must be non-baryonic. The only possible candidate having such characteristics is Dark Matter, which accounts for 26.8% of the critical density in the Universe against 4.9% of the critical density of baryonic matters, in other words, the critical density of Dark Matter is almost $(8\pi)^{1/2}$ times higher than the critical density of baryonic matters – it exactly matches with Eq. (66).

Again, returning to Eq. (60) and using $m^2\,(\mathrm{d}s^2/\mathrm{d}t^2)$ of Eq. (48) so as $m^2\,(\mathrm{d}s^2/\mathrm{d}t^2) = m^2\,(1 - v^2) \equiv m_0^2$ for the rest mass m_0, we can get,

$$\begin{aligned} m_0^2\, \psi(\vec{r}, t) &= -\hbar^2\, g_{\mu\nu}\, \vec{\nabla}_{\mu}\, \vec{\nabla}_{\nu}\, \psi(\vec{r}, t) \\ \therefore \quad & \hbar^2\, g_{\mu\nu}\, \vec{\nabla}_{\mu}\, \vec{\nabla}_{\nu}\, \psi(\vec{r}, t) + m_0^2\, \psi(\vec{r}, t) = 0\,. \end{aligned}$$

Then, considering $\vec{\nabla}_{\nu} = g_{\mu\nu}\, \vec{\nabla}^{\mu}$ and d'Alembertian operator $\Box = \vec{\nabla}_{\mu}\, \vec{\nabla}^{\mu}$, we have,

$$\hbar^2\, g_{\mu\nu}^2\, \Box\, \psi(\vec{r}, t) + m_0^2\, \psi(\vec{r}, t) = 0\,.$$

Thus, we can immediately develop a fermionic field equation as,

$$i\hbar\, \gamma^{\mu}\, g_{\mu\nu}\, \vec{\nabla}_{\mu}\, \psi(\vec{r}, t) - m_0\, \psi(\vec{r}, t) = 0\,. \tag{67}$$

Here, the real fermions exist only in temporal dimension. Thus, Eq. (67) gives us the Dirac equations for fermions, but with an extension that antifermions, those are exist in spatial dimensions, are thrice in number than real fermions in nature. As the motion in temporal dimension is the basic consideration of relativity, the '+'-ve signature of $g_{\mu\nu}$ in Eq. (67) explains us the reason of the forward expansion of the Universe in temporal dimension.

Now, let us replace m^2 of Eq. (48) with the Planck mass $m_P^2 = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2$ for a certain particle with mass $\mathfrak{m}_{(\Lambda)}$, where $N_{(\Lambda)}$ is a very large constant number and $\left\{N_{(\Lambda)}, \mathfrak{m}_{(\Lambda)}\right\} \neq \left\{N_{(\varrho)}, \mathfrak{m}_{(\varrho)}\right\}$ as $\mathfrak{m}_{(\Lambda)} \ll \mathfrak{m}_{(\varrho)}$; when $m_P^2 = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2$ satisfies as [12]: $m_P^2\,\Lambda = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2 \Lambda = \frac{1}{2}\,\langle T \rangle$, where the

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cosmological constant $\Lambda = 8\pi\, G\, \rho_\Lambda$, so as we have $\left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2 (\mathrm{d}s^2/\mathrm{d}t^2) = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2 (1 - v^2) \equiv \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}\right)^2$ for the Planck rest mass $m_{P\,0} = N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}$ and $c = 1$, thus, for,

$$\left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2 \Lambda\,(1 - v^2) = \frac{1}{2}\,\langle T\rangle\,(1 - v^2) \qquad \Longleftrightarrow \qquad \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}\right)^2 \Lambda = \frac{1}{2}\,\langle T_0\rangle\,,$$

where, $\langle T\rangle\,(1 - v^2) \equiv \langle T_0\rangle$, and following the argument cited above,

$$\begin{aligned} \hbar^2\, \mathrm{g}_{\mu\nu}\, \vec{\nabla}_\mu\, \vec{\nabla}_\nu\, \psi(\vec{r},t) + \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}\right)^2 \psi(\vec{r},t) &= 0\,, \\ \therefore \quad 2\,\hbar^2\,\Lambda\,\mathrm{g}^2_{\mu\nu}\,\square\,\psi(\vec{r},t) + \langle T_0\rangle\,\psi(\vec{r},t) &= 0\,, \\ \text{and,} \quad \hbar^2\,\Lambda\,\mathrm{g}^2_{\mu\nu}\,\square\,\psi(\vec{r},t) + \rho_{\Lambda\,0}\,\psi(\vec{r},t) &= 0\,, \end{aligned} \tag{68}$$

by replacing $\frac{1}{2}\,\langle T\rangle$ with ρ_Λ for the cosmological constant $\Lambda = 8\pi\,G\,\rho_\Lambda = m_P{}^{-2}\,\rho_\Lambda = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^{-2} \rho_\Lambda$. So, by using $8\pi\,G = m_P{}^{-2} = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^{-2}$ and by switching $(1 - v^2)^{1/2}$ right to left in the term: $N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\,\langle T_0\rangle^{1/2} = N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\,\langle T\rangle^{1/2}\,(1 - v^2)^{1/2} \equiv N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}\,\langle T\rangle^{1/2}$, we can develop a fermionic field equation as follows,

$$\begin{aligned} i\,\hbar\,\sqrt{2\Lambda}\,\gamma^\mu\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\psi(\vec{r},t) - \langle T_0\rangle^{1/2}\,\psi(\vec{r},t) &= 0\,, \\ \therefore \quad i\,\hbar\,\left(2\,\rho_\Lambda\right)^{1/2}\gamma^\mu\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\psi(\vec{r},t) - N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}\,\langle T\rangle^{1/2}\,\psi(\vec{r},t) &= 0\,. \end{aligned} \tag{69}$$

The interesting thing in Eq. (69) is that Dark Energy has a direct relationship with gravity. In other words, Dark Energy would be obtainable from the breaking of particle symmetry where gravity counts (see, Subsection 3.1 below).

The last equation of Eq. (68) is definitely applicable simultaneously whether the matter is baryonic or non-baryonic.

Adding the last equation of Eq. (68) with Eq. (46) (since Eq. (61) and Eq. (42) are exactly the same), we can get a field equation for Eq. (59) as.

$$\begin{aligned} \hbar^2\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R} + \Lambda\,\mathrm{g}^2_{\mu\nu}\right]\square\,\psi(\vec{r},t) + \left[\mathcal{U}\,\mathcal{P}^2 + \rho_{\Lambda\,0}\right]\psi(\vec{r},t) &= 0\,, \\ \text{i.e.,} \quad \hbar^2\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R} + \Lambda\,g^2_{\mu\nu}\right]\square\,\psi(\vec{r},t) + \left[\mathcal{U}\,\mathcal{P}^2 + \rho_{\Lambda\,0}\right]\psi(\vec{r},t) &= 0\,. \end{aligned} \tag{70}$$

This field equation, which is actually a Klein-Gordon-type equation, is slightly different form the Classical Einstein field equation. Note that gravity and cosmological constant Λ are originated from different spacetimes in Eq. (70). From a particle's point of view, Λ is generated from the kernel of the core of a particle, whereas, gravity is emerged from the outer surface of the same core.

Again, either from the first equation of Eq. (68), or by placing $\frac{1}{2}\,\langle T\rangle = \rho_\Lambda$ in Eq. (69), we can find,

$$i\,\hbar\,\gamma^\mu\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\psi(\vec{r},t) - N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}\,\psi(\vec{r},t) = 0, \tag{71}$$

which is the Planck scale counterpart of Eq. (67), in other words, Eq. (67) and Eq. (71) counterbalance each other's actions of the forward expansion of the Universe in temporal dimension due to their $\mathrm{g}_{\mu\nu}$.

Since, the cosmological constant $\Lambda = 8\pi\,G\,\rho_\Lambda = m_P{}^{-2}\,\rho_\Lambda = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^{-2} \rho_\Lambda$, then again replacing m^2 of Eq. (48) with $m_P^2 = \left(N_{(\Lambda)}\mathfrak{m}_{(\Lambda)}\right)^2 = (\rho_\Lambda/\Lambda)$ gives us, $(\rho_\Lambda/\Lambda)\left(\mathrm{d}s^2/\mathrm{d}t^2\right) = (\rho_\Lambda/\Lambda)\,(1 - v^2) \equiv (\rho_{\Lambda\,0}/\Lambda)$, for $c = 1$. But, we can say, $E_\Lambda^2 = (\rho_{\Lambda\,0}/\Lambda)\,c^4$, i.e., $E_\Lambda = (\rho_{\Lambda\,0}/\Lambda)^{1/2}\,c^2$, as the *rightful and lawful* 'Dark Energy' for relativistic ρ_Λ.

The grate difference between Eq. (66) and Eq. (71) is that the nature of the former one is non-baryonic, whereas, the later one is independent of matter's constructive property, i.e., its effects can be observable simultaneously both in the cases of baryonic and non-baryonic matters. Another difference is that Eq. (66) is effective at $m_P = N_{(\varrho)}\mathfrak{m}_{(\varrho)}$ scale, whereas, Eq. (71) is effective at $m_{P\,0} = N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}$ scale, i.e., Dark Energy had originated at much earlier cosmological epochs than Dark Matter. Similarly, Dark Matter had originated at much earlier cosmological epochs than baryonic matters of Eq. (67) at m_0 scale. Thus, we have a quite fair chronology of the formation of cosmological matters in the Universe. Note it here that gravity was not observable at the cosmological epochs at $m_{P\,0} = N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}$ scale where Dark Energy had originated. At this scale, gravitons just behaved as energy states rather than real particles due to Remark 4. Gravity was also not observable at the next cosmological epochs started out at $m_P = N_{(\varrho)}\mathfrak{m}_{(\varrho)}$ scale where Dark Matter had originated. Gravity became observable first time only in the energy zone $E_{\text{IR}} \leq E_\star$ as we had claimed in Remark 4.

The bosonic and fermionic fields for baryonic matters, Dark Energy and Dark Matter, which are obtainable from GQG, are listed in the Table 1.

Table 1. In General Quantum Gravity, we have twelve staple bosonic and fermionic field equations in two different orders.

Semi-quantum Minkowski GQG:	First Order Equations
	$i\hbar\,\frac{\partial}{\partial t}\,\psi(\vec{r},t) + i\hbar\,\mathrm{v}^j\,\vec{\nabla}_i\,\psi(\vec{r},t) = 0$ $i\hbar\,\mathrm{v}^\nu\,\vec{\Delta}_\mu\,\psi(\vec{r},t) + i\hbar\,g_{\mu\nu}\,\mathrm{v}^\nu\,\vec{\nabla}_\mu\,\psi(\vec{r},t) = 0$ $i\hbar\,\gamma^\mu\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]^{1/2}\vec{\nabla}_\mu\,\psi(\vec{r},t) - \mathcal{U}^{1/2}\,\mathcal{P}\,\psi(\vec{r},t) = 0$
	Second Order Equation
	$\hbar^2\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]\Box\,\psi(\vec{r},t) + \mathcal{U}\,\mathcal{P}^2\,\psi(\vec{r},t) = 0$
Quantum Minkowski GQG:	First Order Equation
	$i\hbar\left(\frac{\ell_P^\mu}{\left(N_{(\varrho)}\mathfrak{m}_{(\varrho)}\right)^3}\right)^{1/2}\gamma^\mu\,\vec{\nabla}_{P\,\mu}\,\psi(\vec{r}_P,t_P) - \mathscr{C}\,\psi(\vec{r}_P,t_P) = 0$ $i\hbar\,\gamma^\mu\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\psi(\vec{r},t) - m_0\,\psi(\vec{r},t) = 0$ $i\hbar\,\sqrt{2\Lambda}\,\gamma^\mu\,\mathrm{g}_{\mu\nu}\,\vec{\nabla}_\mu\,\psi(\vec{r},t) - \langle T_0\rangle^{1/2}\,\psi(\vec{r},t) = 0$
	Second Order Equations
	$-\hbar^2\,\vec{\nabla}_0\,\vec{\nabla}^0\,\psi(\vec{r},t) + \hbar^2\,\vec{\nabla}_i\,\vec{\nabla}^i\,\psi(\vec{r},t) = 0$ $-\hbar^2\,\vec{\Delta}_\mu\,\vec{\Delta}_\nu\,\psi(\vec{r},t) + \hbar^2\,g_{\mu\nu}\,\vec{\nabla}_\mu\,\vec{\nabla}_\nu\,\psi(\vec{r},t) = 0$ $\hbar^2\,\frac{\ell_P^\mu}{\left(N_{(\varrho)}\mathfrak{m}_{(\varrho)}\right)^3}\,\Box_P\,\psi(\vec{r}_P,t_P) + \mathscr{C}^2\,\psi(\vec{r}_P,t_P) = 0$ $\hbar^2\,\mathrm{g}_{\mu\nu}^2\,\Box\,\psi(\vec{r},t) + m_0^2\,\psi(\vec{r},t) = 0$ $2\,\hbar^2\,\Lambda\,\mathrm{g}_{\mu\nu}^2\,\Box\,\psi(\vec{r},t) + \langle T_0\rangle\,\psi(\vec{r},t) = 0$

2.3. *Quantum Non-Minkowski GQG*

Now, let us try to develop an Einstein field equation, which is "completely" Quantum Mechanical (i.e., it has neither a Minkowski spacetime and not its metric is Lorentzian) in comparison to Eq. (42) and Eq. (61).

Eq. (55) immediately tells us that Eq. (49) is possible to be written as follows,

$$\mathrm{d}\,\mathcal{S}_{\mathcal{P}}^2\,\psi(\vec{r},t) = g_{\mu\nu}\,\hat{\boldsymbol{\mathcal{P}}}^\mu\hat{\boldsymbol{\mathcal{P}}}^\nu\,\psi(\vec{r},t) = \left(\hat{m}\,\hat{E} - \hat{\boldsymbol{p}}^i\,\hat{\boldsymbol{p}}^j\right)\psi(\vec{r},t)\,. \tag{72}$$

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After using the mass operator $\hat{m}$ from Eq. (7) and then inputting the value of $\hat{m}\,\hat{E}$ from Eq. (56) into Eq. (72) and then using Eq. (8), we have,

$$\begin{aligned} \mathrm{d}\,\mathcal{S}^2_{\mathcal{P}}\,\psi(\vec{r},t) = g_{\mu\nu}\,\hat{\boldsymbol{\mathcal{P}}}^{\mu}\hat{\boldsymbol{\mathcal{P}}}^{\nu}\psi(\vec{r},t) &= -\,\hbar^2\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^i\,\partial x^j}\right)\psi(\vec{r},t) = -\,\hbar^2\,\frac{\partial^2}{\partial s^2}\,\psi(\vec{r},t) \\ &= -\,\hbar^2\,g^{\mu\nu}\partial_\mu\partial_\nu\,\psi(\vec{r},t)\,. \end{aligned} \tag{73}$$

This line element has neither a Minkowski spacetime and not its metric is Lorentzian, since $g_{\mu\nu}$ is satisfying the last two terms of Eq. (58). Note here that, the LHS of Eq. (73), i.e., $\mathrm{d}\,\mathcal{S}^2_{\mathcal{P}}$, is relativistic, whereas, the RHS of Eq. (73), i.e., $-\,\hbar^2\,\frac{\partial^2}{\partial s^2}$, is Quantum Mechanical. We can compare this line element with Eq. (10), but this time, the relativity-quantum relation in Eq. (73) is explained more uniquely and explicitly than our previous findings which have defined in the above two subsections.

Now [13], let $\mathcal{H}$ be a Hilbert space. Let (M^n, g) be a manifold, where M^n is an n-dimensional differentiable manifold and g is a metric satisfying the last two terms of Eq. (58), which is either as a positive-definite section of the bundle of symmetric (covariant) 2-tensors $T^*M \otimes_S T^*M$ or as positive-definite bilinear maps, $g\left((i\hbar)^{-1}x\right) : T_{\left((i\hbar)^{-1}x\right)}M \times T_{\left((i\hbar)^{-1}x\right)}M \to \mathcal{H}$ for all $(i\hbar)^{-1}x \in M$. Here, $T^*M \otimes_S T^*M$ is the subspace of $T^*M \otimes T^*M$ generated by elements of the form $X \otimes Y + Y \otimes X$. Let $\left\{(i\hbar)^{-1}x^i\right\}_{i=1}^n$ be local coordinates in a neighborhood U of some point of M. In U the vector fields $\left\{\left(i\hbar\,\vec{\nabla}_i\right)^{-1}\right\}_{i=1}^n$ form a local basis for TM and the 1-forms $\left\{i\hbar\,\vec{\nabla}_i\right\}_{i=1}^n$ form a dual basis for T^*M, that is, $i\hbar\,\vec{\nabla}_j\left(i\hbar\,\vec{\nabla}_i\right)^{-1} = \delta^i_j$. Let ∇^g denote the Levi-Civita connection of the metric g. The Christoffel symbols are the components of the Levi-Civita connection and are defined in U by $\nabla_{\left(i\hbar\,\vec{\nabla}_i\right)}\left(i\hbar\,\vec{\nabla}_j\right) \doteqdot \Gamma^k_{ij}\left(i\hbar\,\vec{\nabla}_k\right)$, and for $\left[\left(i\hbar\,\vec{\nabla}_i\right),\left(i\hbar\,\vec{\nabla}_j\right)\right] = 0$, we see that they are given by,

$$\Gamma^k_{ij}\,\psi(\vec{r},t) = \frac{1}{2}\,g^{k\ell}\left[\left(i\hbar\,\vec{\nabla}_i\right)g_{j\ell} + \left(i\hbar\,\vec{\nabla}_j\right)g_{i\ell} - \left(i\hbar\,\vec{\nabla}_\ell\right)g_{ij}\right]\psi(\vec{r},t)\,. \tag{74}$$

Let the curvature $(3,1)$-tensor Rm is defined by, $\mathrm{Rm}(X,Y)Z \doteqdot \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. Thus, the curvature tensor, $R^\ell_{ijk} = \partial_i\,\Gamma^\ell_{jk} - \partial_j\,\Gamma^\ell_{ik} + \Gamma^p_{jk}\,\Gamma^\ell_{ip} - \Gamma^p_{ik}\,\Gamma^\ell_{jp}$, is purely Quantum Mechanical due to Eq. (74). Let the tensor Rc is the trace of Rm curvature tensor: $\mathrm{Rc}(Y,Z) \doteqdot \mathrm{trace}\,(X \mapsto \mathrm{Rm}(X,Y)Z)$, defined by $R_{ij} \doteqdot \mathrm{Rc}\left(\left(i\hbar\,\vec{\nabla}_i\right),\left(i\hbar\,\vec{\nabla}_j\right)\right)$, and the scalar curvature R is the trace of Rc tensor: $\mathrm{R} \doteqdot \sum_{a=1}^n \mathrm{Rc}(e_a, e_a)$ where $e_a \in T_{\left((i\hbar)^{-1}x\right)}M^n$ is a unit vector spanning $L \subset T_{\left((i\hbar)^{-1}x\right)}M^n$. Then, the Einstein-like purely quantum tensor $\mathrm{Rc} - \frac{1}{2}\,\mathrm{R}g$ directly acts on a quantum space. Thus, Einstein-like purely quantum field equation $\left[\mathrm{Rc} - \frac{1}{2}\,\mathrm{R}g\right]\psi(\vec{r},t) = \mathcal{U}\,\psi(\vec{r},t)$, where $\mathcal{U} = 8\pi\,G\,\mathrm{T}_{ij}$, is "completely" Quantum Mechanical for Eq. (74) in comparison to Eq. (42) and Eq. (61).

Instead of considering $\mathfrak{X}^\mu$ of Eq. (34), let us develop $\mathfrak{X}^\mu$ with purely quantum spacetime axes as follows,

$$\begin{aligned} &\Bigg[\left(\widehat{\partial}^2_{[0,1]} \otimes \Psi_{[1,0]}\right), a_i\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right), b_i\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right), \\ &\qquad \left\{\left(a_i a_j\right)^{1/2}\left(\widehat{\partial}^2_{[0,2]} \otimes \Psi_{[2,0]}\right) + \left(b_i b_j\right)^{1/2}\left(\widehat{\partial}^2_{[0,3]} \otimes \Psi_{[3,0]}\right)\right\}\Bigg] \\ &\mapsto \left((i\hbar)^{-1}\mathfrak{x}^0,\,(i\hbar)^{-1}\mathfrak{x}^1,\,(i\hbar)^{-1}\mathfrak{x}^{(1+i)},\,(i\hbar)^{-1}\mathfrak{x}^{(1+(i+\ell))},\,(i\hbar)^{-1}\mathfrak{x}^{(1+(i+\ell)+j)}\right) \in \mathfrak{X}^\mu\,, \end{aligned} \tag{75}$$

for $i, j = 1,2,3$, $i \neq j$ and $\ell = \max i$, where,

$$(i\hbar)^{-1}\mathfrak{x}^{(1+(i+\ell)+j)} = \left((i\hbar)^{-1}\mathfrak{x}_a^{(1+(i+\ell)+\frac{1}{2}j)} + (i\hbar)^{-1}\mathfrak{x}_b^{(1+(i+\ell)+\frac{1}{2}j)}\right).$$

For the selection of the axis from Eq. (75), we use fully democratic way, e.g., if $\left\{(a_3 a_1)^{1/2}\left(\widehat{\partial}^2_{[0,2]}\otimes\Psi_{[2,0]}\right)+(b_3 b_1)^{1/2}\left(\widehat{\partial}^2_{[0,3]}\otimes\Psi_{[3,0]}\right)\right\}$ gives the 11$^{\text{th}}$ dimension, then $a_3\left(\widehat{\partial}^2_{[0,2]}\otimes\Psi_{[2,0]}\right)$ and $b_3\left(\widehat{\partial}^2_{[0,3]}\otimes\Psi_{[3,0]}\right)$ should give us the 9$^{\text{th}}$ and 10$^{\text{th}}$ dimensions, and so on.

2.3.1. Superstring/M-theory

For this sub-subsection, we will switch to the Minkowski signature $(-,+,\ldots,+)$ to express a point particle and a string propagating in a D-dimensional curved spacetime.

The bosonic part of the action of the $N=1$ supergravity theory in 11 dimensions should be (we follow Ref. [14] hereafter),

$$S=\frac{1}{2\kappa^2}\int \mathrm{d}^{11}(i\hbar)^{-1}\mathfrak{x}\sqrt{-g}\left(\mathrm{Rc}-\frac{1}{48}F_4^2\right)-\frac{1}{12\kappa^2}\int F_4\wedge F_4\wedge A_3\,, \tag{76}$$

where $F_4=\mathrm{d}\,A_3$. Definitely, this $D=11$ supergravity is now in $\mathfrak{X}^\mu$ spacetime of Eq. (75) with purely quantum gravity g, which is satisfying the last two terms of Eq. (58) but now with the signature $(-,+,\ldots,+)$. And the overall scenario is condensed inside the observable $(3+1)D$ spacetime, i.e., $\left(\partial^2_{[0,i]}\otimes\Psi_{[0,i]}\right)$ for $i=1,2,3$.

Similarly, the bosonic part of the action of type IIA theory ($N=2$, $d=10$) should be (in the string frame),

$$\begin{aligned} S_{\mathrm{IIA}} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}(i\hbar)^{-1}\mathfrak{x}\sqrt{-g}\left(\exp(-2\phi)\left[\mathrm{Rc}+4(\partial\phi)^2-\frac{1}{12}H_3^2\right]-\frac{1}{4}F_2^2-\frac{1}{48}F_4^2\right)-\\ &\quad -\frac{1}{4\kappa^2}\int F_4\wedge F_4\wedge B_2\,, \end{aligned} \tag{77}$$

where $F_4=\mathrm{d}\,C_3+H_3\wedge C_1$, $F_2=\mathrm{d}\,C_1$, $H_3=\mathrm{d}\,B_2$ and ϕ is the dilaton. And the bosonic part of the action of type IIB theory ($N=2$, $d=10$) should be (in the string frame),

$$\begin{aligned} S_{\mathrm{IIB}} &= \frac{1}{2\kappa^2}\int \mathrm{d}^{10}(i\hbar)^{-1}\mathfrak{x}\sqrt{-g}\Big(\exp(-2\phi)\left[\mathrm{Rc}+4(\partial\phi)^2-\frac{1}{12}H_3^2\right]-\frac{1}{2}(\partial\phi)^2-\\ &\quad -\frac{1}{12}[F_3+aH_3]^2-\frac{1}{480}F_5^2\Big)+\frac{1}{4\kappa^2}\int\left(C_4+\frac{1}{2}B_2\wedge C_2\right)\wedge F_3\wedge H_3\,, \end{aligned}$$

where $F_5=\mathrm{d}C_4+H_3\wedge C_2$, $F_3=\mathrm{d}C_2$, $H_3=\mathrm{d}B_2$, while a is the RR axion and ϕ is the dilaton. Whereas, the bosonic part of the type I action ($N=1$, $d=10$) should be,

$$S_{\mathrm{I}}=\frac{1}{2\kappa^2}\int \mathrm{d}^{10}(i\hbar)^{-1}\mathfrak{x}\sqrt{-g}\left(\exp(-2\phi)\left[\mathrm{Rc}+4(\partial\phi)^2\right]-\frac{1}{12}\tilde{F}_3^2-\frac{1}{4}\exp(-\phi)\,\mathrm{Tr}F^2\right),$$

where $\tilde{F}_3$ is the modified field strength for the two form, ϕ is the dilaton and $F_2=\mathrm{d}A+A\wedge A$, where A is the gauge potential in the adjoint representation of SO(32). As the two heterotic supergravity theories are obtained as the low energy limit of heterotic string theory with gauge group SO(32) and $E_8\times E_8$, respectively, the bosonic part of the actions should be,

$$S_{\mathrm{Heterotic}}=\frac{1}{2\kappa^2}\int \mathrm{d}^{10}(i\hbar)^{-1}\mathfrak{x}\sqrt{-g}\,\exp(-2\phi)\left(\mathrm{Rc}+4(\partial\phi)^2-\frac{1}{12}\tilde{H}_3^2-\frac{1}{4}\mathrm{Tr}F^2\right),$$

where $\tilde{H}_3$ is the modified field strength for the two form, ϕ is the dilaton and $F_2=\mathrm{d}A+A\wedge A$, where A is the gauge potential in the adjoint representation of SO(32) or $E_8\times E_8$, respectively. In the similar

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way, a solitonic supergravity solution for p-branes in 11 dimensions which interpolates between a vacuum with SO(1,2)×SO(8) symmetry yields,

$$\begin{aligned} ds^2 &= H^{-2/3}\,\eta_{\mu\nu}\,(i\hbar)^{-2} d\,\mathfrak{x}^{\mu}\,d\,\mathfrak{x}^{\nu} + H^{1/3}\,\delta_{mn}\,(i\hbar)^{-2} d\,\mathfrak{x}^{m}\,d\,\mathfrak{x}^{n}\,, \\ A_3 &= H^{-1}\,\eta_{\mu\nu}\,(i\hbar)^{-3} d\,\mathfrak{x}^{0}\wedge d\,\mathfrak{x}^{1}\wedge d\,\mathfrak{x}^{2}\,, \\ H &= 1+\frac{R^6}{r^6}\,, \end{aligned}$$

where $\mu,\nu = 0,1,2,\, m,n = 3,\ldots,10$, and H is a harmonic function on the transverse space, whereas r is the radius for the eight-dimensional space transverse to the membrane. Hence, the complete scenario of the Superstring/M-theory is condensed again inside the observable $(3+1)D$ spacetime, i.e., $\left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right)$ for $i = 1,2,3$. But M-theory on $\mathbb{R}^{11}$ does not contain any strings, however, if we replace Minkowski spacetime with $\mathfrak{X}^{\mu}$ spacetime of Eq. (75), then we should find that M-theory on $\mathfrak{X}^{\mu}$ is now contain strings which is inside the observable $(3+1)D$ spacetime, i.e., $\left(\partial^2_{[0,i]} \otimes \Psi_{[0,i]}\right)$ for $i = 1,2,3$.

Let Planck mass $m_P = N\mathfrak{m}$ for a certain particle with mass $\mathfrak{m}$, where N is a very large constant number. Since M-theory is the strong coupling limit of the type IIA string theory, it must be an inherently non-perturbative theory, with no arbitrary coupling constant, but only a length scale ℓ_p, then the relation between this length scale and the IIA length scale and coupling can be obtained by comparing the 11 and 10-dimensional gravitational constants κ_{11} and κ_{10}. But we know $G = -\hbar^2 \frac{\ell_P^{\mu}}{(N\mathfrak{m})^3} \vec{\nabla}_{P\mu} \vec{\nabla}_P^{\mu}$ from Eq. (62). Then, $\kappa_{11}^2 = 2\pi R\,\kappa_{10}^2$, where, by accepting $\ell_p \mapsto \ell_P^{\mu}$,

$$\begin{aligned} 2\kappa_{11}^2 &= (2\pi)^8 \left(-\hbar^{-2} G\,(N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^9 , \\ \therefore \quad & 2\hbar^{18}\kappa_{11}^2\,\Box^9 + (2\pi)^8 G^9\,(N\mathfrak{m})^{27} = 0\,, \\ \text{which yields,} \quad & i\sqrt{2}\hbar^9\kappa_{11}\,\gamma^{\mu}\,\vec{\nabla}_{P\mu}^{9/2} - (2\pi)^4 G^{9/2}\,(N\mathfrak{m})^{27/2} = 0\,, \end{aligned}$$

when $\Box = \vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}$ and γ^{μ} are Dirac's gamma matrices, whereas $2\kappa_{10}^2 = (2\pi)^7 g^2\alpha'^4$. So, we obtain,

$$\left(-\hbar^{-2} G\,(N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^2 = g^{2/3}\alpha'\,,$$

that,

$$\begin{aligned} \alpha' &= \frac{1}{R}\left(-\hbar^{-2} G\,(N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^3 , \\ g &= R^{3/2}\left(-\hbar^{-2} G\,(N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^{-3/2} . \end{aligned} \tag{78}$$

Hence, when the 11-dimensional radius is much smaller than the 11-dimensional length scale we effectively have a 10-dimensional theory, which is type IIA string theory. But this idea should be applicable for any dimensional radious R_{ν} and any dimensional length scale since we have taken $\ell_p \mapsto \ell_P^{\mu}$; thus, we can rewrite Eq. (78) as follows,

$$\begin{aligned} \alpha' &= \frac{\Theta_{\mu}^3}{R_{\nu}}\left(-\hbar^{-2} G\,(N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^3 , \\ g &= \left(\Theta_{\mu}^{-1} R_{\nu}\right)^{3/2}\left(-\hbar^{-2} G\,(N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu}\vec{\nabla}_P^{\mu}\right]^{-1}\right)^{-3/2} , \end{aligned}$$

if $\mu = \nu$, and if $\mu \neq \nu$ then $\alpha' = 0$, $g = 0$, where Θ_μ is a coefficient for the various energy lavels of $\ell_P^{\,\mu}$, for example, $\Theta_{11} = 1$. This suggests that the low energy limit of the strong coupling limit of IIA string theory (which is M-theory) must be the 11-dimensional supergravity for any dimensional radius R_ν and any dimensional length scale. Let no fields depend on the $10^{\,\text{th}}$ space direction $(i\hbar)^{-1}\mathrm{d}\,\mathfrak{x}^{10}$, then splitting the metric and having the three form as,

$$\begin{aligned} \mathrm{d}s_{11}^2 &= \exp\left(-2\phi/3\right)\mathrm{d}s_{\mathrm{IIA}}^2 + \exp\left(4\phi/3\right)(i\hbar)^{-1}\left(\mathrm{d}\,\mathfrak{x}^{10} + C_\mu\,\mathrm{d}\,\mathfrak{x}^{\mu}\right)^2 , \\ A_{\mu\nu,10} &= B_{\mu\nu} , \\ A_{\mu\nu\rho} &= C_{\mu\nu\rho} , \end{aligned}$$

after that inserting these relations into the 11-dimensional supergravity action Eq. (76), a straightforward calculation gives the type IIA supergravity action Eq. (77), in the string frame. If no fields depend on the $n^{\,\text{th}}$ space direction $(i\hbar)^{-1}\mathrm{d}\,\mathfrak{x}^{n}$, then we have,

$$\begin{aligned} \mathrm{d}s_{11}^2 &= \exp\left(-2\phi/3\right)\mathrm{d}s_{\mathrm{IIA}}^2 + \Theta_n\,\exp\left(4\phi/3\right)(i\hbar)^{-1}\left(\mathrm{d}\,\mathfrak{x}^{n} + C_\mu\,\mathrm{d}\,\mathfrak{x}^{\mu}\right)^2 , \\ A_{\mu\nu,n} &= B_{\mu\nu} , \\ A_{\mu\nu\rho} &= C_{\mu\nu\rho} . \end{aligned}$$

Hence, type IIA supergravity can be obtainable from a dimensional reduction of 11-dimensional supergravity (M-theory), where the reduced dimension should be depended on the $n^{\,\text{th}}$ space direction $(i\hbar)^{-1}\mathrm{d}\,\mathfrak{x}^{n}$. So, obtaining type IIA string theory from M-theory by dimensional reduction is now non-restricted for the $11^{\,\text{th}}$ direction but universally for any $n^{\,\text{th}}$ direction. This will be more clear if we consider the relations between branes in type IIA string theory and M-theory as descrbe in the following tensions,

M-brane:

$$\begin{aligned} \mathrm{MW}: &\quad \frac{1}{R_\nu} , \\ \mathrm{M2}: &\quad \frac{1}{(2\pi)^2}\left[\Theta_\mu\left(-\hbar^{-2}\,G\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\,\mu}\right]^{-1}\right)\right]^{-3} , \\ \mathrm{M5}: &\quad \frac{1}{(2\pi)^5}\left[\Theta_\mu\left(-\hbar^{-2}\,G\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\,\mu}\right]^{-1}\right)\right]^{-6} , \\ \mathrm{KK6}: &\quad \frac{(2\pi\,R_\nu)^2}{(2\pi)^8}\left[\Theta_\mu\left(-\hbar^{-2}\,G\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}_P^{\,\mu}\right]^{-1}\right)\right]^{-9} , \end{aligned}$$

Type IIA-brane:

$$
\begin{aligned}
\mathrm{D0}: \quad \frac{1}{R_\nu} &= \left[\left(\Theta_\mu^{-1} R_\nu \right)^{3/2} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^{-3/2} \right]^{-1} \times \\
&\times \left[\frac{\Theta_\mu^3}{R_\nu} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^3 \right]^{-1/2} , \\
\mathrm{F1}: \quad & \frac{2\pi R_\nu}{(2\pi)^2} \Theta_\mu^{-3} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^{-3} \\
&= \left[(2\pi) \frac{\Theta_\mu^3}{R_\nu} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^3 \right]^{-1} , \\
\mathrm{D2}: \quad & \frac{1}{(2\pi)^2} \left[\Theta_\mu \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right) \right]^{-3} \\
&= \frac{1}{(2\pi)^2} \left[\left(\Theta_\mu^{-1} R_\nu \right)^{3/2} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^{-3/2} \right]^{-1} \times \\
&\times \left[\frac{\Theta_\mu^3}{R_\nu} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^3 \right]^{-3/2} , \\
\mathrm{D4}: \quad & \frac{2\pi R_\nu}{(2\pi)^5} \left[\Theta_\mu \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right) \right]^{-6} \\
&= \frac{1}{2\pi} \left[\left(\Theta_\mu^{-1} R_\nu \right)^{3/2} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^{-3/2} \right]^{-1} \times \\
&\times \left[\frac{\Theta_\mu^3}{R_\nu} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^3 \right]^{-5/2} , \\
\mathrm{NS5}: \quad & \frac{1}{(2\pi)^5} \left[\Theta_\mu \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right) \right]^{-6} \\
&= \frac{1}{(2\pi)^5} \left[\left(\Theta_\mu^{-1} R_\nu \right)^{3/2} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^{-3/2} \right]^{-2} \times \\
&\times \left[\frac{\Theta_\mu^3}{R_\nu} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^3 \right]^{-3} , \\
\mathrm{D6}: \quad & \frac{(2\pi R_\nu)^2}{(2\pi)^8} \left[\Theta_\mu \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right) \right]^{-9} \\
&= \frac{1}{(2\pi)^5} \left[\left(\Theta_\mu^{-1} R_\nu \right)^{3/2} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^{-3/2} \right]^{-1} \times \\
&\times \left[\frac{\Theta_\mu^3}{R_\nu} \left(-\hbar^{-2} G \, (N\mathfrak{m})^3 \left[\vec{\nabla}_{P\mu} \vec{\nabla}_P^\mu \right]^{-1} \right)^3 \right]^{-7/2} .
\end{aligned}
$$

Since, type IIA and IIB string theory are T-dual when compactified on a small and large circle, respectively, then it must be possible to relate M-theory and type IIB string theory by the same

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consideration as above. Thus, we obtain the following relations between M-theory and type IIB string theory parameters as,

$$\begin{aligned} g_{\mathrm{B}} &= \frac{R_\nu}{R_{(\nu-1)}}\,, \\ R_{\mathrm{B}} &= \frac{1}{R_{(\nu-1)}\,R_\nu}\left[\Theta_\mu\left(-\hbar^{-2}G\,(N\mathfrak{m})^3\left[\vec{\nabla}_{P\,\mu}\,\vec{\nabla}^{\mu}_{P}\right]^{-1}\right)\right]^3 . \end{aligned}$$

We have skipped a very interesting thing in our above discussion that would be occurred if we construct flux compactifications of M-theory to three-dimensional $\mathfrak{X}$ spacetime preserving $\mathcal{N}=2$ supersymmetry. Here, a scalar function depending on the coordinates of the internal dimensions $\Delta(y)$ (called the warp factor) is included as the explicit form for the metric ansatz as,

$$\mathrm{d}s^2 = \Delta(y)^{-1}\eta_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\mu} + \Delta(y)^{1/2}g_{mn}(y)\mathrm{d}y^{m}\mathrm{d}y^{u}\,,$$

where,

$$x^{\mu} = \left((i\hbar)^{-1}\mathfrak{x}^{0},\,(i\hbar)^{-1}\mathfrak{x}^{1},\,(i\hbar)^{-1}\mathfrak{x}^{(1+i')}\right)\,, \qquad \text{for } i'=1\,,$$

are the coordinates of the three-dimensional spacetime M_3 and,

$$y^{m} = \left((i\hbar)^{-1}\mathfrak{x}^{(1+i'')},\,(i\hbar)^{-1}\mathfrak{x}^{(1+(i+\ell))},\,(i\hbar)^{-1}\mathfrak{x}^{(1+(i+\ell)+j)}\right)\,, \qquad \text{for } \begin{cases} i''=2,3\,, \\ i=1,2,3\,, \end{cases}$$

are the coordinates of the internal eight-manifold M.

Relating M-theory on a line interval and $E_8 \times E_8$ heterotic string theory is quite obvious now and has been omitted here.

3. Dark Energy Scenarios

If we try to develop a Lagrangian for gravity, we can either choose Eq. (47) or Eq. (67). Though, gravity is feeblest in Electroweak or Quantum Chromodynamic interactions, but gravity is always related to their interacting particles, thus, we should like to go with Eq. (47) for Electroweak or Quantum Chromodynamic interactions as $\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right] \to 1$ for Remark 4, whereas, we should choose Eq. (67) for gravity. Since the last equation of Eq. (68) is applicable simultaneously whether the matter is baryonic or non-baryonic, then we should like to choose Eq. (71) to include Dark Energy in the Lagrangian along with gravitational and Electroweak or Quantum Chromodynamic sectors.

Here, we only touch upon the bare minima of Electroweak or Quantum Chromodynamic interactions in presence of gravity and Dark Energy. We have left a number of features for the interested readers to check them out with their own interests.

Following fundamental interactions are considered to take place between the kernel energy of the core, i.e. E_{PLANCK}, and the outer surface energy of the core, i.e. $E_{\star}$, of the particles. Note it that, at this energy state $E_{\text{PLANCK}} > E_{\star}$, gravitons are not true particles but mere energy states.

3.1. Gravitational Electroweak Dark Energy Interactions

Let $\hbar = c = 1$. For baryonic matter, let us include gravity and Dark Energy in the Yang-Mills Lagrangian of the electroweak symmetry $SU(2)_L \otimes U(1)_Y$. We will consider $U(1)_G$ as the symmetry associated with gravity group and it is unbroken, since it does not directly interact with the Higgs.

Let us consider Casimir energy is associated with the right-handed fermions, so by choosing the isospin quantum numbers of different Standard Model fermions and by considering the $U(1)_{\mathscr{C}}$ is the symmetry associated with the Casimir hypercharge, $Y_{\mathscr{C}} = -1$, we can consider the Casimir hypercharge field χ. So, for the overall invariance $SU(2)_L \otimes U(1)_Y \otimes U(1)_G \otimes U(1)_{\mathscr{C}}$, we can assume the unbroken Gravitational Electroweak Dark Energy (GED) gauge group should be at least $SU(3)_{GED}$. Thus, mathematically, Casimir energy must be unified with the Dark Energy interactions minimally as a Yang-Mills field with an $SU(2)_D \otimes U(1)_{\mathscr{C}}$ gauge group, where $SU(2)_D$ is gauged Dark Energy isospin and the $U(1)_{\mathscr{C}}$ is the symmetry associated with the Casimir hypercharge, $Y_{\mathscr{C}}$. So, the Gravitational Electroweak Dark Energy interactions (GED) can trigger the symmetry breaking,

$$\begin{aligned} SU(3)_{GED} &\rightarrow SU(2)_L \otimes U(1)_Y \otimes U(1)_G \otimes SU(2)_D \otimes U(1)_{\mathscr{C}} \\ &\rightarrow U(1)_{em} \otimes U(1)_G \otimes SU(2)_D \otimes U(1)_{\mathscr{C}}\,, \end{aligned}$$

which describes the formal operations that can be applied to the Electroweak, gravitational and Dark Energy gauge fields without changing the dynamics of the system. Let $SU(2)_D \otimes U(1)_{\mathscr{C}}$ fields are the Dark Energy isospin fields Y_1, Y_2, and Y_3, and the Dark Energy hypercharge field χ. We need to remember here that fermionic isospin states in reactions/decays governed by the Dark Energy interaction are conserved, i.e., the transition from $|00\rangle$ to $|10\rangle$ is not allowed in Dark Energy interaction, which is quite familiar with Electromagnetic sector of Electroweak interaction, whereas, no conservation of isospin is occurred in GED. Note here, the structures of these fermionic isospin states of the model are considered to satisfy Eq. (45) of Remark 4, i.e., Dark Energy interaction must take place at the kernel of the fermionic isospin states of the model. We should remember that bosons do not give Dark Energy by nature. On the other hand, all bosons gravitate (e.g., opposite moving photons gravitate).

Let the Yang-Mills Lagrangian of the Gravitational Electroweak Dark Energy (GED) interaction is,

$$\begin{aligned} \mathscr{L}_{\mathrm{GED}} &= \bar{\ell}\, i\, \gamma^{\mu} D_{\mu}\, \ell + \bar{e}_R\, i\, \gamma^{\mu} D_{\mu}\, e_R + \mathrm{g}_{\mu\nu} \left[\bar{f}\, i\, \gamma^{\mu} \mathcal{D}_{\mu\,(G)}\, f + \right. \\ &+ \bar{\ell}\, i\, \gamma^{\mu} \mathscr{D}^{L}_{\mu\,(D)}\, \ell + \bar{e}_R\, i\, \gamma^{\mu} \mathscr{D}^{R}_{\mu\,(D)}\, e_R \left. \right] - \\ &- \frac{1}{4} \left[\mathbf{W}_{\mu\nu}\, \mathbf{W}^{\mu\nu} + B_{\mu\nu}\, B^{\mu\nu} + \mathcal{G}_{\mu\nu}\, \mathcal{G}^{\mu\nu} + \mathbf{Y}_{\mu\nu}\, \mathbf{Y}^{\mu\nu} + \chi_{\mu\nu}\, \chi^{\mu\nu} \right], \end{aligned} \tag{79}$$

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where, f indicates all kind of fermions. By using Eq. (47) for Electroweak interaction, Eq. (67) for gravity and Eq. (71) for Dark Energy interaction, here, covariant derivatives and field tensors are given by,

$$\begin{aligned}
D_\mu\,\ell &= \left\{\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]^{1/2}\vec{\nabla}_\mu + i\,g_W\,\boldsymbol{W}_\mu\cdot\frac{\boldsymbol{\tau}}{2} + i\,g_B\left(\frac{Y_B}{2}\right)B_\mu\right\}\ell\\
&= \left\{\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]^{1/2}\vec{\nabla}_\mu + i\,g_W\,\boldsymbol{W}_\mu\cdot\frac{\boldsymbol{\tau}}{2} + i\,g_B\left(-\frac{1}{2}\right)B_\mu\right\}\ell\\
D_\mu\,e_R &= \left\{\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]^{1/2}\vec{\nabla}_\mu + i\,g_B\left(-\frac{1}{2}\right)B_\mu\right\}e_R,\\
\mathcal{D}_{\mu\,(G)}\,f &= \left\{\vec{\nabla}_\mu + i\,g_G\left(\frac{Y_G}{2}\right)\mathsf{G}_\mu\right\}f = \left\{\vec{\nabla}_\mu + i\,g_G\left(+\frac{2}{2}\right)\mathsf{G}_\mu\right\}f,\\
\mathscr{D}^L_{\mu\,(D)}\,\ell &= \left\{\vec{\nabla}_\mu + i\,g_D\,\mathbf{Y}_\mu\cdot\frac{\boldsymbol{\kappa}}{2} + i\,g_{\mathscr{C}}\left(\frac{Y_{\mathscr{C}}}{2}\right)\chi_\mu\right\}\ell\\
&= \left\{\vec{\nabla}_\mu + i\,g_D\,\mathbf{Y}_\mu\cdot\frac{\boldsymbol{\kappa}}{2} + i\,g_{\mathscr{C}}\left(-\frac{1}{2}\right)\chi_\mu\right\}\ell,\\
\mathscr{D}^R_{\mu\,(D)}\,e_R &= \left\{\vec{\nabla}_\mu + i\,g_{\mathscr{C}}\left(-\frac{1}{2}\right)\chi_\mu\right\}e_R,\\
W^i_{\mu\nu} &= \left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]^{1/2}\left(\vec{\nabla}_\mu\,W^i_\nu - \vec{\nabla}_\nu\,W^i_\mu\right) + g_W\,\epsilon^{jik}\,W^j_\mu\,W^k_\nu,\\
B_{\mu\nu} &= \left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]^{1/2}\left(\vec{\nabla}_\mu\,B_\nu - \vec{\nabla}_\nu\,B_\mu\right),\\
\mathcal{G}_{\mu\nu} &= \left(\vec{\nabla}_\mu\,\mathsf{G}_\nu - \vec{\nabla}_\nu\,\mathsf{G}_\mu\right),\\
\mathrm{Y}^i_{\mu\nu} &= \left(\vec{\nabla}_\mu\,\mathrm{Y}^i_\nu - \vec{\nabla}_\nu\,\mathrm{Y}^i_\mu\right) + g_D\,\xi^{jik}\,\mathrm{Y}^j_\mu\,\mathrm{Y}^k_\nu,\\
\chi_{\mu\nu} &= \left(\vec{\nabla}_\mu\,\chi_\nu - \vec{\nabla}_\nu\,\chi_\mu\right),
\end{aligned} \tag{80}$$

here, the components $Y_{\mathscr{C}}$ and $g_{\mathscr{C}}$ are belonged to Casimir fields, where $Y_B = -1$ and $Y_{\mathscr{C}} = -1$ are the diagonal matrices with the hypercharges for electromagnetic and Casimir fields, respectively, while $Y_G = 2$ is an operator for gravity field, in their diagonal entries, and $i = 1, 2, 3$. In Eq. (79), fermions and vector fields all are massless.

It is necessary to be mentioned that $\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right] \to 1$ for Remark 4 in the Gravitational Electroweak Dark Energy interactions (and it is also true for the Gravitational Chromodynamic Dark Energy interactions, see Subsection 3.2 below). Remember that gravitons are only observable as real particles in the energy zone $E_{\mathrm{IR}} \le E_\star$ and in $E_{\mathrm{UV}} > E_\star$ level, all gravitons appear as non-particle energy states. Only in the energy zone $E_{\mathrm{IR}} \le E_\star$, gravitons behave as real particles.

Now, to introduce spontaneous symmetry-breaking and to generate masses for the gauge bosons by the choice of Higgs vacuum, let us consider a complex scalar field with a quartic interaction, where the Lagrangian has the form, $\mathscr{L} = \vec{\nabla}_\mu\,\phi^\dagger\,\vec{\nabla}^\mu\,\phi - \vartheta^2\,\phi^\dagger\phi - \lambda\,(\phi^\dagger\phi)^2$, from which we find the equation of motion as, $\{\Box + \vartheta^2\}\,\phi + 2\,\lambda\,\phi\,(\phi\,\phi^\dagger) = 0$. So, we have the non-vanishing one, $\Xi = \{(-\vartheta^2)/(2\,\lambda)\}^{1/2} = \{v^2/2\}^{1/2}$.

Let us introduce the matrices in such a way,

$$\begin{bmatrix} M \\ N \\ O \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} v \\ w \\ x \end{bmatrix} = v \begin{bmatrix} a \\ d \\ g \end{bmatrix} + w \begin{bmatrix} b \\ e \\ h \end{bmatrix} + x \begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} av + bw + cx \\ dv + ew + fx \\ gv + hw + ix \end{bmatrix},$$
$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} c & f & i \\ g & h & i \end{bmatrix} \begin{bmatrix} y \\ z \\ x \end{bmatrix} = y \begin{bmatrix} c \\ g \end{bmatrix} + z \begin{bmatrix} f \\ h \end{bmatrix} + x \begin{bmatrix} i \\ i \end{bmatrix} = \begin{bmatrix} cy + fz + ix \\ gy + hz + ix \end{bmatrix}. \tag{81}$$

Let the Lagrangian with gauge-boson masses is,

$$\mathscr{L}_m = \frac{v^2 g_W^2}{4} W_\mu^+ W_\nu^- + \frac{v^2 g_D^2}{4} \Upsilon_\mu^a \Upsilon_\nu^b + + \begin{pmatrix} W_\mu^3 & B_\mu & \mathrm{G}_\mu & \Upsilon_\mu^3 & \chi^\mu \end{pmatrix} \mathcal{M} \begin{pmatrix} W_\mu^3 \\ B_\mu \\ \mathrm{G}_\mu \\ \Upsilon_\mu^3 \\ \chi^\mu \end{pmatrix},$$

where we have written the masses for the charged $W_\mu^\pm$ and $\Upsilon_\mu^{a,b}$ fields defined as $W_\mu^\pm = \left(W_\mu^1 \mp i W_\mu^2\right)/\sqrt{2}$ and $\Upsilon_\mu^{a,b} = \left(\Upsilon_\mu^1 \mp i \Upsilon_\mu^2\right)/\sqrt{2}$, respectively, and where,

$$\mathcal{M} = \frac{v^2}{8} \begin{pmatrix} g_W^2 & -g_W g_B & g_W g_D \\ -g_W g_B & g_B^2 & -g_B g_{\mathscr{C}} \\ g_W g_D & -g_B g_{\mathscr{C}} & 4 g_G^2 \end{pmatrix}, \tag{82}$$

having $det(\mathcal{M}) = 0$, hence allowing a massless photon and a massless graviton. Now, omitting $W_\mu^\pm$ as they are quite obvious, we choose only $\Upsilon_\mu^{a,b}$, whose masses can be seen from $\mathscr{L}_m$ are given as,

$$M_{\Upsilon^{a,b}} = \frac{v}{2} (g_W g_D)^{1/2}.$$

The neutral gauge bosons mix after symmetry-breaking and the mass eigenstates are the neutral weak boson Z_μ, photon A_μ, Dark Energy boson $\mathfrak{D}_\mu$, the Casimir energy $\mathscr{C}_\mu$ and graviton $\mathcal{G}_\mu$ – first four of which are given in terms of W_μ^3, B_μ, Υ_μ^3 and χ_μ as,

$$\begin{pmatrix} Z_\mu \\ A_\mu \\ \mathcal{G}_\mu \\ \mathfrak{D}_\mu \\ \mathscr{C}_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W & \sin\theta_D \\ \sin\theta_W & \cos\theta_W & -\sin\theta_D \\ \sin\theta_D & \sin\theta_D & 4\cos\theta_{W,D} \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \\ \mathrm{G}_\mu \\ \Upsilon_\mu^3 \\ \chi^\mu \end{pmatrix},$$

i.e., alike Eq. (81),

$$\begin{aligned} Z_\mu &= \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu + \sin\theta_D \mathrm{G}_\mu, \\ A_\mu &= \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu - \sin\theta_D \mathrm{G}_\mu, \\ \mathcal{G}_\mu &= \sin\theta_D W_\mu^3 + \sin\theta_D B_\mu + 4\cos\theta_W \mathrm{G}_\mu, \\ \mathfrak{D}_\mu &= \sin\theta_D \Upsilon_\mu^3 - \sin\theta_D \chi^\mu + 4\cos\theta_D \mathrm{G}_\mu, \\ \mathscr{C}_\mu &= \sin\theta_D \Upsilon_\mu^3 + \sin\theta_D \chi^\mu + 4\cos\theta_D \mathrm{G}_\mu, \end{aligned} \tag{83}$$

with the Electroweak mixing angle θ_W and the Dark Energy mixing angle θ_D. Let us check their values. The condition that the fields A_μ and $\mathscr{C}_\mu$ should be the eigenvector of Eq. (82) with zero eigenvalue is written as,

$$0 = \begin{pmatrix} g_W^2 & -g_W g_B & g_W g_D \\ -g_W g_B & g_B^2 & -g_B g_{\mathscr{C}} \\ g_W g_D & -g_B g_{\mathscr{C}} & 4 g_G^2 \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \\ 1 \end{pmatrix} = \begin{pmatrix} g_W^2 \sin\theta - g_W g_B \cos\theta + g_W g_D \\ -g_W g_B \sin\theta + g_B^2 \cos\theta - g_B g_{\mathscr{C}} \\ g_W g_D \sin\theta - g_B g_{\mathscr{C}} \cos\theta + 4 g_G^2 \end{pmatrix},$$

and the vanishing of its RHS requires,

$$\tan\theta = \frac{g_B}{g_W} - \frac{2 g_G}{g_W \cos\theta}, \tag{84}$$

where, $\tan\theta_W = \frac{g_B}{g_W}$, then, $\theta_W = \tan^{-1}\left(\frac{g_B}{g_W}\right)$. Let us consider $\tan\theta$ of Eq. (84) as, $\tan\theta \equiv \tan\theta_D$, then $\theta_D = \tan^{-1}\left(\frac{g_B}{g_W} - \frac{2 g_G}{g_W \cos\theta_W}\right)$ and the Electroweak masses are found to be $M_Z = (v/2)\left(g_W^2 + g_B^2\right)^{1/2}$, while $M_\gamma = 0$, similarly, the Dark Energy masses are found to be $M_{\mathfrak{D}} = (v/2)\left(g_W^2 + g_B^2 + 4 g_G^2\right)^{1/2}$, while $M_{\mathscr{C}} = 0$. This gives

$$\frac{M_\Upsilon}{M_{\mathfrak{D}}} = \frac{(g_W g_D)^{1/2}}{\left(g_W^2 + g_B^2 + 4 g_G^2\right)^{1/2}} = \cos\theta_D .$$

Now, if we write down the charged-current and neutral-current interactions in GED theory, we can see the condition for which the field A_μ couples to the electron via the electromagnetic current is $g_W \sin\theta_W = g_B \cos\theta_W = e$ in Electroweak symmetry breaking, whereas, the condition for which the field $\mathscr{C}_\mu$ couples to the fermion via the Casimir current is $(g_W g_D)^{1/2} \sin\theta_D = g_{\mathscr{C}} \cos\theta_D = \chi$ in Dark Energy symmetry breaking.

By the way, the last two terms of Eq. (83) make it clear that Dark energy field is non-decaying.

3.2. Gravitational Chromodynamic Dark Energy Interactions

In the gauge theory of Gravitational Chromodynamic Dark Energy interactions (GCD), let the gauge symmetry is, $SU(4)_{GCD} \to SU(3)_C \otimes U(1)_G \otimes SU(2)_{D'} \otimes SU(2)_{D''} \otimes U(1)_{\mathscr{C}}$, i.e., the $SU(3)_C$ symmetry of the colour degree of freedom is now with gravitational and Dark Energy symmetry parts, much like what we have already discussed in GED. But apart from Casimir gauge group $U(1)_{\mathscr{C}}$, the non-abelian gauge groups of Dark Energy symmetry, i.e., $SU(2)_{D'} \otimes SU(2)_{D''}$, are quite different than the previously discussed non-abelian Dark Energy gauge group $SU(2)_D$ of GED (we will discuss below for details).

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Using Eq. (47), Eq. (67) and Eq. (71), let the Gravitational Chromodynamic Dark Energy (GCD) Lagrangian, which describes quarks and gluons in interactions in the presence of gravity and the Dark Energy fields for baryonic matter, is as follows,

$$\begin{aligned}
\mathscr{L}_{GCD} &= \sum_q \bar{\psi}_j^{(q)} \left\{ i\,\gamma^\mu \left(D_\mu\right)_{jk} - \left(\mathcal{U}^{1/2}\,\mathcal{P}\right)^{(q)} \delta_{jk} \right\} \psi_k^{(q)} + \\
&+ \sum_q \bar{\psi}_j^{(q)}\, i\,\gamma^\mu\, \mathrm{g}_{\mu\nu} \left(\mathcal{D}_{\mu\,(G)}\right)_{jk} \psi_k^{(q)} + \\
&+ \sum_q \bar{\psi}_j^{(q)} \left\{ i\,\gamma^\mu\, \mathrm{g}_{\mu\nu} \left(\mathscr{D}_{\mu\,(D)}\right)_{jk} - N_{(\Lambda)}\mathfrak{m}^{(q)}_{(\Lambda)\,0}\,\delta_{jk} \right\} \psi_k^{(q)} + \\
&+ \sum_{q_R} \bar{\psi}_j^{(q_R)}\, i\,\gamma^\mu\, \mathrm{g}_{\mu\nu} \left(\mathscr{D}^R_{\mu\,(D)}\right)_{jk} \psi_k^{(q_R)} - \\
&- \frac{1}{4}\sum_a G^a_{\mu\nu}\left(G^a\right)^{\mu\nu} - \frac{1}{4}\,\mathcal{G}_{\mu\nu}\,\mathcal{G}^{\mu\nu} - \\
&- \frac{1}{4}\sum_q \Upsilon^q_{\mu\nu}\left(\Upsilon^q\right)^{\mu\nu} - \frac{1}{4}\,\chi_{\mu\nu}\,\chi^{\mu\nu},
\end{aligned} \tag{85}$$

where, $q = u, d, s, \ldots$ is the flavor label, $j = 1, 2, 3$ is the quark color index, $a = 1, 2, \ldots, 8$ is the gluon color index, where, q_R indicates right-handed quarks and

$$\begin{aligned}
D_\mu &= \left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]^{1/2} \vec{\nabla}_\mu + i\,g_s\,\boldsymbol{A}_\mu \cdot \frac{\boldsymbol{\lambda}}{2} \\
\mathcal{D}_{\mu\,(G)} &= \vec{\nabla}_\mu + i\,g_G \left(\frac{Y_G}{2}\right) \mathsf{G}_\mu = \vec{\nabla}_\mu + i\,g_G \left(+\frac{2}{2}\right) \mathsf{G}_\mu, \\
\mathscr{D}_{\mu\,(D)} &= \vec{\nabla}_\mu + i\,g_D\,\mathbf{Y}_\mu \cdot \frac{\boldsymbol{\kappa}_D}{2} + i\,g_{\mathscr{C}} \left(\frac{Y_{\mathscr{C}}}{2}\right) \chi_\mu \\
&= \vec{\nabla}_\mu + i\,g_D\,\mathbf{Y}_\mu \cdot \frac{\boldsymbol{\kappa}_D}{2} + i\,g_{\mathscr{C}} \left(-\frac{1}{2}\right) \chi_\mu, \\
\mathscr{D}^R_{\mu\,(D)} &= \vec{\nabla}_\mu + i\,g_{\mathscr{C}} \left(-\frac{1}{2}\right) \chi_\mu, \\
G^a_{\mu\nu} &= \left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right]^{1/2} \left(\vec{\nabla}_\mu\,A^a_\nu - \vec{\nabla}_\nu\,A^a_\mu\right) + g_s\,f^{abc}\,A^b_\mu\,A^c_\nu, \\
\mathcal{G}_{\mu\nu} &= \left(\vec{\nabla}_\mu\,\mathsf{G}_\nu - \vec{\nabla}_\nu\,\mathsf{G}_\mu\right), \\
\Upsilon^q_{\mu\nu} &= \left(\vec{\nabla}_\mu\,\Upsilon^q_\nu - \vec{\nabla}_\nu\,\Upsilon^q_\mu\right) + g_D\,\xi^{qrs}\,\Upsilon^r_\mu\,\Upsilon^s_\nu, \\
\chi_{\mu\nu} &= \left(\vec{\nabla}_\mu\,\chi_\nu - \vec{\nabla}_\nu\,\chi_\mu\right),
\end{aligned}$$

where $\boldsymbol{\kappa}_D \neq \boldsymbol{\kappa}$, A^a_μ is the field of gluon and, g_s is the coupling constant, $\boldsymbol{\lambda}$ the Gell-Mann matrices in the space of colour whereas f^{abc} is the $SU(3)_C$ structure constant, here, Υ^q_μ is the Dark Energy fields and $q = 1, 2, \ldots, 6$ for six quark flavours. These Dark Energy fields are completely different from GED Dark Energy fields unlike χ_μ field, and they give two different kind of symmetry groups for two different families of quarks, namely, for light quarks u, d and s, we have Dark Energy gauge group $SU(2)_{D'}$, whereas, for heavy quarks c, t and b, we have Dark Energy gauge group $SU(2)_{D''}$. With abelian gauge group $U(1)_{\mathscr{C}}$, they give the overall Dark Energy symmetry as $SU(2)_{D'} \otimes SU(2)_{D''} \otimes U(1)_{\mathscr{C}}$. Remark 4 confirms here that $\left[\mathrm{R}_{\zeta\eta} - \frac{1}{2}\,g_{\zeta\eta}\,\mathrm{R}\right] \to 1$. Note that $m_{P0} = N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}$ scale for Dark Energy is intending here the kernel energy state of the core (i.e., the spacetime x^μ) what we have already cited above in Remark 4. In Eq. (85), Dark Energy fields are massless and they require Higgs mechanism to gain masses.

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4. Dark Matter Scenario

If a matter satisfies Eq. (21) and Eq. (22) so as it yields the internal hidden spacetime of ρ which has 'proper' spacetime arrangement for its ∂-dependency, whereas, the overall (observable) system's spacetime arrangement is quite 'improper' as it is $\widehat{\partial}$-dependent, then this kind of matter must be Dark Matter.

Let us consider a renormalizable Lagrangian, $\mathcal{L} = \mathcal{L}_{\text{(GED+GCD)}} + \mathcal{L}_{\varrho_i} + \mathcal{L}_{\text{mix}}$, where $\mathcal{L}_{\text{(GED+GCD)}}$ is for baryonic GED and GCD interactions, $\mathcal{L}_{\varrho_i}$ is the Lagrangian for Dark Matter particles ϱ_i, and $\mathcal{L}_{\text{mix}}$ contains possible non-gravitational interactions coupling between baryonic matters and ϱ_i particles. Let these ϱ_i particles are stable and behave as almost collisionless matter of Dark light quarks and/or antiquarks because of Eq. (21) and Eq. (22). This additional set of ϱ_i particles must be associated with a Dark Matter gauge group, $\mathrm{SU}(3)'_C$, denoted with a prime ($'$). Candidates for $\mathrm{SU}(3)'_C$ gauge group are almost non-luminous and non-absorbing matter, otherwise if they interact/decay with any of the baryonic particles, these non-baryonic particles would ensue the Universe very unstable with unbelievably high acceleration rate. Thus,

1. The non-baryonic Dark Matter sector, which corresponds to the gauge group $\mathrm{SU}(3)'_C$, is *not* an exact copy of the Standard Model baryonic sector $\mathrm{SU}(3)_C$. Though, in both cases, the symmetry interchanges the common type of gauge boson(s) and can be a full invariance of the two (almost) phenomenologically equivalent theories, although they are originated from fundamentally different physics maybe at very different cosmological epochs.
2. The particles for $\mathrm{SU}(3)'_C$ gauge group would have only felt the gravitational attraction of other baryonic/non-baryonic objects.

So, the Dark sector particles are solely graviton, neutral Dark weak boson, darkgluons and darkquarks only, viz. $(\mathcal{G}, Z'^0, g', u', d', \ldots)$, but not leptonic $(e', \gamma', W'^{\pm}, \ldots)$, even no Casimir $\mathscr{C}'$ particle, and neither any charged Higgs fields. Colored and/or electrically charged particles are prevented from mixing with their Dark analogues by color and electric charge conservation laws, though, the physical gluons may couple to antidarkquarks with extremely weak gluonic strength (to be discussed shortly, vide Eq. (87) and its following discussion).

Let the above Lagrangian $\mathcal{L}$ respects an exact parity symmetry, which we also refer to as mirror symmetry [15]:

$$\begin{array}{ccc} (\overrightarrow{x}, t) \to (-\overrightarrow{x}, t)\,, & g^{\mu} \leftrightarrow g'_{\mu}, & \Phi_H \leftrightarrow \Phi'_H, \\ q_{\Bbbk L} \leftrightarrow \gamma_0\, q'_{\Bbbk R}, & u_{\Bbbk R} \leftrightarrow \gamma_0\, u'_{\Bbbk L}, & d_{\Bbbk R} \leftrightarrow \gamma_0\, d'_{\Bbbk L}, \end{array}$$

where g_{μ} are the SU(3)$_C$ gluons, Φ_H is the Standard Model neutral Higgs doublet with its Dark partner Φ'_H, the fermion fields are quarks $q_{\Bbbk L} \equiv (u_{\Bbbk}, d_{\Bbbk})_L$, $u_{\Bbbk R}$, and $d_{\Bbbk R}$ whereas darkquarks $q'_{\Bbbk R} \equiv (u'_{\Bbbk}, d'_{\Bbbk})_R$, $u'_{\Bbbk L}$, and $d'_{\Bbbk L}$ represent the (anti-) darkquark families, while $\Bbbk = 1, 2, 3$ is the generation index, and γ_0 is a Dirac gamma matrix.

Suppose, $\mathcal{L}_{\varrho_i}$ is the combination of Dark Matter GED and GCD interactions, and suppose, these fundamental interactions are considered to take place between kernel of the core energy and outer surface energy of a Dark particle, i.e., at $E_{\text{PLANCK}} > E_\star$, where gravitons are not particles but mere energy states. Thus, applying Eq. (66) into Eq. (79) and Eq. (85), we get,

$$\begin{aligned}
\mathscr{L}_{\varrho_i} = & \sum_{q'_L} \bar{\psi}_j^{(q'_L)}\, i\,\gamma^\mu \left(D'_\mu\right)^W_{jk} \psi_k^{(q'_L)} + \sum_{q'_R} \bar{\psi}_j^{(q'_R)}\, i\,\gamma^\mu \left(D'_\mu\right)^W_{jk} \psi_k^{(q'_R)} + \\
& + \sum_{q'} \bar{\psi}_j^{(q')} \left\{ i\,\gamma^\mu \left(D'_\mu\right)_{jk} + N_{(\varrho)}\mathfrak{m}_{(\varrho)}^{(q')}\,\delta_{jk} \right\} \psi_k^{(q')} + \\
& + \sum_{q'} \bar{\psi}_j^{(q')}\, i\,\gamma^\mu\, \mathrm{g}_{\mu\nu} \left(\mathcal{D}_{\mu\,(G)}\right)_{jk} \psi_k^{(q')} + \\
& + \sum_{q'_L} \bar{\psi}_j^{(q'_L)}\, i\,\gamma^\mu\, \mathrm{g}_{\mu\nu} \left(\mathscr{D}^L_{\mu\,(D)}\right)_{jk} \psi_k^{(q'_L)} + \sum_{q'_R} \bar{\psi}_j^{(q'_R)}\, i\,\gamma^\mu\, \mathrm{g}_{\mu\nu} \left(\mathscr{D}^R_{\mu\,(D)}\right)_{jk} \psi_k^{(q'_R)} + \\
& + \sum_{q'} \bar{\psi}_j^{(q')} \left\{ i\,\gamma^\mu\, \mathrm{g}_{\mu\nu} \left(\mathscr{D}_{\mu\,(D)}\right)_{jk} - N_{(\Lambda)}\mathfrak{m}_{(\Lambda)\,0}^{(q')}\,\delta_{jk} \right\} \psi_k^{(q')} - \\
& - \frac{1}{4} W'_{\mu\nu} \left(W'\right)^{\mu\nu} - \frac{1}{4} \sum_a G'^a_{\mu\nu} \left(G'^a\right)^{\mu\nu} - \frac{1}{4} \mathcal{G}_{\mu\nu}\, \mathcal{G}^{\mu\nu} - \\
& - \frac{1}{4} \sum_i \mathrm{Y}'^i_{\mu\nu} \left(\mathrm{Y}'^i\right)^{\mu\nu} - \frac{1}{4} \sum_{q'} \mathrm{Y}^{q'}_{\mu\nu} \left(\mathrm{Y}^{q'}\right)^{\mu\nu}, \qquad (86)
\end{aligned}$$

where $\{N, \mathfrak{m}\} \neq \{M, \mathfrak{n}\}$, $\mathfrak{m} \gg \mathfrak{n}$ and where Dark Matter weak and Dark Matter chromodynamic covariant derivatives are,

$$\begin{aligned}
\left(D'_\mu\right)^W &= (8\pi)^{1/2}\, \vec{\nabla}_{P\,\mu} + i\, g'_W\, \boldsymbol{W}'_\mu \cdot \frac{\boldsymbol{\tau}'}{2}, \\
D'_\mu &= (8\pi)^{1/2}\, \vec{\nabla}_{P\,\mu} + i\, g'_s\, \boldsymbol{\mathcal{A}}'_\mu \cdot \frac{\boldsymbol{\lambda}'}{2},
\end{aligned}$$

when other covariant derivatives are as analogous as before, but they do not contain either photon or Casimir energy field. In Eq. (86), both Dark Matter weak fields and Dark Energy fields are massless and they require (neutral) Higgs mechanism to gain masses. Here we also presume that Dark Energy gauge groups $SU(2)_D$ and $SU(2)_{D'} \otimes SU(2)_{D''}$ are analogous to the GED and GCD Dark Energy gauge groups.

So, due to Eq. (86), the Dark Matter gauge symmetry is,

$$SU(5)_{DM} \longrightarrow SU(3)_{C'} \otimes SU(2^\star)_L \otimes U(1)_G \otimes SU(2)_D \otimes SU(2)_{D'} \otimes SU(2)_{D''},$$

here $SU(2^\star)_L$ gives only Z'^0 boson but no other (charged) weak vector bosons. As a policy of desperation, we can consider that one of the Dark Matter particles is definitely *axion*.

Considering that both of the ordinary (i.e., baryonic) and Dark particles are too close to interact in spacetime no matter what their cosmological epochs are, then for the physical effects of gluon-darkgluon kinetic mixing, let us take account of $\mathrm{SU}(3)_C \otimes \mathrm{SU}(3)'_C$ quantum chromodynamics for the quark $\psi^{(q)}$ and gluon field $\mathcal{A}_\mu$, antidarkquark $\psi^{(q')}$ and antigluon field $\mathcal{A}'_\mu$ as follows by using

Eq. (85) and Eq. (86), when gravitational and Dark Energy interactions are quite obvious and unaltered, so as omitted here, then we have the Lagrangian as,

$$\begin{aligned}\mathcal{L} &= \sum_q \bar{\psi}_j^{(q)} \left\{ i\,\gamma^\mu \left(D_\mu\right)_{jk} - \left(\mathcal{U}^{1/2}\,\mathcal{P}\right)^{(q)} \delta_{jk} \right\} \psi_k^{(q)} + \\ &+ \sum_{q'} \bar{\psi}_j^{(q')} \left\{ i\,\gamma^\mu \left(D'_\mu\right)_{jk} + N_{(\varrho)}\mathfrak{m}_{(\varrho)}^{(q')}\,\delta_{jk} \right\} \psi_k^{(q')} - \\ &- \frac{1}{4}\sum_a G^a_{\mu\nu}\left(G^a\right)^{\mu\nu} - 2\pi \sum_a G'^a_{\mu\nu}\left(G'^a\right)^{\mu\nu} - \frac{\epsilon}{2\,(8\pi)^{1/2}}\sum_a G^a_{\mu\nu}\left(G'^a\right)^{\mu\nu} - \\ &- \frac{g_s^2\,\Theta_s}{64\,\pi^2}\, f^{\mu\nu\lambda\rho} \sum_a \left(G^a\right)^{\mu\nu}\left(G^a\right)^{\lambda\rho} - \frac{g_s'^2\,\Theta_s}{8\,\pi}\, f^{\mu\nu\lambda\rho} \sum_a \left(G'^a\right)^{\mu\nu}\left(G'^a\right)^{\lambda\rho} - \\ &- \frac{\epsilon\left(g_s^2\,g_s'^2\right)^{1/2}\Theta_s}{32\,\pi^2 (8\pi)^{1/2}}\, f^{\mu\nu\lambda\rho} \sum_a \left(G^a\right)^{\mu\nu}\left(G'^a\right)^{\lambda\rho},\end{aligned}$$

involving a sufficiently small dimensionless parameter ϵ, and there is no symmetry reason for suppressing this term, with the minimal particle content [16]. Here $\mathcal{A}^C_\mu \equiv A^C_\mu$ and $\mathcal{A}'^C_\mu \equiv \frac{1}{2}\,A'^C_\mu$. Note here, that the $\mathrm{SU}(3)_C$-$\mathrm{SU}(3)'_C$ kinetic mixing term is gauge invariant. The kinetic mixing can be removed with a non-orthogonal transformation [17],

$$\mathcal{A}_\mu \to \tilde{\mathcal{A}}_\mu \equiv \mathcal{A}_\mu + \epsilon\,\mathcal{A}'_\mu \;\;, \qquad \mathcal{A}'_\mu \to \tilde{\mathcal{A}}'_\mu \equiv \mathcal{A}'_\mu \sqrt{1-\epsilon^2} \;\; .$$

We can transform to a basis where only one of these states couples to gluons. By the orthogonal state we call the *sterile gluon* $\mathcal{A}^\mu_2$,

$$\mathcal{A}^\mu_1 = \mathcal{A}^\mu \sqrt{1-\epsilon^2}, \qquad\qquad \mathcal{A}^\mu_2 = \mathcal{A}'^\mu + \epsilon\,\mathcal{A}^\mu,$$

the Lagrangian for an ordinary matter environment shows (to leading order in ϵ) that the physical gluon couples to antidarkquarks with gluonic strength $g_s\epsilon$, which is extremely weak for the sufficiently small dimensionless parameter ϵ, while g'_s doesn't couple to ordinary matter at all. Similarly, considering the *sterile darkgluon* $\mathcal{A}'^\mu_1$,

$$\mathcal{A}'^\mu_1 = \mathcal{A}^\mu + \epsilon\,\mathcal{A}'^\mu, \qquad\qquad \mathcal{A}'^\mu_2 = \mathcal{A}'^\mu \sqrt{1-\epsilon^2},$$

the Lagrangian for a dark matter environment becomes (to leading order in ϵ),

$$\begin{aligned}\mathcal{L} &= \sum_q \bar{\psi}_j^{(q)} \left\{ i\,\gamma^\mu \left(D_\mu\right)_{jk} - \left(\mathcal{U}^{1/2}\,\mathcal{P}\right)^{(q)} \delta_{jk} \right\} \psi_k^{(q)} + \\ &+ \sum_{q'} \bar{\psi}_j^{(q')} \left\{ i\,\gamma^\mu \left(D'_\mu\right)_{jk} + N_{(\varrho)}\mathfrak{m}_{(\varrho)}^{(q')}\,\delta_{jk} \right\} \psi_k^{(q')} + \\ &+ \sum_q \bar{\psi}_j^{(q)}\, g_s \gamma^\mu\, \lambda^C_{jk} \left(\mathcal{A}'^{C\mu}_1 - \epsilon\,\mathcal{A}'^{C\mu}_2\right) \psi_k^{(q)} + \\ &+ \sum_{q'} \bar{\psi}_j^{(q')}\, g'_s \gamma^\mu\, \lambda'^C_{jk}\, \mathcal{A}'^{C\mu}_2\, \psi_k^{(q')}, \end{aligned} \tag{87}$$

in terms of the ordinary matter physical states,

$$\mathcal{A}'^\mu_2 = \left(\mathcal{A}^\mu_2 - \epsilon\,\mathcal{A}^\mu_1\right),$$

(to leading order in ϵ) and suppressing the $G^a_{\mu\nu}\left(G^a\right)^{\mu\nu}$ and $G'^a_{\mu\nu}\left(G'^a\right)^{\mu\nu}$ related components in $\mathcal{L}$. Evidently, an antidarkquark would emit the state $\mathcal{A}'^\mu_2$, thus, the flux of g'_s detectable in an ordinary matter detector is reduced by a factor ϵ^2. Since ϵ is sufficiently small, this makes such emission should not be detected in the present state collider experiments.

Remark 6. *So, the above inspection implies that, antiquark-quark → antidarkquark-darkquark or quark-quark (or quark-antiquark) → darkquark-antidarkquark annihilation channels may occur within the nucleons and gives us the effects something like Refs. [18–20], but antidarkquark-darkquark → quark-antiquark or darkquark-antidarkquark → quark-antiquark annihilation channels should never take place on the inside of a nucleon (baryonic or dark), and neither any decay $q'_1 \to q'_2 + (f\bar{f})$ through intermediated spin-1 gauge bosons, where f and $\bar{f}$ stand for light Standard Model particles (assuming the conservation of a dark-parity) unless $(f\bar{f})$ are leptons.*

Remark 6 is sufficient to explain the effectively more collisionless bow shock phenomena of the mass component of the merging galaxy clusters like as the 1E 0657-56 cluster [21], commonly known as the Bullet Cluster, and the similar collisionless behavior that has been observed in other merging galaxy clusters, for example, two high-redshift clusters, CL 0152-1357 [22] and MS 1054-0321 [23], and several other merging clusters, viz. A754, A1750, A1914, A2034, A2142 [24], A2744 [25], A2163 [26], MACS J0025.4-1222 [27], and A1758 [24,28], etc.

5. Conclusion and Discussion

In this work, we have quantized the classical theory of General Relativity and contributing a very natural geometric way, we have wrote a fundamental theory of quantum gravity coupled to matter.

Present physics is unable to provide us a more acceptable scenario of Einstein field equation which is developed in a quantum spacetime. Moreover, it is commonly believed in contemporary physics that gravity is the bending of spacetime, but in GQG, Einstein field equations Eq. (44) and Eq. (61), and then GED and GCD interactions, assure us that, *gravity is the bending of spacetime intermediated by gravitons in its quantum gravity field, whose geometric part bends spacetime, whereas its quantum part interacts with the spacetime by exchanging gravitons*.

Three different aspects of quantum gravity in Subsection 2.1, Subsection 2.2 and Subsection 2.3, respectively, are developed in different spacetimes, viz. one Einstein field equation Eq. (44) is developed in a semi-quantum Minkowski spacetime, while, another Einstein field equation Eq. (61) is developed in a purely quantum Minkowski spacetime, whereas the last one is developed in a quantum Non-Minkowski spacetime. This is a remarkable achievement of GQG.

Contrary to Einsteinian's General Relativity, we can say that LHS of Eq. (44) or Eq. (61) is strongly depended upon different matter fields that exchange different types of vector bosons causing either positive pressure or negative pressure by bending spacetime.

Except this work, there has no evidence of simple bosonic and fermionic fields (i.e., neither supersymmetric nor stringy) that provides us Dark Energy and Dark Matter. GQG fields give us a gauge picture of Dark Matter through Eq. (66), which assists us in Section 4, where Dark Matter appears quite naturally in the same GQG fields for Dark Energy.

From GED, GCD and Dark Matter scenarios, we have a Universal Model as $SU(3)_{GED} \otimes SU(4)_{GCD} \otimes SU(5)_{DM} \subset SU(7)_{UM}$, where it is clear that Dark energy field is homogeneous, as well as non-decaying, in all kind of matter fields. So, it is also clear that whether the matter is baryonic or Dark, or their mixture, the effective universal relativistic cosmological constant Λ_{eff} at the surrounding is positive – that is why the Universe is expanding and accelerating, even at the present epoch. Dark Energy particles, that the fundamental interactions possessed in abundance due to the Universal Model $SU(3)_{GED} \otimes SU(4)_{GCD} \otimes SU(5)_{DM} \subset SU(7)_{UM}$, give an excellent fit to observations with the present day $\sim 68.3\%$ content of the Dark Energy (see the text immediate after Eq. (66) for baryonic and Dark Matter abundance), i.e., it is providing us the solution of the 'coincidence problem': *Why is the energy density of matter nearly equal to the Dark Energy density today?* So, from GQG, we have a suitable solution that why Dark Energy has become dominant after galaxy formation.

The overall scenarios of unification of gravity, Dark Energy, Dark Matter with fundamental interactions of particles or fields in GQG have left behind sufficient amount of calculations in Section 3 and Section 4 that give us a prospective opportunity for their future uses and they ensure us that graviton, Dark Energy and Dark Matter now become possible to be testable (directly or indirectly) at laboratory scales (i.e., in a standard particle collider) without regarding Planck scales.

In our present work, we have not chosen Superstring/M-theory, but it has come up quite naturally in GQG, whereas we have intentionally omitted the formalism of Loop Quantum Gravity since its construction looks quite artificial in comparison to Superstring/M-theory in GQG, though we can easily develop an effective formalism of Loop Quantum Gravity with the help of Subsection 2.3.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. K. Arun, C. Sivaram and A. Prasad, *Phys. Sci. Forum* **7**, 5 (2023). https://doi.org/10.3390/ECU2023-14018
2. J. Frieman and O. Lahav, in *The Dark Energy Survey : the story of a cosmological experiment*, ed. by O. Lahav et al. (World Scientific, Singapore, 2021), p. 5.
3. W. Greiner, *Quantum Mechanics: An Introduction* (Springer-Verlag, Berlin, 2001), p. 217.
4. J. Nagata, *Modern Dimension Theory* (P. Noordhoff N.V., Groningen, 1965), Chapter II and Chapter III.
5. G. Lambiase and G. Papini, *The Interaction of Spin with Gravity in Particle Physics: Low Energy Quantum Gravity* (Springer Nature, Switzerland, 2021), p.7.
6. Y. Choquet-Bruhat , *General Relativity and Einstein's Equations* (Oxford University Press , Oxford, 2009), p. 12.
7. J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addison Wesley, San Francisco, 2003), p. 171.
8. D. Radhakrishna Reddy and Y. Aditya, *Introduction to General Relativity, Cosmology and Alternative Theories of Gravitation* (Cambridge Scholars Publishing, Newcastle, 2022), p. 41.
9. S. Carlip, *General Relativity: A Concise Introduction* (Oxford University Press, Oxford, 2019), p. 5.
10. C. P. Burgess, in *Foundations of Space and Time: Reflections on Quantum Gravity*, ed. by J. Murugan, A. Weltman and G. F. R. Ellis (Cambridge University Press, Cambridge, 2012), p. 52 and p. 59.
11. S. Raychaudhuri and K. Sridhar, *Particle Physics of Brane Worlds and Extra Dimensions* (Cambridge University Press, Cambridge, 2016), p. 101.
12. L. Lombriser, *Phys. Lett.* **B797**, 134804 (2019). DOI: https://doi.org/10.1016/j.physletb.2019.134804
13. B. Chow, P. Lu and L. Ni, *Hamilton's Ricci Flow* (American Mathematical Society and Science Press, Rhode Island, 2010), pp. 2-6.
14. H. Larsson, *Aspects of M-theory: Supergravity, duality and noncommutativity*, Thesis for the degree of Doctor of Philosophy, Department of Theoretical Physics, Chalmers University of Technology and Göteborg University (Göteborg 2004).
15. R. Foot, *Int. J. Mod. Phys.* **D13**, 2161 (2004). DOI: https://doi.org/10.1142/S0218271804006449
16. Z. Berezhiani, *Int. J. Mod. Phys.* **A19** 3775 (2004). DOI: https://doi.org/10.1142/S0217751X04020075
17. R. Foot, *Int. J. Mod. Phys.* **A29**, 1430013 (2014). DOI: https://doi.org/10.1142/S0217751X14300130
18. B. Aubert et al., *Phys. Rev. Lett.* **93** 091802 (2004). DOI: https://doi.org/10.1103/PhysRevLett.93.091802
19. M. Ablikim et al., *Phys. Rev. Lett.* **97** 202002 (2006). DOI: https://doi.org/10.1103/PhysRevLett.97.202002
20. O. Tajima et al., *Phys. Rev. Lett.* **98** 132001 (2007). DOI: https://doi.org/10.1103/PhysRevLett.98.132001
21. D. Clowe et al., *Astrophys. J.* **648** L109 (2006). DOI: https://doi.org/10.1086/508162
22. M. J. Jee et al., *Astrophys. J.* **618** 46 (2005). DOI: https://doi.org/10.1086/425912
23. M. J. Jee et al., *Astrophys. J.* **634** 813 (2005). DOI: https://doi.org/10.1086/497001
24. N. Okabe, K. Umetsu, *Publ. Astron. Soc. Jpn.* **60** 345 (2008). DOI: https://doi.org/10.1093/pasj/60.2.345
25. J. Merten et al., *Mon. Not. Roy. Astron. Soc.* **417** 333 (2011). DOI: https://doi.org/10.1111/j.1365-2966.2011.19266.x
26. G. Soucail, *Astron. Astrophys.* **540** A61 (2012). DOI: https://doi.org/10.1051/0004-6361/201118283
27. M. Bradač et al., *Astrophys. J.* **687** 959 (2008). DOI: https://doi.org/10.1086/591246

Preprints (www.preprints.org) | NOT PEER-REVIEWED | Posted: 17 October 2023 doi:10.20944/preprints202306.1472.v3

28. B. Ragozzine et al., *Astrophys. J.* **744** 94 (2012). DOI: https://doi.org/10.1088/0004-637X/744/2/94